Arithmetic overconvergent differential operators

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Notations-Review of basic definitions

Let p be a prime number. In this talk, $\mathfrak{o}$ denotes a discrete valuation ring, of inequal char. $(0,p)$ with fraction field K, residue field k, π a uniformizer of $\mathfrak{o}$. Let $\mathcal{P}$ be a smooth formal $\mathfrak{o}$-scheme. Denote by X the special fiber of a formal scheme $\mathfrak{X}$ over $\mathfrak{o}$.

If $\mathcal{P}$ is affine and has some local coordinates $x_1,\dots,x_M$, note $\partial_1,\dots,\partial_M$ the corresponding derivations, then we recall that Berthelot contructed the ring of arithmetic differential operators locally defined by

$$\mathscr{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}(\mathcal{P}) = \left\{ \sum_{\underline{\nu}} a_{\underline{\nu}} \underline{\partial}^{[\underline{\nu}]} \mid a_{\underline{\nu}} \in \mathcal{O}_{\mathcal{P},\mathbb{Q}}(\mathcal{P}), \text{ and } \exists C>0, \eta<1 \mid \|a_{\underline{\nu}}\| < C\eta^{|\underline{\nu}|} \right\} ,$$

where the norm is any spectral norm on the Tate algebra $\mathcal{O}_{\mathcal{P}}(\mathcal{P}) \otimes K$.

Take $Z \subset P$ a divisor, then Berthelot defines over $\mathcal{P}$ a sheaf of overconvergent functions $\mathcal{O}_{\mathcal{P}}({}^\dagger Z)$. If locally on $\mathcal{P}$, Z is defined by the equation $f=0$ then

$$\mathcal{O}_{\mathcal{P},\mathbb{Q}}({}^\dagger Z)(\mathcal{P}) = \left\{ \sum_{l\in\mathbb{N}} \frac{a_l}{f^l} \mid a_l \in \mathcal{O}_{\mathcal{P},\mathbb{Q}}(\mathcal{P}), \text{ and } \exists C>0, \eta<1 \mid \|a_l\| < C\eta^{l} \right\} .$$

Motivation

Berthelot defines also a sheaf of rings $\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z)$, with overconvergent singularities along Z. Locally on $\mathcal{P}$

$$\mathscr{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}({}^{\dagger}Z) = \left\{ \sum_{l,\underline{\nu}} \frac{a_{l,\underline{\nu}}}{f^l} \underline{\partial}^{[\underline{\nu}]} \mid a_{l,\underline{\nu}} \in \mathcal{O}_{\mathcal{P},\mathbb{Q}}(\mathcal{P}), \exists C > 0, \eta < 1 \mid \|a_{l,\underline{\nu}}\| < C\eta^{|\underline{\nu}|+l} \right\} .$$

Among the different reasons to introduce this sheaf let us mention the following facts. If $\mathcal{P}_K$ is the (rigid analytic) generic fiber of $\mathcal{P}$, we have a specialization map $sp : \mathcal{P}_K \to \mathcal{P}$. Denote by $U = P \backslash Z$, and consider E an overconvergent isocrystal on U along Z. Then we have

Theorem (Berthelot) : The functor sp_* gives an equivalence of categories between the category of overconvergent isocrystal and the coherent $\mathscr{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}({}^{\dagger}Z)$-modules, which are coherent $\mathcal{O}_{\mathcal{P},\mathbb{Q}}({}^{\dagger}Z)$-modules.

If, furthermore, E is endowed with a Frobenius structure, then the statement holds with the category of coherent $\mathscr{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-modules, which are coherent $\mathcal{O}_{\mathcal{P},\mathbb{Q}}({}^{\dagger}Z)$-modules (**Theorem of Caro-Tsuzuki**).

Denote $\mathcal{U} = \mathcal{P} \backslash Z$, Berthelot proved the following caracterization of overconvergent isocrystals :

Theorem (Berthelot) : A coherent $\mathscr{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}({}^{\dagger}Z)$ is associated (via *sp*) to an overconvergent isocrystal if and only if $\mathcal{E}_{|\mathcal{U}}$ is a coherent $\mathcal{O}_{|\mathcal{U},\mathbb{Q}}$-module.

Moreover Berthelot proved a localisation exact sequence, that allows to proceed on induction on the support of a $\mathscr{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-module. If $\mathcal{E}$ is a coherent $\mathscr{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-module there is a triangle in the derived categories of $\mathscr{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-modules

$$R\Gamma_Z(\mathcal{E}) \to \mathcal{E} \to \mathcal{E}({}^{\dagger}Z) \stackrel{+1}{\to} .$$

In other words the sheaf $\mathcal{E}({}^{\dagger}Z)$ can be considered as the localization of $\mathcal{E}$ to the formal open subscheme $\mathcal{U}$.

Caro's contruction of the derived category of overholonomic $\mathscr{D}^\dagger_{\mathcal{P}}$-modules involves also this ring of arithmetic differential operators with overconvergent coefficients.

Finally, it is possible to prove that this ring $\mathscr{D}^\dagger_{\mathcal{P},\mathbb{Q}}({}^\dagger Z)$ only depends on $\mathcal{U}$, not on $\mathcal{P}$.

Caro's approach

Caro gave a global version of Berthelot's theorem.

This approach is based on the following result : take $\mathcal{P} = \hat{\mathbb{P}}^N$ the formal projective space of dim. N, Z the divisor at infinity, $t_1, \ldots, t_N$ coordinates on $\hat{\mathbb{A}}^N = \hat{\mathbb{P}}^N \backslash Z$, and denote

$$A_N(K)^\dagger = \left\{ \sum_{\underline{\nu}} a_{\underline{\mu},\underline{\nu}} \underline{t}^{\underline{\mu}} \underline{\partial}^{[\underline{\nu}]} \mid a_{\underline{\mu},\underline{\nu}} \in K \text{ and } \exists C > 0, \eta < 1 \mid \|a_{\mu,\underline{\nu}}\| < C\eta^{|\underline{\mu}|+|\underline{\nu}|} \right\}.$$

Proposition (H) :

1. $$\Gamma(\mathcal{P}, \mathcal{O}_{\mathcal{P},\mathbb{Q}}({}^\dagger Z)) = K < t_1, \ldots, t_n >^\dagger,$$
2. $$\Gamma(\mathcal{P}, \mathscr{D}^\dagger_{\mathcal{P},\mathbb{Q}}({}^\dagger Z)) = A_N(K)^\dagger.$$

Caro's approach

Caro considers a **complete frame** situation : let Y be a smooth k-scheme, $Y_{\mathfrak{o}}$ a smooth lifting of Y over $\mathrm{Spec}\ \mathfrak{o}$ (by Elkik's theorem), $Y_{\mathfrak{o}} = \mathrm{Spec}\, \mathfrak{o}[t_1, \ldots, t_N]/I$, that gives a closed immersion $Y \hookrightarrow \mathbb{A}^N_{\mathfrak{o}}$. Denote by $U^\dagger = (\mathbb{A}^N_{\mathfrak{o}})^\dagger$ the weak completion (in the sense of Meredith) of the affine space of dim. N. We can take the weak completion of these schemes to find a closed immersion of weak formal schemes v : $Y^\dagger \hookrightarrow U^\dagger$.

In this situation, Mebkhout-Narvaez-Maccaro defined weakly complete rings of differential operators over these weakly complete schemes : $\mathscr{D}^\dagger_{Y^\dagger}$ and $\mathscr{D}^\dagger_{U^\dagger}$. Caro defined the cohomological operations for these sheaves of rings.

Let E be an overconvergent isocrystal over $Y^\dagger$, $\mathcal{E}^{(0)}$ the associated coherent $\mathcal{D}^{(0)}_{U^\dagger}$-module, then we have

Definition (Caro) :

$$sp_{Y^\dagger \hookrightarrow U^\dagger, Z, +} E := \mathscr{D}^\dagger_{\mathcal{P}}({}^\dagger Z) \otimes_{\mathscr{D}^{(0)}_{U^\dagger}} v_+(\mathcal{E}^{(0)}).$$

Caro denotes by $Isoc^{\dagger\dagger}(Y, \mathcal{P})/K$ the essential image of this functor. Then

Theorem (Caro) : The functor $sp_{Y^\dagger \hookrightarrow U^\dagger, Z, +}$ is an equivalence of categories between the category of overconvergent isocrystals on Y_s and the category $Isoc^{\dagger\dagger}(Y, \mathcal{P})/K$.

The local frame situation

We want to work with a local frame, which is a triple $(Y, X, \mathcal{P})$ with

1. $\mathcal{P}$ is a smooth affine formal scheme over $\mathfrak{o}$,
2. X is a closed scheme of P,
3. Y is a smooth affine k-scheme, with local coordinates (for this talk),
4. smooth outside a divisor : there exists a divisor $Z \hookrightarrow P$ such that $X \backslash Y = X \bigcap Z$.

Denote $\mathcal{U} = \mathcal{P} \backslash Z$, j the open immersion $\mathcal{U} \subset \mathcal{P}$. The affine scheme Y (resp. P) can be lifted to a smooth $\mathfrak{o}$-scheme $Y_{\mathfrak{o}} = \mathrm{Spec}\, C$ (resp. $\mathrm{Spec}\, \mathsf{A}$). Denote by $\mathcal{Y}$ and $Y^{\dagger}$ the completion and the weak completion of $Y_{\mathfrak{o}}$.

Why considering this situation ? Because if you work on induction on the dimension of the support of a coherent $\mathscr{D}^{\dagger}_{\mathcal{P}}$-module, you can always restrict to this situation.

Main problem : the scheme X is usually not smooth so that you can not work with differential operators on X. To solve this problem, one usually applies de Jong's theorem. We can also observe that in the rigid analytic situation, there is a basis of smooth strict neighborhoods of $]Y[_{\mathcal{P}}$ into $]X[_{\mathcal{P}}$.

The construction

Idea : try to construct a ring of overconvergent singularities on $\mathcal{Y}$. You first need overconvergent functions. We work with $\mathcal{O}_{\mathcal{P}}({}^{\dagger}Z)$ locally defined by

$$\mathcal{O}_{\mathcal{P}}({}^{\dagger}Z)(\mathcal{P}) = \left\{ \sum_{l\in\mathbb{N}} \frac{a_l}{f^l} \,|\, a_l \in \mathcal{O}_{\mathcal{P}}(\mathcal{P}), \text{ and } \exists C > 0, \eta < 1 \,|\, \|a_l\| < C\eta^l \right\}.$$

This sheaf is obtained as the weak completion of the sheaf of meromorphic functions along Z, and $(\mathcal{O}_{\mathcal{P}}(*Z))^{\dagger}$, also denoted $\mathcal{O}_{\mathcal{P}}({}^{\dagger}*Z)$. Over affine opens, this sheaf has weakly complete, noetherian, sections. Moreover

$$\mathcal{O}_{\mathcal{P}}({}^{\dagger}*Z)/\pi\mathcal{O}_{\mathcal{P}}({}^{\dagger}*Z) \simeq j_*\mathcal{O}_{U_0},\ \mathcal{O}_{\mathcal{P}}({}^{\dagger}*Z) \otimes \mathbb{Q} \simeq \mathcal{O}_{\mathcal{P},\mathbb{Q}}({}^{\dagger}Z).$$

Moreover we have :

Proposition : (Caro) The canonical morphism : $\mathcal{O}_{\mathcal{P}}({}^{\dagger}*Z) \to j_*\mathcal{O}_{\mathcal{U}}$ is faithfully flat.

We now consider the following sheaf $\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}{*}Z)$ which is by definition $\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z)(*Z)$.

Proposition : (H) The sheaf $\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}{*}Z)$ is a sheaf of rings.

As before, we have :

$$\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}{*}Z)/\pi\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}{*}Z) \simeq j_*\mathscr{D}_{U_0},\ \mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}{*}Z)\otimes\mathbb{Q} \simeq \mathscr{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}({}^{\dagger}Z).$$

This construction was needed to prove that the ring $\mathscr{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}Z)$ has finite cohomological dimension. Moreover we have :

Proposition : (Caro) The canonical morphism : $\mathscr{D}_{\mathcal{P}}({}^{\dagger}{*}Z) \to j_*\mathscr{D}^{\dagger}_{\mathcal{U}}$ is faithfully flat.

Coming back to the frame situation, assume that $Z = V(f)$, denote by $A^{\dagger}$ and $C^{\dagger}$ the weak completions of A et C resp.. Then $U^{\dagger}$ corresponding to $(A^{\dagger}[1/f])^{\dagger}$ is a weak formal scheme lifting U. The closed immersion of special fibers $Y \hookrightarrow U$ can be lifted to a closed immersion of the associated formal schemes $Y^{\dagger} \hookrightarrow U^{\dagger}$. Denote by $I^{\dagger} = Ker(A^{\dagger}[1/f]^{\dagger} \to C^{\dagger})$.

We have then the following inclusions

$$(A^{\dagger}[1/f])^{\dagger} \subset \hat{A}[1/f]^{\dagger} = \Gamma(\mathcal{P}, \mathcal{O}_{\mathcal{P}}({}^{\dagger}{*}Z)),$$

so that we can consider

$$\mathcal{O}_{\mathcal{Y}^{\dagger}} := \mathcal{O}_{\mathcal{P}}({}^{\dagger}{*}Z)/I^{\dagger}\mathcal{O}_{\mathcal{P}}({}^{\dagger}{*}Z).$$

Denote by s the surjection : $\mathcal{O}_{\mathcal{P}}({}^{\dagger}{*}Z) \to \mathcal{O}_{\mathcal{Y}^{\dagger}}$.

We have the following properties :

$$\mathcal{O}_{\mathcal{Y}^{\dagger}}/\pi\mathcal{O}_{\mathcal{Y}^{\dagger}} \simeq \mathcal{O}_{Y},\ \mathcal{O}_{\mathcal{Y}^{\dagger},\mathbb{Q}} \simeq sp_{*}j^{\dagger}\mathcal{O}_{]X[},$$

and

Proposition : The canonical morphism : $\mathcal{O}_{\mathcal{Y}^{\dagger}} \to \mathcal{O}_{\mathcal{Y}}$ is faithfully flat. The sheaf $\mathcal{O}_{\mathcal{Y}^{\dagger}}$ has noetherian sections over affine opens.

Overconvergent Differential operators

We apply now constructions of Mebkhout-Narvaez-Maccaro to construct differential operators. Denote by $t_1, \dots, t_n$ liftings in $\mathcal{O}_{\mathcal{Y}^\dagger}$ of a system of coordinates on Y, we have :

Proposition : The sheaf

$$\Omega^1_{\mathcal{Y}^\dagger} := \mathcal{O}_{\mathcal{Y}^\dagger} dt_1 \bigoplus \dots \bigoplus \mathcal{O}_{\mathcal{Y}^\dagger} dt_n / \{dh \,|\, h \in I^\dagger\},$$

is free on Y.

Finally, we can follow Mebkhout-Narvaez-Maccaro to construct arithmetic differential operators $\mathscr{D}^\dagger_{\mathcal{Y}^\dagger}$. By definition, if $Y = U$, then $\mathscr{D}^\dagger_{\mathcal{Y}^\dagger} \simeq \mathscr{D}^\dagger_{\mathcal{P}}({}^\dagger Z)$.

We have the following properties : this sheaf is coherent, and

$$\mathscr{D}^\dagger_{\mathcal{Y}^\dagger} / \pi \mathscr{D}^\dagger_{\mathcal{Y}^\dagger} \simeq \mathscr{D}_Y.$$

Proposition : The canonical map : $\mathscr{D}^\dagger_{\mathcal{Y}^\dagger} \to \mathscr{D}^\dagger_{\mathcal{Y}}$ is faithfully flat.

Local description

We use derivations $\partial_, \dots, \partial_n$ given by the dual basis of previous $dt_1, \dots, dt_n$.

$$\mathscr{D}^{\dagger}_{\mathcal{Y}^{\dagger},\mathbb{Q}} = \left\{ \sum_{l,\underline{\nu}} s\left(\frac{a_{l,\underline{\nu}}}{f^l}\right) \underline{\partial}^{[\underline{\nu}]} \mid a_{l,\underline{\nu}} \in \mathcal{O}_{\mathcal{P},\mathbb{Q}}, \text{ and } \exists C > 0, \eta < 1 \mid \|a_{l,\underline{\nu}}\| < C\eta^{|\underline{\nu}|+l} \right\}$$

Possible applications

Applications, Work in Progress

Let E be an overconvergent isocrystal on Y, along Z, then $sp_* E$ is a coherent $\mathscr{D}^\dagger_{\mathcal{Y}^\dagger,\mathbb{Q}}$-module, which is a coherent $\mathcal{O}_{\mathcal{Y}^\dagger,\mathbb{Q}}$-module. We should get this way an equivalence of categories. And we have

Proposition :(H) A coherent $\mathscr{D}^\dagger_{\mathcal{Y}^\dagger,\mathbb{Q}}$-module $\mathcal{E}$ is associated to an overconvergent isocrystal if and only if

$$\mathcal{E}_{|\mathcal{Y}} := \mathscr{D}^\dagger_{\mathcal{Y}} \otimes_{\mathscr{D}^\dagger_{\mathcal{Y}^\dagger}} \mathcal{E},$$

is a coherent $\mathcal{O}_{\mathcal{Y},\mathbb{Q}}$-module.

This also should simplify some constructions in Caro's theory.

Still to be done : do classical constructions of Berthelot using this sheaf.

Thank you for your attention !