Integral p-adic cohomology theories

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Introduction

Suppose Λ is a complete regular local ring with fraction field K. Can we expect that there is a "reasonable" cohomology theory $H^{\cdot}(X, \Lambda)$ with its values in Λ-modules for separated schemes of finite type over a field k? Of course this depends on what "reasonable" means but for most definitions the answer is "yes" unless k and the residue field of Λ have the same positive characteristic. One of the basic requirements of "reasonable" should be that the $H^n(X, \Lambda)$ should be finitely generated Λ-modules and vanish for sufficiently large n.

This is the case for example when $\ell \neq p = char(k)$ and the theory is ℓ-adic étale cohomology $H^{\cdot}(X, \mathbb{Z}_\ell)$. When $\ell = p$ these requirements are met if X/k is *proper*, but not otherwise. But even in the proper case p-adic étale cohomology does not satisfy a more stringent version what "reasonable" should mean for a cohomology theory $H^{\cdot}(X)$ with Λ-coefficients, namely that

$$H^{\cdot}(X) \otimes_{\Lambda} K \simeq H^{\cdot}_{rig}(X)$$

for X/k separated and of finite type. Of course this is not to say that the $H^{\cdot}_{et}(X, \mathbb{Z}_p)$, or more generally the $H^{\cdot}_{et,c}(X, \mathbb{Z}_p)$ are without interest.

When k is perfect and X/k is smooth Davis, Langer and Zink have defined an overconvergent version of the de Rham-Witt complex $W^{\dagger}\Omega^{\cdot}_{X/W(k)}$ and showed that that

$$H^{\cdot}(X, W^{\dagger}\Omega^{\cdot}_{X/W(k)}) \otimes \mathbb{Q} \simeq H^{\cdot}_{rig}(X)$$

when X is quasiprojective. One might entertain the faint hope that the $H^{\cdot}(X, W^{\dagger}\Omega^{\cdot}_{X/W(k)})$ are finitely generated $W(k)$-modules. We will see that this is *never* true if X is a smooth affine curve. Davis, Langer and Zink make the more reasonable conjecture that the image of $H^{\cdot}(X, W^{\dagger}\Omega^{\cdot}_{X/W(k)})$ in $H^{\cdot}_{rig}(X)$ is finitely generated as a $W(k)$-module, but but Ertl and Shiho have produced counterexamples to this assertion as well.

One should also keep in mind the original motivation for crystalline cohomology: the study of p-torsion phenomena. Killing all p-torsion seriously reduces the interest of any proposed theory.

The folk wisdom is that there is no "reasonable integral p-adic cohomology theory" for separated schemes of finite type of a field of characteristic $p > 0$. The theory I wish to explain today makes the prospect of this rather unlikely, though it does not completely exclude it. This is joint work with Tomoyuki Abe.

Globally Perfect Models

For the rest of this talk Λ is a complete local ring with residue field k and fraction field K. As usual $D_{perf}(\Lambda)$ is the triangulated category of perfect complexes of Λ-modules, which since Λ is local means that an object of $D_{perf}(\Lambda)$ is quasi-isomorphic to a bounded complex of *free* Λ-modules.

The following requirements could be viewed as "reasonable" requirements on a cohomology theory $H^{\cdot}(X)$:

- $H^{\cdot}(X)$ can be used to compute rigid cohomology (or rigid cohomology with compact supports) via a suitable comparison theorem.
- $H^{\cdot}(X)$ may be computed as the cohomology of an object of $D_{perf}(\Lambda)$;
- The $H^{\cdot}(X)$ are compatible with finite étale descent, in a sense to be explained in a moment.

When Λ is regular the second requirement reduces to the condition that the $H^n(X)$ be finitely generated Λ-modules, and vanish for $n \gg 0$.

To explain the last condition we use the *finite étale site* whose definition we now recall. If X is a scheme the site $Fet(X)$ is the category of finite étale morphisms $Y \to X$, and the coverings are surjective morphisms. The associated topos will be written X_{fet}. A morphism $\pi : Y \to X$ induces a morphism $\pi_{fet} : Y_{fet} \to X_{fet}$ of topoi, whence functors π^* and $R\pi_*$. There is also a projection $\alpha : X_{et} \to X_{fet}$ compatible with the morphisms π_{fet} and $\pi_{et} : Y_{et} \to X_{et}$.

Definition

A object M of $D(X_{fet}, \Lambda)$ is *globally perfect* if

- M has finite Tor-dimension, and
- for every $\pi : Y \to X$ in $Fet(X)$, $R\Gamma(Y, \pi^* M)$ is in $D_{perf}(\Lambda)$.

Suppose for example that X/k is proper and smooth and V is a Cohen ring for k. If $\mathcal{O}_{X/V}$ is the structure sheaf of the crystalline-étale topos of X and $u : (X/V)_{crys-et} \to (X/V)_{et}$ is the usual projection, $R(\alpha u)_* \mathcal{O}_{X/V}$ is a globally perfect object of X_{fet}.

If X/k is separated and of finite type one can show that there is a globally perfect object M of $D_{perf}(X_{fet}, \mathbb{Z}_p)$ such that

$$H^{\cdot}(\pi^* M) \simeq H^{\cdot}_c(Y, \mathbb{Z}_p)$$

for all $\pi : Y \to X$ in $Fet(X)$.

Let $H^{\cdot}$ be a functor from $Fet(X)$ to the category $\mathrm{Mod}^{\cdot}_K$ of $\mathbb{N}$-graded K-vector spaces (e.g. the restriction to $Fet(X)$ of your favorite K-valued cohomology theory).

Definition

A *globally perfect model of* $H^{\cdot}$ is a globally perfect M in $D_{perf}(X_{fet}, \Lambda)$ and a functorial isomorphism

$$H^{\cdot}(Y) \xrightarrow{\sim} H^{\cdot}(R\Gamma(Y, \pi^* M)) \otimes_{\Lambda} K$$

for all $Y \to X$ in $Fet(X)$.

The requirement that $H^\cdot$ have a globally perfect model is slightly stronger than what was earlier described as reasonable since we now require that M have finite Tor-dimension. If Λ is regular this is not too serious.

The key technical result is

Theorem

Suppose Λ is noetherian and X is affine. If M is a globally perfect object of $D(X_{fet}, \Lambda)$ and $\pi : Y \to X$ is finite étale Galois with group G then $R\Gamma(Y, \pi^* M)$ is a perfect complex of $\Lambda[G]$-modules.

The sense of the assertion is that there is an object in $D_{perf}(\Lambda[G])$ whose image in $D_{perf}(\Lambda)$ computes $R\Gamma(Y, \pi^* M)$ together with the action of G.

We will need some technical business about cohomological dimension. Recall that if R is a ring and X is a topos, the R-cohomological dimension of X, i.e. the smallest integer d such that $H^n(X, F) = 0$ for all $n > d$ and every R-module F in X.

Lemma

Suppose k has characteristic $p > 0$ and X is a k-scheme of finite type.

- *The $\mathbb{Z}$-cohomological dimension of X_{et} is finite.*
- *If X is affine the $\mathbb{Z}_{(p)}$-cohomological dimension of X_{fet} is finite.*

Proof (sketch): The first assertion is actually a theorem of Gabber. For the second we begin with a result of Abbes-Gros-Tsuji asserting that when X is affine the topos X_{fet} is equivalent to the classifying topos $B_{\pi_1(X)}$. It therefore suffices to show that $H^n(\pi_1(X), F) = 0$ for $n \gg 0$ and all continuous $\pi_1(X)$-modules F.

Back in the day Serre showed that $H^n(\pi_1(X), F)$ is torsion for $n > 0$, so if F is a $\mathbb{Q}$-module, $H^n(\pi_1(X), F) = 0$ for $n > 0$. Again by Abbes-Gros-Tsuji X_{fet} is a coherent topos and thus $H^n(\pi_1(X), \_)$ commutes with inductive limits. A simple devissage using the exact sequence

$$0 \to F_{tor} \to F \to F \otimes \mathbb{Q} \to F \otimes (\mathbb{Q}/\mathbb{Z}) \to 0$$

reduces to the case when F is p-torsion. But in this case Achinger showed that there are isomorphisms

$$H^{\cdot}(X_{et}, F) \simeq H^{\cdot}(\pi_1(X), F)$$

and the assertion follows from the étale case. ∎

In what follows we will use the symbol Λ to denote the sheafification in $Fet(X)$ of the constant presheaf with value Λ.

Lemma

For any $\pi : Y \to X$ in $Fet(X)$ and M in $D^+(X_{fet}, \Lambda)$ there is a functorial isomorphism

$$R\pi_*\pi^*(M) \xrightarrow{\sim} M \otimes^L_\Lambda R\pi_*\Lambda.$$

Proof: The adjunction $\pi^* R\pi_*(\Lambda) \to \Lambda$ yields morphisms

$$\begin{aligned}\pi^*(M \otimes^L_\Lambda R\pi_*(\Lambda)) &\simeq \pi^*(M) \otimes^L_\Lambda \pi^* R\pi_*(\Lambda) \\ &\to \pi^*(M) \otimes^L_\Lambda \Lambda \simeq \pi^*(M)\end{aligned}$$

and applying the adjunction to this yields the morphism in the lemma. It will be an isomorphism if it is after pulling it back by a covering morphism $g : W \to X$ in $Fet(X)$. I claim g^* and $R\pi_*$ commute: in fact Abbes-Gros-Tsuji have shown that g^* has an exact right adjoint, so g^* sends injectives to injectives, which proves the claim. Since g^* and $R\pi_*$ commute we can replace π by its base change $W \times_X Y \to W$. Now there is a finite étale $g : W \to X$ such that $W \times_X Y$ is a disjoint sum of copies of W, and in this case the proof is easy.

Proof of the theorem: we must show that

$$R\Gamma(Y, \pi^*(M)) \simeq R\Gamma(X, R\pi_*\pi^*(M))$$

is a perfect complex of $\Lambda[G]$-modules. The $\Lambda[G]$-module $R\pi_*\pi^*(\Lambda) \simeq R\pi_*\Lambda$, which I will denote by $\mathcal{G}$ is a flat by the argument of the previous lemma. The lemma then yields an isomorphism

$$R\pi_*\pi^*(M) \xrightarrow{\sim} M \otimes^L_\Lambda \mathcal{G}.$$

Since $\mathcal{G}$ is $\Lambda[G]$-flat and M is in $D_{ftd}(\Lambda)$, $R\pi_*\pi^*(M)$ is in $D_{ftd}(\Lambda[G])$ and it follows that $R\Gamma(X_{fet}, R\pi_*\pi^*(M))$ is also in $D_{ftd}(\Lambda[G])$.

Since X is affine the lemma on cohomological dimension shows that

$$R\Gamma(Y_{fet}, \pi^*(M)) \simeq R\Gamma(X_{fet}, R\pi_*\pi^*(M))$$

lies in $D^b(\Lambda[G])$; since the $H^n(Y_{fet}, \pi^*(M))$ are finitely generated Λ-modules they are finitely generated $\Lambda[G]$-modules. Since Λ is noetherian $R\Gamma(Y_{fet}, \pi^*(M))$ is a perfect complex of $\Lambda[G]$-modules. ∎

The Main Theorem

Denote by K_0 the fraction field of a Cohen ring of k, and recall that K is the fraction field of Λ, which we can make into an extension field of K_0. In what follows rigid cohomology has coefficients in K_0.

Theorem

Suppose k is a field of characteristic $p > 0$, X is a smooth affine curve over k and Λ is a complete noetherian local ring with residue field k and fraction field K. The functors

$$H^{\cdot}_{rig}(\_) \otimes_{K_0} K,\ H^{\cdot}_{rig,c}(\_) \otimes_{K_0} K : Fet(X) \to \mathrm{Mod}^{\cdot}_K$$

do not have a globally perfect model.

Proof: Any smooth affine curve X has an Artin-Schreier cover $\pi : Y \to X$ that ramifies at infinity. So it suffices to invoke the following lemma:

Lemma

Suppose

- X *is an smooth affine curve over a field* k *of characteristic* $p > 0$;
- $\pi : Y \to X$ *is a finite étale Galois cover whose group* G *is a p-group;*
- M *is a globally perfect model of the functor* $H^{\cdot}_{rig}(\_) \otimes_{K_0} K : Fet(X) \to \mathrm{Mod}^{\cdot}_K$ *or of* $H^{\cdot}_{rig,c}(\_) \otimes_{K_0} K : Fet(X) \to \mathrm{Mod}^{\cdot}_K$.

Then $\pi : Y \to X$ *extends to a finite étale morphism* $\bar{\pi} : \bar{Y} \to \bar{X}$ *of smooth projective curves.*

Proof: Denote by

$$\chi(Z) = \sum_i (-1)^i \dim_{K_0} H^i_{rig}(Z)$$

the rigid Euler characteristic of a separated k-scheme Z of finite type.

If Z/k is smooth,

$$\chi(Z) = \sum_i (-1)^i \dim_{K_0} H^i_{rig,c}(Z)$$

by duality, so the third hypothesis says that

$$\chi(Y) = \sum_i (-1)^i \dim_K H^i(Y)$$

for all $Y \to X$ in $Fet(X)$. The isomorphism

$$H^n(R\Gamma(\pi^* M \otimes_\Lambda K)) \simeq H^n_{rig,c}(Y) \otimes_{K_0} K$$

is functorial, so by the previous theorem there is a perfect complex $M^{\cdot}_Y$ of $\Lambda[G]$-modules such that

$$H^n(M^{\cdot}_Y \otimes_\Lambda K) \simeq H^n_{rig,c}(Y) \otimes_{K_0} K$$

and then

$$H^n(M^{\cdot}_Y)^G \simeq H^n_{rig,c}(X) \otimes_{K_0} K$$

since $\pi : Y \to X$ is finite étale.

Since G is a p-group, the group ring $\Lambda[G]$ is local, so a finitely generated projective $\Lambda[G]$-module is free. Therefore these last isomorphisms imply that

$$\chi(X) = |G|\chi(Y).$$

Comparing this with the Grothendieck-Ogg-Shafarevich formula, we see that the ramification of $\pi : Y \to X$ is tame at infinity. But since G is a p-group, π must totally wild at infinity if it is ramified at all. It is therefore unramified at infinity, which is the conclusion of the lemma. ■

An Example

Suppose k is perfect and consider the functors

$$H^n(X) = H^n(X, W^\dagger\Omega^\cdot_{X/W})$$

where $W = W(k)$ and $W^\dagger\Omega^\cdot_{X/W}$ is the overconvergent de Rham-Witt complex of Davis, Langer and Zink. One can show that if the $H^n(X)$ are finitely generated W-modules for all X and $n \geq 0$ then $R\Gamma(X, W^\dagger\Omega^\cdot_{X/W})$ would be a globally perfect model of $H^\cdot_{rig}(\_) : Fet(X) \to \mathrm{Mod}^\cdot_K$. The theorem says that this is false for *every* smooth affine curve X. In fact Ertl and Shiho have constructed smooth affine curves for which $H^1(X, W^\dagger\Omega^\cdot_{X/W})$ is not finitely generated modulo torsion.

Thank you.