Tropical differential equations and their solutions

Alex Fink, QMUL 8 Dec 2020

"Tropical Geometry, Berkovich Spaces, Arithmetic D-Modules and p-adic Local Systems".

- Setup.

char $\mathbb{K} = 0$.

$R_m := \mathbb{K}[\![t_1, \dots, t_m]\!] = \mathbb{K}[\![\underline{t}]\!]$

$(R_m, (d_1, \dots, d_m))$ is a differential ring, $d_i = \frac{d}{dt_i}$.

$R_m\{x_1, \dots, x_n\} = R_m\{\underline{x}\} := R_m[x_{i,J} : 1 \leq i \leq n,\ J \in \mathbb{N}^m]$

is a diff. ring. $d_i\, x_{kJ} = x_{k,(J+e_i)}$.

Let $(\mathbb{B}, +, \cdot)$ be the boolean idempotent semifield,

$\mathbb{B} = \{0, 1\}$, $0 = 0_{\mathbb{B}}$, $1 = 1_{\mathbb{B}}$, $1 + 1 = 1$.

$\mathbb{B}[\![t_1, \dots, t_m]\!]$ is an idem. semiring

$\cong$

$(\{\text{subsets of } \mathbb{N}^m\}, \cup, \text{Minkowski sum})$.

$\mathbb{B}[\![t_1, \dots, t_m]\!]\{x_1, \dots, x_n\} = \mathbb{B}[\![t_1, \dots, t_m]\!]\{x_{i,J}\}$.

$$d_i : \mathbb{B}[\![t_1, \dots, t_m]\!] \longrightarrow \mathbb{B}[\![t_1, \dots t_m]\!]$$

$$d_i\Big(\sum_{a \in \mathbb{N}^m} \alpha_a t^a\Big) = \sum_{a_i \geq 1} \alpha_a \, t^{a-e_i}$$

$$\text{supp} : R_m = \mathbb{K}[\![\underline{t}]\!] \longrightarrow \mathbb{B}[\![\underline{t}]\!]$$

$$\text{supp}\Big(\sum_{a \in \mathbb{N}^m} \alpha_a t^a\Big) = \sum_{\alpha_a \neq 0} t^a$$

Then.
$$\begin{tikzcd} R_m \arrow[r, "\text{supp}"] \arrow[d, "d_i"'] & \mathbb{B}[\![\underline{t}]\!] \arrow[d, "d_i"'] \\ R_m \arrow[r, "\text{supp}"] & \mathbb{B}[\![\underline{t}]\!] \end{tikzcd}$$
commutes.

<u>Examples.</u> • In R_1. $\quad x_1 x_2 \big|_{x_1 = 1-t,\ x_2 = 1+t} = 1 - t^2$

$$\Big\downarrow \text{supp}$$

In $\mathbb{B}[\![t]\!]$, $\quad x_1 x_2 \big|_{x_1 = 1+t,\ x_2 = 1+t} = 1 + t + t^2$

• In R_1, $\quad t x_{1,1} - a . x_{1,0} \big|_{x_1 = t^a} = 0$

$a \geq 0, \in \mathbb{N}$

$$\Big\downarrow \text{supp.}$$

In $\mathbb{B}[\![t]\!]\{x_1\}$, $\quad t\, x_{1,1} + x_{1,0} \big|_{x_1 = t^a} = t^a$

Evaluate a tropical diff. poly?

- We need a second semiring:

- $\mathbb{S}_m := \mathbb{B}[\![t_1, \dots, t_m]\!]/\sim$ ← to be defined.

s.t. $\mathrm{val}: R_m \to \mathbb{S}_m$, $\mathrm{val}(\varphi) := [\mathrm{supp}(\varphi)]_\sim$,

satisfies $\mathrm{val}(\varphi\psi) = \mathrm{val}(\varphi)\,\mathrm{val}(\psi)$.

E.g. $\mathbb{S}_1 = (\mathbb{N} \cup \{\infty\}, \min, +) \subseteq \mathbb{T}$,

$\mathrm{val}: R_1 \to \mathbb{S}_1$ is the t-adic valuation.

Given $\varphi \in (R_m)^n$, let $S = \mathrm{supp}(\varphi) = (\mathrm{supp}(\varphi_1), \dots, \mathrm{supp}(\varphi_n)) \in \mathbb{B}[\![\underline{t}]\!]^n$.

Let $T_M = \underset{\substack{\text{\rotatebox{90}{$\in$}}\\ R_m}}{\alpha_M} \prod_{i,J} x_{i,J}^{M_{i,J}} \in R_m\{\underline{x}\}$.

<u>Fact</u> S determines.

$$\mathrm{val}(T_M|_{\underline{x}=\varphi}) = \mathrm{val}(\alpha_M) \prod_{i,J} [d^J s_i]_\sim^{M_{i,J}}. \quad (*)$$

Let $\operatorname{trop}(T_M) = \operatorname{val}(\alpha_M) \prod_{i,J} x_{i,J}^{M_{i,J}} \in \mathbb{B}[\![\underline{t}]\!]\{\underline{x}\}$.

and define $\operatorname{trop}(T_M)|_{\underline{x}=S} = (*)$.

Extend the def. of trop linearly: for $f \in R_m\{\underline{x}\}$, write $f = \sum_M T_M$, and let $\operatorname{trop}(f) = \sum_M \operatorname{trop}(T_M)$
$\in \mathbb{B}[\![\underline{t}]\!]\{\underline{x}\}$.

To be defined: when does a sum of tropical terms vanish?

E.g. $m = 1$. $\mathbb{S}_1 = \mathbb{N} \cup \{\infty\}$.

$\sum_M \operatorname{trop}(T_M)|_{\underline{x}=S}$ vanishes if the minimum is attained at least twice.

<u>Def.</u> $S \in \mathbb{B}[\![t_1, \ldots, t_m]\!]^n$ is a <u>solution</u> to

$F = \sum_M T_M \in \mathbb{S}_m\{x_1, \ldots, x_n\}$ if

$F|_{\underline{x}=S}$ vanishes.

Prop. If $\varphi \in (R_m)^n$, $f \in R_m\{\underline{x}\}$ satisfy $f|_{\underline{x}=\varphi} = 0$

then $\mathrm{supp}(\varphi) \in \mathbb{B}[\![\underline{t}]\!]^n$ and $\mathrm{trop}(f) \in \mathbb{S}_m\{\underline{x}\}$ φ is a solution to f.

satisfy $\mathrm{trop}(f)|_{\underline{x} = \mathrm{supp}(\varphi)} = 0$. $\leftarrow$ $\mathrm{supp}(\varphi)$ is a solution to $\mathrm{trop}(f)$.

Theorem $\left(\begin{array}{ll} m=1: & \text{Aroca – Garay – Toghani} \quad 2016 \\ m \in \mathbb{N}: & \text{FGHNTB} \quad 2020 \end{array}\right)$

Let $\mathbb{K} = \overline{\mathbb{K}}$, and $\mathbb{K}$ uncountable. Let $I \subseteq R_m\{\underline{x}\}$ be a differential ideal. If $S \in \mathbb{B}[\![\underline{t}]\!]^n$ is a solution to $\mathrm{trop}(I)$, then $\exists\, \varphi \in (R_m)^n$ such that φ is a solution to I.

Proposition (Macpherson?) Define, for $a, b \in \mathbb{B}[\![\underline{t}]\!]$,

$a \sim b$ if $\exists c$ s.t. $ac = bc$. for some $c \in \mathbb{B}[\![\underline{t}]\!]$.

Then $\mathbb{S}_m = \mathbb{B}[\![\underline{t}]\!]/\sim$ is isomorphic to. the semiring

$$(**)\left(\left\{ y \mapsto \min_i \left\{ \sum_{j=1}^{m} a_{ij} y_j \right\},\ a_{ij} \in \mathbb{N} \right\},\ \min,\ + \right)$$

of functions $\mathbb{N}^m \to \mathbb{N} \cup \{\infty\}$

$$\left[\sum_{a\in A} t^{a}\right]_\sim \longmapsto \left(y \mapsto \min\left\{\sum_j a_j y_j : a \in A\right\}\right)$$

$(A \subseteq \mathbb{N}^m)$

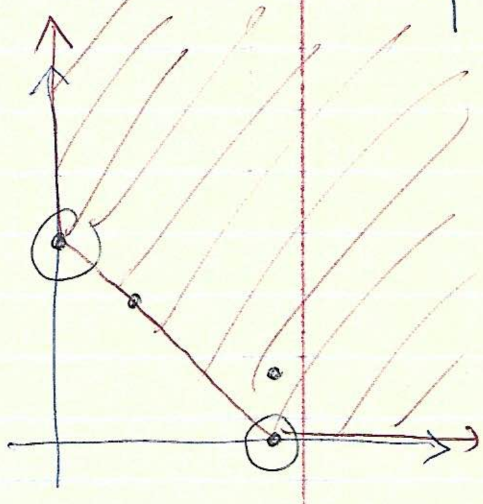
Also

$S_m \simeq (\{$lattice polyhedra in $\mathbb{N}^m$ with recession cone $\mathbb{N}^m\}$, $\mathrm{conv}(- \cup -)$, Minkowski sum$)$. (†)

e.g. $m = 2$,

$[t_1^3 + t_1^3 t_2 + t_1 t_2^2 + t_2^3]_\sim =$

$[t_1^3 + t_2^3]_\sim =$



A sum of terms vanishes in S_m if

- the minimum is attained twice pointwise in (**).
- every vertex appears twice in (†).