$\mathcal{D}$-modules and irreducibility results for locally analytic representations

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(jt. work with K. Ardakov)

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$F/\mathbb{Q}_p$ finite ext , $F \subseteq K$ complete

$\mathbb{G}_F$ connected reductive$/_F$ s.t. $\mathbb{G} = \mathbb{G}_F \times_F K$ split

$$\begin{array}{l} \mathbb{G} \\ \cup \\ \mathbb{B} \quad \mathfrak{b} = \mathfrak{t} + \mathfrak{n} \end{array}$$

$G := \mathbb{G}_F(F)$, $D(G,K)$ locally analytic distributions

§1. <u>The parabolic BGG category</u>

$P \subseteq G$, $\mathfrak{p} \subseteq \mathfrak{g}$ Lie algebras$/_K$ (!)

$\mathcal{O}$ BGG category $\subseteq \{\mathfrak{g}\text{-modules}\}$

* f.g. over $\mathcal{U}(\mathfrak{g})$
* $\mathfrak{t}$-semisimple
* locally $\mathfrak{n}$-finite ($\mathcal{U}(\mathfrak{n})v$ fin. dim$/K$ $\forall v$)

Ex: • $\left\{ \begin{array}{c} \text{fin. dim}/K \\ \mathfrak{g}\text{-modules} \end{array} \right\} \subseteq \mathcal{O}$

• $\lambda : \mathfrak{t} \to K$, $M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}), \lambda} K \in \mathcal{O}$

"Vermamodule"

$\mathcal{O}^{\mathfrak{p}} \subseteq \mathcal{O}$ full subcategory of $M \in \mathcal{O}$ s.t. $\mathfrak{p}$ acts locally finite on M

Ex: $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_{\mathfrak{p}}$, $\mathfrak{l} \to \mathrm{End}_K(V)$ fin. dim. simple

$$M(V) := \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{p})}{\otimes} V \in \mathcal{O}^{\mathfrak{p}},$$

parabolic Verma module

$$\mathfrak{m}_0 := \mathcal{Z}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})\mathfrak{g} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$$

<u>Thm</u> (Orlik-Strauch, 2014): $\exists$ exact functor

$$\mathcal{F}'^{G}_{P} : \mathcal{O}_0^{\mathfrak{p}} \longrightarrow \mathcal{C}_{D(G,K)_0}$$ with the property:

$M \in \mathcal{O}_0^{\mathfrak{p}}$ irred. & $\mathfrak{p}$ "max" for M

$\Longrightarrow$ $\mathcal{F}'^{G}_{P}(M)$ irreducible.

Ex: $\mathcal{F}^{G}_{P}(M(V)) = \mathrm{Ind}^{G}_{P}(V')$ parabolic induction

<u>Aim</u>: Reinterpret $\mathcal{F}'^{G}_{P}$ geometrically

Draw some consequences.

§2. Equivariant D-modules on the flag variety

$X = (G/B)^{an}$, D_X alg. diff operators

$G \curvearrowright$

G

$\mathcal{C}_{X/P}$ = coadmissible P-equiv. D_X-modules

I. Equivariant localization theorem
(Huyghe - Sch. - Strauch , Ardakov)

$$\Gamma(X,-) : \mathcal{C}_{X/G} \xrightarrow{\cong} \mathcal{C}_{D(G,K)_0} .$$

II. Induction equivalence (Ardakov), $P \leq G$ closed

$\exists$ functor $\mathrm{ind}_P^G : \mathcal{C}_{X|P} \longrightarrow \mathcal{C}_{X|G}$ "adjoint"

to res_P^G s.t.

$Y \leq X$ closed, irreducible s.t. $(*)$ $\begin{cases} G_Y := \mathrm{Stab}_G(Y) \text{ cocompact} \\ {}^gY \cap Y \neq \emptyset \Rightarrow {}^gY = Y \ \forall g \end{cases}$

then

$$\mathcal{C}^{Y}_{X|G_Y} \xrightarrow[\mathrm{ind}_{G_Y}^G]{\cong} \mathcal{C}^{G\cdot Y}_{X|G} .$$

Ex: W Weyl gp of $\mathbb{G}$, $w \in W$

simple roots of G

$$X_w := \left(\overline{\mathbb{B} w \mathbb{B} / \mathbb{B}} \right)^{an}$$ analytic Schubert variety

$$W_I = \langle s_\alpha \mid \alpha \in I \rangle \subset W \qquad I \subset \Delta = \{\alpha_1, \dots, \alpha_\ell\}$$

$w_{0,I} \in W_I$ longest element

$Y := X_{w_{0,I}}$ sat. $(*)$

$$GY = \mathbb{B} W_I \mathbb{B}(K) \cap G$$

§3. Main results

: X analytic flag variety of G

$\widehat{D}_X$ infinite order diff. op. on X $\supseteq D_X$

locally $\mathcal{U} \subset X$ $\quad \partial_1, \dots, \partial_n$

$x_1, \dots, x_n$

$$\widehat{D}_X(\mathcal{U}) = \Big\{ \sum_{\underline{\alpha}} f_{\underline{\alpha}} \underline{\partial}^{\underline{\alpha}} : \|f_{\underline{\alpha}}\| r^{|\underline{\alpha}|} \xrightarrow[|\alpha| \to \infty]{} 0 \quad \forall\, r > 0 \Big\}.$$

$E(\mathcal{M}) := \widehat{D}_X \otimes_{D_X} \mathcal{M}$ for $\mathcal{M} \in D_X$-mod.

$\mathrm{Loc}(M) := D_X \otimes_{\mathcal{U}(\mathfrak{g})} M$ classical localization $M \in \mathcal{U}(\mathfrak{g})_0$-mod

(Beilinson–Bernstein) 1981, over $\mathbb{C}$

$$\mathrm{Coh}(D_X) \xleftarrow{\cong} \mathcal{U}(\mathfrak{g})_0\text{-mod}_{fg}$$

<u>Thm A</u> (in progress): $w \in W$, $P := G_{\chi_w}$

Then $\mathrm{ind}_P^G(E \circ \mathrm{Loc}(L_w)) \in \mathcal{C}_{X/G}$ is simple.

Here: $\mathrm{Irr}(\mathcal{O}_0) = \{L_w\}_{w \in W}$ where

$$M(w) := M(-w(\rho)-\rho) \underset{\text{unique simple quotient}}{\twoheadrightarrow} L(-w(\rho)-\rho) = L_w .$$

$$\rho = \frac{1}{2}\sum_{\alpha \in \phi^+} \alpha$$

(Thm. is o.k. for $G = GL_2, GL_3$ or

$w = w_{0,I}$ ~~from~~ longest element in $W_{\underline{I}}$)

<u>Thm B</u>: The diagram of functors

$$\begin{tikzcd} \mathcal{O}_0^{\mathfrak{p}} \arrow[r, "\mathcal{F}_P^{\prime G}"] \arrow[d, "\mathcal{F}\circ \mathrm{Loc}"'] & \mathcal{C}_{D(G,K)_0} \\ \mathcal{C}_{X|P} \arrow[r, "\mathrm{ind}_P^G"] & \mathcal{C}_{X|G} \arrow[u, "\cong"'] \end{tikzcd}$$

is commutative.

<u>Rmk</u>: Thm. A + Thm B (re)prove irreducibility of $\mathcal{F}^{\prime G}_{G_{X_w}}(L_w) \in \mathcal{C}_{D(G,K)}$ geometrically (Orlik-Strauch 2014 for G_F split).