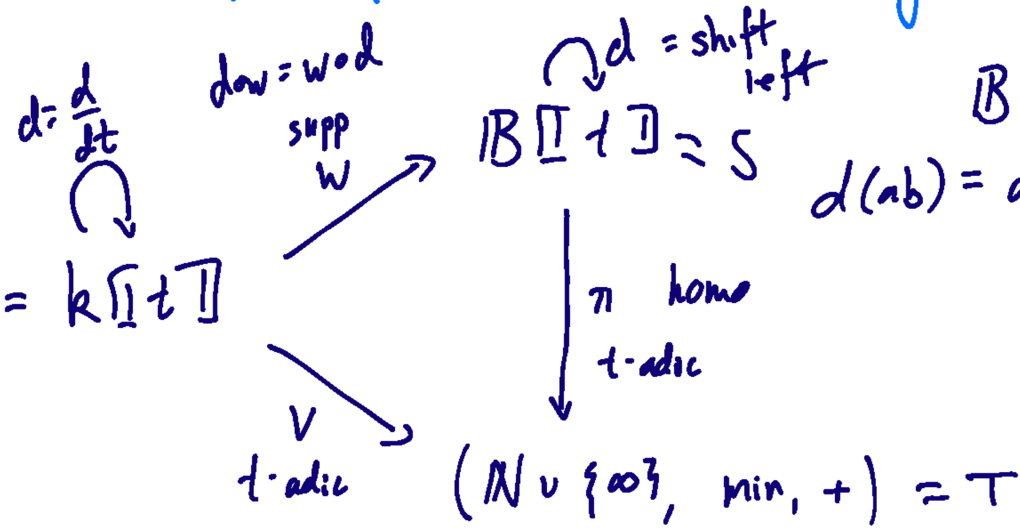
A GENERAL THEORY OF TROPICAL DIFFERENTIAL EQUATIONS

Jef Giansiracusa joint with Stefano Mereta

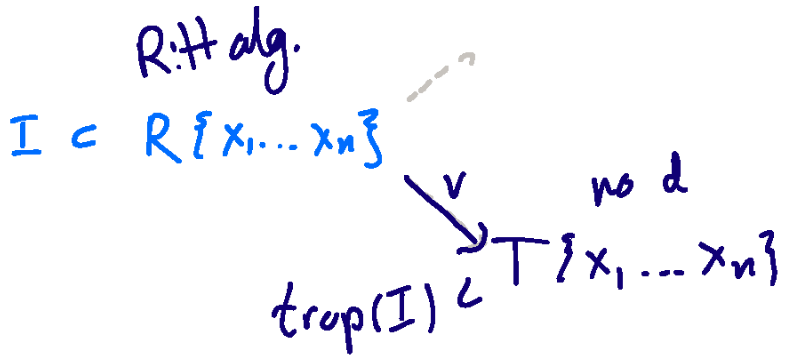
Grigoriev's setup (in terms of semirings)



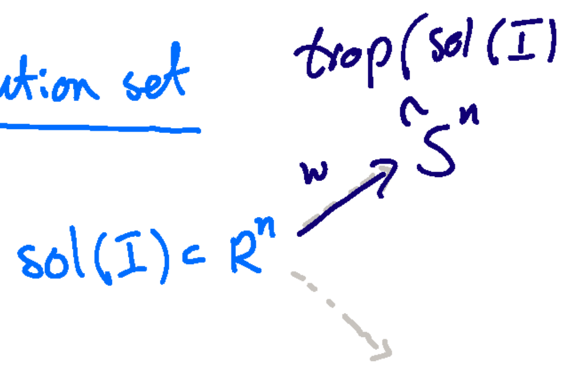
$\mathbb{B} = \{0,1\}$ $1+1=1$

$d(ab) = da \cdot b + a\,db$

Differential ideal



Solution set



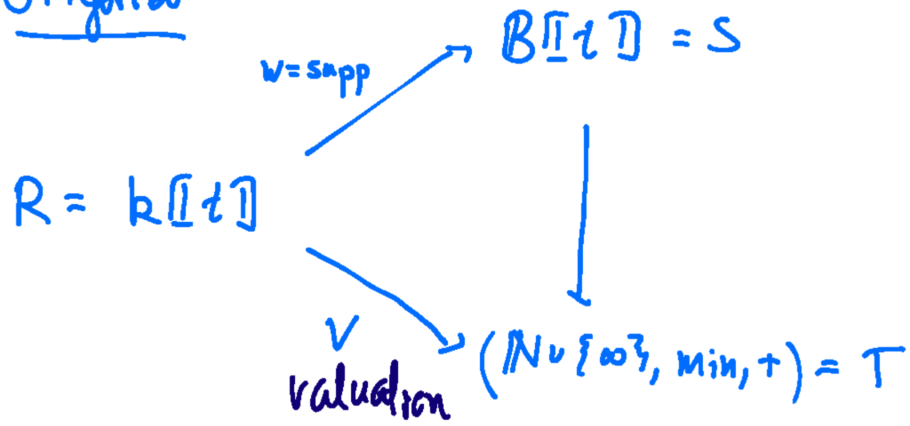
Why?

We can evaluate $f \in T\{x_1 \ldots x_n\}$ on $p \in S^n$

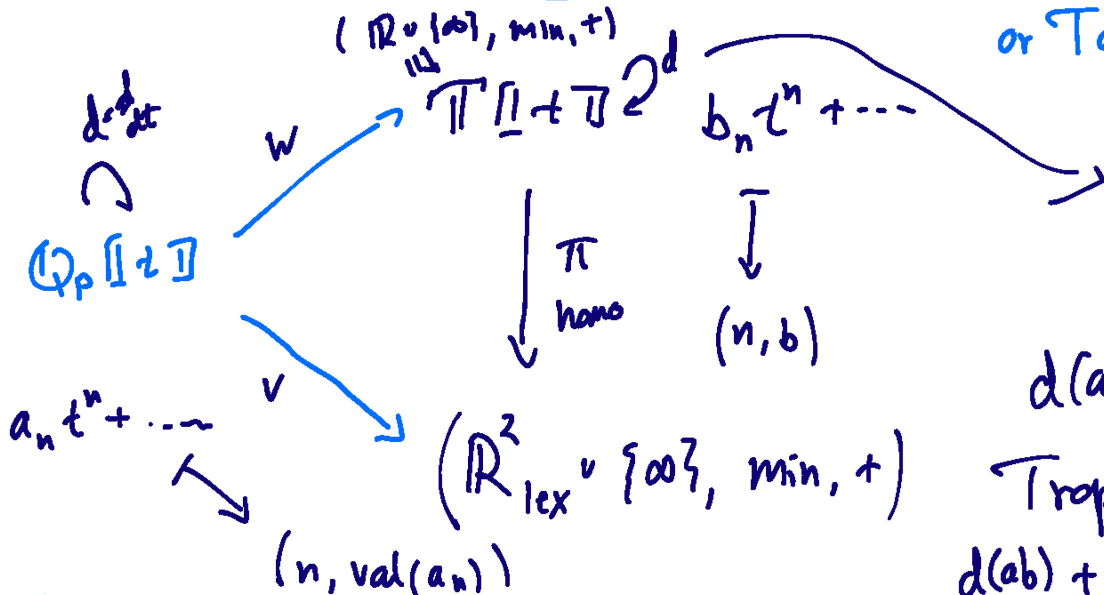
$$d^j x_i \longmapsto \pi(d^j p_i) \in T$$

$$f(p) \in T$$

Grigoriev



For p-adic differential equations, replace $k[\![t]\!]$ with $\mathbb{Q}_p[\![t]\!]$ $\cup$ or Tate algebra $\mathbb{Q}_p\langle t \rangle$



$d(t^n) = \text{val}(n) t^{n-1}$

$d \circ w = w \circ d$

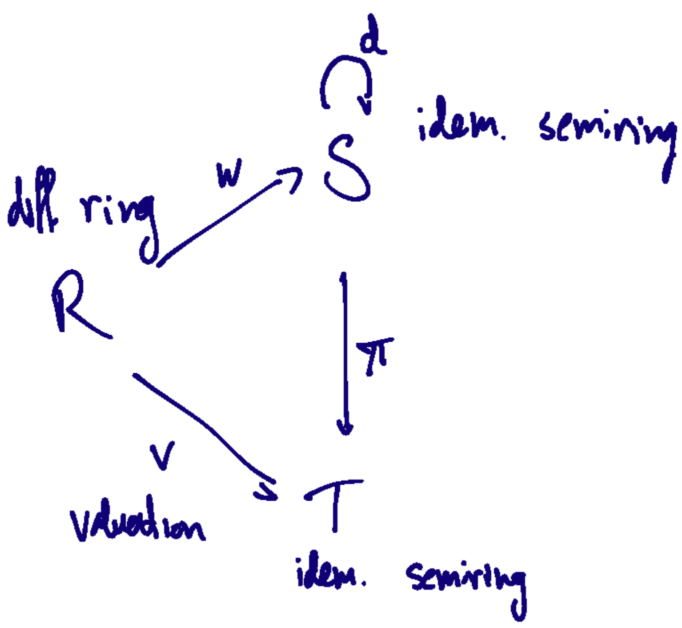
$d(ab) \neq da \cdot b + adb$

Tropical Leibniz

$d(ab) + da \cdot b + adb$
$= \quad da \cdot b + adb$
$= d(ab) + da \cdot b$
$= d(ab) + \quad adb$

} Bend relations

Abstraction



$w \circ d = d \circ w$

d_S satisfies tropical Leibniz

differential enhancement (cf v)

evaluate

$f \in T\{x_1 \ldots x_n\}$ and $p \in S^n$

$f(p) \in T$

Tropical solutions

$$f = \sum_\alpha a_\alpha x^\alpha$$

Given $f \in T\{x_1 \dots x_n\}$

$$\mathrm{sol}(f) = \{ p \in S^n \text{ s.t. } f(p) = \sum_\alpha a_\alpha \pi(p^\alpha) \text{ tropically vanishes} \}$$

Tropicalizing

Differential ideal

$I \subset R\{x_1 \dots x_n\} \xrightarrow{v} T\{x_1 \dots x_n\}$

$\mathrm{trop}(I) \subset$

Solution set

$\mathrm{sol}(I) \subset R^n \xrightarrow{w} \mathrm{trop}(\mathrm{sol}(I)) \subset S^n$

Prop

$$\{\text{Tropicalizations of classical solutions}\} \subset \{\text{Tropical solutions}\}$$

$$\mathrm{trop}(\mathrm{sol}(I)) \subset \mathrm{sol}(\mathrm{trop}(I))$$

Question

Fundamental ~~Theorem~~

equality?

3 facts from non-differential tropical/algebraic geometry

① Fundamental Theorem

② Varieties are functors, and so are tropical varieties

$I \subset k[x_1 \dots x_n]$ $\qquad$ $\mathrm{trop}(I) \subset \mathbb{T}[x_1 \dots x_n]$

$k \longrightarrow K$ $\qquad$ $\mathrm{Hom}\left(\mathbb{T}[x_i] / \text{bend relation of } f \in \mathrm{trop}(I), \mathbb{T}\right) \cong \mathrm{trop}(V(I))$

$V(I) \subset k^n$

$$K \mapsto V(I \otimes K) = \mathrm{Hom}_{k\text{-alg}}\left(k[x_1 \dots x_n]/I, K\right)$$

③ Payne's Inverse limit theorem

$$\begin{array}{ccc} & & \mathbb{A}^n \\ & \overset{\varphi}{\nearrow} & \big\downarrow \text{monomial} \\ X & & \\ & \underset{\psi}{\searrow} & \\ & & \mathbb{A}^m \end{array} \qquad \begin{array}{c} \mathrm{trop}_\varphi(X) \subset \mathbb{T}^n \\ \big\downarrow \\ \mathrm{trop}_\psi(X) \subset \mathbb{T}^m \end{array}$$

$$X^{an} \xrightarrow{\cong} \lim_{\text{affine embedding}} \mathrm{trop}_\varphi(X)$$

If $X = \mathrm{spec}\, A$, $X^{an} = \{\text{valuations on } A \text{ extending the valuation on } k\}$

For tropical differential equations

1. Fundamental Theorem ?
2. Functors ✓
3. Inverse limit theorem ✓

Given a differential R-algebra $R \longrightarrow A$ with

$$\begin{array}{ccc} R & \xrightarrow{w} & S \\ & \searrow^{v} & \downarrow \\ & & T \end{array}$$

diff enh.

affine embeddings $\rightsquigarrow$ presentations

$\operatorname{Spec} A \hookrightarrow \mathbb{A}^n$

$R\{x_1, \ldots, x_n\} \xrightarrow{\alpha} A$

$A \cong R\{x_i\} / \ker \alpha$

Tropicalize the equations

$T\{x_1, \ldots x_n\} \supset \operatorname{trop}(\ker \alpha)$

$S\{x_1 \dots x_n\} \circlearrowleft d$ differential?

$i \neq j \qquad d(x_i x_j) = dx_i \cdot x_j + x_i \, dx_j$

$$d(s x^u) = ds \cdot x^u + s\, d(x^u) \qquad u \in \mathbb{N}^n,\ s \in S$$

$$d(x_i^m) = w(m)\, x_i^{m-1} dx_i$$

Constructing the tropicalization as a pair

$S\{x_1 \dots x_n\}_{red}$ reduction = maximal quotient

$\pi \downarrow$

$T\{x_1 \dots x_n\} / $ Bend rel. of trop(ker α)

$\Big\}\ Trop_\alpha(A)$

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ S' & \longrightarrow & T' \end{array}$$

Prop Solutions of $J \subset T\{x_1 \dots x_n\}$ in $S' \to T'$

$\iff$ homomorphisms of pairs

$S\{x_1 \dots x_n\}_{red} \leftarrow S \to S'$

$\downarrow \qquad\qquad S \downarrow T$

$T\{x_1 \dots x_n\} /$ bend relations of $J \leftarrow T \rightleftarrows \to T$, $\quad S' \downarrow T$

Affine schemes $\overset{\cong}{\longleftrightarrow}$ commutative rings

Tropical affine schemes^{op} $\hookrightarrow$ idempotent semirings

affine tropical differential schemes^{op} $\hookrightarrow$ Reduced Pairs $\begin{array}{c} S \\ \downarrow \\ T \end{array}$

Inverse limit theorem

$$\begin{array}{ccc} & \overset{w}{\nearrow} & S \\ R & & \downarrow \\ & \underset{v}{\searrow} & T \end{array}$$

$R \to A$

"Analytification of A"

$\left\{ \text{valuations with differential enhancement} \quad \begin{array}{ccc} & R & \to S \\ A & \swarrow \nearrow \searrow & \downarrow \\ & \searrow & T \end{array} \quad \text{extending the one on } R \right\}$

Theorem (G-Mereta)

$$A^{an} \cong \lim_{\left\{\begin{array}{l}\text{presentations} \\ R\{x_1,\dots x_n\} \xrightarrow{\varphi} A\end{array}\right\}} \mathrm{Sol}(\mathrm{trop}(\ker \varphi))$$