EX 22

a) $\lim_{x\to+\infty}(2x+5) = 2\cdot(+\infty)+5 = +\infty$

b) $\lim_{x\to+\infty}(x^2-5x-2) = \lim_{x\to+\infty}\left(x^2-2\frac{5}{2}\cdot x+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2-2\right) =$

$= \lim_{x\to+\infty}\left(\left(x-\frac{5}{2}\right)^2-\frac{33}{4}\right) = +\infty$

c) $\lim_{x\to+\infty}\frac{1}{x^3-x^2+1} = \lim_{x\to+\infty}\frac{\frac{1}{x^3}}{1-\frac{1}{x}+\frac{1}{x^3}} = \frac{0}{1} = 0$ $\quad \frac{1}{+\infty}=0$

d) $\lim_{x\to+\infty}(x-\sqrt{x+1}) = \lim_{x\to+\infty}\frac{(x-\sqrt{x+1})(x+\sqrt{x+1})}{x+\sqrt{x+1}} =$

$= \lim_{x\to+\infty}\frac{x^2-(x+1)}{x+\sqrt{x+1}} = \lim_{x\to+\infty}\underbrace{\frac{x^2}{x}}_{\to+\infty}\underbrace{\frac{\left(1-\frac{1}{x}-\frac{1}{x^2}\right)}{\left(1+\sqrt{\frac{1}{x}+\frac{1}{x^2}}\right)}}_{\to 1} = +\infty$

e) $\lim_{x\to+\infty}(x-\sqrt{x^3+1}) = \lim_{x\to+\infty}\frac{(x-\sqrt{x^3+1})(x+\sqrt{x^3+1})}{x+\sqrt{x^3+1}} =$

$= \lim_{x\to+\infty}\frac{x^2-(x^3+1)}{x+\sqrt{x^3+1}} = \lim_{x\to+\infty}\underbrace{\frac{x^3}{x^{3/2}}}_{\to+\infty}\underbrace{\frac{\left(\frac{1}{x}-1-\frac{1}{x^3}\right)}{\left(\frac{1}{x^{1/2}}+\sqrt{1+\frac{1}{x^3}}\right)}}_{\to -1} = -\infty$

f) $\lim_{x\to+\infty}\frac{x+3}{x^2+x+6} = \lim_{x\to+\infty}\frac{\frac{1}{x}+\frac{3}{x^2}}{1+\frac{1}{x}+\frac{6}{x^2}} = \frac{0}{1} = 0$

g) $\lim_{x\to+\infty}\frac{1}{\sqrt{x^2+x}-\sqrt{x^2+1}} = \lim_{x\to+\infty}\frac{\sqrt{x^2+x}+\sqrt{x^2+1}}{(\sqrt{x^2+x}-\sqrt{x^2+1})(\sqrt{x^2+x}+\sqrt{x^2+1})} =$

$= \lim_{x\to+\infty}\frac{\sqrt{x^2+x}+\sqrt{x^2+1}}{(x^2+x)-(x^2+1)} = \lim_{x\to+\infty}\frac{\sqrt{x^2+x}+\sqrt{x^2+1}}{x-1} = \lim_{x\to+\infty}\underbrace{\frac{x}{x}}_{\to 1}\underbrace{\frac{\left(\sqrt{1+\frac{1}{x}}+\sqrt{1+\frac{1}{x^2}}\right)}{\left(1-\frac{1}{x}\right)}}_{\to 2} =$

$= 2$

h) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} \underset{\text{la règle de l'Hôpital}}{=} \lim\limits_{x\to 0} \dfrac{(\sin x)'}{x'} = \lim\limits_{x\to 0} \dfrac{\cos x}{1} = \cos 0 = 1$

i) $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x^2} \underset{\text{la règle de l'Hôpital}}{=} \lim\limits_{x\to 0} \dfrac{(\cos x - 1)'}{(x^2)'} = \lim\limits_{x\to 0} -\dfrac{\sin x}{2x} = -\dfrac{1}{2} \lim\limits_{x\to 0} \dfrac{\sin x}{x} =$

$= -\dfrac{1}{2}$

j) $\lim\limits_{x\to 0+} \dfrac{|x| - 1}{\sqrt{x} - 1} = \dfrac{0-1}{0-1} = 1$

k) $\lim\limits_{x\to 4+} \dfrac{\sqrt{x-4}}{x+2} = \dfrac{\sqrt{4-4}}{4+2} = 0$

l) $\lim\limits_{x\to 3-} \dfrac{|x-3|}{x-3} = \lim\limits_{x\to 3-} \dfrac{-(x-3)}{x-3} = \lim\limits_{x\to 3-} -1 = -1.$

car $x\to 3- \Rightarrow x-3<0 \Rightarrow |x-3| = -(x-3)$

m) $\lim\limits_{x\to 1-} \ln\left(\dfrac{1+x}{1-x}\right) = \ln\left(\dfrac{2}{0+}\right) = \ln(+\infty) = +\infty$

n) $\lim\limits_{x\to -\infty} \dfrac{e^x + e^{-x}}{2x} = \lim\limits_{x\to -\infty} \dfrac{e^x}{2x} - \dfrac{1}{2} \lim\limits_{x\to -\infty} \dfrac{e^{-x}}{-x} = \dfrac{0}{-\infty} - \dfrac{1}{2}\cdot(+\infty) = -\infty,$

car $\lim\limits_{y\to +\infty} \dfrac{e^y}{y} = \lim\limits_{x\to -\infty} \dfrac{e^{-x}}{-x} = +\infty$

o) $\lim\limits_{x\to +\infty} \dfrac{x^3 + 2x - 1}{2x^2 - x - 2} = \lim\limits_{x\to +\infty} \underbrace{\dfrac{x^3}{x^2}}_{+\infty} \underbrace{\dfrac{\left(1 + \frac{2}{x^2} - \frac{1}{x^3}\right)}{\left(2 - \frac{1}{x} - \frac{2}{x^2}\right)}}_{\frac{1}{2}} = +\infty$

p) $\lim\limits_{x\to +\infty} \left(\sqrt{2x+3} - \sqrt{2x-1}\right) = \lim\limits_{x\to +\infty} \dfrac{\left(\sqrt{2x+3} - \sqrt{2x-1}\right)\left(\sqrt{2x+3} + \sqrt{2x-1}\right)}{\sqrt{2x+3} + \sqrt{2x-1}} =$

$= \lim\limits_{x\to +\infty} \dfrac{(2x+3) - (2x-1)}{\sqrt{2x+3} + \sqrt{2x-1}} = \lim\limits_{x\to +\infty} \dfrac{4}{\sqrt{2x+3} + \sqrt{2x-1}} = \dfrac{4}{+\infty} = 0$

q) $\lim\limits_{x\to +\infty} \sqrt{x}\left(\sqrt{x+1} - \sqrt{x}\right) = \lim\limits_{x\to +\infty} \dfrac{\sqrt{x}\left(\sqrt{x+1} - \sqrt{x}\right)\left(\sqrt{x+1} + \sqrt{x}\right)}{\sqrt{x+1} + \sqrt{x}} =$

$= \lim\limits_{x\to +\infty} \dfrac{\sqrt{x}(x+1-x)}{\sqrt{x+1} + \sqrt{x}} = \lim\limits_{x\to +\infty} \dfrac{\sqrt{x}}{\sqrt{x}\left(\sqrt{1 + \frac{1}{x}} + 1\right)} = \lim\limits_{x\to +\infty} \dfrac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \dfrac{1}{2}$

r) $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{4x+1}-\sqrt{x}}{\sqrt{x+1}} = \lim_{x\to+\infty} \frac{\left(\sqrt{4x+1}-\sqrt{x}\right)\left(\sqrt{4x+1}+\sqrt{x}\right)}{\sqrt{x+1}\left(\sqrt{4x+1}+\sqrt{x}\right)} =$

$\displaystyle= \lim_{x\to+\infty} \frac{4x+1-x}{\sqrt{x+1}\left(\sqrt{4x+1}+\sqrt{x}\right)} = \lim_{x\to+\infty} \frac{3x+1}{\sqrt{x+1}\left(\sqrt{4x+1}+\sqrt{x}\right)} =$

$\displaystyle= \lim_{x\to+\infty} \frac{x\left(3+\frac{1}{x}\right)}{x\sqrt{1+\frac{1}{x}}\left(\sqrt{4+\frac{1}{x}}+1\right)} = \frac{3}{\sqrt{1}\cdot\left(\sqrt{4}+1\right)} = \frac{3}{3} = 1$

s) $\displaystyle\lim_{x\to-\infty} \frac{x^2-2x-1}{3x^3-2x^2-x-2} = \lim_{x\to-\infty} \underbrace{\frac{x^2}{x^3}}_{0}\,\underbrace{\frac{\left(1-\frac{2}{x}-\frac{1}{x^2}\right)}{\left(3-\frac{2}{x}-\frac{1}{x^2}-\frac{2}{x^3}\right)}}_{\frac{1}{3}} = 0$

t) $\displaystyle\lim_{x\to+\infty} \frac{-x^4+2x-1}{x^4-x-2} = \lim_{x\to+\infty} \frac{\cancel{x^4}\left(-1+\frac{2}{x^3}-\frac{1}{x^4}\right)}{\cancel{x^4}\left(1-\frac{1}{x^3}-\frac{2}{x^4}\right)} = \frac{-1}{1} = -1$

u) $\displaystyle\lim_{x\to+\infty} \left(\sqrt{x+4}-\sqrt{x-3}\right) = \lim_{x\to+\infty} \frac{\left(\sqrt{x+4}-\sqrt{x-3}\right)\left(\sqrt{x+4}+\sqrt{x-3}\right)}{\sqrt{x+4}+\sqrt{x-3}} =$

$\displaystyle= \lim_{x\to+\infty} \frac{x+4-(x-3)}{\sqrt{x+4}+\sqrt{x-3}} = \lim_{x\to+\infty} \frac{7}{\sqrt{x+4}+\sqrt{x-3}} = \frac{7}{+\infty} = 0$

v) $\displaystyle\lim_{x\to0} \frac{\ln(1+ax)}{x+ax^3} = \lim_{x\to0} \frac{\left(\ln(1+ax)\right)'}{\left(x+ax^3\right)'} = \lim_{x\to0} \frac{a}{1+ax}\cdot\frac{1}{1+3ax^2} = a$

la règle de l'Hôpital

w) $\displaystyle\lim_{x\to0} \frac{\ln(1+x)-xe^x}{x^2} = \lim_{x\to0} \frac{\left(\ln(1+x)-xe^x\right)'}{\left(x^2\right)'} = \lim_{x\to0} \frac{\frac{1}{1+x}-e^x-xe^x}{2x} =$

l'Hôpital: 1 fois

$\displaystyle= \lim_{x\to0} \frac{1-e^x(1+x)^2}{2x(1+x)} = \lim_{x\to0} \frac{1-e^x(1+x)^2}{2x+2x^2} = \lim_{x\to0} \frac{\left(1-e^x(1+x)^2\right)'}{\left(2x+2x^2\right)'} =$

l'Hôpital: 2 fois

$\displaystyle= \lim_{x\to0} \frac{-e^x(1+x)^2-2(1+x)e^x}{2+4x} = \frac{-1-2}{2} = -\frac{3}{2}$

x) $\displaystyle\lim_{x\to+\infty} \frac{e^{2x}}{x+7} = +\infty$, car $\displaystyle\lim_{x\to+\infty} \frac{e^{\beta x}}{x^\alpha} = +\infty \quad \forall \alpha>0,\ \beta>0$