K-stability from a non-Archimedean perspective

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Yau–Tian–Donaldson conjecture

(X, L) polarized smooth complex variety of dim $n \geq 1$.

Question: Does X admit cscK metric $\omega \in c_1(L)$?

$$\mathrm{Ric}\,\omega \wedge \omega^{n-1} = \overline{S}\omega^n$$

True when $n = 1$. Obstructions when $n > 1$.

YTD conjecture: $\exists$ cscK metric $\Leftrightarrow$ (X, L) "K-stable".

Much activity in recent years:

- Chen–Donaldson–Sun, Tian: true when X Fano, $L = -K_X$.
- Other proofs in Fano case:
  - Datar–Székelyhidi.
  - Chen-Sun-Wang.
  - Berman–Boucksom–J: variational method.
  - Li-Tian–Wang: X possibly singular.

This talk: NA interpretation of K-stability.

Test configurations

(X, L) polarized variety over alg closed field k of char 0.

A *test configuration* $(\mathcal{X}, \mathcal{L})$ for (X, L) consists of

flat projective morphism $\mathcal{X} \to \mathbb{A}^1$;
semiample $\mathbb{Q}$-lb $\mathcal{L}$ on $\mathcal{X}$;
$\mathbb{G}_m$-action on $(\mathcal{X}, \mathcal{L})$ lifting action on $\mathbb{A}^1$;
$\mathbb{G}_m$-equivariant isom $(\mathcal{X}, \mathcal{L})|_{\mathbb{A}^1 \setminus \{0\}} \simeq (X, L) \times (\mathbb{A}^1 \setminus \{0\})$.

Invariants and definitions:

$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = a(\overline{\mathcal{L}}^{n+1}) + b(\overline{\mathcal{L}}^n \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1})$.
$\mathrm{J}(\mathcal{X}, \mathcal{L}) =$ "norm" of $(\mathcal{X}, \mathcal{L})$.

Say that (X, L) is

K-semistable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for all $(\mathcal{X}, \mathcal{L})$;
uniformly K-stable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \varepsilon \mathrm{J}(\mathcal{X}, \mathcal{L})$ for all $(\mathcal{X}, \mathcal{L})$.

Conjecturally, need to generalize test configurations. Proposals:

Filtrations of $R(X, L) = \bigoplus_m H^0(X, mL)$.
A certain class of functions on X^{an}.

Goal: explain these from NA analytic point of view.

Plurisubharmonic functions

X^{an} = Berkovich analytification wrt trivial absolute value on k.

View elements $v \in X^{\mathrm{an}}$ additively as semivaluations.

Any tc $(\mathcal{X}, \mathcal{L})$ induces a function $\varphi_{\mathcal{L}} \in \mathrm{C}^0(X^{\mathrm{an}})$.

($\approx$ model metric on L^{an}.)

Write $\mathcal{H} \subset \mathrm{C}^0(X^{\mathrm{an}})$ for the set of such functions. Analogies:

- $\mathcal{H} \approx$ convex $\mathbb{Q}$-PL functions on $\mathbb{R}^n$.
- $\mathcal{H} \approx$ smooth positive Hermitiean metrics on L^h if $k = \mathbb{C}$.

Construct new classes: $\mathcal{H} \subset \mathcal{E}^\infty_\uparrow \subset \mathcal{E}^\infty \subset \mathcal{E}^1 \subset \mathrm{PSH}$.

- $\mathrm{PSH} := \{$decreasing limits of functions in $\mathcal{H}\}$.
- $\mathcal{E}^\infty := \{$bounded functions in $\mathrm{PSH}\}$.
- $\mathcal{E}^\infty_\uparrow := \{$increasing limits of functions in $\mathcal{H}\}$.

Rmk: Can view functions in PSH as semipositive singular metrics on L^{an}, cf Zhang, Gubler, Chambert–Loir, Boucksom–Favre–J, C-L–Ducros, G–Künnemann,...

Plurisubharmonic functions of finite energy

Define *Monge–Ampère energy* $\mathrm{E}\colon \mathcal{H} \to \mathbb{R}$ by

$$\mathrm{E}(\varphi_{\mathcal{L}}) = \frac{(\mathcal{L}^{n+1})}{(n+1)(L^n)}.$$

It naturally extends to $\mathrm{E}\colon \mathsf{PSH} \to \mathbb{R} \cup \{-\infty\}$. The space

$$\mathcal{E}^1 := \{\mathrm{E} > -\infty\} \subset \mathsf{PSH}$$

is a NA analogue of a space considered in complex geometry by Guedj–Zeriahi, Darvas,...

Technical assumption: *continuity of envelopes*: if $f \in \mathrm{C}^0$, then

$$\mathrm{P}(f) := \sup\{\varphi \in \mathsf{PSH} \mid \varphi \leq f\}$$

is continuous. This is true if X smooth.

Equip $\mathcal{E}^1$ with the *Darvas metric* d_1. Given $\varphi, \psi \in \mathcal{E}^1$, set

$$d_1(\varphi, \psi) = \mathrm{E}(\varphi) + \mathrm{E}(\psi) - 2\mathrm{E}(P(\varphi \wedge \psi))$$

Thm [Boucksom-J] The space $(\mathcal{E}^1, d_1)$ is a complete metric space. It contains $\mathcal{H}$ as a dense subset.

Filtrations and norms

Section ring $R = R(X, L) = \bigoplus_{m \geq 0} R_m = \bigoplus_{m \geq 0} H^0(X, mL)$.
Consider norm $\| \cdot \|$ on R (submultiplicative and NA)
Additively: $\chi = -\log \| \cdot \|$.
Norms are in bijection with filtrations: $\mathcal{F}^\lambda R = \{\chi \geq \lambda\}$.
Will restrict to *graded* norms:

$$\chi(\sum_m s_m) = \min_m \chi(s_m)$$

Any tc $(\mathcal{X}, \mathcal{L})$ induces a norm/filtration $\chi_{\mathcal{L}}$ s.t.

$$\mathcal{F}^0 H^0(X, mL) \simeq H^0(\mathcal{X}, m\mathcal{L})$$

Such a norm is of *finite type* i.e. determined by $\chi|_{R_m}$, $m \leq m_0$.
Converse also true (up to replacing L by multiple).

The space of graded norms

Write $\mathcal{N}$ for the space of graded norms on R that satisfy the boundedness condition

$$-Cm \leq \chi \leq Cm \text{ on } R_m \setminus \{0\}.$$

The restriction of χ to $R_m \setminus \{0\}$ takes values (with multiplicity)

$$\lambda_{m,1} \geq \lambda_{m,2} \geq \cdots \geq \lambda_{m,N_m}.$$

Boucksom-Chen '10: the following limit exists

$$\mathrm{vol}(\chi) := \lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{N_m} \lambda_{m,j}$$

Can show that we get a semi-distance d_1 on $\mathcal{N}$ by

$$d_1(\chi, \chi') := \mathrm{vol}(\chi) + \mathrm{vol}(\chi') - 2\,\mathrm{vol}(\chi \wedge \chi') \geq 0$$

Say that χ, χ' are *asymptotically equivalent* of $d_1(\chi, \chi') = 0$.
Get a metric space $(\mathcal{N}/\sim, d_1)$. Not complete.

Comparing graded norms and psh functions

Have generalized test configurations in two ways: psh functions on X^{an} and graded norms on R. Now compare the constructions.

Define a *Fubini–Study operator*

$$\mathrm{FS}\colon \mathcal{N} \to \mathcal{E}^{\infty}_{\uparrow}$$

by $\mathsf{FS}(\chi) = \lim_m m^{-1}\,\mathsf{FS}_m(\chi)$, where

$$\mathsf{FS}_m(\chi)(v) = \sup\{\chi(s) - v(s) \mid s \in R_m \setminus \{0\}\}.$$

Thm [Boucksom-J]. We have $\mathrm{FS}(\mathcal{N}) = \mathcal{E}^{\infty}_{\uparrow}$. Moreover:

(i) if $\chi, \chi' \in \mathcal{N}$, then $\mathrm{FS}(\chi) = \mathrm{FS}(\chi')$ iff $\chi \sim \chi'$;

(ii) $\mathsf{FS}\colon (\mathcal{N}/\sim, d_1) \to (\mathcal{E}^1, d_1)$ is an isometry with dense image.

Key ingredient in proof:

$$\mathrm{E}(\mathrm{FS}(\chi)) = \mathrm{vol}(\chi)$$

This uses Boucksom–J–Hisamoto '17 and Okounkov bodies.

Energy functionals

Can formulate K-stability in terms of functionals on $\mathcal{H}$ or $\mathcal{E}^1$.

These functionals are modeled on their Archimedean cousins used in Kähler geometry.

Two types of functionals:

Energy functionals: $\mathrm{E}, \mathrm{I}, \mathrm{J}, \dots$
Entropy functionals: $\mathrm{H}, \mathrm{L}. \dots$

The energy functionals involve mixed Monge–Ampère measures.

For example:

$$\mathrm{E}(\varphi) = \frac{1}{n+1} \sum_{j=0}^{n} \int \varphi \, \mathsf{MA}(\varphi, \dots, \varphi, 0, \dots, 0)$$

When $\varphi = \varphi_{\mathcal{L}} \in \mathcal{H}$, this becomes an intersection number.

Energy functionals are continuous under monotone limits.

Entropy functionals

The entropy functionals also use the *log discrepancy function*

$$A_X \colon X^{\mathrm{an}} \to \mathbf{R} \cup \{+\infty\}$$

for X is smooth (or klt). Related to Temkin's canonical metric.

For example,

$$\mathrm{H}(\varphi) = \int A_X \, \mathsf{MA}(\varphi)$$

Here H is *not* continuous under monotone limits.

Entropy regularization conj: given $\varphi \in \mathcal{E}^1$ there exists a decreasing sequence $(\varphi_n)_n$ in $\mathcal{H}$ s.t. $\varphi_n \to \varphi$ and $\mathrm{H}(\varphi_n) \to \mathrm{H}(\varphi)$.

The *Mabuchi functional* is defined by

$$\mathrm{Mab}(\varphi) = \mathrm{H}(\varphi) + \mathrm{F}(\varphi)$$

for a suitable energy functional F.

Stability and cscK metrics

Are these spaces and functionals useful?

Chi Li '20 gave a sufficient stability criterion for the existence of cscK metrics.

Thm. Assume X smooth, $k = \mathbb{C}$, and that (X, L) satisfies

$$\mathrm{Mab} \geq \varepsilon \mathrm{J} \quad \text{on } \mathcal{E}^1 \tag{*}$$

Then there exists a cscK metric $\omega \in c_1(L)$.

Idea of proof:

(1) Using geodesic rays in the Archimedean version $\mathcal{E}^1$, propagate (*) above to a similar inequality there.

(2) Use deep results by Darvas–Rubinstein and Chen–Cheng to conclude.

The entropy regularization conjecture implies that (*) is also a *necessary* condition, by work of Boucksom–Hisamoto–J.