Bornological $\widehat{\mathcal{D}}$-modules on rigid analytic spaces

A. Bode

UMPA
ENS Lyon

Outline

$\mathcal{D}$-modules are $\mathcal{O}$-modules with a compatible action of the tangent sheaf.
They arise in various contexts, e.g.

- complex varieties, complex manifolds
- arithmetic $\mathcal{D}$-modules on char p schemes or formal schemes (Berthelot)
- $\widehat{\mathcal{D}}$-modules on p-adic rigid analytic varieties (Ardakov, Wadsley)

Key features in the complex world:

- Can be used to study representations geometrically.

Theorem (BB-Localization)

Let G be a semisimple algebraic group over $\mathbb{C}$, and let $X = G/B$ be its flag variety.
Then there is an equivalence

$$\{\textit{coherent } \mathcal{D}_X\textit{-modules}\} \cong \{\textit{f. g. } U(\mathfrak{g})_0\textit{-modules}\}$$
$$\mathcal{M} \mapsto \mathcal{M}(X)$$

- On $\mathrm{D}^b(\mathcal{D}\text{-mod})$, can define six functors: f_+, f^+, $f_!$, $f^!$, $\otimes^L_{\mathcal{O}}$, $\mathbb{D}$.
- 3 notions of finiteness:

$$\{\text{coh. / } \mathcal{O}\} \subset \{\text{holonomic}\} \subset \{\text{coh. / } \mathcal{D}\},$$

and $\mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}\text{-mod})$ is stable under the six functors above.

$\Longrightarrow$ great for geometric constructions/manipulations of representations!

From now on, fix

- K a finite extension of $\mathbb{Q}_p$
- R the valuation ring of K
- $\pi \in R$ a uniformizer.

Ardakov–Wadsley introduced coadmissible $\widehat{\mathcal{D}}$-modules on smooth rigid analytic K-varieties and proved an analogue of BB Localization.
In this talk, we give an analogue of the second point: a derived category with all six functors.

If $X = \mathbb{A}^1_{\mathbb{C}}$, we have

$$\mathcal{D}(X) = \mathbb{C}[x; \partial] = \left\{ \sum a_{ij} x^i \partial^j : a_{ij} \in \mathbb{C} \right\} \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x]),$$

where $\partial = \frac{\mathrm{d}}{\mathrm{d}x}$ satisfies $\partial \cdot x = x\partial + 1$ (product rule).
Think: functions on the cotangent space $X \times \mathbb{A}^1$.

Now let

$$K\langle x\rangle = \left\{\sum_{i=0}^{\infty} a_i x^i : a_i \in K, a_i \to 0\right\}$$

be the ring of analytic functions on the closed disk X of radius 1.
Then

$$\widehat{\mathcal{D}}(X) = \varprojlim \mathcal{D}_n(X) = \varprojlim K\langle x; \pi^n \partial\rangle.$$

Think: functions on the cotangent space $X \times \mathbb{A}^{1,\mathrm{an}}$.

Definition

*A K-algebra A is called **Fréchet–Stein** if*

$$A \cong \varprojlim A_n$$

for A_n Noetherian Banach K-algebras, with each transition map $A_{n+1} \to A_n$ flat with dense image.
*An A-module M is called **coadmissible** if*

$$M \cong \varprojlim M_n$$

for M_n a finitely generated A_n-module, s.t. the natural morphism $A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ is an isomorphism.

The category $\mathcal{C}_A$ of coadmissible A-modules is abelian.

Theorem (Ardakov–Wadsley, B.)

Let X be a smooth affinoid K-variety. Then $\widehat{\mathcal{D}}(X)$ is a Fréchet–Stein algebra, and the functor

$$\widehat{\mathcal{D}}(U)\widehat{\otimes}_{\widehat{\mathcal{D}}(X)} - : \mathcal{C}_{\widehat{\mathcal{D}}(X)} \to \mathcal{C}_{\widehat{\mathcal{D}}(U)}$$

is exact for $U \subset X$ an affinoid subdomain.

$\Longrightarrow$ can form the category of coadmissible $\widehat{\mathcal{D}}_X$-modules completely analogously to coherent $\mathcal{O}_X$-modules.

Definition

A (convex) ***bornology*** *on a K-v.s. V is a collection $\mathcal{B}$ of subsets of V such that*

- $\{v\} \in \mathcal{B}$ *for all* $v \in V$.
- $\mathcal{B}$ *is closed under finite unions.*
- *if* $B \in \mathcal{B}$, *then* $R \cdot B \in \mathcal{B}$.
- *if* $B \in \mathcal{B}$, $\lambda \in K$, *then* $\lambda B \in \mathcal{B}$
- *if* $B \in \mathcal{B}$ *and* $B' \subset B$, *then* $B' \in \mathcal{B}$.

If $B \subset V$ such that $B \in \mathcal{B}$, we say that B is bounded.
A K-linear map between bornological vector spaces is bounded if it sends bounded subsets to bounded subsets.

Example:
Let V be a locally convex topological K-vector space, whose topology is given by a family of seminorms q_i.
Then the rule

$$B \in \mathcal{B} \iff q_i(B) \subset K \text{ is bounded } \forall i$$

defines a bornology on V.

Proposition

The category $\widehat{\mathcal{B}c}_K$ of complete bornological K-vector spaces is a complete, cocomplete, quasi-abelian category with enough projectives. Its 'abelian envelope' (left heart) has enough projectives and enough injectives.
The completed tensor product $\widehat{\otimes}_K$ gives $\widehat{\mathcal{B}c}_K$ the structure of a closed symmetric monoidal category.

Theorem (B., 2020)

Let X be a smooth rigid analytic K-variety. Then $\widehat{\mathcal{D}}_X$ is a sheaf of K-algebras in $\widehat{\mathcal{B}}c_K$.
The category $\widehat{\mathcal{B}}c(\widehat{\mathcal{D}}_X)$ of complete bornological $\widehat{\mathcal{D}}_X$-modules is a complete, cocomplete, quasi-abelian category admitting flat resolutions. Its left heart has enough injectives.

$\implies$ can form the derived category $\mathrm{D}(\widehat{\mathcal{D}}_X)$, and define f_+, f^+, $f_!$, $f^!$, $\widehat{\otimes}^{L}_{\mathcal{O}}$, $\mathbb{D}$ as in the complex algebraic case.

Theorem (B., 2020)

Let X be a smooth rigid analytic K-variety. There is an exact fully faithful functor

$$\{\text{coadmissible } \widehat{\mathcal{D}}_X\text{-modules}\} \to \widehat{\mathcal{B}}c(\widehat{\mathcal{D}}_X).$$

Can now define an analogue of $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{D})$.

Definition

*Let $M \in \mathrm{D}(\widehat{\mathcal{D}}_X)$ for X affinoid. We say M is a $\mathcal{C}$-**complex** if*

- $M_n := \mathcal{D}_n \widehat{\otimes}^L_{\widehat{\mathcal{D}}_X} M \in \mathrm{D}^b_{\mathrm{coh}}(\mathcal{D}_n)$ *for all n,*
- $\mathrm{H}^i(M) \to \varprojlim \mathrm{H}^i(M_n)$ *is an isomorphism.*

We denote the full subcategory of $\mathcal{C}$-complexes by $\mathrm{D}_{\mathcal{C}}(\widehat{\mathcal{D}}_X)$.

Theorem

- $\mathrm{D}_{\mathcal{C}}(\widehat{\mathcal{D}}_X)$ *is a triangulated subcategory.*
- *If* $M \in \mathrm{D}_{\mathcal{C}}(\widehat{\mathcal{D}}_X)$*, then* $\mathrm{H}^i(M)$ *is coadmissible.*
- *If* M *is a complete bornological* $\widehat{\mathcal{D}}_X$*-module, then* M *is coadmissible if and only if* $M \in \mathrm{D}_{\mathcal{C}}(\widehat{\mathcal{D}}_X)$ *when viewed as a complex concentrated in one degree.*

Theorem

Let $f : X \to Y$ be a morphism between smooth rigid analytic K-varieties.

- *If f is smooth, then f^+ sends $\mathrm{D}_{\mathcal{C}}(\widehat{\mathcal{D}}_Y)$ to $\mathrm{D}_{\mathcal{C}}(\widehat{\mathcal{D}}_X)$.*
- *If f is projective, then f_+ sends $\mathrm{D}_{\mathcal{C}}(\widehat{\mathcal{D}}_X)$ to $\mathrm{D}_{\mathcal{C}}(\widehat{\mathcal{D}}_Y)$.*