EX. 8

$$u_{n+2} = \frac{1}{6}\left(u_{n+1} + u_n\right) \qquad \boxed{u_0, u_1 \geq 0} \Rightarrow u_n \geq 0$$

$$c := \max(u_0, u_1).$$

1) Recurrence :

$$u_2 = \frac{1}{6}(u_1 + u_0) \leq \frac{2}{6} c = \frac{1}{3} c \leq \frac{1}{2} c. \checkmark$$

$$u_{n+2} = \frac{1}{6}(u_{n+1} + u_n)$$

$$\leq \frac{1}{6}\left(c\, 2^{1-(n+1)} + c\, 2^{1-n}\right)$$

$$-n \qquad 1-n$$

$$= c\, 2^{1-(n+2)} \cdot \frac{1}{6}\left(2 + 2^2\right)$$

$$-n-1 \qquad 1 \qquad 2$$

$$= c\, 2^{1-(n+2)} \quad \checkmark$$

$\Rightarrow \sum u_n$ DOMINÉE PAR $2c \cdot \sum 2^{-n}$ CONV. (GEOM.) $\left(\frac{1}{2}\right)^n$

2) $S = \sum_{k=0}^{\infty} u_k$

$S' = \sum_{k=1}^{\infty} u_k$

$$\boxed{S'' = \frac{1}{6} S' + \frac{1}{6} S}$$

$$S'' = \sum_{k=2}^{\infty} u_k = \sum_{k=0}^{\infty} u_{k+2} = \sum_{k=0}^{\infty} \frac{1}{6}(u_{k+1} + u_k) = \frac{1}{6}\sum_{k=0}^{\infty} u_{k+1} + \frac{1}{6}\sum_{k=0}^{\infty} u_k$$

$\circledast \quad S'' = \frac{1}{6}(S' + S)$

ON A AUSSI $\begin{cases} S' = S - u_0 \\ S'' = S - u_0 - u_1 \end{cases}$

$\Rightarrow \circledast$ DONNE $\boxed{S - u_0 - u_1 = \frac{1}{6}\left(S - u_0 + S\right)}$

$\Rightarrow \frac{2}{3} S = \frac{5 u_0}{6} + u_1 \ldots$

$\Rightarrow S = \frac{3}{2}\left(\frac{5}{6} u_0 + u_1\right) = \frac{5}{4} u_0 + \frac{3}{2} u_1$

EX. 9 $|x|<1$

1) • $\sum x^n$ EST GEOMETRIQUE ET $|x|<1 \Rightarrow$ CONVERGE ABSOLUMENT

• $\sum n x^n$ D'ALEMBERT :

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)|x|^{n+1}}{n\,|x|^n} = |x| \cdot \frac{n+1}{n} \longrightarrow |x| < 1$$

$\Rightarrow$ CONVERGE ABSOLUMENT

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2) $$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$$ (BIEN CONNU).

$\sum_{n=1}^{N} n x^{n-1}$ EST LA <u>DERIVÉE</u> DE $\sum_{n=0}^{N} x^n$

$$\Rightarrow \sum_{n=1}^{N} n x^{n-1} = \left(\frac{1-x^{N+1}}{1-x}\right)' = \frac{-(N+1)x^N(1-x) + (1-x^{N+1})}{(1-x)^2}$$

$$= \frac{-(N+1)x^N + (N+1)x^{N+1} + 1 - x^{N+1}}{(1-x)^2}$$

$$= \frac{N x^{N+1} - (N+1)x^N + 1}{(1-x)^2}$$

3) $\sum_{m=1}^{+\infty} m \cdot x^{m-1} = \lim_{N\to+\infty} \frac{N \cdot x^{N+1} - (N+1)\, x^{N} + 1}{(1-x)^2}$ $\quad N \cdot x^{N+1} \to 0, \quad x^{N} \to 0$

$$= \frac{1}{(1-x)^2}$$

Remarque : $\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$

DERIVATA

$\sum_{m=1}^{\infty} m\, x^{m-1} = \frac{1}{(1-x)^2}$

$\Downarrow$

PARLARE DI FUNZIONI ANALITICHE

4) (e) $\sum_{m=1}^{\infty} \left((m-2)\, 3^{-m} + (m-3)\, 2^{-m} \right)$

PAR D'ALEMBERT $\left(\sum (m-2)\, 3^{-m}\right)$ et $\left(\sum (m-3)\, 2^{-m}\right)$ CONVERGENT

$\Rightarrow$ NOTRE SERIE CONVERGE

$$\sum_{m=1}^{\infty} (m-2)\, 3^{-m} + (m-3)\, 2^{-m} = \sum_{m=1}^{\infty} \left((m+1) \cdot 3^{-m} - 3 \cdot 3^{-m} \right) + \sum_{m=1}^{\infty} \left((m+1)\, 2^{-m} - 4\, 2^{-m} \right)$$

$$= \sum_{m=1}^{\infty} (m+1)\, 3^{-m} - \sum_{m=1}^{\infty} 3^{-m+1} + \sum_{m=1}^{\infty} (m+1)\, 2^{-m} - \sum_{m=1}^{\infty} 2^{-m+2}$$

$2 + 1 + \frac{1}{2} + \frac{1}{4} \ldots$

$$= \sum_{m=0}^{\infty} (m+1)\, 3^{-m} - 1 - \sum_{m=0}^{\infty} 3^{-m} + \sum_{m=0}^{\infty} (m+1)\, 2^{-m} - 1 - \sum_{m=0}^{\infty} 2^{-m} - 2$$

$$= \frac{1}{\left(1 - \frac{1}{3}\right)^2} - 1 - \frac{1}{1 - \frac{1}{3}} + \frac{1}{\left(1 - \frac{1}{2}\right)^2} - 1 - \frac{1}{1 - \frac{1}{2}} - 2$$

4) (b) $\sum_{n=1}^{+\infty} (n^2 - 2n)\, 3^{-n}$

CONVERGENCE AVEC D'ALEMBERT :

$$\frac{\left((n+1)^2 - 2(n+1)\right)(1/3)^{n+1}}{(n^2 - 2n)\,(1/3)^n} = \frac{1}{3} \cdot \frac{(n^2 + \cancel{2n} + \cancel{1} - \cancel{2n} - \cancel{1})}{(n^2 - 2n)} \longrightarrow \frac{1}{3} < 1$$

$\Rightarrow$ CONVERGE

$$\sum_{n \geq 1} (n^2 - 2n)\, 3^{-n} = \; ?$$

$$\left(\sum_{n=0}^{N} x^n\right)'' = \left(\sum_{n=1}^{N} n\, x^{n-1}\right)' = \sum_{n=2}^{N} n(n-1)\, x^{n-2}$$

$$\left(\frac{1 - x^{N+1}}{1-x}\right)'' = \left(\frac{N x^{N+1} - (N+1)x^N + 1}{(1-x)^2}\right)' = \frac{\left(N(N+1)x^N - N(N+1)x^{N-1}\right)(1-x) + 2\cdot\overbrace{\left(Nx^{N+1} - (N+1)x^N + 1\right)}}{(1-x)^3}$$

$$\left(\frac{P(x)}{(1-x)^n}\right)' = \frac{P'(x)\cdot(1-x) + P(x)\cdot n}{(1-x)^{n+1}}$$

$$\lim_{N \to +\infty} (\,\cdot\,)$$

$$\frac{2}{(1-x)^3} = \left(\frac{1}{(1-x)^2}\right)'$$

$$\Rightarrow \boxed{\sum_{n \geq 2} n(n-1)\, x^{n-2} = \frac{2}{(1-x)^3}}$$

$$\Rightarrow \sum_{n=1}^{+\infty} (n^2 - 2n)\, 3^{-n} = \sum_{n\geq 1} (n^2-n)\, 3^{-n} - \sum_{n=1}^{\infty} n\, 3^{-n}$$

$$= 3^{-2} \sum_{n\geq 1} (n^2-n)\, 3^{-n+2} - \sum_{n\geq 1} (n+1)\, 3^{-n} + \sum_{n\geq 1} 3^{-n}$$

$$n=1 \Rightarrow n^2-n=0$$

$$= 3^{-2} \cdot \sum_{n\geq 2} (n^2-n)\, 3^{-n+2} - \sum_{n\geq 0} (n+1)\, 3^{-n} + \cancel{1} + \sum_{n\geq 0} 3^{-n} - \cancel{1}$$

$$= \frac{1}{3^2} \cdot \frac{2}{\left(1-\frac{1}{3}\right)^3} - \frac{1}{\left(1-\frac{1}{3}\right)^2} + \frac{1}{1-\frac{1}{3}}.$$

EX. 10 — CRITÈRE D'ABEL ($a_n, b_n \in \mathbb{C}$)

$$\sum_{n=0}^{N} a_n b_n \quad , \quad A_n = \sum_{m=0}^{n} a_m \quad , \quad B_n = \sum_{m=0}^{n} b_m$$

$$\boxed{\delta_n = b_n - b_{n+1}}$$

$$\Rightarrow \sum_{n=0}^{N} a_n b_n = A_N\, b_{N+1} - \sum_{n=0}^{N} A_n \cdot \delta_n$$

C'EST UNE FORMULE SIMILAIRE À L'INTEGRATION PAR PARTIES

$$\int_a^b fG = [FG]_a^b - \int_a^b Fg \quad \text{avec } F' = f,\ G' = g.$$

DANS CETTE ANALOGIE
$$\begin{cases} A_n = \sum a_n & \text{JOUE LE ROLE D'UNE PRIMITIVE DE } (a_n)_n \\ \delta_n = b_n - b_{n+1} & \text{JOUE LE ROLE DE LA DERIVÉE DE } (b_n)_n \end{cases}$$

$\Rightarrow$ **CRITERE D'ABEL:** Si

1) $\exists M > 0,\ \exists n_0 \in \mathbb{N}$, t.q. $\forall N \geq n_0$, $|A_N| \leq M$
2) $b_n \longrightarrow 0$
3) $\sum |\delta_n| = \sum (b_n - b_{n+1})$ conv.

$\Rightarrow \sum a_n b_n$ conv.

Proof: 1)+2) $\Rightarrow \lim_N A_N b_{N+1} = 0$, 1)+3) $\Rightarrow \sum |A_n \delta_n| \leq M \cdot \sum |\delta_n|$ conv. ABS $\Rightarrow \sum A_n \delta_n$ conv.

PARFOIS ON REMPLACE 3) PAR

3') $b_n \in \mathbb{R}$, $\exists n_0$ t.q. $\forall n \geq n_0$, $b_{n+1} \leq b_n$

$$\Downarrow$$

$$\sum_{n \geq n_0} |\delta_n| = \sum_{n \geq n_0} |b_n - b_{n+1}| = \sum_{n \geq n_0} b_n - b_{n+1}$$

TELESCOPES

<u>COROLLAIRE : (LEIBNIZ)</u> : Si $a_n = (-1)^n$, $b_n \geq 0$, $b_{n+1} \leq b_n$

$\Rightarrow A_n \in \{0, 1\}$

$\Rightarrow |A_n| \leq 1$

APPLICATION SI a_n EST SOUS FORME DE <u>COSINUS</u> OU SINUS OU EXPONENTIELLE

EX 10.1

$$\sum_{n \geq n_0} \frac{\cos(2n)}{2n}$$

$\underline{n_0 \text{ à choisir}}$

(LA CONVERGENCE NE DEPEND PAS DE n_0)

$a_n = \cos(2n)$, $b_n = \frac{1}{2n}$, $\begin{cases} (b_n)_n \text{ decroissante} \\ b_n \longrightarrow 0 \end{cases}$

$$|A_n| = \left|\sum_{k=n_0}^{n} \cos(2k)\right|$$

$$= \left|\sum_{k=n_0}^{n} Re\left(e^{i2k}\right)\right| = \left|Re\left(\sum_{k=n_0}^{n} \left(e^{2i}\right)^k\right)\right|$$

$$= \left|Re\left(\frac{\left(e^{2i}\right)^{n_0} - \left(e^{2i}\right)^{n+1}}{1 - e^{2i}}\right)\right| \underset{|Re(z)| \leq |z|}{\leq} \left|\frac{\left(e^{2i}\right)^{n_0} - \left(e^{2i}\right)^{n+1}}{1 - e^{2i}}\right| \leq \frac{2}{|1 - e^{2i}|} = M$$

$\Rightarrow$ Abel s'APPLIQUE CAR

1) $\forall n \geq 0$, $|A_n| \leq M$

2) $b_n = \frac{1}{2n} \longrightarrow 0$

3) $\sum |\delta_n| = \sum (b_n - b_{n+1}) = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \ldots = b_1$ converge.

$|\delta_n| = \delta_n$ car $\underline{\forall n,\ b_n \geq b_{n+1}}$

Donc $\sum \frac{\cos(2n)}{2n}$

EX. 10.2

$$\sum_{n \geq n_0} \frac{\sin^2(n)}{n} \quad , \quad n_0 \text{ à choisir}$$

Remarque : $A_n = \sum_{k=n_0}^{n} \sin^2(k)$ n'EST PAS BORNÉ

~~Car~~ $\sin^2(k) = \frac{1 - \cos(2k)}{2}$ LINEARISATION

$$\Rightarrow \sum_{n_0}^{n} \sin^2(k) = \underbrace{\frac{1}{2}(n - n_0)}_{\text{NON BORNÉ}} - \underbrace{\frac{1}{2}\sum_{k=n_0}^{n} \cos(2k)}_{\text{BORNÉ (VOIR 10.1)}}$$

$\Rightarrow$ MONTRONS QUE $\sum \frac{\sin^2(n)}{n}$ DIVERGE.

$$\frac{\sin^2(n)}{n} = \frac{1 - \cos(2n)}{2n} = \frac{1}{2n} - \frac{\cos(2n)}{2n}$$

$\sum \frac{1}{2n}$ DIVERGE $\qquad \sum \frac{\cos(2n)}{2n}$ CONVERGE PAR 10.1

$$\Rightarrow \boxed{\sum \frac{\sin^2(n)}{n} \text{ DIVERGE}}$$

EXEMPLE : (ABEL)

$$\sum_{k=n_0}^{n} \frac{\sin^3(k)}{\ln(k)} \qquad \xrightarrow{} n_0 \text{ CONVENABLE}$$

$$\begin{cases} a_n = \sin^3(n) \\ b_n = \dfrac{1}{\ln(n)} \end{cases}$$

$b_n \to 0$ et $b_{n+1} \leq b_n, \forall n$

$$A_n = \sum_{k=n_0}^{n} a_k = \sum_{k=n_0}^{n} \sin^3(k)$$

LINEARISATION : $\sin^3(\cancel{x}) = \frac{3}{4}\sin(\cancel{x}) - \frac{\sin(3x)}{4}$

(PREUVE : $\sin^3(x) = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 = \dots$)

$$\Rightarrow |A_n| = \left| \frac{3}{4} \sum_{k=n_0}^{n} \sin(k) - \frac{1}{4} \sum_{k=n_0}^{n} \sin(3k) \right|$$

$$= \left| \frac{3}{4} \sum_{k=n_0}^{n} \mathrm{Re}(e^{ik}) - \frac{1}{4} \sum_{k=n_0}^{n} \mathrm{Re}(e^{i3k}) \right|$$

$$= \left| \mathrm{Re}\left(\frac{3}{4} \frac{e^{in_0} - e^{i(n+1)}}{1 - e^{i}} - \frac{1}{4} \frac{e^{3in_0} - e^{3i(n+1)}}{1 - e^{3i}} \right) \right|$$

$$\leq \frac{3}{4} \left| \frac{e^{in_0} - e^{i(n+1)}}{1 - e^{i}} \right| + \frac{1}{4} \left| \frac{e^{3in_0} - e^{3i(n+1)}}{1 - e^{3i}} \right|$$

$$\leq \frac{3}{4} \cdot \frac{2}{|1 - e^{i}|} + \frac{1}{4} \frac{2}{|1 - e^{3i}|} = M$$

ALORS $\forall n, n_0$, $A_n = \sum_{k=n_0}^{n} a_k$ VERIFIE

$$|A_n| \leq M$$

$\Rightarrow$ $\displaystyle\sum_{k \geq n_0} \frac{\sin^3(k)}{\ln(k)}$ converge

EX 11

1) VOIR PROPOSITION 4.6 DU POLY

$0 \in I$, $f \in C^\infty$, $x \in I$

RECURRENCE SUR k :

$$\int_0^x f'(t)\,dt = f(x) - f(0). \Rightarrow \text{ok}$$

INTEGRATION PAR PARTIES : $\int_0^x f' \cdot 1 = \left[-f'(t)\cdot(x-t)\right]_0^x - \int_0^x -f''(t)(x-t)$

$= +f'(0)\,x + \int_0^x f''(t)(x-t)\,dt$

RECURRENCE : SUPPOSONS AVOIR DEMONTRÉ QUE

$$f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2} + \dots + f^{(k-1)}(0)\cdot\frac{x^{k-1}}{(k-1)!} + \int_0^x f^{(k)}(t)\frac{(x-t)^{k-1}}{(k-1)!}\,dt$$

$$\int_0^x f^{(k)}(t)\frac{(x-t)^{k-1}}{(k-1)!}\,dt = \left[-\frac{f^{(k)}(t)}{k!}(x-t)^k\right]_{x_0}^x - \int_0^x - f^{(k+1)}(t)\frac{(x-t)^k}{k!}\,dt$$

$$= + f^{k}(0)\frac{x^k}{k!} + \int_0^x f^{(k+1)}(t)\frac{(x-t)^k}{k!}\,dt$$

$\Rightarrow$ ok

11,2) $e^x = \underbrace{\sum_{k=0}^{n} \frac{x^k}{k!}}_{S_n} + \int_0^x e^t \frac{(x-t)^n}{n!}\,dt$

$$\Rightarrow |e^x - S_n| = \left| \int_0^x e^t \frac{(x-t)^n}{n!}\,dt \right|$$

$$\leq \int_0^x \left| e^t \frac{(x-t)^n}{n!} \right| dt$$

$$= \int_0^x e^t \frac{|x-t|^n}{n!}\,dt$$

Maintenant remarquons que x peut être NEGATIF

$$\Rightarrow \quad x \geq 0,\ t \in [0,x] \Rightarrow e^t \leq e^x$$

$$x \leq 0,\ t \in [x,0] \Rightarrow e^t \leq e^0 = 1 \leq e^{-x}$$

$$\Rightarrow \forall x \in \mathbb{R},\ 0 \leq |t| \leq |x| \Rightarrow e^t \leq \underset{\displaystyle e^{|x|}}{\underset{\wedge\!\!|}{\max(1, e^x)}}$$

$$\Rightarrow |e^x - S_n| \leq \int_0^x e^{|x|} \frac{|t-x|^n}{n!}\,dt$$

$$= e^{|x|} \cdot \int_0^x \frac{|t-x|^n}{n!}\,dt$$

PRIMITIVE

$x \geq 0 \Rightarrow |t-x| = x-t \Rightarrow F(t) = -\frac{(x-t)^{n+1}}{(n+1)!}$, $F'(t) = \frac{|t-x|^n}{n!}$

$x \leq 0 \Rightarrow |t-x| = t-x \Rightarrow F(t) = \frac{(t-x)^{n+1}}{(n+1)!}$

DANS LES DEUX CAS $F(x)=0$

Si $x \geq 0$, $F(0) = -\dfrac{x^{n+1}}{(n+1)!} \Rightarrow F(x) - F(0) = \dfrac{|x|^{n+1}}{(n+1)!}$

$x \leq 0$, $F(0) = (-1)^{n+1} \dfrac{x^{n+1}}{(n+1)!} = \dfrac{|x|^{n+1}}{(n+1)!} \Rightarrow \int_0^x = -\int_x^0 = F(0) - F(x)$
$= \dfrac{|x|^{n+1}}{(n+1)!}$

DANS LE DEUX CAS ON TROUVE

$$|e^x - S_n| \leq e^{|x|} \frac{|x|^{n+1}}{(n+1)!}$$

$$\Rightarrow \forall x \in \mathbb{R}, \quad \lim_n |e^x - S_n| = \lim_n e^{|x|} \frac{|x|^{n+1}}{(n+1)!} = 0$$

11.3) $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} \cdots + (-1)^{n+1}\dfrac{x^n}{n} + \displaystyle\int_0^x \ln(1+t)^{(n+1)} \frac{(x-t)^n}{n!}\,dt$

$\ln(1+t)' = \dfrac{1}{1+t}$, $\ln(1+t)'' = \dfrac{-1}{(1+t)^2}$, $\ln(1+t)''' = \dfrac{(-1)(-2)}{(1+t)^3}$

$$\ln(1+t)^{(n+1)} = \frac{(-1)^n \, n!}{(1+t)^{n+1}}$$

$x \geq 0$, $t \in [0,x] \Rightarrow \left|\ln(1+t)^{(n+1)}\right| = \dfrac{n!}{|1+t|^{n+1}} \leq \dfrac{n!}{1} \leq n!$

$-1 < x \leq 0$, $t \in [x,0] \Rightarrow \left|\ln(1+t)^{(n+1)}\right| = \dfrac{n!}{|1+t|^{n+1}} \leq \dfrac{n!}{|1+x|^{n+1}}$

$\Rightarrow$ EN GENERAL $\left|\ln(1+t)^{(n+1)}\right| \leq \dfrac{n!}{\min(1, |1+x|^{n+1})} = \dfrac{n!}{M}$

$$\Rightarrow \left|\int_0^x \ln(1+t)^{(n+1)} \frac{(x-t)^n}{n!}\,dt\right| \leq \int_0^x \left|\ln(1+t)^{(n+1)}\right| \frac{|x-t|^n}{n!}\,dt$$

$$\leq \int_0^x \frac{n!}{M} \cdot \frac{|x-t|^n}{n!}\,dt$$

$$= \frac{n!}{M} \cdot \int_0^x \frac{|x-t|^n}{n!}\,dt$$

Comme avant si $F(t)$ est une primitive de $|x-t|^n/n!$ sur $[0,x]$ alors

$$\begin{cases} x \geq 0 \Rightarrow F(t) = -\dfrac{(x-t)^{n+1}}{(n+1)!} \Rightarrow \displaystyle\int_0^x \frac{|x-t|^n}{n!}\,dt = F(x) - F(0) = \frac{x^{n+1}}{(n+1)!} \geq \\ x \leq 0 \Rightarrow F(t) = \dfrac{(t-x)^{n+1}}{(n+1)!} \Rightarrow \displaystyle\int_0^x \frac{|x-t|^n}{n!}\,dx = -\int_x^0 \frac{(x-t)^n}{n!}\,dt = -F(0) + F(x) = \frac{(-x)^{n+1}}{(n+1)!} \geq 0 \end{cases}$$

$$\int_0^x \frac{|x-t|^{n+1}}{(n+1)!}\,dt = \frac{|x|^{n+1}}{(n+1)!}$$

Donc : $$\left|\int_0^x \ln(1+t)^{(n+1)} \frac{|x-t|^n}{n!}\,dt\right| \leq \frac{n!}{M} \cdot \frac{|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{M \cdot (n+1)}$$

Or : $$\lim_n \frac{|x|^{n+1}}{M \cdot (n+1)} = \begin{cases} 0 & \text{si } x \in]-1, 1] \\ +\infty & \text{si } x > 1 \end{cases}$$

(Remarque : $M = \min(1, |1+x|^{n+1})$ EST > 0 SEULEMENT SI $x > -1$)

$$\Rightarrow \left|\ln(1+x) - \sum_{k=1}^{n} (-1)^{n-1} \frac{x^n}{n}\right| = \left|\int_0^x \ln(1+t)^{(n+1)} \frac{(x-t)^n}{n!}\,dt\right| \leq \frac{|x|^{n+1}}{M \cdot (n+1)} \longrightarrow 0$$

si $x \in]-1, 1]$

$x \in]-1, 1]$: $$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

11.3

- $$\sum_{n\geq 1}^{\infty} \frac{(-1)^n}{n} = -\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \cdot 1 = -\ln(1+1) = -\ln(2)$$

- $$\sum_{n\geq 2}^{\infty} \frac{1}{n \cdot 2^n} = \sum_{n\geq 2} \frac{(-1)^n}{n} \cdot \left(\frac{-1}{2}\right)^n = -\sum_{n\geq 2} \frac{(-1)^{n-1}}{n} \cdot \left(-\frac{1}{2}\right)^n$$

$$= \left(-\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \left(-\frac{1}{2}\right)^n\right) + \left(-\frac{1}{2}\right)$$

$$= \ln\left(1-\frac{1}{2}\right) - \frac{1}{2} = \boxed{\ln\left(\frac{1}{2}\right) - \frac{1}{2}}$$