Analytic spaces

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k non-archim. analytic (complete, $|\,| : k \to \mathbb{R}_{\geq 0}$) fld

$k = k^a$ alg. cl., k° = integers, $\tilde{k} = k^\circ / m_{k^\circ}$ res. fld.

Berkovich k-analytic spaces, $X = \mathcal{M}(A_i)$,

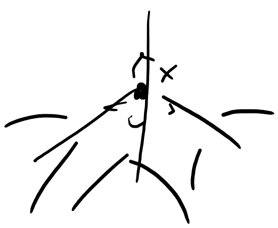
$A_i = k\{t_1 .. t_n\}/I$

Spec $\sim$ $\{\, |\ |_x : A \to \mathbb{R}_{\geq 0}$ semival. $\}$.

pts $\sim$ $\{ A \to \mathcal{H}(x)$ fld, k-hom. $\}/\sim$

k-analyt. fld

Always work with k-curves top. ind. graphs, like Bruhat-Tits 1-dim bld.



vertices pts of type 2

$\widetilde{\mathcal{H}(x)} / \tilde{k}$ is a funct. field of a curve C_x $Br(x) = C_x$.

Tropical reduction

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Geom. $/k$ → Geom $/\tilde{k}$ (reduction)

Trop

→ Geom $/|k^\times|$ (tropicaliz.)

polyhedral

?

Tropical red.

The example: $\mathfrak{X}$ a s.s. model of X, $\Gamma_{\mathfrak{X}}$ incid. graph of $\mathfrak{X}_s$ ← me

$\Gamma_{\mathfrak{X}} \hookrightarrow \mathfrak{X}$

X

tropical

$\Gamma_{\mathfrak{X}}$ — $(\Gamma_{\mathfrak{X}} + PL)$

$\mathfrak{X}_s$ — $\mathfrak{X}_s + \mathcal{M}_{\mathfrak{X}_s}$ ← vert. log str.

diff. form

$p = 0$!

(X, φ) : $X \xrightarrow{f} Y$

$p > 0$

$p = \mathrm{char}\,\tilde{k}$ cover

Curve complexes

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Amini-Baker $\quad X$ curve (nice), $\underset{\text{div.}}{D \subset X}$ $\{p_1 \dots p_n\}$

$(\mathcal{X}, D)$ a s.s. model $\quad D_s = D \cap (\mathcal{X}_s)_{sm}$.

$\Gamma_\mathcal{X}$ finite edges $\overrightarrow{uv} = \vec{e}$, u,v - gen. pts on $\mathcal{X}_s$

node

length $= -\log|\pi|$, node is $\{xy = \pi\}$ (form. loc.)

$\pi \in m_{k^\circ}$; inf. edges (legs) $\overrightarrow{u,p}$ $\quad u$ gen. pt.

$p \in C_u$

$\Gamma_\mathcal{X} \hookrightarrow X$ $\quad$ norm. comp. of $\mathcal{X}_s$ $\quad \hat{D}$

u a type 2 pt $\quad \mathcal{B}r(u) = C_u$

$\underbrace{(\Gamma_\mathcal{X}, \{C_u\})}_{\forall u}$ $\quad e = \overrightarrow{uv} \mapsto \underline{\underline{p_u \in C_u}}$

Rem: $X \setminus \underline{\underline{V(\Gamma_\mathcal{X})}} = \sqcup$ open discs, annuli, punct. discs

$X \xrightarrow{\pi} \mathcal{X}_s$ $\quad$ smooth $\quad$ nodes $\quad$ marked pts

Functions

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X, f merom funct. on X $\quad (X \xrightarrow{f} \mathbb{P}^1)$

$D \supset$ zeroes & poles of f. $(\mathcal{X}, D)$ a s.s. model of (X, D).

$f \mapsto |f| : X \to \mathbb{P}$ $\qquad |f|(x) = |f|_x$

$\forall x$ of type 2 $\quad c_x \in k \quad |c_x| = |f|_x^{-1}$, $\quad \underline{|c_x f_x| = 1}$:

$0 \neq \widetilde{c_x f_y} = \tilde{f}_x$ it is a merom. funct. on C_x.

$\widetilde{\mathcal{H}(x)}$

$(\mathcal{X}, D)$ $\qquad$ metr. curve complex

$(X, f) \longmapsto (\Gamma_{\mathcal{X}}, \{C_u\}, |f||_{\Gamma_{\mathcal{X}}}, \tilde{f}_u)$

PL $-\log|f|$

$\forall \; \overset{\vec{e}}{\underset{u \quad\quad v}{\bullet\!\!\longrightarrow\!\!\bullet}}$ $\qquad \underline{\text{slope}_{\vec{e}} \log|f| = \text{or}} \; \tilde{f}_u \implies \text{ord}_e \tilde{f}_u =$

$\text{ord}_e \qquad\qquad = -\text{ord}_{e^{op}} \tilde{f}_v$

$\overset{e}{\underset{u \quad\quad p}{\longrightarrow}}$ $\qquad \text{ord}_p f = \text{ord}_p \tilde{f}_u$.

Diff forms

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$(X, \underline{\Phi})$, $D \ni$ zeroes, poles of Φ, $(\mathcal{X}_1, D_1)$

$\updownarrow$

$(\Gamma_{\mathcal{X}}, \{C_v\})$

Prop: (i) $\exists!$ maximal metric on Ω_X s.t.

$\mathcal{O}_X \xrightarrow{d} \Omega_X$ is non-expand., in fact, the the unit ball $\Omega^\circ_X = \mathrm{Sheafif.}(\mathcal{O}^\circ_X \, d\, \mathcal{O}^\circ_X)$ $\leftarrow$ funct. $|\,| \leq 1$

forms ≤ 1

(ii) The reduction $\Omega^\circ / m_{k^\circ} \Omega^\circ$ restricts to $\underline{\Omega^{\log}_{C_x}}$ on C_x for each x of type 2.

$\Omega_{C_x}(\sum_{P \in C_x} P)$

Rem: $\|\Phi\| : X \to \mathbb{R}_{\geq 0}$ not contin. but on each $\Gamma \subset X$ fin graph PL

$(X, \Phi) \rightsquigarrow (\Gamma_{\mathcal{X}}, \{C_v\}, \|\Phi\||_{\Gamma_{\mathcal{X}}}, \widetilde{\Phi}_v)$

PL

merom. forms on C_v

$\mathrm{slope}_e(\log\|\Phi\|) = -\mathrm{logord}_e \widetilde{\Phi}_v$ compat. for legs.

[illegible] compat. for legs.

$e = \overrightarrow{uv}$ $\qquad -\mathrm{ord}+1 \qquad \Rightarrow$

$$\mathrm{ord}_e \widetilde{\varphi}_u + \mathrm{ord}_{e^{op}} \widetilde{\varphi}_v = -2$$

Residue

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$A = \mathcal{M}(k\{k\{r_1 t^{-1}, r_2^{-1} t\}\})$ $r_1 \le |t| \le r_2$

$\varphi = \sum a_i t^i \frac{dt}{t_u}$ φ has no zeros/poles on A

$\exists u$ $\|\varphi\| = |a_u t^u|$ on the skeleton. $a_0 = \mathrm{Res}_A \varphi$

is indep. of coord.

Res_φ: edges of $\Gamma_X \to k$ it is harmonic

$\sum \mathrm{Res}_e \varphi = 0$ $\forall x$ type 2

$\underset{u}{\bullet}\xrightarrow{\bar{e}}\underset{p}{}$ $\mathrm{Res}^{Br(x)}$

$_\varphi(e) = \mathrm{Res}_p(\varphi)$

$\widetilde{\varphi_u} = \widetilde{c_u \varphi}$

Ω_{c_u}

$\widetilde{\mathrm{Res}_\varphi(e)} \cdot c_u = \mathrm{Res}_e \widetilde{\varphi}_u$

Th: if char $\tilde{k} = 0$ $\Rightarrow$ any compat.

$(\Gamma_X, \{C_u\}, \|\varphi\||_{\Gamma_X}, \widetilde{\varphi}_u, \mathrm{Res}_\varphi)$ lifts to

a (X, φ) $/k$.

Covers

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$X \xrightarrow{f} Y$ finite, gen. etale

Th: $\exists$ $\mathrm{Ram}(f) \subset \Gamma_X = f^{-1}(\Gamma_Y)$.

$u \in \Gamma_X \quad v \in \Gamma_Y$

$(\Gamma_f : \Gamma_X \to \Gamma_Y, \tilde{f}_u)$ is a nat.

$C_u \xrightarrow{\tilde{f}_u} C_v$

trop. red./datum for f.

simple compat. (degrees).

Th: [ABBR]: if $(\Gamma_f, \{\tilde{f}_u\})$ tame (i.e. all $\tilde{f}_u$ are separable) then $\exists$ a lifting.

Def: $\delta_f : X \to [0,1]$ $\quad \delta_f(x) = \delta_{\mathcal{H}(x)/\mathcal{H}(y)}$

PL on finite graphs. $\quad$ wild

Prop: $\delta_f = \|\tau\|$ for $\tau \in \Gamma(\omega_f) = (f^*\Omega_Y)^{-1} \otimes \Omega_X$

$f^*\Omega_Y \to \Omega_X$

naturally metrized

$y = f(x)$ $\quad (f^*\Omega_Y^{\circ})^{-1} \otimes \Omega_X^{\circ}$

Proof ... log

Prop: $\tilde{\omega}_{f,x} = \omega^{\log}_{\tilde{f}_x} = (f^*\Omega_{C_y})^{-1} \otimes \Omega_{C_x}$

$\cup$

$\tilde{\tau}_u$ for $u \in \Gamma(\mathfrak{X})$

Th: assume $(\Gamma_f, \tilde{f}_u, \delta_f|_{\Gamma_f}, \tilde{\tau}_u)$ compat.

and it is minimally wild, i.e. $\forall v \in \Gamma_y$ $\exists$ at most one $x \to y$ with insep. $\tilde{f}_x$ and then $\deg(C_x/C_y) = p$

Then $\exists$ a lift $(f: X \to Y)$.

Proofs

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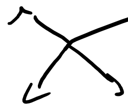
lift stars  $\in T_*$

Gluing is done by classific. / annuli.

<u>Lemma</u>

φ on A is binomial $\varphi = a_0 \frac{dt}{t} + a_n t^n \frac{dt}{t}$

for some t on A $\quad \operatorname{Res}_\Delta \varphi \quad \|\varphi\|, n, |a_n|$.

deg = p cover $A_x \xrightarrow{f} A_y$ is binomial

$t_y = t_x^p + a_n t_x^n \quad$ n, $|a_n|$ given by $f|_A$.