p-adic Tate conjectures and abeloid varieties

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Bibliography

Oliver Gregory, Christian Liedtke:

p-adic Tate conjectures and abeloid varieties,

Doc. Math. 24 (2019), 1879 - 1934.

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The classical Tate conjecture

X smooth and proper variety over a field F

$G_F := \mathrm{Gal}(\overline{F}/F)$ absolute Galois group

$\ell \neq p := \mathrm{char}(F)$

cycle class map $NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^2_{ét}(X_{\overline{F}}, \mathbb{Q}_\ell(1))^{G_F}$

Question: is this map surjective?

The classical Tate conjecture

Conjecture (Tate): If F is finitely generated over its prime field, then

$$NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^2_{\acute{e}t}\left(X_{\overline{F}}, \mathbb{Q}_\ell(1)\right)^{G_F}$$

is surjective.

The classical Tate conjecture

Known for:

abelian varieties: André, Faltings, Tate, Tankeev, Zarhin

hyperkähler varieties in characteristic zero: André, Tankeev

K3 surfaces: Charles, Kim, Madapusi-Pera, Maulik, Nygaard, Ogus

surfaces with $p_g = 1$ in characteristic zero and with sufficiently non-trivial VHS: Moonen

Raskind's Tate conjecture

Question: is there a Tate conjecture for p-adic fields, that is, if F is a finite extension of $\mathbb{Q}_p$?

Conjecture (Raskind): If F is a p-adic field, if $\ell = p$, and if X has totally degenerate reduction, then

$$NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^2_{\acute{e}t}\left(X_{\overline{F}}, \mathbb{Q}_\ell(1)\right)^{G_F}$$

is surjective.

Raskind's Tate conjecture

Why $\ell = p$?

if $\ell \neq p$ and X has good reduction with special fibre $\mathcal{X}_0$, then

$$H^n_{ét}(X_{\overline{F}}, \mathbb{Q}_\ell(i))^{G_F} \quad \cong \quad H^n_{ét}(\mathcal{X}_{0,\overline{k}}, \mathbb{Q}_\ell(i))^{G_k}$$

which computes invariants of $\mathcal{X}_0$

if X has semi-stable reduction: conjectures of Consani

Raskind's Tate conjecture

Why totally degenerate reduction?

examples of Lubin-Tate and Oort of
elliptic curves with good reduction, where

$$\mathrm{End}(E) \otimes \mathbb{Q}_p \quad \to \quad \mathrm{End}(V_p(E))$$

is not surjective. Then, $E \times E$ would be a counter-example
to a more general Raskind's conjecture.

see also Appendix A of our article for more examples

Raskind's Tate conjecture

Known cases of Raskind's conjectures:

<u>products of Tate elliptic curves</u>: Raskind-Xarles, Gregory-L.

<u>varieties that are uniformised by Drinfeld's upper half plane:</u> Ito-Rapoport

Evidence is rather thin, but a p-adic approach to the classical Tate conjecture via Raskind's Tate conjecture would be very interesting!

II. Translating Raskind's conjecture into semi-linear algebra

1. A semi-linear algebra translation
2. Rational structures
   (on log-crystalline cohomology)
3. Application to Raskind's conjecture
4. A theorem of Berthelot, Ogus, and Yamashita
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A semi-linear algebra translation

K p-adic field

X smooth and proper over K

$\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ proper and semi-stable model

p-adic Hodge theory gives an isomorphism

$$H^n_{ét}(X_{\overline{K}}, \mathbb{Q}_p(m))^{G_K}$$

$$\cong \quad H^n_{log-cris}(\mathcal{X}_0/K_0)^{\varphi=p^m, N=0} \cap \operatorname{Fil}^m H^n_{dR}(X/K)$$

A semi-linear algebra translation

relevant for Raskind's conjecture:

$$H^2_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p(1))^{G_K}$$

which is isomorphic to

$$H^2_{log-cris}(\mathcal{X}_0/K_0)^{\varphi=p, N=0} \cap \operatorname{Fil}^1 H^2_{dR}(X/K)$$

assume that special fibre is SNCD:

$$\mathcal{X}_0 = \bigcup_i Y_i$$

A semi-linear algebra translation

$$\mathcal{X}_0 = \bigcup_i Y_i \qquad \text{SNCD}$$

$$Y^{[m]} := \coprod \bigcap_{\text{m-fold intersections}} Y_i$$

Steenbrink - Rapoport-Zink - Mokrane - Nakkajima spectral sequence

$$E_1^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H_{cris}^{h-2j-k}(Y^{[2j+k]}/K_0)(-j-k)$$

$$\Rightarrow \quad H^h_{log-cris}(\mathcal{X}_0/K_0)$$

Rational structures

total degeneration:

$$H^n_{cris}(Y^{[m]}/K_0) \quad = \quad \begin{cases} \text{zero} & \text{if } n \text{ is odd} \\ \text{spanned by algebraic cycles} & \text{if } n \text{ is even} \end{cases}$$

then, we obtain an explicit description of $H^2_{log-cris}(\mathcal{X}_0/K_0)$
as follows:

Rational structures

rational structure

$$V = A \oplus B_0 \oplus B_1 \oplus C$$

a direct sum of $\mathbb{Q}$ - vector spaces and
two linear operators

$$\begin{aligned}
\varphi_V \quad : \quad & \mathrm{id} && \text{on } A \\
& p \cdot \mathrm{id} && \text{on } B_0 \oplus B_1 \\
& p^2 \cdot \mathrm{id} && \text{on } C
\end{aligned}$$

$$\begin{aligned}
N_V \quad : \quad & \text{zero} && \text{on } A \oplus B_1 \\
& C \cong && N(C) = B_0 \\
& B_0 \cong && N(B_0) = A
\end{aligned}$$

Rational structures

given such a decomposition and operators

$$V = A \oplus B_0 \oplus B_1 \oplus C, \quad \varphi_V, \quad N_V$$

get an associated (φ, N)-module over K_0

$$(V \otimes_{\mathbb{Q}} K_0, \quad \varphi_V \otimes \sigma, \quad N_V \otimes \mathrm{id})$$

where σ is the Frobenius on K_0

Rational structures

given X with total degenerate special fibre $\mathcal{X}_0$

the SRZMN-spectral sequence plus the cycle class maps of the $Y^{[m]}$ equip

$$H^2_{log-cris}(\mathcal{X}_0/K_0)$$

with a rational structure

$$V \;=\; A \oplus B_0 \oplus B_1 \oplus C, \quad \varphi_V\,, \quad N_V$$

Back to Raskind's conjecture

want to understand

$$H^2_{log-cris}(\mathcal{X}_0/K_0)^{\varphi=p,N=0} \cap \operatorname{Fil}^1 H^2_{dR}(X/K)$$

use the rational structure

$$V \;=\; A \oplus B_0 \oplus B_1 \oplus C, \quad \varphi_V, \quad N_V$$

and find

$$H^2_{log-cris}(\mathcal{X}_0/K_0)^{\varphi=p,N=0} \quad \cong \quad B_1 \otimes_{\mathbb{Q}} \mathbb{Q}_p$$

Back to Raskind's conjecture

$$H^2_{log-cris}(\mathcal{X}_0/K_0)^{\varphi=p,N=0} \quad \cong \quad B_1 \otimes_{\mathbb{Q}} \mathbb{Q}_p$$

first (logarithmic) Chern class map induces

$$\mathrm{Pic}(\mathcal{X}_0) \otimes F \quad \to \quad \mathrm{Pic}^{log}(\mathcal{X}_0) \otimes F \quad \to \quad H^2_{log-cris}(\mathcal{X}_0/K_0)^{\varphi=p,N=0}$$

<u>Proposition:</u>

$$\begin{array}{llll} \text{if} & F = \mathbb{Q}, & \text{then the image is } B_1 \\ \text{if} & F = \mathbb{Q}_p, & \text{then the map is surjective} \end{array}$$

Back to Raskind’s conjecture

$$H^2_{ét}(X_{\overline{K}}, \mathbb{Q}_p(1))^{G_K} \quad \cong \quad \underbrace{H^2_{log-cris}(\mathcal{X}_0/K_0)^{\varphi=p, N=0}}_{\mathrm{Pic}^{log}(\mathcal{X}_0)\otimes\mathbb{Q}_p} \quad \cap \quad \mathrm{Fil}^1 H^2_{dR}(X/K)$$

<u>Conjecture (equivalent to Raskind’s conjecture):</u> a class in

$\mathrm{Pic}^{log}(\mathcal{X}_0) \otimes \mathbb{Q}_p$ lifts to $\mathrm{Pic}(X) \otimes \mathbb{Q}_p$

if and only if its first crystalline Chern class lies in

$\mathrm{Fil}^1 H^2_{dR}(X/K)$

Back to Raskind's conjecture

upshot: Raskind's conjecture is (equivalent to) a sort of

variational log-Tate conjecture

<u>Theorem (Berthelot-Ogus, Yamashita):</u> a class in

$\mathrm{Pic}^{log}(\mathcal{X}_0) \otimes \mathbb{Q}$ lifts to $\mathrm{Pic}(X) \otimes \mathbb{Q}$

if and only if its first crystalline Chern class lies in

$$\mathrm{Fil}^1\, H^2_{dR}(X/K)$$

A theorem of Berthelot, Ogus, and Yamashita

Theorem (Berthelot-Ogus, Yamashita):

a class in

$\mathrm{Pic}^{log}(\mathcal{X}_0) \otimes \mathbb{Q}$ lifts to $\mathrm{Pic}(X) \otimes \mathbb{Q}$

if and only if its first crystalline Chern class lies in

$$\mathrm{Fil}^1\, H^2_{dR}(X/K)$$

Raskind - admissibility

given a rational structure $V = A \oplus B_0 \oplus B_1 \oplus C, \quad \varphi_V, \quad N_V$

have an associated (φ, N)-module over K_0

$$(V \otimes_{\mathbb{Q}} K_0, \quad \varphi_V \otimes \sigma, \quad N_V \otimes \mathrm{id})$$

<u>Definition:</u> a filtration Fil^* on $V \otimes_{\mathbb{Q}} K$ is called Raskind-admissible if the natural inclusion

$$(\mathrm{Fil}^1 \cap B_1) \otimes_{\mathbb{Q}} \mathbb{Q}_p \quad \subseteq \quad \mathrm{Fil}^1 \cap (B_1 \otimes_{\mathbb{Q}} \mathbb{Q}_p)$$

is an equality

A reformulation of Raskind's conjecture

Theorem: Given X over K with totally degenerate reduction, the following are equivalent:

1) Raskind's conjecture holds true, that is,

$$\mathrm{NS}(X) \quad \to \quad H^2_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p(1))^{G_K}$$

is surjective

2) the Hodge filtration $\mathrm{Fil}^*(X/K)$ is Raskind-admissible with respect to the natural rational structure on $H^2_{log-cris}(\mathcal{X}_0/K_0)$

III. Raskind's conjecture for abelian varieties

1. Abeloid varieties

2. Homomorphisms between abeloid varieties and their p-adic Tate modules

3. Raskind's conjecture for abeloid varieties

4. Counter-examples

Abeloid varieties

goal: establish/disprove Raskind's conjecture for abelian varieties

K p-adic field with valuation $\nu_p : K^\times \to \mathbb{Q}$

period matrix

$$Q = (q_{i,j}) \quad \in \quad \mathrm{Mat}_{g\times g}(K)$$

such that

$$\nu_p(q_{i,j}) > 0 \quad \forall\, i, j$$

$$\mathrm{ord}_p(Q) \quad := \quad (\nu_p(q_{i,j})) \quad \in \quad \mathrm{GL}_g(\mathbb{Q})$$

Abeloid varieties

given a period matrix

$$Q \;=\; (q_{i,j}) \quad \in \quad \mathrm{Mat}_{g\times g}(K)$$

there is a lattice (generated by the columns of Q)

$$\Lambda \quad \subset \quad (K^{\times})^g$$

there is a proper rigid variety over K,
an <u>abeloid variety</u>

$$(K^{\times})^g \,/\, \Lambda$$

Abeloid varieties

Example: If $g = 1$, then the abeloid variety

$$K^\times \,/\, q^{\mathbb{Z}}$$

is the Tate elliptic curve.

Remarks:

1) In general and if $g \geq 2$, then an abeloid variety is not algebraisable.

2) The special fibre of its Néron model is a split torus (totally degenerate reduction).

Abeloid varieties

Iwasawa's p-adic logarithm $\log_p : K^\times \to \mathbb{C}_p$

<u>Definition:</u>

$$\mathcal{L}_Q \quad := \quad \mathrm{ord}_p(Q)^{-1} \cdot \log_p(Q) \quad \in \quad \mathrm{Mat}_{g\times g}(K)$$

<u>Remark:</u> this generalises the classical invariant

$$\mathcal{L}(q) \quad = \quad \frac{\log_p(q)}{\mathrm{ord}_p(q)}$$

for the Tate elliptic curve $K^\times/q^{\mathbb{Z}}$

Abeloid varieties

Given two abeloid varieties $A,\ B$ of dimensions $g,\ h$

associated to period matrices $Q_A,\ Q_B$

want to understand/describe homomorphisms between

1) the two abeloid varieties $A,\ B$

3) their p-adic Tate modules $T_p(A),\ T_p(B)$

Abeloid varieties

(rational) homomorphisms between the varieties

$$
\begin{aligned}
& \mathrm{Hom}(A, B) \otimes \mathbb{Q} \\
\cong\quad & \mathrm{Hom}(A_{\overline{K}}, B_{\overline{K}}) \otimes \mathbb{Q} \\
\cong\quad & \{M \in \mathrm{Mat}_{g\times h}(\mathbb{Q}) \,|\, \mathcal{L}(Q_A) \cdot M = M \cdot \mathcal{L}(Q_B)\}
\end{aligned}
$$

(rational) homomorphisms between their p-adic Tate modules

$$
\begin{aligned}
& \mathrm{Hom}(T_p(A), T_p(B)) \otimes \mathbb{Q} \\
\cong\quad & \mathrm{Hom}(T_p(A_{\overline{K}}), T_p(B_{\overline{K}})) \otimes \mathbb{Q} \\
\cong\quad & \{M \in \mathrm{Mat}_{g\times h}(\mathbb{Q}_p) \,|\, \mathcal{L}(Q_A) \cdot M = M \cdot \mathcal{L}(Q_B)\}
\end{aligned}
$$

Raskind's conjecture for abeloid varieties

<u>Theorem:</u> Given an abeloid variety A over K, the following are equivalent:

1) Raskind's conjecture holds true, that is,

$$\mathrm{NS}(A) \quad \to \quad H^2_{\acute{e}t}(A_{\overline{K}}, \mathbb{Q}_p(1))^{G_K}$$

is surjective.

2) the natural inclusion

$$\mathrm{End}(A) \otimes \mathbb{Q}_p \quad \to \quad \mathrm{End}(T_p(A_{\overline{K}}) \otimes \mathbb{Q})^{G_K}$$

is surjective (the ``other'' Tate conjecture).

Raskind's conjecture for abeloid varieties

if Q is a period matrix for A,

then Raskind's conjecture for A is equivalent to the surjectivity of

$$\{M \in \mathrm{Mat}_{g\times g}(\mathbb{Q}) \,|\, \mathcal{L}(Q) \cdot M = M \cdot \mathcal{L}(Q)\} \otimes \mathbb{Q}_p$$
$$\to$$
$$\{M \in \mathrm{Mat}_{g\times g}(\mathbb{Q}_p) \,|\, \mathcal{L}(Q) \cdot M = M \cdot \mathcal{L}(Q)\}$$

Remark: again, an interplay between $\mathbb{Q}$ and $\mathbb{Q}_p$!

Counter-examples to Raskind's conjecture

<u>Theorem:</u> There exists an algebraisable abeloid surface over $\mathbb{Q}_p$ with $p \geq 5,\ p \equiv 1 \mathrm{mod} 3$, for which Raskind's conjecture is false.

idea of the counter-examples:

choose period matrix

$$Q \quad := \quad S^{-1} \odot \begin{pmatrix} p & 1 \\ 1 & \varepsilon \cdot p \end{pmatrix} \odot S$$

where $\varepsilon \in 1 + p\mathbb{Z}_p$ is a non-trivial p-adic unit and

where S is a ``well-chosen'' symmetric matrix