Derived Cartier transform, derived Satake and cohomology.

(p is large enough)

X – smooth variety over k of char p ($k = \bar{k}$).

Deligne–Illusie: $H^*_{DR}(X) \simeq H^*_{Hodge}(X)$ if X lifts to $W_2(k)$.

In the form of Ogus–Vologodsky the argument is as follows.

By definition $\underbrace{H^*_{DR}(X) = \mathrm{Ext}^*_{D_X\text{-mod}}(\mathcal{O}_X, \mathcal{O}_X)}$

D_X – sheaf of crystalline diff. operators.

$\shortparallel$

$\langle \mathcal{O}, \mathrm{Vect} \rangle / \ldots$

$$H^*_{Hodge}(X) = \mathrm{Ext}_{Coh(T^*X)}(\mathcal{O}_X, \mathcal{O}_X)$$

Can pass from one to the other via the Cartier transform, a functor some D-modules to $QCoh(T^*X^{(1)})$.

To define it observe that f^p, $\xi^p - \xi^{[p]} \in Z(D_X)$ moreover $Z(D_X) \simeq \mathcal{O}(T^*X^{(1)})$. and D_X is an Azumaya algebra over $T^*X^{(1)}$.

The restr-n of A.A to $X^{(1)} \underset{0\text{ sect}}{\hookrightarrow} T^*X^{(1)}$ splits,

$Fr_* \mathcal{O}$ is the splitting bundle.

If it was true that the splitting extends to $T^*X^{(1)}$ (never true $\dim X > 0$) or at least to the formal nbhd of X in T^*X (will be the case for us) we would done. b/c of the eq-nce

$$QCoh \simeq \text{modules over a split Azumaya}$$

$$\mathcal{O}_X \longleftarrow (\mathcal{O}_X, d)$$

DI $\Leftarrow$ applying the eq-nce.

OV show under the above condition the A.A. on pd nbhd of X in T^*X, that's turns out to suffice.

We used $H^*_{Hodge}(X) = Ext^*_{T^*X}(\mathcal{O}_X, \mathcal{O}_X) =$

$$= R\Gamma\,(R\,\underline{Hom}_{T^*X}(\mathcal{O}_X, \mathcal{O}_X)).$$

$$\cong$$

$$\bigoplus \Omega^i_X[-i].$$

The ~~..~~ forms also appear in a different way via HKR Thm.

$$\mathcal{O}_X \overset{L}{\underset{\mathcal{O}_{T^*X}}{\otimes}} \mathcal{O}_X \simeq \bigoplus \Omega^i_X[i].$$

A modern name: small loops space

$\omega(X) := X \times_{X^2} X$ – derived fiber product

We are interested in $X = G/B$. G – reductive alg-c grp/k.

$B = T\cdot U$, U – max. unipotent.

Claim [BMR] For $X = G/B$ the above A.A splits on the formal nbhd of the zero section.

$\omega(X)$ arises in the following construction.

Recall $T^*(X/G) = T^*(X)//G$ – Ham. red-n

$$\overset{\shortparallel}{\mu^{-1}(0)/G}$$

~~Co~~ This is a composition of 2 steps.

$T^*(X) \underset{\mu}{/} G := \mu^{-1}(0)$ and taking q-nt

Consider $G \curvearrowright T^*(G/U)\underset{\mu}{/}T \simeq \omega(G/B)$

$$T^*(G/U)/T = \tilde{\mathfrak{g}} = \{(\underset{\substack{\uparrow \\ G/B}}{\mathfrak{b}}, x) \mid x \in \mathfrak{b}\}$$

$\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \{0\} \simeq \omega(G/B)$ (since $\mathfrak{b}^\perp = \mathfrak{n}$).

Motivation from RT.

Thm [BMR] $D^b(\mathfrak{g}\text{-mod}_0) \simeq D^b(D\text{-mod}^T_{G/U\ \mathrm{nilp}})$

modules with generalized central char of the trivial repr-n.

nilp. p-curv

Cor of the splitting: $\mathrm{RHS} \simeq D^b(\mathrm{Coh}_{G/B}(\tilde{\mathfrak{g}}^{(1)}))$.

Cor $D^b(G_1\text{-mod}_{block}) \simeq D\,\mathrm{Coh}(\omega(G/B)^{(1)})$.

Frob. kernel

$$G_1\text{-mod} = U\mathfrak{g}\text{-mod} \otimes_{Z_{Fr}} k.$$

Thm $D^b(G\text{-mod}_{block}) \simeq D\,Coh^{G^{(1)}}(\omega(G/B)^{(1)})$

$$D^b(G_2\text{-mod}_{block}) \simeq D\,Coh^{G_{2-1}^{(1)}}(\omega(G/B)^{(1)}).$$

"block" - means principal block.

Applications to understanding $H^*(G_2)$.

$G_1 \to G \to G^{(1)}$

This connects to geometric Langlands.

$F = \mathbb{C}((t)) \supset O = \mathbb{C}[[t]]$

${}^L G$ - Langlands dual group / $\mathbb{C}$.

$G_F = {}^L G((t)) \supset G_0 \supset I \supset I^0$ - prounip. radical.

I Iwahori $\longrightarrow B$

$G_0 \to G$.

$\mathcal{Y} = G_F / G_0$.

Thm (Mirkovic - Vilonen)

$$Rep(G) \simeq Perv_{G_0}(\mathcal{Y})$$ - coefficients in k.

(geom. Satake equivalence)

Derived Satake eq-nce.

$$D_{G_0}(\mathcal{Y}) \simeq D\,Coh^{G \times G}(\omega(G))$$

(Drinfeld; in char 0 B. Finkelberg, Gaitsgory, Lurie (unp.)

in char p, in progress with Arinkin).

Finkelberg - Mirk. conjecture (Thm in progress w. S. Riche, L. Rider)

$$\mathrm{Rep}(G)_{block} \simeq \mathrm{Perv}_{I^\circ}(\mathcal{Gr})$$

<u>Rmk</u> $D_{G(O)}(\mathcal{Gr}) \circlearrowright D_{I^\circ}(\mathcal{Gr})$

~~This~~ Can see such an action in the coherent descr-n.

Things to understand: S^1 action and S^1 local-n on the 2 sides (in char 0 discussed by Ben-Zvi–Nadler; in char p related to Smith theory)

Treumann–"

Relation $H^*(G_2) \longleftrightarrow$ det bundle on $\mathcal{Gr}$ and its char classes.

(Friedlander–Suslin)

□

$$\mathrm{Perv}_{G(O)}(\mathcal{Gr}) \circlearrowright \mathrm{Perv}_{I^\circ}(\mathcal{Gr})$$

$$\wr\wr$$

$$\mathrm{Rep}(G^{(1)}) \circlearrowright \mathrm{Rep}(G)_{block}.$$