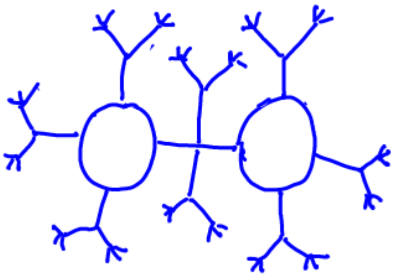
# Non-Archimedean Uniformization & Tropicalization: Teichmüller space & $\mathcal{M}_g$

arXiv: 2004.07508

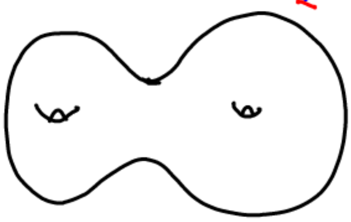
## A. Motivation

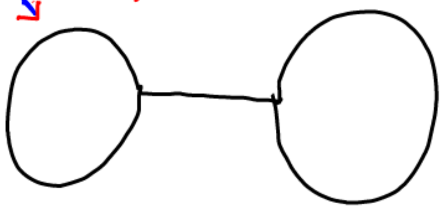


Smooth proj. curves over $\mathbb{C}((t))$

is a family of

is a skeleton of





Compact Riemann surfaces $X$

$=$

Smooth proj. curves over $\mathbb{C}$

$g = \frac{1}{2} b_1(X)$

$=$ "number of holes"

Metric graphs $\Gamma$

$\subseteq$

tropical curves $(\Gamma, h: \Gamma \to \mathbb{Z}_{\geq 0})$ with $\# \mathrm{supp}(h) < \infty$

$g(\Gamma) = b_1(\Gamma) + \sum_{p \in \Gamma} h(p)$

### Moduli spaces

$g \geq 2$

Berkovich analytification

$\mathcal{M}_{g,n}^{an}$

is a skeleton of [Abramovich-Caporaso-Payne '15]

$\mathcal{M}_g$

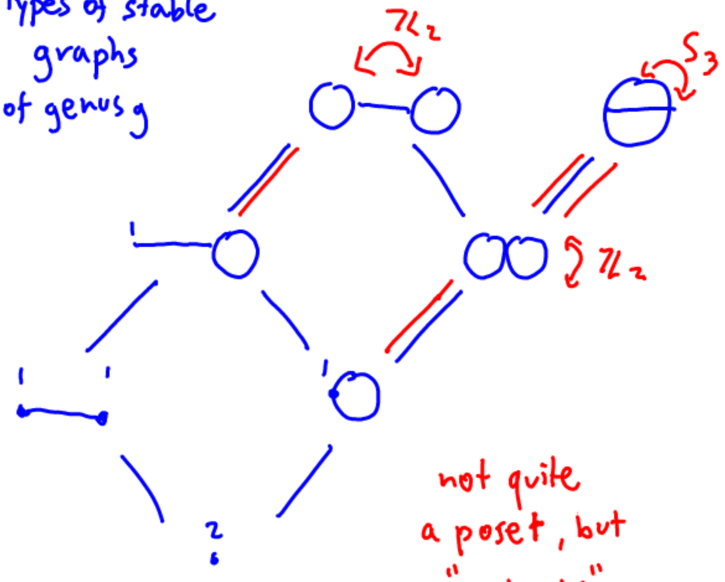
moduli space of smooth algebraic curves of genus $g$

$\mathcal{M}_g^{trop}$

moduli space of stable tropical curves of genus $g$

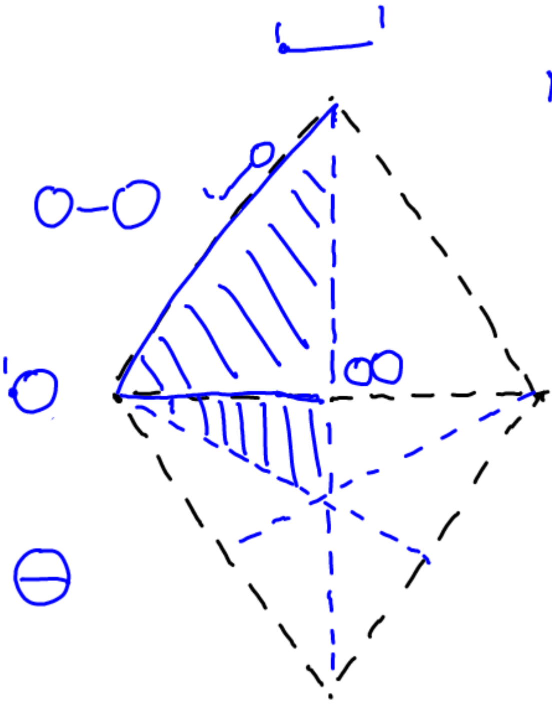
Example : $\mathcal{M}_2^{trop}$

Types of stable graphs of genus g




not quite a poset, but a "postack"

↳ Cavalieri - Chan - U - Wise '20



$M_2^{trop}$

## Uniformization (à la Mumford)

$K$ = non-Arch field

$X$ = smooth proj. curve $/K$

$\Gamma$ = skeleton of $X^{an}$

$G = \pi_1(X^{an}) = \pi_1(\Gamma) = F_b$ (where $b = b_1(\Gamma) = b_1(X^{an})$)

$$\begin{tikzcd} \mathbb{P}^{1,an} \supseteq \Omega \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{``}\square\text{''}"] & \tilde{\Gamma} \arrow[d] \\ X^{an} \arrow[r, "\rho"'] & \Gamma \end{tikzcd}$$

universal cover of $\Gamma$ $\rightsquigarrow$ $\tilde{\Gamma}/G = \Gamma$

$\Omega/G = X^{an}$

not really in the same category

$\rho$: retraction to the skeleton

## Uniformization of moduli spaces

Gerritzen-Herrlich $p$-adic Teichmüller space '80s $\not\cong$

[U'20]: $\mathcal{T}_g^{an} := \left\{ (X, \varphi) \;\middle|\; \begin{array}{l} X \in \mathcal{M}_g^{an} \\ \varphi: \pi_1(X^{an}) \xrightarrow{\sim} F_b \end{array} \right\}$

$\mathbb{C}$-analytic not algebraic

?? ??

is skeleton of

Teichmüller space

$$\mathcal{T}_g := \left\{ (X, \varphi) \;\middle|\; \begin{array}{l} X \in \mathcal{M}_g \\ \varphi: \pi_1(X) \xrightarrow{\sim} \Pi_g \end{array} \right\}$$

Tropical Teichmüller space

$$\mathcal{T}_g^{trop} := \left\{ (\Gamma, \varphi) \;\middle|\; \begin{array}{l} \Gamma \in \mathcal{M}_g^{trop} \\ \varphi: \pi_1(\Gamma) \xrightarrow{\sim} F_b \end{array} \right\}$$

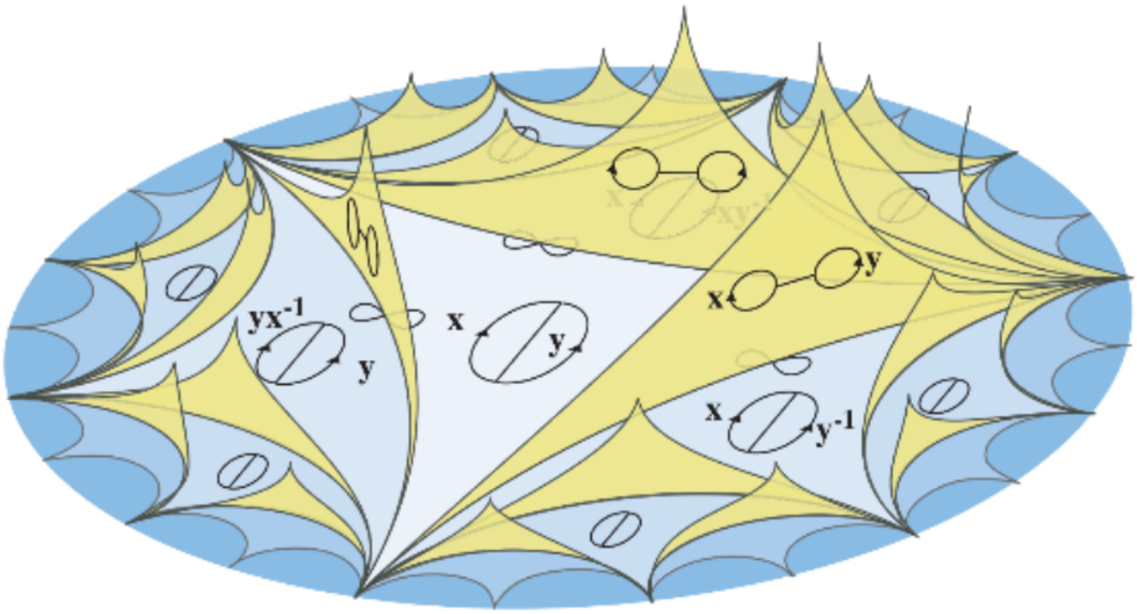
where $b = b_1(\Gamma) = g(\Gamma) - \sum_{p \in \Gamma} h(p)$

$\Pi_g := \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$

$\cup\!\!|$

Culler-Vogtmann Outer Space '86

Example: $\mathcal{J}_2^{trop}$




Picture by K. Vogtmann '08

## B, Main Result

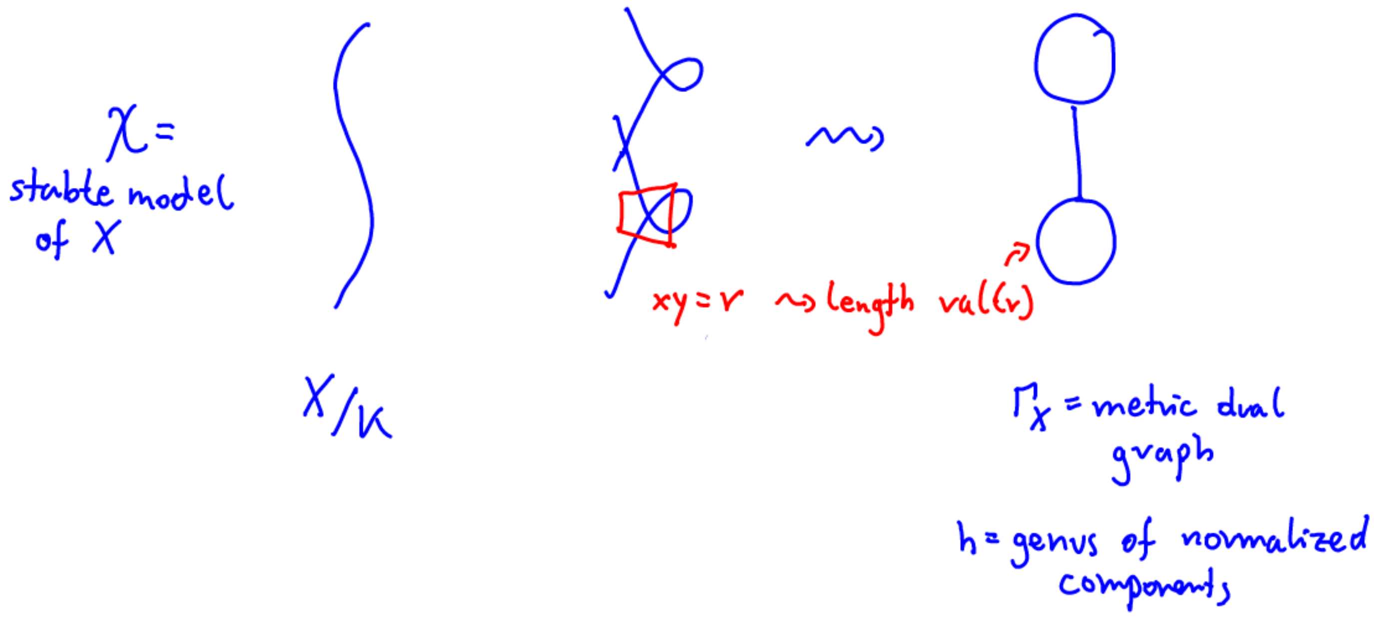
**Thm** ( U'20)

Let $K = \bar{K}$ a non-Arch. field & $g \geq 2$.

$\mathcal{J}_g^{an}$ is a smooth separated $K$-analytic Deligne-Mumford stack without boundary. The natural tropicalization map

$$\begin{aligned} \mathrm{trop}_g : \mathcal{J}_g^{an} &\longrightarrow \mathcal{X}_g^{trop} \\ (X, \varphi) &\longmapsto \left[ \begin{array}{l} \Gamma_X = \text{skeleton of } X^{an},\ h : \Gamma_X \longrightarrow \mathbb{Z}_{\geq 0} \\ \qquad \varphi^{trop} := \left( \pi_1(\Gamma_X) \xrightarrow{\sim} \pi_1(X^{an}) \xrightarrow[\varphi]{\sim} F_b \right) \end{array} \right] \end{aligned}$$

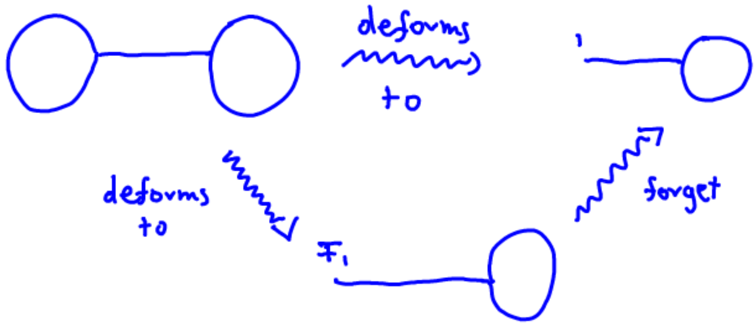
has a continuous section that makes $\mathcal{X}_g^{trop}$ into a strong deformation retract of $\mathcal{J}_g^{an}$.




**Cor.:** Culler-Vogtmann Outer Space is a strong deformation retract of the locus of Mumford curves in $\mathcal{J}_g$

**Strategy:** 1) Construct $\widetilde{J}_g^{trop}$ as a "postack"/cone stack

↳ What is $\pi_1(\Gamma, h)$ ? (∗)




metric graph of groups [Bass-Serre]

There are other answers to (∗) !

$\Rightarrow \pi_1(\Gamma) = F_g$

$g = b_1(\Gamma) + \sum_p h(p)$ genus of $(\Gamma, h)$

2) Lift $\widetilde{J}_g^{trop}$ to an algebraic stack $a^*\widetilde{J}_g^{trop}$ with log structure

↳ Lift the "postack" diagram

$$(G,h) \text{ stable graph} \longmapsto \left[ \mathbb{A}^{E(G)} / \mathbb{G}_m^{E(G)} \right]$$

↳ Take 2-colimit

3) Take fiber product

$$\begin{tikzcd} J_g^{log} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & a^*\widetilde{J}_g^{trop} \arrow[d] \\ M_g^{log} \arrow[r] & a^* M_g^{trop} \end{tikzcd}$$

log-smooth DM-stack loc. of finite type/$\mathbb{Z}$ & univ. closed
not separated !

in the same category !

4) Apply Raynaud's generic fiber functor

$$J_g^{an} \simeq \widetilde{J}_g^{an} := \left(\widetilde{J}_g^{log}\right)^{\wedge}_{K^\circ, \eta} \qquad \boxed{/}$$