Fourier-Mukai transform for formal schemes

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Objectives

In 1981, Mukai constructed the Fourier-Mukai transform for abelian varieties over an algebraically closed field (4), which gives an equivalence of categories between quasi-coherent sheaves over A and the ones over $A^\vee$, its dual variety. Laumon generalized these results for abelian varieties over a locally noetherian basis (2).

One can then ask the following question: can these results be generalized even more? What about formal abelian varieties? And abelian rigid analytic varieties?

The generalization of the Fourier-Mukai transform's construction is based on a simple idea: make the classical construction commute with (derived) inverse limit. Even if the idea seems simple, it implies to clearly understand quasi-coherent sheaves and functors defined over formal varieties.

When the formal Fourier-Mukai transform is constructed and its fundamental results has been proved, one can then obtain these results over its generic fiber.

Classical Fourier-Mukai transform

Given an abelian variety A over S locally noetherian, one can define the following functor from the category of S-schemes to the one of abelian groups:

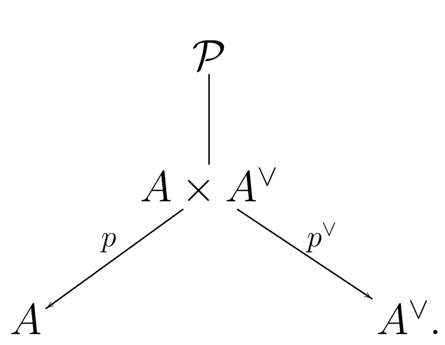
$$Pic^0(A \times \bullet/\bullet) : T \mapsto Pic^0(A \times T/T),$$

where $Pic^0(X/T)$ is the abelian group of invertible $\mathcal{O}_X$-modules $\mathcal{L}$ such that

$$m^*\mathcal{L} \simeq p_1^*\mathcal{L} \otimes p_2^*\mathcal{L},$$

with $m, p_1, p_2 : X \times_T X \to X$ the multiplication and the canonical projections.

The functor $Pic^0(A \times \bullet/\bullet)$ is representable by an abelian variety $A^\vee$, called the dual variety of A. The representability of this functor also gives a universal element $\mathcal{P} \in Pic^0(A \times A^\vee/A^\vee)$, called the Poincaré sheaf.



Using these notations, the Fourier-Mukai transform is the functor $\mathcal{F} : D^b_{qcoh}(\mathcal{O}_A) \to D^b_{qcoh}(\mathcal{O}_{A^\vee})$ defined by

$$\mathcal{F}(\mathcal{E}^\cdot) = Rp^\vee(\mathcal{P} \overset{\mathbb{L}}{\otimes} p^*\mathcal{E}^\cdot).$$

The dual Fourier-Mukai transform $\mathcal{F}^\vee : D^b_{qcoh}(\mathcal{O}_{A^\vee}) \to D^b_{qcoh}(\mathcal{O}_A)$ is defined in the same way.

The three following properties of Fourier-Mukai transform are the most important ones, and the ones we want to preserve when generalizing the construction:

Theorem (Mukai)

1. Preserves coherence: If $\mathcal{E}^\cdot \in D^b_{coh}(\mathcal{O}_A)$, then $\mathcal{F}(\mathcal{E}^\cdot) \in D^b_{coh}(\mathcal{O}_{A^\vee})$.
2. Equivalence of categories: $\mathcal{F} : D^b_{qcoh}(\mathcal{O}_A) \to D^b_{qcoh}(\mathcal{O}_{A^\vee})$ is an equivalence of categories.
3. Involutivity: $\mathcal{F}^\vee \circ \mathcal{F} \simeq \langle -1\rangle^* \bullet [-d]$, with $\langle -1\rangle : A \to A$ the inverse morphism and $d = dim(A)$.

First approach

Let V be a discrete valuation ring, π its uniformizer, $V_i = V/_{\pi^i V}$ and $S_i = Spec(V_i)$. Given (A_i) a direct system of abelian varieties over the schemes S_i, one can obtain a formal abelian variety $\mathcal{A} = \varinjlim A_i$ over $\mathcal{S} = Spf(V)$.

The dual abelian variety of $\mathcal{A}$ is then $\mathcal{A}^\vee = \varinjlim A_i^\vee$ and the Poincaré sheaf associated is $\mathcal{P} = \varprojlim \mathcal{P}_i$, where $\mathcal{P}_i$ is the Poincaré sheaf over $A_i \times A_i^\vee$.

The only thing missing to extend the construction of the Fourier-Mukai is a 'good notion' of quasi-coherent sheaves over $\mathcal{A}$, and the inverse and direct image functors associated.

Formal quasi-coherent sheaves

Let $\mathcal{E}^\cdot \in D^b(\mathcal{O}_\mathcal{A})$ be a bounded complex of $\mathcal{O}_\mathcal{A}$-module and $\mathcal{E}_i^\cdot = \mathcal{O}_{A_i} \overset{\mathbb{L}}{\otimes}_\mathcal{A} \mathcal{E}^\cdot$.

$\mathcal{E}^\cdot$ is said quasi-coherent (or more precisely, $\mathcal{E}^\cdot \in D^b_{qcoh}(\mathcal{O}_\mathcal{A})$) if $\mathcal{E}_0^\cdot \in D^b_{qcoh}(\mathcal{O}_{A_0})$ and $\mathcal{E}^\cdot \simeq R\varprojlim \mathcal{E}_i^\cdot$.

We also have the following characterization proved by Berthelot (1):

$$\mathcal{E}^\cdot \in D^b_{qcoh}(\mathcal{O}_\mathcal{A}) \Leftrightarrow \mathcal{E}_0^\cdot \in D^b_{qcoh}(\mathcal{O}_{A_0}) \text{ and } \forall i, \mathcal{O}_{A_i} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{A_{i+1}}} \mathcal{E}_{i+1}^\cdot \simeq \mathcal{E}_i^\cdot.$$

Remark that the definition of formal quasi-coherent sheaves is more rigid than the classical one, hence, it isn't clear that the usual direct and inverse images preserve quasi-coherence of formal sheaves. Actually, there are examples of quasi-coherent sheaves $\mathcal{E}$ and (flat) morphisms $f : \mathcal{X} \to \mathcal{Y}$ such that $\mathcal{O}_\mathcal{X} \otimes_{f^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{E}$ isn't quasi-coherent, so this cannot be a good definition of inverse image.

The idea is to define the (derived) inverse image in the following way:

$$f^!\mathcal{E}^\cdot = R\varprojlim Lf_i^*\mathcal{E}_i^\cdot,$$

where $f_i : X_i \to Y_i$ is the reduction modulo π^i of $f : \mathcal{X} \to \mathcal{Y}$. With this definition, the inverse image of a quasi-coherent sheaf is a quasi-coherent sheaf.

For the direct image however, there isn't any problem.

Formal Fourier-Mukai transform

With the previous definitions, one can easily generalize the definition of Fourier-Mukai transform in the following way:

$$\forall \mathcal{E}^\cdot \in D^b_{qcoh}(\mathcal{O}_\mathcal{A}), \mathcal{F}(\mathcal{E}^\cdot) = Rp_*^\vee(\mathcal{P} \overset{\mathbb{L}}{\otimes} p^!\mathcal{E}^\cdot).$$

Given the particular form of the definition of $p^!$, one obtain the following result:

$$\mathcal{F}(\mathcal{E}^\cdot) \simeq R\varprojlim \mathcal{F}_i(\mathcal{E}_i^\cdot),$$

where $\mathcal{F}_i$ is the Fourier-Mukai transform over A_i. In other words, Fourier-Mukai transform commutes with inverse limit.

Thanks to this result and Berthelot's characterization of formal quasi-coherent sheaves, the main properties of Fourier-Makai transforms can be proven in the formal case:

Theorem (V.)

$\mathcal{F}$ is an equivalence of categories between $D^b_{qcoh}(\mathcal{O}_\mathcal{A})$ and $D^b_{qcoh}(\mathcal{O}_{\mathcal{A}^\vee})$ that preserves coherence and which is involutive.

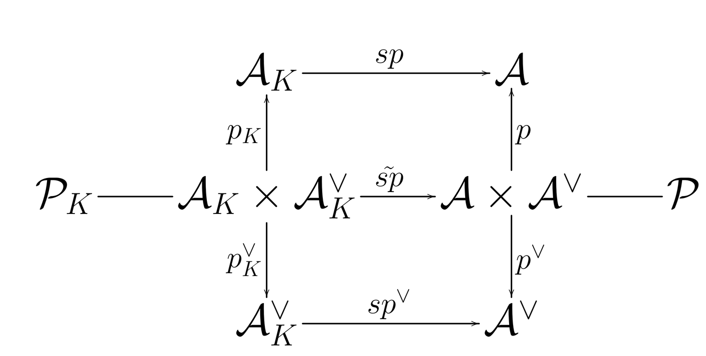
The rigid analytic case

Let K be the fraction field of V, $\mathcal{A}_K$ be the generic fiber of the formal abelian variety $\mathcal{A}$ and $sp : \mathcal{A}_K \to \mathcal{A}$ be the specialization morphism.

Note that sp induces a group scheme structure over $\mathcal{A}_K$, coming from the one over $\mathcal{A}$.

We will define Fourier-Mukai transform for coherent sheaves over $\mathcal{A}_K$. This is the first step of a potential construction of Fourier-Mukai transform over an abeloid rigid analytic variety.

As explained in (3), the dual variety of $\mathcal{A}_K$ is isomorphic to the generic fiber of $\mathcal{A}^\vee$. It will be denoted by $\mathcal{K}^\vee$ in the following. Moreover, if $sp^\vee : \mathcal{A}_K^\vee \to \mathcal{A}^\vee$ is the specialization morphism of $\mathcal{A}^\vee$, then $\tilde{sp} = sp \times sp^\vee$ is the one of $\mathcal{A} \times \mathcal{A}^\vee$ and the Poincaré sheaf over $\mathcal{A}_K \times \mathcal{A}_K^\vee$ is $\mathcal{P}_K \simeq \tilde{sp}^*\mathcal{P}$. By construction, the following diagram is commutative:



Rigid analytic Fourier-Mukai transform

One can define Fourier-Mukai transform over $\mathcal{A}_K$ in the following way: given $\mathcal{E}^\cdot \in D^b_{coh}(\mathcal{O}_{\mathcal{A}_K})$, the Fourier-Mukai transform of $\mathcal{E}^\cdot$ is

$$\mathcal{F}_K(\mathcal{E}^\cdot) = Rp_{K*}^\vee(\mathcal{P}_K \otimes p_K^*\mathcal{E}^\cdot).$$

Then, we have the following isomorphism of functors from $D^b_{coh}(\mathcal{O}_{\mathcal{A}_K})$ to $D^b_{coh}(\mathcal{O}_{\mathcal{A}^\vee,\mathbb{Q}})$:

$$sp_*^\vee \circ \mathcal{F}_K \simeq \mathcal{F} \circ sp_*.$$

In other words, Fourier-Mukai transform commutes with the specialization.

With this result, it is possible to obtain the mains properties of Fourier-Mukai transforms in the rigid analytic case.

Theorem (V.)

$\mathcal{F}_K$ is an equivalence of categories between $D^b_{coh}(\mathcal{O}_{\mathcal{A}_K})$ and $D^b_{coh}(\mathcal{O}_{\mathcal{A}_K^\vee})$. Moreover, it is involutive.

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