Hodge Theory of p-adic analytic varieties

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Based on a joint work with Pierre Colmez

Algebraic comparison theorem

Notation: K - cdvf, char $(0,p)$, $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$, $C = \widehat{\overline{K}}$,
$K \supset \mathscr{O}_K \to k$, k – perfect, $F = W(k)$.

Theorem (Algebraic comparison theorem) X/K – algebraic variety. There exists a natural $\mathbf{B}_{\mathrm{st}}$-linear, $\mathscr{G}_K$-equivariant period isomorphism $(r \geq 0)$

$$\begin{aligned} \alpha_{pst}: \quad & H^r_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}} \simeq H^r_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{F^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}, \quad (\varphi, N, \mathscr{G}_K), \\ \alpha_{\mathrm{dR}}: \quad & H^r_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} \simeq H^r_{\mathrm{dR}}(X_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}_{\mathrm{dR}}, \quad Fil, \end{aligned}$$

where $\alpha_{\mathrm{dR}} = \alpha_{pst} \otimes \mathbf{B}_{\mathrm{dR}}$.

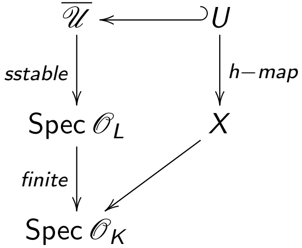
Here:
(1) $H^r_{\mathrm{dR}}(X_{\overline{K}})$ – Deligne de Rham cohomology (uses resolution of singularities)
(2) $H^r_{\mathrm{HK}}(X_{\overline{K}})$ – Beilinson Hyodo-Kato cohomology (uses de Jong's alterations)

Hyodo-Kato theory for algebraic varieties

Based on crystalline cohomology: **Hyodo-Kato**, **Beilinson**

X/K -algebraic variety (**sic** !)

(i) **locally**: in h-topology alterations allow



$$(a) \quad \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{U}_0/\mathscr{O}^0_{F_L}), \quad H^*\text{- finite rank}/F_L, \quad (\varphi, N),$$

$$(b) \quad \iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{U}_0/\mathscr{O}^0_{F_L}) \otimes^L_{F_L} L \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(U).$$

(ii) **globalization**: make (i) geometric and glue in h-topology. Get

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K}}), \quad H^*\text{- finite rank}/F^{\mathrm{nr}}, \quad (\varphi, N, \mathscr{G}_K),$$

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{F^{\mathrm{nr}}} \overline{K} \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})$$

Restated algebraic comparison theorem

(i) **de Rham-to-étale comparison**:

$$H^r_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \simeq (H^r_{\text{HK}}(X_{\overline{K}}) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}})^{\varphi=1, N=0} \cap F^0(H^r_{\text{dR}}(X) \otimes_K \mathbf{B}_{\text{dR}}), \quad \mathscr{G}_K,$$

or: we have a bicartesian diagram ($r \geq 0$)

$$\begin{array}{ccc} H^r_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r)) & \longrightarrow & (H^r_{\text{HK}}(X_{\overline{K}}) \otimes_{F^{\text{nr}}} \mathbf{B}^+_{\text{st}})^{\varphi=p^r, N=0} \\ \downarrow & & \downarrow \\ F^r(H^r_{\text{dR}}(X) \otimes_K \mathbf{B}^+_{\text{dR}}) & \longrightarrow & H^r_{\text{dR}}(X) \otimes_K \mathbf{B}^+_{\text{dR}} \end{array}$$

We will write it as (upper index refers to cohomology degree)

$$\begin{array}{ccc} H^r_{\text{ét},r} & \longrightarrow & \text{HK}^r_r \\ \downarrow & & \downarrow \\ H^r(F^r) & \longrightarrow & \mathrm{DR}^r \end{array}$$

or: there exists an exact sequence

$$0 \to H^r_{\acute{e}t,r} \to H^r(F^r) \oplus \mathrm{HK}^r_r \to \mathrm{DR}^r \to 0$$

(ii) **étale-to-de Rham comparison**:

$$\mathrm{Hom}(H^r_{\acute{e}t}(X_{\overline{K}}, \mathbf{Q}_p), \mathbf{B}_{st})^{\mathscr{G}_K - sm} \simeq H^r_{\mathrm{HK}}(X_{\overline{K}})^*, \quad (\varphi, N, \mathscr{G}_K),$$

$$\mathrm{Hom}_{\mathscr{G}_K}(H^r_{\acute{e}t}(X_{\overline{K}}, \mathbf{Q}_p), \mathbf{B}_{dR}) \simeq H^r_{dR}(X_{\overline{K}})^*, \quad Fil$$

Analytic varieties

X/K - smooth rigid analytic variety

Case 1 : X proper,

(A) Scholze:

(i) $H^r_{\text{ét}}(X_C, \mathbf{Q}_p)$ is finite rank over $\mathbf{Q}_p$:

- Artin-Schreier to pass to coherent cohomology
- Cartier-Serre argument for finitness of coherent cohomology

(ii) Hodge-de Rham spectral sequence degenerates

$\Rightarrow$ get **de Rham comparison isomorphism**:

$$\alpha_{\mathrm{dR}} : \quad H^r_{\text{ét}}(X_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} \simeq H^r_{\mathrm{dR}}(X) \otimes_K \mathbf{B}_{\mathrm{dR}}, \quad Fil,$$

(B) Colmez-Nizioł: **Algebraic comparison theorem** holds

(1) X/K – smooth, proper, rigid analytic

$$\alpha_{pst}: \quad H^r_{\text{ét}}(X_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}} \simeq H^r_{\text{HK}}(X_C) \otimes_{F^{\text{nr}}} \mathbf{B}_{\text{st}}, \quad (\varphi, N, \mathscr{G}_K),$$
$$\alpha_{\text{dR}}: \quad H^r_{\text{ét}}(X_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}} \simeq H^r_{\text{dR}}(X) \otimes_K \mathbf{B}_{\text{dR}}, \quad \mathit{Fil}.$$

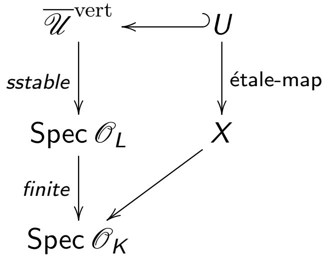
(2) X/C – smooth, proper, rigid analytic

$$\alpha_{pst}: \quad H^r_{\text{ét}}(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}} \simeq H^r_{\text{HK}}(X) \otimes_{\breve{C}} \mathbf{B}_{\text{st}}, \quad (\varphi, N),$$
$$\alpha_{\text{dR}}: \quad H^r_{\text{ét}}(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}} \simeq H^r_{\text{dR}}(X/\mathbf{B}^+_{\text{dR}}), \quad \mathit{Fil}.$$

Hyodo-Kato theory for analytic spaces

Grosse-Klönne; X/K -smooth dagger variety

(i) **locally**: in *étale*-topology alterations allow



$(a)\quad \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{U}_0/\mathscr{O}^0_{F_L}),\quad H^*\text{- finite rank}/F_L,\quad (\varphi, N),$

$(b)\quad \iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{U}_0/\mathscr{O}^0_{F_L}) \otimes^L_{F_L} L \simeq \mathrm{R}\Gamma_{\mathrm{rig}}(U).$

(ii) **globalization**: make (i) geometric and glue in *étale*-topology.

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_C),\quad H^*\text{- finite rank}/F^{\mathrm{nr}},\quad (\varphi, N, \mathscr{G}_K),$$

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \otimes_{F^{\mathrm{nr}}} \overline{K} \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X_C)$$

Case 2:

X/C Stein, smooth:

1. there exists an admissible covering by affinoids
$$\cdots \Subset U_n \Subset U_{n+1} \Subset \cdots$$
2. $H^i(X, \mathscr{F}) = 0$, $\mathscr{F}$-coherent, $i > 0$
3. $\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p) \simeq \mathrm{R}\lim_n \mathrm{R}\Gamma_{\text{ét}}(U_n, \mathbf{Q}_p)$
4. $H^r_{\text{proét}}$ is infinite dimensional
5. Hodge- de Rham spectral sequence does not degenerate

Theorem (Colmez-Dospinescu-N) X/C Stein smooth rigid space (or a dagger affinoid). There exists a map of exact sequences (all cohomologies are of X)

$$\begin{array}{ccccccccc}
0 \to & \Omega^{r-1}/\ker d & \to & H^r_{\text{proét}}(\mathbf{Q}_p(r)) & \to & (H^r_{\text{HK}} \widehat{\otimes}^R_{\breve{C}} \mathbf{B}^+_{\text{st}})^{\varphi=p^r, N=0} & \to 0 \\
& \| & & \downarrow \alpha & & \downarrow \iota_{\text{HK}} \otimes \theta & \\
0 \to & \Omega^{r-1}/\ker d & \longrightarrow & \Omega^{r, d=0} & \longrightarrow & H^r_{\text{dR}} & \longrightarrow 0
\end{array}$$

Main theorem

Theorem (Colmez-N) X/K smooth dagger variety.
(i) **de Rham-to-étale**: there exists a bicartesian diagram

$$\begin{array}{ccc} H^r_{\text{proét}}(X_C, \mathbf{Q}_p(r)) & \longrightarrow & (H^r_{\text{HK}}(X_C)\widehat{\otimes}^R_{F^{\text{nr}}}\mathbf{B}^+_{\text{st}})^{\varphi=p^r, N=0} \\ \downarrow & & \downarrow {\scriptstyle \iota_{\text{HK}}\otimes\iota} \\ H^r(F^r(\mathrm{R}\Gamma_{\text{dR}}(X)\widehat{\otimes}^R_K\mathbf{B}^+_{\text{dR}})) & \longrightarrow & H^r_{\text{dR}}(X)\widehat{\otimes}^R_K\mathbf{B}^+_{\text{dR}} \end{array}$$

(ii) **étale-to-de Rham**: $([K:\mathbf{Q}_p]<\infty)$

$$\text{Hom}(H^r_{\text{proét}}(X_C, \mathbf{Q}_p), \mathbf{B}_{\text{st}})^{\mathscr{G}_K-prosm} \simeq H^r_{\text{HK}}(X_C)^* \quad (\varphi, N, \mathscr{G}_K),$$

$$\text{Hom}_{\mathscr{G}_K}(H^r_{\text{proét}}(X_C, \mathbf{Q}_p), \mathbf{B}_{\text{dR}}) \simeq H^r_{\text{dR}}(X)^*, \quad Fil???$$

Remark (i) holds also for X/C.

Remarks

(1) X is **proper** then (degeneration of Hodge-de Rham sp. seq.)

$$H^r(F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}_K^R\mathbf{B}_{\mathrm{dR}}^+)) \simeq F^r(H^r_{\mathrm{dR}}(X)\widehat{\otimes}_K^R\mathbf{B}_{\mathrm{dR}}^+)$$

and the horizontal arrows are injective
(2) X is **Stein or an affinoid** then the two horizontal arrows are surjective and their kernels are $\Omega^{r-1}(X_C)/\ker d$.
(3) **Topology**: We work in the category of locally convex spaces (quasi-abelian).

- Tensor products are projective (commute with limits) and (right) derived.
- Overconvergence implies "good properties":
  1. higher derived functors of tensor products vanish,
  2. cohomology is "classical".

Digression: Banach-Colmez spaces

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \simeq (H^n_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{K^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}})^{\mathbf{N}=0,\varphi=1} \cap F^0(H^n_{\mathrm{dR}}(X_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}_{\mathrm{dR}})$$

What structure can we put on:

$$(H^n_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{K^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{\mathbf{N}=0,\varphi=1} \simeq (H^n_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{K^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{cr}})^{\varphi=1}$$

Example

- $H^n_{\mathrm{HK}}(X_{\overline{K}}) \simeq K^{\mathrm{nr}}\{-1\} \Rightarrow (H^n_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{K^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{cr}})^{\varphi=1} \simeq \mathbf{B}^{+,\varphi=p}_{\mathrm{cr}}$.
  Have:
  $$0 \to \mathbf{Q}_p t \to \mathbf{B}^{+,\varphi=p}_{\mathrm{cr}} \to C \to 0$$
  So $\mathbf{B}^{+,\varphi=p}_{\mathrm{cr}} \sim C \oplus \mathbf{Q}_p$.
- More generally, we have $\mathbf{B}^{+,\varphi=p^m}_{\mathrm{cr}} \sim C^m \oplus \mathbf{Q}_p$ because:
  $$(FES): \quad 0 \to \mathbf{Q}_p t^m \to \mathbf{B}^{+,\varphi=p^m}_{\mathrm{cr}} \to \mathbf{B}^+_{\mathrm{dR}}/t^m\mathbf{B}^+_{\mathrm{dR}} \to 0$$

Digression, cont.

In which reasonable category

$$\boxed{\mathbf{B}_{\mathrm{cr}}^{+,\varphi=p^m} \sim C^m \oplus \mathbf{Q}_p}$$

Remark The category of topological vector spaces is not good: $C \oplus \mathbf{Q}_p \simeq C$!

Colmez, **Fontaine**: $\exists$ abelian category $\mathscr{BC}$ of Banach-Colmez vector spaces $\mathbb{W}$:

- $\mathbb{W} \simeq C^n \pm \mathbf{Q}_p^m$
- $\mathrm{Dim}(\mathbb{W}) := (\dim_C \mathbb{W}, \dim_{\mathbf{Q}_p} \mathbb{W})$
- $\mathrm{Dim}(\mathbb{W})$ is additive on short exact sequences

Example

1. $\mathbf{B}_{\mathrm{dR}}^+/t^m$ is $\mathbb{B}_m$ with $\mathrm{Dim}(\mathbb{B}_m) = (m, 0)$.
2. $\mathbf{B}_{\mathrm{cr}}^{+,\varphi^a=p^b}$ is $\mathbb{U}_{a,b}$ with $\mathrm{Dim}(\mathbb{U}_{a,b}) = (b, a)$.
3. $C/\mathbf{Q}_p$ has $\mathrm{Dim} = (1, -1)$.

Digression, qBC spaces

qBC space: A Vector Space $\mathbb{W}$ of the form

$$0 \to \mathbb{W}_0 \to \mathbb{W} \to \mathbb{W}/\mathbb{W}_0 \to 0$$

such that:

- $\mathbb{W}_0$ is a $\mathbb{B}_m$-module, $m \geq 0$,
- $\mathbb{W}/\mathbb{W}_0$ is a BC space

Typical example of $\mathbb{B}_m$-module: $(\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_K \mathbf{B}_{\mathrm{dR}}^+)/F^m$

Everything stated below for BC spaces extends to qBC spaces

Proof of the main theorem

Step 1: equip everything in sight with BC structure
Step 2: reduce to X quasi-compact: write

$$X = \cup_n U_n, \quad U_n \subset U_{n+1}, \quad U_n\text{-quasi-compact}$$

$$C(X): \quad 0 \to H^r_{\text{proét},r}(X_C) \to H^r(F^r)(X_C) \oplus \mathrm{HK}^r_r(X_C) \to \mathrm{DR}^r(X_C) \to 0$$

Claim Have $C(X) = \varprojlim_n C(U_n)$: to control $\mathrm{R}^1 \varprojlim_n$ use:
(i) Mittag-Leffler in BC category and
(ii)Mittag-Leffler in loc. conv. top. vs for coh. cohomology
Step 3: Assume X quasi-compact
Lemma Main Theorem is equivalent to the following:

1. The pair $(H^r_{\text{HK}}(X_C), H^r_{\text{dR}}(X_C))$, $r \geq 0$, is *acyclic*.
2. $H^r_{\text{proét}}(X_C, \mathbf{Q}_p)$ is effective, i.e., has curvature ≥ 0, for all r.
3. For all r,

$$\mathrm{ht}(H^r_{\text{proét}}(X_C, \mathbf{Q}_p)) = \dim_K H^r_{\text{dR}}(X).$$

Acyclicity and curvature

An (M, M_K)- filtered (φ, N)-module is called **acyclic** if (equivalently):

- the associated vector bundle $\mathscr{E}$ on X_{FF} is acyclic, i.e., $H^1(X_{\mathrm{FF}}, \mathscr{E}) = 0$
- $\mathscr{E}$ has HN slopes ≥ 0
- $(M \otimes \mathbf{B}_{\mathrm{st}})^{\varphi=1, N=0} \to (M \otimes \mathbf{B}_{\mathrm{dR}})/F^0$ is surjective

Remark If (M, M_K) is a weakly admissible filtered (φ, N)-module then it is acyclic: all Harder-Narasimhan slopes of $\mathscr{E}$ are 0.

(1) **Curvature** BC $\mathbb{W}$ has curvature:

- > 0 if $\mathrm{Hom}(\mathbb{W}, \mathbb{V}^1) = 0$; $\Leftarrow H^1(X_{FF}, \mathscr{E})$, $\mathscr{E}$ a vector bundle
- $= 0$ if it is affine, i.e., it is a successive extension of $\mathbb{V}^1$; think $H^0(X_{FF}, \mathscr{F}_\infty)$, $\mathscr{F}_\infty$ coherent sheaf, supported at ∞, torsion
- < 0 if it injects into $\mathbf{B}_{\mathrm{dR}}^d$; think $H^0(X_{FF}, \mathscr{E})$.

Example (i) $\mathbb{V}^1$ curvature 0 and height 0
(ii) $\mathbb{V}^1/\mathbf{Q}_p$ curvature > 0 and height $-1 < 0$
(iii) $\mathbb{U} = (\mathbf{B}_{\mathrm{cr}}^+)^{\varphi=p}$ curvature < 0 and height 1
(iv) Compare:

- $\mathbb{U}/\mathbf{Q}_p t \simeq \mathbb{V}^1$ curvature 0 and height 0
- if $x \in \mathbb{U}(C) \setminus \mathbf{Q}_p t$ then $\mathbb{U}/\mathbf{Q}_p x$ curvature > 0 and height 0

Fact: Every BC space $\mathbb{W}$ has a unique filtration

$$\mathbb{W}_{>0} \subset \mathbb{W}_{\geq 0} \subset \mathbb{W}$$

such that

- $\mathbb{W}_{>0}$ has curvature > 0,
- $\mathbb{W}_{\geq 0}/\mathbb{W}_{>0}$ has curvature 0,
- $\mathbb{W}/\mathbb{W}_{\geq 0}$ has curvature < 0

Curvature and Harder-Narasimhan filtration

(v) **Le Bras**: category $\mathscr{BC}$ is Harder-Narasimhan:

- there is a notion of slope $\mu(\mathbb{W})$ such that $\mu(\mathbb{U}_\lambda) = -1/\lambda$ (if $\lambda = d/h$ in lowest terms then $\mathbb{U}_\lambda := \mathbb{U}_{h,d}$). Have:

$$\mathbb{U}_\lambda = \begin{cases} \mathbb{H}^0(X_{\mathrm{FF}}, \mathscr{O}(\lambda)), & \text{if } \lambda \geq 0, \\ \mathbb{H}^1(X_{\mathrm{FF}}, \mathscr{O}(\lambda)), & \text{if } \lambda < 0 \end{cases}$$

- there is canonical HN filtration; it splits (nocanonically):

$$\mathbb{W} = \mathbb{U}_{-1/\lambda_1} \oplus \cdots \oplus \mathbb{U}_{-1/\lambda_r} \oplus (\oplus_x \mathbb{H}^0(X_{\mathrm{FF}}, \mathscr{F}_x)),$$

where $\mathscr{F}_x$ is a torsion sheaf supported on x (and $\mathbb{H}^0(X_{\mathrm{FF}}, \mathscr{F}_x)$ is of slope 0).

(vi) **Curvature and Harder-Narasimhan filtration** :

- $\mathbb{W}_{>0} \simeq (\oplus_{\lambda_i > 0} \mathbb{U}_{-1/\lambda_i}) \oplus (\oplus_{x \neq \infty} \mathbb{H}^0(X_{\mathrm{FF}}, \mathscr{F}_x))$,
- $\mathbb{W}_{\leq 0} \simeq (\oplus_{\lambda_i < 0} \mathbb{U}_{-1/\lambda_i}) \oplus \mathbb{H}^0(X_{\mathrm{FF}}, \mathscr{F}_\infty)$,
- $\mathbb{W}_{<0} \simeq (\oplus_{\lambda_i < 0} \mathbb{U}_{-1/\lambda_i})$,
- $\mathbb{W}_{=0} \simeq \mathbb{H}^0(X_{\mathrm{FF}}, \mathscr{F}_\infty)$.

Proof of the main theorem

We will prove claim (3) of the lemma: X quasi-compact over K. For all r,

$$\mathrm{ht}(H^r_{\text{proét}}(X_C, \mathbf{Q}_p)) = \dim_K H^r_{\mathrm{dR}}(X).$$

(i) Note that this is true for affinoids.
(ii) We will show that it is true for a union of two affinoids (the general case is similar). So, assume that U_1, U_2 are affinoids, let

$$U = U_1 \cup U_2, \quad U_{12} = U_1 \cap U_2.$$

Note that

$$\mathrm{ht}(\mathrm{HK}^r_r) = \dim_K H^r_{\mathrm{dR}}(X)$$

$\Rightarrow$ it suffices to show that

$$\mathrm{ht}(H^r_{\text{proét},r}) = \mathrm{ht}(\mathrm{HK}^r_r).$$

(iii) Consider the map

$$g : H^r_{\text{proét},r} \to \mathrm{HK}^r_r$$

and let us pretend that

$$\mathrm{ht} : BC \text{ spaces } \to \text{an abelian category}$$

that is exact.

Show that

$$\mathrm{ht}(g) : \mathrm{ht}(H^r_{\text{proét},r}) \to \mathrm{ht}(\mathrm{HK}^r_r)$$

is an isomorphism.

It is clear what to do: Mayer-Vietoris yields the following map of exact sequences

$$\begin{array}{ccccccccc}
\mathrm{ht}^{r-1}_{\text{ét}}(U_1 \oplus U_2) & \to & \mathrm{ht}^{r-1}_{\text{ét}}(U_{12}) & \to & \mathrm{ht}^{r}_{\text{ét}}(U) & \to & \mathrm{ht}^{r}_{\text{ét}}(U_1 \oplus U_2) & \to & \mathrm{ht}^{r}_{\text{ét}}(U_{12}) \\
\wr\downarrow g & & \wr\downarrow g & & \downarrow g & & \wr\downarrow g & & \wr\downarrow g \\
\mathrm{ht}^{r-1}_{\mathrm{HK}}(U_1 \oplus U_2) & \to & \mathrm{ht}^{r-1}_{\mathrm{HK}}(U_{12}) & \to & \mathrm{ht}^{r}_{\mathrm{HK}}(U) & \to & \mathrm{ht}^{r}_{\mathrm{HK}}(U_1 \oplus U_2) & \to & \mathrm{ht}^{r}_{\mathrm{HK}}(U_{12})
\end{array}$$

Use five lemma.

(iv) But ht does not have these properties so we consider a partial

Categorification of height

Consider $h : BC \to C(\mathbf{B}_{\mathrm{dR}} - modules)$,

$$\mathbb{W} \mapsto \mathrm{Hom}(\mathbb{W}, \mathbf{B}_{\mathrm{dR}}).$$

Facts:

(1) if $\mathbb{W}$ is effective then

$$\mathrm{rk}(h(\mathbb{W})) = \mathrm{ht}(\mathbb{W});$$

in general

$$\mathrm{rk}(h(\mathbb{W})) = \mathrm{ht}(W) + \mathrm{rk}(\mathrm{Ext}(\mathbb{W}, \mathbf{B}_{\mathrm{dR}})).$$

(2) h is an exact functor on effective BC's.

(v) It suffices to show that everything in sight is effective:

- we know it for all the affinoids
- it is clear for $\mathrm{HK}_r^r(U)$
- for $H^r_{\mathrm{pro\acute{e}t}}(U_C)$ we argue by induction on r using the fact

$$\text{acyclicity of } (H^{r-1}_{\mathrm{HK}}(X), H^{r-1}_{\mathrm{dR}}(X_K)) \Rightarrow \text{ effectiveness of } H^r_{\mathrm{pro\acute{e}t}}(U_C)$$
$$\Rightarrow \text{acyclicity of } (H^r_{\mathrm{HK}}(X), H^r_{\mathrm{dR}}(X_K)).$$

Thank you !