A colimit theorem in tropical differential algebra

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Main Goal

Extend the setting of the theory of differential tropical equations as introduced in [1] and [2] to encompass the non-trivially valued case, introduce a differential tropicalization functor and prove an inverse limit theorem in a similar fashion to [3] and [5] .

Basic definitions

- A *pre-differential semiring* is a pair $(\mathcal{S}, d)$ where $\mathcal{S}$ is an idempotent semiring and $d : \mathcal{S} \to \mathcal{S}$ is a linear map satisfying tropical Leibniz rule, i.e.
$$d(xy) + d(x)y = d(xy) + xd(y) = d(x)y + xd(y)$$
- A *differential semiring* $(\mathcal{S}, d, w)$ is a pre-differential semiring $(\mathcal{S}, d)$ equipped with a valuation $w : \mathbb{Z} \to \mathcal{S}$ (as in [4]). Let us denote the category of differential semirings as **DSR**. A subset X of $\mathcal{S}$ is a *differential primitive set* if $d(xy) = (dx)y + x(dy)$ and $d(x^n) = w(n)x^{n-1}dx$ for all $x, y \in X$, $x \neq y$ such that they generate a multiplicative monoid free of rank 2;
- A *differential* $\mathbb{F}_1$*-module* is a pair $(\mathcal{M}, d_{\mathcal{M}})$ where $\mathcal{M}$ is a pointed set and $d_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a map of pointed sets. Let us denote the category of differential $\mathbb{F}_1$-modules as $\mathbf{D}\mathbb{F}_1$**-Mod**.

Given a differential ring (R, d_R) we define the functor:
$$\begin{array}{rcl} R\{-\} : \mathbf{D}\mathbb{F}_1\textbf{-Mod} & \to & \mathbf{D}R\textbf{-Alg} \\ \mathcal{M} & \mapsto & (R[F(\mathcal{M})], d) \end{array}$$
where $F(\mathcal{M})$ is the free monoid with zero generated by $\mathcal{M}$ and the differential $d : R\{\mathcal{M}\} \to R\{\mathcal{M}\}$ is given as the unique linear and Leibniz extension of $d_{\mathcal{M}}$ extending d_R.

Given a differential semiring $(\mathcal{S}, d, w)$ we define the functor:
$$\begin{array}{rcl} \mathcal{S}\{-\}_w : \mathbf{D}\mathbb{F}_1\textbf{-Mod} & \to & \mathbf{D}\mathcal{S}\textbf{-Alg} \\ \mathcal{M} & \mapsto & (\mathcal{S}[F(\mathcal{M})], d_w) \end{array}$$
where the differential $d_w : \mathcal{S}\{\mathcal{M}\} \to \mathcal{S}\{\mathcal{M}\}$ is given as the unique linear and (classic) Leibniz extension of $d_{\mathcal{M}}$ extending $d_{\mathcal{S}}$, such that $d_w(x_m^n) = w(n)d(x_m^{n-1})$ for all $m \in \mathcal{M}$.

The category of pairs

Let **Pairs** be the category with objects semiring homomorphisms $\Phi : S \to T$, S a pre-differential semiring, and morphisms of the form $f = (f_1, f_2) : \Phi \to \Phi'$ where f_1 is a map of differential semirings and f_2 is a map of semirings such that the following diagram commutes:
$$\begin{array}{ccc} S & \xrightarrow{f_1} & S' \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi'} \\ T & \xrightarrow[f_2]{} & T' \end{array}$$
Given an object $\Phi : S \to T$ in **Pairs** we can associate to it its reduction $\Phi^r : S^r \to T$, where $S^r := S/_\sim$ with $\sim$ the largest differential congruence contained in $\ker\Phi$. A pair Φ such that $\Phi = \Phi^r$ is said reduced.

Denoting by **RedPairs** the full subcategory of **Pairs** of reduced pairs:

Reduction

Reduction is a functor:
$$(-)^r : \textbf{Pairs} \to \textbf{RedPairs}$$

Differential enhancements and tropicalization

Let us fix a valued differential ring of characteristic 0 $v : (R, d) \to \mathcal{T}$ and a *differential enhancement* of it, i.e. a reduced pair $\Phi : \mathcal{S} \to \mathcal{T}$, together with a differential map $w : R \to \mathcal{S}$ making the triangle commute, and such that $w(R) \subset \mathcal{S}$ is a differential primitive set with respect to the composition $\mathfrak{w} : \mathbb{Z} \to R \to \mathcal{S}$ (which makes $\mathcal{S}$ into a differential semiring):
$$\begin{array}{ccccc} & & R & & \\ & {\scriptstyle v}\swarrow & & \searrow{\scriptstyle \omega} & \\ \mathcal{T} & & \xleftarrow[\Phi]{} & & \mathcal{S} \end{array}$$
A solution (in the sense of Grigoriev) to a differential polynomial $P \in \mathcal{S}\{X_1, \ldots, X_n\}$ is an n-tuple $S \in \mathcal{S}^n$ such that $\Phi(P|_{X_i^{(j)} = d^j S_i})$ is attained at least twice in $\mathcal{T}$.

Given a finitely generated differential R-algebra A, let $\mathcal{C}$ be the category whose objects are differential $\mathbb{F}_1$-modules $\mathcal{M}$ together with a surjective differential homomorphisms $R\{\mathcal{M}\} \twoheadrightarrow A$.

Given an element $f \in \mathcal{T}[F(\mathcal{M})]$ and $m \in F(\mathcal{M})$, let $f_{\hat{m}}$ be the result of deleting the m-term from f.

For an ideal $I \subset \mathcal{T}[F(\mathcal{M})]$, let $\mathcal{B}(I)$ be the semiring congruence generated by (see [4], 5.1.1):
$$\{f \sim f_{\hat{m}} : f \in I, m \in supp(f)\}$$
Given an object $\psi \in \mathcal{C}$, we define its differential tropicalization as the reduced pair:
$$\mathcal{T}rop^d(\psi) : \mathcal{S}\{\mathcal{M}\}_{\mathfrak{w}}/_{\sim_{\Phi\circ\pi}} \longrightarrow \mathcal{T}[F(\mathcal{M})]/\mathcal{B}(v(\ker\psi))$$
i.e. the reduction of the composition:
$$\mathcal{S}\{\mathcal{M}\}_{\mathfrak{w}} \to \mathcal{T}[F(\mathcal{M})] \twoheadrightarrow \mathcal{T}[F(\mathcal{M})]/\mathcal{B}(v(\ker\psi))$$
of coefficientwise Φ and the quotient map π.

Tropicalization

Differential tropicalization is a functor:
$$\mathcal{T}rop^d(-) : \mathcal{C} \to \textbf{RedPairs}$$

Given a pair φ over Φ and a presentation $\psi : R\{\mathcal{M}\} \twoheadrightarrow A$ of A, solutions for its tropicalization over φ, are morphisms from $\mathcal{T}rop^d(\psi)$ to φ.

The colimit theorem

Let $DM : \mathbf{D}R\textbf{-Alg} \to \mathbf{D}\mathbb{F}_1\textbf{-Mod}$ be the forgetful functor sending a differential R-algebra to the corresponding differential $\mathbb{F}_1$-module, this functor is adjoint to $R\{-\}$. In analogy with the non-differential case treated in [5], let's now consider $\tilde{A} := R\{DM(A)\}$. It gives an element of $\mathcal{C}$, namely:
$$\begin{array}{rcl} \boldsymbol{ev} : \tilde{A} & \twoheadrightarrow & A \\ x_a & \mapsto & a \end{array}$$
as the differential on $\tilde{A}$ is defined as $d(x_a) = x_{d(a)}$.

Let $\mathcal{C}_{aff}$ be the subcategory of $\mathcal{C}$ whose objects are of the form $\mathcal{M} = \wedge_n \overline{\mathbb{N}}$ for some n ($\wedge$ denotes the smash product of pointed sets), thus giving presentations $R\{\mathcal{M}\} = R\{X_1, \ldots, X_n\} \twoheadrightarrow A$. Then:

Colimit theorem

$$\underset{\varphi \in \mathcal{C}_{aff}}{\underrightarrow{\operatorname{colim}}}\, \mathcal{T}rop^d(\varphi) = \mathcal{T}rop^d(\boldsymbol{ev})$$

This result can be looked at as a differential version of the inverse limit theorem of [3], as stated in [5], Theorem 4.1.1.

Furthermore, in analogy to [5], Theorem 3.3.6, we can look at the universal tropicalization as parametrising the differential enhancement of the differential algebra A compatible with the fixed one on $v : (R, d) \to \mathcal{T}$ in a suitable way.

The differential enhancement:
$$\begin{array}{ccccc} & & A & & \\ & {\scriptstyle v_{\text{univ}}}\swarrow & & \searrow{\scriptstyle w_{\text{univ}}} & \\ \mathcal{T}[F(DM(A))]/\mathcal{B}(v(\ker\boldsymbol{ev})) & & \xleftarrow[\Phi_{\text{univ}}]{} & & \mathcal{S}\{DM(A)\}_{\mathfrak{w}}/_\sim \end{array}$$
of A given by $\mathcal{T}rop^d(\boldsymbol{ev})$ is called the universal differential enhancement of A compatible with the fixed one on R, where both v_{univ} and w_{univ} send $a \in A$ to x_a and Φ_{univ} is the map $\Phi : \mathcal{S} \to \mathcal{T}$ applied coefficientwise. There is a bijection:
$$\left\{ \begin{array}{c} \text{Differential enhancements of } A \text{ compatible} \\ \text{with the fixed one on } R \end{array} \right\}$$
$$\updownarrow$$
$$\left\{ \begin{array}{c} \text{Morphisms of pairs } \Phi_{\text{univ}} \to \varphi \\ \text{over } \Phi \end{array} \right\}$$

References

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