Twisted local Simpson correspondence and crystals on the q-crystalline and prismatic sites

Michel Gros (CNRS & Université de Rennes I)
(joint work in progress with A. Quirós and B. Le Stum)

Imperial College of London, December 8-10, 2020

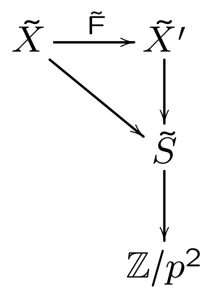
Tropical Geometry, Berkovich Spaces, Arithmetic $\mathscr{D}$-Modules and p-adic Local Systems

I. Untwisted model : Simpson correspondence in characteristic $p > 0$ and its generalizations

1. The result

S a $\mathbb{Z}/p\mathbb{Z}$-scheme, F_S its absolute Frobenius, X a smooth S-scheme, , $X' := X \times_{S,\mathsf{F}_S} S$, $\mathsf{F} : X \to X'$ the *relative* Frobenius.

$\widetilde{S}$ a flat $\mathbb{Z}/p^2\mathbb{Z}$-scheme and a smooth lift over $\widetilde{S}$ of the previous data :



Theorem (Berthelot, Ogus-Vologodsky) *To such lift over $\mathbb{Z}/p^2\mathbb{Z}$ is canonically associated :*

(i) *an equivalence of categories*

$$[\widetilde{\mathsf{F}}] : \mathscr{O}_{X'} - \mathsf{HIG}^{\mathsf{qn}} \to \mathscr{O}_X - \mathsf{MCI}^{\mathsf{qn}}$$

*between the one of $\mathscr{O}_{X'}$-Higgs modules $(\mathscr{F}, \theta)$ ($\theta : \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_{X'}} \Omega^1_{X'/S}$ a $\mathscr{O}_{X'}$-**linear** morphism such that $\theta \wedge \theta = 0$) with quasi-nilpotent Higgs field and the one of $\mathscr{O}_X$-modules with quasi-nilpotent integrable connection $(\mathscr{E}, \nabla)$.*

(ii) *a quasi-isomorphism between the Higgs complex of $(\mathscr{F}, \theta)$ and the direct image by* F *of the de Rham complex of $(\mathscr{E}, \nabla)$*

$$(\mathscr{F} \xrightarrow{\theta} \mathscr{F} \otimes_{\mathscr{O}_{X'}} \Omega^1_{X'/S} \xrightarrow{\theta\wedge -} \mathscr{F} \otimes_{\mathscr{O}_{X'}} \Omega^2_{X'/S} \to \ldots] \xrightarrow{\approx} \mathsf{F}_* \mathsf{DR}^{\bullet}(\mathscr{E}, \nabla)$$

2. Elements of Berthelot's proof

$\mathscr{D}_X := \mathscr{D}_{X/S}$: the algebra of *crystalline* (or *level 0* in Berthelot's terminology) differentiel operators (generated locally over $\mathscr{O}_X$ by the derivations ∂) ;

$\mathsf{T}_{X'} := \mathsf{T}_{X'/S}$: the tangent sheaf of X' ; $\mathsf{S}(\mathsf{T}_{X'})$ its symmetric algebra which is also the center $\mathsf{Z}_{\mathscr{D}_X}(\mathscr{D}_X)$ of $\mathscr{D}_X$;

$\mathscr{K}_X := \mathsf{Z}_{\mathscr{D}_X}(\mathscr{O}_X)$: the centralizer of $\mathscr{O}_X$ in $\mathscr{D}_X$;

$\widehat{\mathscr{D}}_X := \varprojlim_{n\geq 0} \mathscr{D}_X/\mathscr{K}_X^n$; $\widehat{\mathsf{S}(\mathsf{T}_{X'})} := \varprojlim_{n\geq 0} \mathsf{S}(\mathsf{T}_{X'})/(\mathscr{K}_X^n \cap \mathsf{S}(\mathsf{T}_{X'}))$ the corresponding completions.

Theorem (Berthelot) *To each lift* $(\widetilde{S}, \widetilde{X}, \widetilde{X}', \widetilde{\mathsf{F}})$ *over* $\mathbb{Z}/p^2\mathbb{Z}$ *as before is canonically associated an isomorphism of* $\widehat{\mathsf{S}(\mathsf{T}_{X'})}$*-algebras :*

$$\widehat{\mathscr{D}}_X \xrightarrow{\sim} \mathscr{E}nd_{\widehat{\mathsf{S}(\mathsf{T}_{X'})}}(\mathscr{O}_X \otimes_{\mathscr{O}_{X'}} \widehat{\mathsf{S}(\mathsf{T}_{X'})}).$$

$\mathscr{I} \subset \mathscr{O}_{X\times_S X}$ the ideal of the diagonal, $\mathscr{P}_{X,(0)}$ the PD-enveloppe of $\mathscr{I}$, $\overline{\mathscr{I}} \subset \mathscr{P}_{X,(0)}$ the PD-ideal generated by $\mathscr{I}$. Sheaf of principal parts (of level 0) and order n :

$$\mathscr{P}^n_{X,(0)} = \mathscr{P}_{X,(0)}/\overline{\mathscr{I}}^{[n+1]}$$

$$\mathscr{D}^{(0)}_{X,n} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{P}^n_{X,(0)}, \mathscr{O}_X) \; ; \; \mathscr{D}_X = \varinjlim_{n\in\mathbb{N}} \mathscr{D}^{(0)}_{X,n}.$$

It exists a well-defined application (*divided Frobenius*)

$$\frac{1}{p!}(\widetilde{\mathsf{F}}^* \times \widetilde{\mathsf{F}}^*) : \tilde{\mathscr{I}}' \to p\mathscr{P}_{\widetilde{X},(0)} \quad \underset{.p!}{\overset{\sim}{\longleftarrow}} \quad \mathscr{P}_{\widetilde{X},(0)}/p\mathscr{P}_{\widetilde{X},(0)}$$
$$\overset{\sim}{\to} \quad \mathscr{P}_{X,(0)}$$

coming from the study of $(\widetilde{\mathsf{F}} \times \widetilde{\mathsf{F}})^* : \tilde{\mathscr{I}}' \to \tilde{\mathscr{I}}$:

Take $x \in \mathscr{O}_X$, $x' := 1 \otimes x \in \mathscr{O}_{X'}$ with liftings $\tilde{x} \in \mathscr{O}_{\tilde{X}}$, $\tilde{x}' \in \mathscr{O}_{\tilde{X}'}$, $\widetilde{\mathsf{F}}^*(\tilde{x}') = \tilde{x}^p + py$.

Let $\tilde{\xi} := 1 \otimes \tilde{x} - \tilde{x} \otimes 1$, $\tilde{\xi}' := 1 \otimes \tilde{x}' - \tilde{x}' \otimes 1$, then

$$\begin{aligned}(\widetilde{\mathsf{F}} \times \widetilde{\mathsf{F}})^*(\tilde{\xi}') &= 1 \otimes \tilde{x}^p - \tilde{x}^p \otimes 1 + p.(1 \otimes y - y \otimes 1) \\ &= \tilde{\xi}^p + \sum_{i=1}^{p-1} \binom{p}{i} (\tilde{x}^{p-i} \otimes 1)\tilde{\xi}^i + p(1 \otimes y - y \otimes 1) \\ &\equiv \tilde{\xi}^p \bmod p\tilde{\mathscr{I}}\end{aligned}$$

Now, going to $\mathscr{P}_{\widetilde{X},(0)}$,

$$(\widetilde{\mathsf{F}} \times \widetilde{\mathsf{F}})^*(\tilde{\xi}') = p!.\tilde{\xi}^{[p]} + p\zeta \in \mathscr{P}_{\widetilde{X},(0)}$$

with $\zeta \in \tilde{\mathscr{I}}$.

Divide by $p!$, factorize by $\Omega^1_{X'/S}$ and finally extend to an isomorphism

$$\sigma_{\widetilde{\mathsf{F}}} : \mathscr{O}_{X \times_{X'} X} \ \otimes_{\mathscr{O}_{X'}} \ \Gamma(\Omega^1_{X'/S}) \overset{\sim}{\to} \mathscr{P}_{X,(0)}.$$

3. Generalization and Topos reinterpretation

Suppose now that $\widetilde{X}, \widetilde{X}' \to \widetilde{S}$ smooth morphisms of formal schemes over $\mathbb{Z}_p$.
$\mathscr{O}_{\widetilde{X}}$-modules $\mathscr{E}$ endowed with a p^m-integrable connection ($m \geq 0$) :

$$\nabla(fe) = f\nabla(e) + p^m e \otimes df \; ; \; f \in \mathscr{O}_{\widetilde{X}}, e \in \mathscr{E}$$

form a category $\mathscr{O}_{\widetilde{X}_n} - \mathrm{MCI}^{(-m)}$.

Theorem (Shiho) *For $m = 1$, $\widetilde{\mathsf{F}}^*$ induces, for every $n \in \mathbb{N}$, an equivalence of categories*

$$[\widetilde{\mathsf{F}}] : \mathscr{O}_{\widetilde{X}'_n} - \mathrm{MCI}^{(-1),\mathrm{qn}} \to \mathscr{O}_{\widetilde{X}_n} - \mathrm{MCI}^{\mathrm{qn}}.$$

Dilatation ("$\xi \mapsto \xi/p^m$") on principal parts allows to construct a $\mathscr{O}_{\widetilde{X}}$-algebra $\mathscr{D}_{\widetilde{X}}^{(-m)}$ of differential operators (of *level* $-m$). Locally a basis over $\mathscr{O}_{\widetilde{X}}$ given by the $(\partial^{<l>_{-m}})_{l \geq 0}$'s acting on $f \in \mathscr{O}_{\widetilde{X}}$ by $\partial^{<l>_{-m}}(f) = l! p^{ml} \partial^{[l]}(f)$, e.g. $\boxed{\partial^{<1>_{-1}} x - x \partial^{<1>_{-1}} = p}$ ("$\partial^{<1>_{-1}} = p\partial$"). Former computation on principal parts as before gives the result.

Let W the ring of Witt vectors over a perfect field of characteristic $p > 0$, $\widetilde{S} := \mathrm{Spf}(W)$. It exists two ringed topos $(\widetilde{\mathscr{E}}, \mathscr{O}_{\mathscr{E}})$, $(\widetilde{\underline{\mathscr{E}}}, \mathscr{O}_{\underline{\mathscr{E}}})$ and

Theorem (Oyama ($n = 1$), Xu) *Shiho's equivalence sits into a commutative diagram :*

$$\begin{array}{ccc} \text{Crystals}(\mathscr{O}_{\mathscr{E}',n} - \text{Mod}) & \xrightarrow[\mathsf{C}^*]{\sim} & \text{Crystals}(\mathscr{O}_{\underline{\mathscr{E}},n} - \text{Mod}) \\ \downarrow & & \downarrow \\ \mathscr{O}_{\widetilde{X}'_n} - \mathrm{MCI}^{(1),\mathrm{qn}} & \xrightarrow[{[\widetilde{\mathsf{F}}]}]{\sim} & \mathscr{O}_{\widetilde{X}_n} - \mathrm{MCI}^{\mathrm{qn}} \end{array}$$

C^* induced by a morphism between sites (Cartier's transform).

II. Twisted Simpson correspondence

R : ring, $q \in R^{\times}$, $R[t]$ endowed with σ s.t. $\sigma(t) = qt$, $f : R[t] \to A$ étale, , A endowed with an automorphism σ s.t. $\sigma(x) = qx$ for $x := f(t)$. For $n \in \mathbb{N}$, $(n)_q = 1 + q + q^2 + ... + q^{n-1}$.

1. Twisted differential operators of level $-m$, $m \geq 0$

Example. If $A = R[t]$, $\mathrm{D}_{A,q} := \mathrm{D}^{(0)}_{A,q}$ is $R[t, \partial_q]_{\mathrm{nc}} / < \partial_q t - qt\partial_q = 1 >$.

$\mathrm{D}^{(-1)}_{A,q}$ is the subalgebra of $\mathrm{D}_{A,q}$ generated by $\partial_q^{<1>_{-1}} := (p)_q \partial_q$. One has $\boxed{\partial_q^{<1>_{-1}} t - qt\partial_q^{<1>_{-1}} = (p)_q}$.

Let $y := (1-q)x$ and $A\langle\xi\rangle_{q,y}$ the free A-module with generators $\xi^{[n]_{q,y}}$ (or simply $\xi^{[n]}$), $n \in \mathbb{N}$ endowed with the structure of unitary commutative A-algebra given by

$$\xi^{[n]}.\xi^{[m]} = \sum_{i=0}^{\min(m,n)} (-1)^i q^{\frac{i(i-1)}{2}} \binom{m+n-i}{m}_q \binom{m}{i}_q y^i \xi^{[m+n-i]}$$

(the $(n)_q! \xi^{[n]} = \prod_{i=0}^{n-1} (\xi + (i)_q y) =: \xi^{(n)}$ form a basis of $A[\xi]$.)

Let $I^{[n+1]_{q,y}}$ be the free$A$-submodule generated by the $\xi^{[k]_{q,y}}$ with $k > n$,

$$\mathrm{P}_{A,q,n} := \mathrm{P}^{(0)}_{A/R,q,n} = A\langle\xi\rangle_{q,y} / I^{[n+1]_{q,y}} \ ; \ \mathrm{P}_{A,q} = \varprojlim_{n \in \mathbb{N}} \mathrm{P}_{A,q,n}.$$

One has a *twisted* Taylor application :

$$\mathscr{T} : A \to \mathrm{P}_{A,q} \ ; \ \mathscr{T}(z) = \sum_{k=0}^{+\infty} \partial_q^k(z) \xi^{[k]}$$

allowing for M a left A-module to define $\mathrm{P}_{A,q,n} \otimes'_A M$ with $\xi^{[k]} \otimes' zs = \mathscr{T}(z)\xi^{[k]} \otimes' s$.

Finally,

$$\mathsf{D}_{A,q,n} = \mathsf{Hom}_R(\mathsf{P}_{A,q,n} \otimes'_A A, A) \; ; \; \mathsf{D}_{A,q} = \bigcup_{n\geq 0} \mathsf{D}_{A,q,n}$$

with ring structure coming from the coproduct $\mathsf{P}_{A,q} \to \mathsf{P}_{A,q} \otimes'_A \mathsf{P}_{A,q}$; $\xi^{[n]} \mapsto \sum_{i=0}^{n} \xi^{[n-i]} \otimes' \xi^{[i]}$.

For $m \geq 0$, let $A\langle\omega\rangle_{q(-m)} := A\langle\omega\rangle_{q^{p^m},y}$; $\omega^{\{n\}_q} := \omega^{[n]_{q^{p^m},y}}$ (Morally : $\omega = \frac{\xi}{(p^m)_q}$; $\omega^{\{n\}_q} = \frac{\xi^{[n]_{q^{p^m}}}}{(p^m)_q^n}$) with algebra structure

$$\omega^{\{n\}_q}.\omega^{\{m\}_q} = \sum_{i=0}^{\min(m,n)} (-1)^i q^{\frac{p^m i(i-1)}{2}} \begin{pmatrix} m+n-i \\ m \end{pmatrix}_{q^{p^m}} \begin{pmatrix} m \\ i \end{pmatrix}_{q^{p^m}} y^i \omega^{\{m+n-i\}_q}$$

Along the same lines as for $m=0$, one define a ring $\mathsf{D}_{A,q}^{(-m)}$ which is essentially the A-subalgebra of $\mathsf{D}_{A,q^{p^m}}$ generated by $\partial_q^{<1>_{-m}} := (p^m)_q \partial_q^{p^m}$ (at least if A has no $(p)_q$-torsion).

Modules over $\mathsf{D}_{A,q}^{(-m)}$ are the same as A-modules endowed with a twisted connection of level $-m$, i.e. R-linear application $\nabla_q : M \to M \otimes \Omega_{A,q^{p^m}}$ ($\Omega_{A,\sigma} = I/I\sigma(I)$) s.t.

$$\nabla_q(fs) = (p^m)_q \, s \otimes \mathsf{d}_{q^{p^m}} f + \sigma^{p^m}(f)\nabla_q(s).$$

3. Divided Frobenius

$\tilde{\mathsf{F}}_R : R \to R$ lifting of the absolute Frobenius of R/p s.t. $\tilde{\mathsf{F}}_R(q) = q^p$;

$\tilde{\mathsf{F}}_A : A \to A$ lifting of the absolute Frobenius of A/p s.t. $\tilde{\mathsf{F}}_A(x) = x^p$;

$\tilde{\mathsf{F}}_{A[\xi]} : \xi \mapsto (\xi + x)^p - x^p$;

Let $A' = A \otimes_{R,\widetilde{F}_R} R$; $y' := y \otimes 1$.

Theorem *The relative Frobenius* $\tilde{\mathsf{F}} : A'[\xi] \to A[\xi]$ *(*$\xi \mapsto (\xi + x)^p - x^p$*) canonically defines a divided Frobenius morphism*

$$[\tilde{\mathsf{F}}]_q : A'\langle\omega\rangle_{q(-1)} \to A\langle\xi\rangle_q$$

explicitly given by :

$$\omega^{\{n\}} \mapsto \sum_{i=n}^{pn} \mathsf{b}_{n,i}(q) x^{pn-i} \xi^{[i]}$$

with $\mathsf{a}_{n,i}(u) := \sum_{j=0}^{n} (-1)^{n-j} u^{\frac{p(n-j)(n-j-1)}{2}} \begin{pmatrix} n \\ j \end{pmatrix}_{u^p} \begin{pmatrix} pj \\ i \end{pmatrix}_u \in \mathbb{Z}[u]$, $\mathsf{b}_{n,i}(u) := \frac{(i)_q!}{(n)_{q^p}!(p)_q^n} \mathsf{a}_{n,i}(u) \in \mathbb{Z}[u]$.

4. Consequences : twisted local Simpson correspondence

Theorem *The divided Frobenius morphism* $[\tilde{\mathsf{F}}]_q$ *induces an equivalence of categories between the category* $A' - \mathrm{MCI}_q^{(-1),\mathrm{qn}}$ *of* A'*-modules of finite presentation endowed with a twisted connection of level -1 topologically quasi-nilpotent and the one* $A - \mathrm{MCI}_q^{\mathrm{qn}}$ *of* A*-modules of finite presentation endowed with a twisted connection (of level 0) topologically quasi-nilpotent.*

For $q = 1$ (“undeformed case”), it’s Shiho equivalence.

Pour $q = \zeta \neq 1$, $\zeta^p = 1$, one can show that, along the same lines as in characteristic $p > 0$, the completion of $\mathrm{D}_{A,q}$ is a matrix algebra. In particular:

Corollary *One has an isomorphism*

$$C_q : \mathrm{H}^i(q\text{-}\mathrm{DR}(A)) \tilde{\to} \Omega^i_{A'}$$

for all $i \geq 0$.

V. Twisted Simpson correspondance and q-crystalline and prismatic sites

1. $(p)_q$-Prisms, q-PD-pairs

δ-ring : $(R, \delta : R \to R)$ with R a $\mathbb{Z}_{(p)}$-algebra and $\delta : R \to R$ s.t. :

$\delta(0) = \delta(1) = 0$; $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$; $\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p+y^p-(x+y)^p}{p}$

Ensure that $\phi(x) := x^p + p\delta(x)$ is a lifting of the absolute Frobenius of R/p.

δ-pair (R, I) : a δ-structure on R and $I \subset R$ an ideal.

Assume moreover from now that R is a $\mathbb{Z}[q]_{(p,q-1)}$-algebra (always endowed with the $(p, q-1)$-topology for a fixed $q \in R$ s.t. $\delta(q) = 0$).

(bounded) $(p)_q$-prism over $(R, (p)_q)$: δ-pair of the form $(B, (p)_q)$ with B a complete R-algebra without $(p)_q$-torsion (+ some "technical" conditions)

Example : $(\mathbb{Z}_p[[q-1]], (p)_q)$.

q-PD-pair : $(B, \mathfrak{b})$ with B a complete δ-R-algebra without $(p)_q$-torsion s.t. $\phi(f) - (p)_q\delta(f) \in (p)_q\mathfrak{b}$ (+ some "technical" conditions) and also s.t. $(B, (p)_q)$ is a (bounded) prism over $(R, (p)_q)$: $(B \twoheadrightarrow \bar{B} := B/\mathfrak{b}$ will be called a q-PD-thickening).

Example : $(\mathbb{Z}_p[[q-1]], (q-1))$.

2. <u>q-crystalline site</u>

Fix $(R,\mathfrak{r}) \to (A,\mathfrak{a})$ a morphism of q-PD-pairs, $\mathscr{X} := \mathrm{Spf}(\bar{A})$ ($\bar{A} := A/\mathfrak{a}$).

$q-\mathrm{CRIS}(\mathscr{X}/R)$: morphism $(R,\mathfrak{r}) \to (B,\mathfrak{b})$ to a q-PD-pair (bounded and complete) and morphism $\mathscr{X} \to \mathrm{Spf}(\bar{B})$ of $\bar{R}$-formal schemes ;

$\mathscr{O}_{q-\widetilde{\mathrm{CRIS}(\mathscr{X}}/R)} : [(R,\mathfrak{r}) \to (B,\mathfrak{b}), \mathscr{X} \to \mathrm{Spf}(\bar{B}))] \mapsto B$;

$\mathscr{O}_{q-\widetilde{\mathrm{CRIS}(\mathscr{X}}/R)}$-module = family of B-modules E_B with compatible family of linear applications $B' \otimes_B E_B \to E_{B'}$;

q-crystal : compatible family made of bijective arrows.

Theorem *If A is endowed with a topological coordinate x s.t. $\delta(x) = 0$, it exists a functor $E \mapsto E_A$ from the category of q-crystals of finite presentation on $\bar{A}/R$ to the one of quasi-nilpotents $\mathrm{D}_{A,q}$-modules topologically of finite type.*

Proposition *For A as in the theorem, $(A\langle\xi\rangle_{q,y}, I^{[1]_{q,y}})$ identifies with the q-PD-enveloppe of the δ-pair $(A[\xi],(\xi))$ with* $\delta(\xi) = \sum_{i=1,\dots,p-1} \frac{1}{p}\binom{p}{i} x^{p-i}\xi^i$ (i.e. is the universal q-PD-pair solution of the problem of unique extension to a q-PD-pair of a morphism $(A[\xi],(\xi)) \to (B,((p)_q))$ with $(B,((p)_q))$ a q-PD-pair s.t. B has no $(p)_q$-torsion)).*

*Remember that $\tilde{\mathrm{F}}_{A[\xi]} : \xi \mapsto (\xi + x)^p - x^p$.

3. <u>Prismatic site</u>

R as before and s.t. $(R,(p)_q)$ is a bounded prism $(R,(p)_q)$, $\mathscr{X}$ a p-adic $R/(p)_q$-formal scheme

$(p)_q - \mathsf{PRIS}(\mathscr{X}/R)$: opposite category to the one of $(p)_q$-prisms B on R endowed with a morphism $\mathsf{Spf}(B/(p)_q)) \to R/(p)_q)$;

$\mathcal{O}_{(p)_q - \widetilde{\mathsf{PRIS}(\mathscr{X}/R)}} : [B \to B/(p)_q \leftarrow A/((p)_q)] \mapsto B$

Theorem *If A is endowed with a topological coordinate x s.t. $\delta(x) = 0$ and $\bar{A} := A/(p)_q$, it exists a functor $E \mapsto E_A$ from the category of prismatic crystals of finite presentation on $\bar{A}/R$ to the one of quasi-nilpotents $\mathrm{D}^{(-1)}_{A,q}$-modules topologically of finite type.*

Proposition *For A as in the theorem, $(A\langle\frac{\xi}{(p)_q}\rangle_{q^p,y},((p)_q)))$ identifies with the prismatic enveloppe of the δ-pair $(A[\xi],(\xi))$ (i.e. is the universal $(p)_q$-prism solution of the problem of unique extension to a $(p)_q$-prism of a morphism $(A[\xi],(\xi)) \to (B,((p)_q))$ to a $(p)_q$-prism $(B,((p)_q))$ (B without $(p)_q$-torsion)).*

4. Prismatic Cartier transform

$(R,\mathfrak{r}) \to (A,\mathfrak{a})$ morphism of q-PD-pairs ; $\phi : \mathfrak{r} \to ((p)_q)$ induces $\bar{\phi} : R/\mathfrak{r} \to R/(p)_q$; $A' = A\widehat{\otimes}_{R,\phi}R$

$\mathscr{X} := \mathsf{Spf}(A/\mathfrak{a})$; $\mathscr{X}' := \mathscr{X}\widehat{\otimes}_{R/\mathfrak{r},\bar{\phi}}R/(p)_q = \mathsf{Spf}(A'/(p)_qA')$

$$\mathsf{C}_q : q-\mathsf{CRIS}(\mathscr{X}'/R) \to (p)_q-\mathsf{PRIS}(\mathscr{X}/R)$$

induced by :

$$\begin{array}{rcl} (B,\mathfrak{b}) & \mapsto & (B,(p)_q) \\ (\mathsf{Spf}(B/\mathfrak{b}) \to \mathscr{X}) & \mapsto & (\mathsf{Spf}(B/(p)_q) \to \mathscr{X}') \end{array}$$

of the composite $\mathsf{Spf}(B/(p)_q) \to \mathsf{Spf}(B/\mathfrak{b}) \to \mathscr{X})$

Theorem *If A is endowed with a topological coordinate x s.t. $\delta(x) = 0$, $\bar{A} := A/\mathfrak{a}$, $\bar{A}' := A'/(p)_q$, twisted Simpson correspondence sits into the commutative diagram*

$$\begin{array}{ccc} \mathsf{Crystals}(\mathcal{O}_{(p)_q-\widetilde{\mathsf{PRIS}(\mathscr{X}'/R)}} - \mathsf{Mod}) & \xrightarrow[\mathsf{C}_q^*]{} & \mathsf{Crystals}(\mathcal{O}_{q-\widetilde{\mathsf{CRIS}(\mathscr{X}/R)}} - \mathsf{Mod}) \\ \downarrow & & \downarrow \\ A' - \mathsf{MCI}_q^{(-1),\mathsf{qn}} & \xrightarrow[{[\tilde{\mathsf{F}}]_q}]{\sim} & A - \mathsf{MCI}_q^{\mathsf{qn}} \end{array}$$

Remarks. (i) Adding flat topologies on the sites should (in preparation) turn the four maps into equivalences.

(ii) Recent related works : A. Chatzistamatiou, M. Morrow-T. Tsuji.