Complete intersections of quadrics and complete intersections on Segre varieties with common specializations

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Abstract

We investigate whether surfaces that are complete intersections of quadrics and complete intersection surfaces in the Segre embedded product $\mathbb{P}^1 \times \mathbb{P}^k \hookrightarrow \mathbb{P}^{2k+1}$ can belong to the same Hilbert scheme. For $k = 2$ there is a classical example: it comes from K3 surfaces in projective 5-space that degenerate into a hypersurface on the Segre threefold. We show that for $k \geq 3$ there is only one more example, but here its (connected) Hilbert scheme has at least two irreducible components.

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1 Introduction

1.1 Motivation

This note is motivated by exercise 11, Ch. VIII in [3], where two types of surfaces of degree 8 in $\mathbb{P}^5$ are compared: those that are complete intersections of three quadrics and those that arise as smooth hypersurfaces of bidegree $(2,3)$ in the Segre embedded $\mathbb{P}^1 \times \mathbb{P}^2$. The exercise asks to show that the latter arise as limits of some well-chosen complete intersection of quadrics. The limit surfaces in the example form a divisor in the boundary of the 19-dimensional family consisting of complete intersections of three quadric hypersurfaces in $\mathbb{P}^5$ and the problem is to make this explicit. We have included a construction in Section 3 where it appears as Theorem 3.1.

This phenomenon is restricted to surfaces: if we want to construct higher dimensional examples in a similar fashion, we are doomed to fail since by the Lefschetz hyperplane theorem complete intersections of dimension $\geq 3$ have the same second

\footnote{We remark that Beauville’s exercise is closely related to Saint-Donat’s work on projective models of K3-surfaces [22]; see also [18] Chap. 7.}
Betti number as the surrounding variety in which they are embedded (see (3)) and so cannot live in a non-trivial product of projective spaces. The simplest generalization for surfaces amounts to a comparison of complete intersection quadrics in \( \mathbb{P}^{2k+1} \) and complete intersection surfaces lying on \( \mathbb{P}^1 \times \mathbb{P}^k \) for \( k \geq 3 \) using the Segre embedding \( \mathbb{P}^1 \times \mathbb{P}^k \hookrightarrow \mathbb{P}^{2k+1} \).

That indeed examples are hard to come by is already apparent within our modest search area: we show that there is exactly one other example where a similar phenomenon occurs: a complete intersection of 5 quadrics in \( \mathbb{P}^7 \) and a complete intersection of type \((0, 4), (4, 2)\) in a Segre embedded \( \mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7 \) belong to the same Hilbert scheme and have common “specialization” to a (possibly singular) surface in \( \mathbb{P}^7 \). Both types of surfaces are simply connected, of general type and have invariants \( c_1^2 = 2^7, c_2 = 2^8 \). This is the first main new result of this paper. See Section 4 for details.

1.2 Moduli

The last 40 years have seen a tremendous development in understanding moduli spaces of surfaces. See e.g. the overviews [21, 26]. For surfaces of general type one usually fixes several numerical invariants and searches for a moduli space for all surfaces with the given invariants. Mumford [19] was the first to find badly behaved Hilbert schemes of smooth curves. This resulted in a principle stating that even the best behaved projective manifolds \( X \) can have local moduli spaces with arbitrary bad singularities at the point corresponding to \( X \) and Ravi Vakil in [24] made this more precise and showed that this instance of Murphy’s law indeed holds in many cases. So there is no reason to believe that complete intersections would behave better in this respect.

In this note we are especially interested in complete intersection surfaces in projective spaces and products of those. Here the story starts with extensive work by E. Horikawa [12, 13] in the seventies; the relevant surfaces still carry his name. The simplest example [12] is a quintic surface in \( \mathbb{P}^3 \). It is simply connected with numerical invariants \( c_1^2 = 5, c_2 = 55 \). Quintic surfaces form a Zariski open subset of one component \( M_1 \) of dimension 40 = \( \binom{8}{3} \) – 16 of the moduli space. There is a second component \( M_2 \) of the same dimension, a Zariski open subset of which parametrizes double covers of a quadric. The intersection \( M_2 \cap M_1 \) is a divisor in both components which corresponds to certain double covers of Hirzebruch surfaces. These give “specializations” of quintic surfaces that are no longer quintic surfaces. The necessary calculations to show this are quite involved.

There are several other studies of moduli spaces involving complete intersections in (weighted) projective space since these come up naturally when describing the canonical rings of surfaces of general type as shown by M. Reid and used by F. Catanese and co-workers. However, this is not the place to comment extensively on those since we are foremost interested in complete intersections in projective space and complete intersections in products of projective spaces. In this direction we mention that Benoist [4] has recently developed a good moduli theory for the first class of varieties.
Our second main result is that our construction gives two irreducible components of the moduli space of simply connected surfaces with invariant \(c_1^2 = 2^7, c_2 = 2^8\) of different dimensions. We calculate these dimensions and show that the Kuranishi spaces are smooth. Since we show (Lemma 2.1) that the latter come from the same connected Hilbert scheme of simply connected surfaces of degree 2 in \(P^7\) and with the invariants \(c_1^2 = 2^7, c_2 = 2^8\) we deduce that the corresponding Hilbert scheme has at least two irreducible components.

1.3 Topology

We continue this introduction with a few remarks on the (differential) topology of complete intersection surfaces. They are simply connected and for these it is well known that the invariants \(c_1^2, c_2\) together with the ”parity” of \(c_1\) completely determine the topological type of the surface. See e.g. [11], Chap. VIII, Lemma 3.1, Chap. IX.1. Recall here that a cohomology class \(c \in H^2(X, \mathbb{Z})\) is ”even” if it is of the form \(c = 2d, d \in H^2(X, \mathbb{Z})\) and ”odd” otherwise. Hence we may indeed speak of the ”parity” of classes in \(H^2\). It is well known that for a simply connected surface \(c_1\) is even precisely if the surface has a spin structure. Hence the terminology ”spin surface”. Other authors also call such surfaces ”even” surfaces. Our construction necessarily gives only spin surfaces and whenever two of those have the same Chern classes, they must be oriented homeomorphic.

The differentiable classification of simply surfaces is far more difficult since the only known computable differentiable invariant is the divisibility of the canonical class (under some extra hypotheses that are usually satisfied). This is often used to show that two surfaces are not diffeomorphic. See for example [7] 20.

As a consequence of Ehresmann’s theorem [8], surfaces that are deformations of each other are diffeomorphic and this is basically the only way to show this. For the surfaces of the second example we don’t know that there is a common specialization which is a smooth surface and this prevents us to decide whether the two types of surfaces are diffeomorphic or not.

1.4 The structure of the paper

Since some standard properties of numerical invariants of surfaces play a central role in the proofs, we start by recalling these in Section 2.1 and use them in Section 2.2 to calculate the basic invariants for the two types of surfaces. Then we solve Beauville’s exercise, and finally we compare the basic invariants and prove the main result. This comparison is quite delicate since the formulas are involved. The reader will be quickly convinced that a general comparison with complete intersection surfaces in products of projective spaces having 3 or more factors, is out of reach with the present approach. The best one could hope for is to find similar examples with a structured computer search. In Remark 4.5 a we briefly address the case \(P^2 \times P^k\), and provide some details for \(k = 2\).
In Section 5 we address the deformation theory of our new class of examples and show that these form two components in moduli. Hence here the situation differs from the $K_3$ example. Contrary to Horikawa’s example of the deformation of the quintic surface, local calculations can be avoided making this example a bit more accessible.

2 Calculation of some numerical invariants

2.1 Surface invariants

If $S$ is a complex projective surface, the following topological invariants

\[ b_1(S) = b_3(S), \quad b_2(S), \quad e(S) = 2 - 2b_1(S) + b_2(S), \]

are related to the Chern classes $c_1(S), c_2(S)$ and the complex invariants through the formulas (see e.g. [11, Chapter I.5]):

\[ c_2(S) = e(S), \quad b_1(S) = 2q(S), \]

\[ \chi(O_S) = p_g(S) - q(S) + 1 = \frac{1}{12}(c_1^2(S) + c_2(S)) \] (Noether’s formula). \hfill (1)

So, the information given by the basic triple invariant \{b_1(S), c_1^2(S), c_2(S)\} completely determines the complex invariants $K_S^2 = c_1^2(S), p_g(S)$ and $q(S)$. In particular, if $S$ is simply connected (and hence $b_1(S) = 0$), the Chern classes suffice for that. However, to determine the topological type, one also needs the intersection form on $H^2(S)$ which is determined by the signature and the parity. The signature is given by the Chern classes, but the parity is determined by the parity of the first Chern class in $H^2(S)$.

Suppose that $S$ comes with a preferred embedding $S \hookrightarrow \mathbb{P}^{n+2}$ as a codimension $n$ submanifold, and $H = \mathcal{O}_{\mathbb{P}^{n+2}}(1)$ is the hyperplane bundle, then the embedding yields two more invariants:

\[ \deg(S) := [S] \cdot H^2 \in H^{2n+4}(\mathbb{P}^{n+2}) = \mathbb{Z}, \quad K_S \cdot H|_S. \] \hfill (2)

Lemma 2.1. The invariants \{2\} together with the basic triple invariant determine the Hilbert scheme of $S \hookrightarrow \mathbb{P}^{n+2}$.

Proof: By Hartshorne [11], p. 366, Exercise 1.2, the Hilbert polynomial for a surface $S$ is

\[ P_S(z) = \frac{1}{2}az^2 + bz + c, \quad a = \deg S, \quad b = \frac{1}{2}\deg S + 1 - \pi, \quad c = \chi(O_S) - 1, \]

\[ \pi = \text{genus of the curve } (S \cap H) = \frac{1}{2}(K_S \cdot H + \deg S + 2). \] \hfill □
Remark 2.2. By [10] the Hilbert scheme $H_S$ of $S \subset P^{n+2}$ is connected. This implies that if $S' \in H_S$, the surface $S$ can be deformed into $S'$. In fact Hartshorne in loc. cit. proves that this deformation can be done via a linear deformation. Suppose that the resulting family is through smooth surfaces, then by [8] they would be diffeomorphic. In general, all one can say is that $S$ and $S'$ deform to the same surface which may or may not be singular.

2.2 Complete intersection surfaces

The general situation is as follows: $P$ is a smooth projective manifold of dimension $n+2$ and $X \subset P$ is a complete intersection surface cut out by hypersurfaces $f_1, \ldots, f_n$. The Whitney product relation between the total Chern class $c = 1 + c_1 + c_2 + \cdots$ of $X$ and $P$:

$$c(X) \cdot (1 + F_1) \cdots (1 + F_n) = c(P), \quad F_j = c_1(f_j)$$

allows to calculate the invariants $c_1(X)$ and $c_2(X)$.

Suppose that the Chern classes of $P$ are in the subring $A(P) \subset H^*(P)$ generated by the Picard group $\text{Pic}(P)$. Then, the Chern classes of $X$ belong to the subring $j^*A(P) \subset H^*(X)$, where $j : X \hookrightarrow P$ is the inclusion, which leads to simple formulas for them. Below this is applied to complete intersections in products of projective spaces.

For the calculation of the Betti numbers $b_j(X)$ the following consequence of Lefschetz' hyperplane theorem will play a role:

$$b_j(Y) = b_j(P), \quad j \leq \dim Y - 1, \quad Y \text{ a complete intersection in } P. \quad (3)$$

For instance, if $P = P^{n+2}$, one has $b_1(X) = 0$. In fact, a sharper version of the Lefschetz hyperplane theorem implies that $X$ is simply connected.

Example 2.3 (Surface complete intersections of quadrics). Let $S \subset P = P^{2k+1}$ be a smooth complete intersection of $2k-1$ quadrics. Its degree is

$$\deg S = 2^{2k-1}.$$ 

Since $\text{Pic}(P) = \mathbb{Z}H$, where $H$ is the class of a hyperplane in $H^*(P, \mathbb{Z})$, and $c(P) = (1 + H)^{2k+2}$, setting

$$h := j^*H, \quad S \hookrightarrow P \text{ the embedding},$$

one finds from the Whitney product relation:

$$c_1(S) = -2(k-2)h \quad (4)$$

$$c_1^2(S) = 4(k-2)^2 \cdot 2^{2k-1} \quad (5)$$

$$c_2(S) = (2k^2 - 5k + 5)h^2 = (2k^2 - 5k + 5) \cdot 2^{2k-1} \quad (6)$$

$$-c_1(S) \cdot h = 2(k-2) \cdot 2^{2k-1}. \quad (7)$$
Example 2.4 (Surface complete intersections in $\mathbb{P}^1 \times \mathbb{P}^k$). If $P = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, the Picard group is given by $\text{Pic}(P) = \mathbb{Z}H_1 + \mathbb{Z}H_2$ where $H_j$ is the pull back of the generator of $\text{Pic}(\mathbb{P}^{n_j})$, $j = 1, 2$. For any complete intersection surface $T \subset P$ we write

$$h_k = j^*H_k, \quad k = 1, 2, \quad j : T \hookrightarrow P \text{ the embedding.}$$

From now on, assume that $n_1 = 1$ and $n_2 = k$ and suppose that $T \subset P$ is a complete intersection of $k - 1$ hypersurfaces of bidegrees $(a_1, b_1), \ldots, (a_{k-1}, b_{k-1})$:

$$[T] = \left( a_1 H_1 + b_1 H_2 \right) \cdots \left( a_{k-1} H_1 + b_{k-1} H_2 \right) \in H^*(P, \mathbb{Z}).$$

One has the intersection table in $H^*(T, \mathbb{Z})$:

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$h_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$h_2$</td>
<td>$b$</td>
</tr>
</tbody>
</table>

Now $h_1 + h_2$ comes from an ample class, so $0 < h_1 \cdot (h_1 + h_2) = b$ implies that all $b_i \geq 1$. Since we wish to compare with a surface of general type (a complete intersection of quadrics), the case $a_i = 0$ ($i = 1, \ldots, k - 1$) is not of interest to us. Indeed, in our computations later on we will assume and use that at least one of the $a_i$ is positive. In summary:

$$b \neq 0, \quad c \neq 0. \quad (8)$$

Let $s : \mathbb{P}^1 \times \mathbb{P}^k \hookrightarrow \mathbb{P}^{2k+1}$ be the Segre embedding and let

$$h := s^*H = h_1 + h_2, \quad \text{Pic}(\mathbb{P}^{2k+1}) = \mathbb{Z} \cdot H.$$

Setting

$$\alpha := \sum_{j=1}^{k-1} a_j, \quad \beta := \sum_{j=1}^{k-1} b_j, \quad \gamma := \sum_{i<j} a_i b_j, \quad \delta := \sum_{i<j} b_i b_j,$$

$$x := (\alpha - 1)(2\beta - k - 1) + (k + 1) - \gamma, \quad y := \beta^2 + (k + 1)(\frac{1}{2}k - \beta) - \delta, \quad$$

$$u := 2(\alpha - 2)(\beta - (k + 1)), \quad v := (\beta - (k + 1))^2,$$

one finds first of all

$$j^*c(P) = (1 + 2h_1)(1 + (k + 1)h_2 + \frac{1}{2}k(k + 1)h_2^2) = 1 + 2h_1 + (k + 1)h_2 + 2(k + 1)h_1 h_2 + \frac{1}{2}k(k + 1)h_2^2,$$

$$j^*(1 + F_1) \cdots j^*(1 + F_{k-1}) = 1 + \alpha h_1 + \beta h_2 + \gamma h_1 h_2 + \delta h_2^2.$$
and thus, by the Whitney formula:

\[\begin{align*}
    c_1(T) &= (-\alpha + 2)h_1 + (-\beta + (k + 1))h_2 \\
    c_1^2(T) &= ub + vc \\
    c_2(T) &= xb + yc \\
    \deg T = h^2 &= 2b + c \\
    -c_1(T) \cdot h &= (\alpha + \beta - (k + 3))b + (\beta - (k + 1))c.
\end{align*}\]

Equation (10), (11), together with the parity of \(\alpha\) and \(\beta\) determine the topological type. By Lemma 2.1, together with the remaining two equations they determine the component of the Hilbert scheme of subvarieties of \(P^{2k+1}\).

3 The case \(k = 2\): degeneration of degree 8 \(K_3\) surfaces in \(P^5\)

A smooth complete intersection \(S\) of three quadrics in \(P^5\) is simply connected (by the Lefschetz hyperplane theorem) and has trivial canonical bundle (by (4)). So it is a \(K_3\) surface. As an intersection of 3 quadrics it has degree 8.

We next consider a smooth hypersurface \(T\) of bidegree \((2, 3)\) in \(P^1 \times P^2\). Again, it is simply connected by Lefschetz, and by (6) it follows that \(K_T\) is trivial: \(T\) is also a \(K_3\)-surface. Consider its Segre image \(s(T)\) in \(P^5\). From from (12) and Table 19 we see that \(\deg s(T) = 8\). The equations describing the image of \(P^1 \times P^2\) in \(P^5\) will appear below after analyzing bihomogeneous polynomials of bidegree \((2, 3)\). The surfaces \(S\) and \(T\) have the same Hilbert polynomial and so by Lemma 2.1 they belong to the same connected Hilbert scheme. The component to which \(T\) belongs has dimension \(3 \cdot 10 - 1 = 29\) with the bi-projective group of dimension \(3 + 8 = 11\) acting, while the component to which \(S\) belongs has bigger dimension \(18 \cdot 3 = 54\). To explain this, we remark that the ideal generated by three quadrics \(Q_k, k = 1, 2, 3\) can be normalized: we may assume that there exist three monomials exactly one of which occurs in one of the \(Q_j\) but not the other two. The projective group of dimension \(6^2 - 1 = 35\) then acts on the Hilbert scheme with 19-dimensional quotient. This calculation shows that in moduli, the surfaces \(T\) give a divisor on the 19-dimensional moduli space of those projective \(K_3\) surfaces that have a genus 5 hyperplane section. So there is only one component of the Hilbert scheme and the following theorem is a consequence. We want however to give a constructive proof.

**Theorem 3.1.** There exists a one parameter family \(\{S_t\}\) whose fibers \(S_t\) with \(t \neq 0\) are complete intersections of three quadrics and whose special fiber \(S_0\) is the given Segre embedded surface \(s(T)\).

**Proof:** Let \(R = \mathbb{C}[u, v] \otimes \mathbb{C}[x_1, x_2, x_3]\), the homogeneous coordinate ring of \(P^1 \times P^2\). Consider a bihomogeneous polynomial of bidegree \((2, 3)\) defining the surface \(T\):

\[F = u^2C_{11} + uvC_{12} + v^2C_{22},\]
where the $C_{ij} \in C[x_1, x_2, x_3]$ are homogeneous cubics. Our first goal is to rewrite $F$ in terms of the products $ux_i$ and $vx_j$ so that the link with the coordinates of $P^5$ can be exploited. Write

$$C_{ij} = \sum_{\alpha} Q_{ij}^\alpha x_\alpha,$$

where the $Q_{ij}^\alpha \in C[x_1, x_2, x_3]$ are quadratic polynomials. Note that the $Q_{ij}^\alpha$ are not uniquely determined by the $C_{ij}$.

Now every homogeneous quadratic polynomial $Q$ corresponds to a bilinear form $q$ such that $Q = q(x, x) = (x_1, x_2, x_3)$. If $q_{ij}^\alpha$ denotes the bilinear form corresponding to $Q_{ij}^\alpha$, then

$$u^2 C_{11} = \sum_{\alpha} x_\alpha q_{11}^\alpha (ux, ux),$$
$$v^2 C_{22} = \sum_{\alpha} x_\alpha q_{22}^\alpha (vx, vx),$$
$$uv C_{12} = \sum_{\alpha} x_\alpha q_{12}^\alpha (ux, vx),$$

so that $F$ can be written in the form

$$F = \sum_{\alpha} x_\alpha \left[ q_{11}^\alpha (ux, ux) + q_{22}^\alpha (vx, vx) + q_{12}^\alpha (ux, vx) \right].$$

Next consider $R' = C[X_1, X_2, X_3, X'_1, X'_2, X'_3]$, the homogeneous coordinate ring of $P^5$, and the homomorphism $h : R' \to R$ induced by the Segre embedding, and which is determined by

$$(X_1, X_2, X_3, X'_1, X'_2, X'_3) \mapsto (x_1 u, x_2 u, x_3 u, x_1 v, x_2 v, x_3 v).$$

If

$$A := \begin{pmatrix} X_1 & X_2 & X_3 \\ X'_1 & X'_2 & X'_3 \end{pmatrix}$$

and if $A_1, A_2, A_3$ are the subdeterminants of $A$ obtained by omitting the first, second and third column, respectively, then the three quadratic equations $A_1 = 0, A_2 = 0, A_3 = 0$ describe the Segre embedded $P^1 \times P^2$ in $P^5$ (and $h$ induces an injective map $R'(A_1, A_2, A_3) \leftrightarrow R$). Now the polynomial $F$ gives rise to two polynomials in the coordinate ring of $P^5$, namely, $uF = \sum X_\alpha Q^\alpha$, which we denote by $C$, and $vF = \sum X'_\alpha Q^\alpha$, which we denote by $C'$. Note furthermore that the quadrics $A_1, A_2, A_3$ obey the two relations $\sum (-1)^\alpha X_\alpha A_\alpha = 0$ and $\sum (-1)^\alpha X'_\alpha A_\alpha = 0$.

**Claim.** It is possible to choose the quadrics $Q^\alpha (\alpha = 1, 2, 3)$ in such a way that for small non-zero values of $t$ the 3 quadrics $Q^\alpha_t = tQ^\alpha + (-1)^\alpha A_\alpha$ (for $\alpha = 1, 2, 3$) define a complete intersection $S_t$ as mentioned in the theorem.
Before proving this claim, first note that \( \sum X_\alpha Q^\alpha = t \sum X_\alpha Q^\alpha = tC \), and similarly for \( C' \). So the zero set of the ideal \( I_t = (Q_1^t, Q_2^t, Q_3^t, C, C') \) is the intersection \( Q_1^t = Q_2^t = Q_3^t = 0 \) of quadrics in \( \mathbb{P}^5 \) for \( t \neq 0 \), whilst for \( t = 0 \) the zero set of \( I_0 \) gives the ‘degenerate’ K3 surface \( s(T) \).

**Proof of the Claim.** The preceding construction regarding polynomials of bidegree \((2, 3)\) can be rephrased as follows. Let \( U \) and \( V \) be complex vector spaces of dimensions 2 and 3, respectively. These ‘model’ the degree 1 parts of \( \mathbb{C}[u,v] \) and \( \mathbb{C}[x_1, x_2, x_3] \). Now put 

\[
W = U \otimes V.
\]

Then the obvious bilinear map \( U \times V \to U \otimes V = W \) induces the Segre embedding. In these terms, we view our bidegree \((2, 3)\) polynomial \( F \) as an element of \( S^2 U \otimes S^3 V \). The maps 

\[
S^2 V \otimes V \to S^3 V, \quad \text{induced by} \quad Q \otimes L \mapsto Q \cdot L,
\]

and 

\[
S^2(U \otimes V) \to S^2 U \otimes S^2 V, \quad \text{induced by} \quad (x \otimes v) \cdot (x' \otimes v') \mapsto x \cdot x' \otimes v \cdot v',
\]

are surjective. These fit in the following commutative diagram:

\[
\begin{array}{c}
S^2 U \otimes S^2 V \otimes V \\
\downarrow \quad \uparrow q \\
S^2(U \otimes V) \otimes V \quad \cong \quad \bigoplus^3 S^2(U \otimes V) = \bigoplus^3 S^2 W
\end{array}
\]

In this diagram the map \( q \) represents the explicit construction described above. Indeed,

\[
q(Q^1(ux, vx), Q^2(ux, vx), Q^3(ux, vx)) = \sum_{\alpha=1}^3 x_\alpha Q^\alpha(ux, vx)
\]

\[
= \left( \sum_{\alpha=1}^3 x_\alpha q_1^\alpha(x, x) \right) uv
\]

\[
+ \left( \sum_{\alpha=1}^3 x_\alpha q_2^\alpha(x, x) \right) u^2
\]

\[
+ \left( \sum_{\alpha=1}^3 x_\alpha q_3^\alpha(x, x) \right) v^2.
\]

Now \( \mathbb{P}(S^2 U \otimes S^3 V) \) contains a Zariski open subset corresponding to smooth surfaces, including the one defined by \( F \). Its preimage under \( q \) gives rise to a Zariski open subset in \( \mathbb{P}(\bigoplus^3 S^2 W) \). This latter open subset meets the Zariski open part of \( \mathbb{P}(\bigoplus^3 S^2 W) \) corresponding to smooth intersections of three quadrics in a Zariski open set. In particular, in our construction with \( F \), a triple \((Q^1, Q^2, Q^3)\) of quadrics in \( \bigoplus^3 S^2 W \) can be chosen defining a complete intersection. This proves the above claim and hence the theorem. \( \square \)
4 Comparison for $k \geq 3$

The following integers come up in the relevant formulae below; for clarity, we also recall some of the previous notation:

$$K_T = ah_1 + bh_2,$$
where $a = \alpha - 2 = \sum a_j - 2$,
$$b = \beta - (k + 1) = \sum b_j - (k + 1),$$
$$b = \prod_{j=1}^{k-1} b_j,$$
$$c = \sum_{j=1}^{k-1} a_j \cdot (b_1 \cdots \hat{b}_j \cdots b_{k-1}),$$
$$x = 2ab + a(k + 1) + 2b + 2(k + 1) - \gamma, \quad \gamma = \sum_{i \neq j} a_i b_j,$$
$$y = b^2 + b(k + 1) + \frac{1}{2}k(k + 1) - \delta, \quad \delta = \sum_{i \neq j} b_i b_j.$$

The calculations in Example 2.4 imply:

**Lemma 4.1.** 1. The topological invariants $c_1^2(T), c_2(T)$ equal those of a smooth complete intersection $S$ of $(2k - 1)$ quadrics in $\mathbb{P}^{2k+1}$ precisely if

$$2ab b + b^2 c = 2^{2k-1} (2(k - 2))^2,$$
$$x b + y c = 2^{2k-1} (2k^2 - 5k + 5).$$

Suppose that $K_T$ is ample. Then $a \geq 0$ and $b \geq 1$. If such a $T$ exists with even $a$ and $b$ it is oriented homeomorphic to a smooth complete intersection of $(2k - 1)$ quadrics in $\mathbb{P}^{2k+1}$.

2. The surfaces $S$ and $T$ belong to the same Hilbert scheme if, moreover,

$$2b + c = 2^{2k-1},$$
$$a + b + b c = 2^{2k-1} (2k - 2).$$

We first consider the case $k = 3$ and then we have:

$$b = b_1 b_2 = \delta, \quad \gamma = a_1 b_2 + a_2 b_1 = c.$$

**Theorem 4.2.** A smooth complete intersection $T$ of two hypersurfaces of type $(4, 2)$ and $(0, 4)$ in $\mathbb{P}^1 \times \mathbb{P}^3$ is oriented homeomorphic to a smooth complete intersection $S$ of 5 quadrics in $\mathbb{P}^7$. This is the only possibility among complete intersections of $\mathbb{P}^1 \times \mathbb{P}^3$. Both surfaces are simply connected, spin and have invariants $c_1 = 2^7, c_2 = 2^8$.

The two types of surfaces belong to the same Hilbert scheme of $\mathbb{P}^7$ when we consider $T$ as embedded in $\mathbb{P}^7$ through the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7$. In particular they deform to the same, possibly singular, surface.

**Proof:** Because of (18), our system of equations reduces to

$$2ab \delta + b^2 \gamma = 2^7,$$
$$(2ab + 4a + 2b + 8 - \gamma) \delta + (b^2 + 4b + 6 - \delta) \gamma = 2^8.$$

\(^2\)If it can be chosen to be smooth, the two types of surfaces are diffeomorphic.
By (8), $\gamma = c \neq 0$. Rewriting the first equation as $b(2a\delta + b\gamma) = 2^7$, we see that $b$ is a power of 2 and that $b^2 \leq b(2a\delta + b\gamma) = 2^7$ so we conclude that $b = 2^\ell$ with $\ell = 0, 1, 2, 3$. Hence

$$2a\delta + b\gamma = 2^{7-\ell}.$$  

Subtracting this twice from the second equation, after some rewriting, yields,

$$(\gamma - (2^\ell + 4))(\delta - (2^\ell + 3)) = (2^\ell + 4)(2^\ell + 3) + 2^{7-\ell} - 2^6.$$  

The right-hand side equals 84, 30, 24, respectively, for $\ell = 0, 1, 2$, respectively.

- **Case** $\ell = 0$, i.e. $b = 1$. Then $(\gamma - 5)(\delta - 4) = 84 = 7 \cdot 4 \cdot 3$. Now $0 \leq \delta = b_1b_2 = b_1(6 - b_1) = b_1(5 - b_1) \leq 6$, and $\gamma \geq 0$, so both factors $\gamma - 5$ and $\delta - 4$ must be positive. But then $\delta$ must be 6 and $\gamma$ must be 47. But the equation $2ab\delta + b^2\gamma = 2^7$ reduces to $12a + 47 = 128$ which has no integer solutions.

- **Case** $\ell = 1$, i.e. $b = 2$. Then $(\gamma - 6)(\delta - 5) = 30 = 2 \cdot 3 \cdot 5$. The solution $\gamma = 0$ and $\delta = 0$ is ruled out, since we saw that $\gamma \neq 0$.

Otherwise $1 \leq \delta = b_1(6 - b_1) \leq 9$ so that $-4 \leq \delta - 5 \leq 4$. For divisibility reasons, the only possibility for $\delta$ is 8 and thus $\gamma = 16$. Then the equation $2ab\delta + b^2\gamma = 2^7$ reduces to $a = 2$. We get

$\delta = b_1b_2 = 8,$

$\gamma = a_1b_2 + a_2b_1 = 16,$

$a + 2 = a_1 + a_2 = 4.$

The first equation has solutions $(b_1, b_2) = (1, 8), (2, 4)$. The first is incompatible with the other two equations. The second leads to the only solution $(a_1, a_2) = (4, 0)$, $(b_1, b_2) = (2, 4)$ compatible with the three equations.

- **Case** $\ell = 2$, i.e. $b = 4$. Then $(\gamma - 8)(\delta - 7) = 24 = 2^3 \cdot 3$. The equation $2ab\delta + b^2\gamma = 2^7$ reduces to $a\delta + 2\gamma = 16$. Now $\delta = b_1(8 - b_1)$ can only assume the values 0, 1, 7, 2, 6, 3, 5 and 4, 4. From divisibility the only possibility left for $\delta$ is 15. But then $a\delta + 2\gamma = 16$ implies $a = 0$ and $\gamma = 8$. But $\gamma \neq 8$ because the factor $\gamma - 8$ must be nonzero.

- **Case** $\ell = 3$. Here we have $8(2a\delta + 8\gamma) = 2^7$ so that $a\delta + 4\gamma = 8$. Since $a \geq 0, \gamma \geq 1$, the only possibilities for $\gamma$ are 1 and 2, but that conflicts with $(\gamma - 12)(\delta - 11) = 84$.

Concluding, we have shown that the only solution to the first two equations is as stated. However, for this solution, $a = b = 2$ the remaining equations are identical to the first equation and so $T$ and $S$ belong to the same Hilbert scheme.

We complete the above result by showing that the phenomenon of Theorem 4.2 does not occur for $k \geq 4$:

**Proposition 4.3.** If $k \geq 4$ there cannot exist two surfaces $S$ and $T$ of the above type which belong to the same Hilbert scheme.
Proof: The idea here is to consider the three equations (14), (16), (17) as a system of equations for b, c with coefficients involving a and b. By (8), if an integer solution exists the rank of the coefficient matrix has to be at most 1. This means that \( ab = b^2, a + b = 2b \) and so \( a = b \). But then the equations imply that \( a = b = 2(k-2) \).

To exclude this solution, argue as follows:

\[
\sum_{j=1}^{k-1} a_j = a + 2 = 2(k-1)
\]

\[
\sum_{j=1}^{k-1} b_j = b + (k+1) = 3(k-1).
\]

The maximal value of b can be computed with the methods of Lagrange multipliers: the maximum for \( b \) occurs for \( b_j = 3 \) and equals \( 3^{k-1} \). Note that there is an extremal value for \( c \) when \( a_j = 2, b_j = 3 \) but this is not a maximum. But we may use that \( \sum_{j \neq i} b_j \leq 3(k-1) - 1 \) since \( b_i \geq 1 \). We then use the Langrange multiplier method for the product of \( (k-2) \) different \( b_j \). This gives:

\[
\prod_{j \neq i} b_j \leq \left( 3 + \frac{2}{k-2} \right)^{k-2}
\]

and hence

\[
c \leq \left( 3 + \frac{2}{k-2} \right)^{k-2} \cdot \left( \sum_{j=1}^{k-1} a_j \right) = \left( 3 + \frac{2}{k-2} \right)^{k-2} \cdot (2(k-1)).
\]

But this would imply

\[
4^{k-1} = \frac{1}{2}(2b + c) \leq 3^{k-2} \cdot \left( 3 + \frac{2}{3(k-2)} \right)^{k-2} \cdot (k-1)
\]

which is false as soon as \( k \geq 6 \).

To exclude \( k = 4, 5 \) we have to use that the \( b_j \) are positive integers summing up to \( 3(k-1) \). For \( k = 5 \), writing down all possibilities for the quadruple \( (b_1, b_2, b_3, b_4) \), we see that the product of three among them can be 48 for \( (1, 3, 4, 4) \), 45 for \( (1, 3, 3, 5) \) and at most 40 for all other quadruples. The first quadruple gives, using that \( a_1 + a_2 + a_3 + a_4 = 8, \)

\[
4^4 = 256 = \frac{1}{2}c + 24a_1 + 8a_2 + 6(a_3 + a_4) + 48 = 18a_1 + 2a_2 + 6 \cdot 8 + 48
\]

and so \( 80 = 9a_1 + a_2 \) which has no solutions since \( a_1 + a_2 \leq 8 \). For \( (1, 3, 3, 5) \) we find

\[
256 = \frac{1}{2}(45a_1 + 15a_2 + 15a_3 + 9a_4) + 45 = \frac{1}{2}(36a_1 + 6a_2 + 6a_3 + 9 \cdot 8) + 45 = 18a_1 + 3(a_2 + a_3) + 81,
\]
which gives a contradiction modulo 3.

In the other cases, we have

\[ 256 = \frac{1}{2}c + b \leq \frac{1}{2}40 \cdot 8 + 3^4 = 241, \]

and hence no solution either.

For \( k = 4 \) there is a solution to \( b + \frac{1}{2}c = 4^3 \), namely \((a_1, a_2, a_3) = (0, 0, 6)\), and \((b_1, b_2, b_3) = (4, 4, 1)\). This can be seen to be the only one: we only have to test whether for each of the values of the triples \((b_1, b_2, b_3) = (1, 1, 7), (1, 2, 6), (1, 3, 5), (1, 4, 4), (2, 2, 5), (2, 3, 4), (3, 3, 3)\), i.e., the positive integral solutions of \( b_1 + b_2 + b_3 = 9 \), one can find a triple \((a_1, a_2, a_3)\) with \( a_1 + a_2 + a_3 = 6 \) such that

\[ \frac{1}{2}(a_1b_2b_3 + a_2b_1b_3 + a_3b_1b_2) + b_1b_2b_3 = 64. \]

This gives for \((1, 4, 4)\) one solution only, which is the one we had (up to renumbering).

For the other triples the argument resembles the one for \( k = 5 \). To test for instance \((b_1, b_2, b_3) = (1, 2, 6)\), one gets

\[ 6a_1 + 3a_2 + a_3 + 12 = 5a_1 + 2a_2 + 18 \leq 5 \cdot (a_1 + a_2) + 18 \leq 48 < 64. \]

It remains to exclude the solution we found, \((a_1, a_2, a_3) = (0, 0, 6)\), \((b_1, b_2, b_3) = (4, 4, 1)\). For this we observe that it does not satisfy the remaining equation \((\ref{eq:15})\) since \( x = 22, y = 10 \) while \( b = 16, c = 96 \) and thus \((\ref{eq:15})\) would give

\[ 22 \cdot 16 + 10 \cdot 96 = 17 \cdot 128, \]

which is false. \( \square \)

**Remark 4.4.** For \( k \geq 4 \), there could still be solutions to the ”topological” equations \((\ref{eq:14}), (\ref{eq:15})\). We have not tested this since these equations become unwieldy. Some experimentation suggests that existence of solutions is very unlikely.

Referring to Remark 2.2 if these do exist, they would give other examples of complete intersection surfaces in \( \mathbb{P}^1 \times \mathbb{P}^k \) oriented homeomorphic to complete intersections of quadrics.

**Remark 4.5.** If we replace \( \mathbb{P}^1 \times \mathbb{P}^k \) by \( \mathbb{P}^2 \times \mathbb{P}^k \), and try to compare complete intersection surfaces in the latter space with complete intersections of quadrics in \( \mathbb{P}^{3k+2} \), we are led to introduce a new variable, since the intersection table corresponding to Table \((\ref{eq:19})\) no longer contains a 0. In this case we need to bring more equations into play than the ones corresponding to \((\ref{eq:16})\) and \((\ref{eq:17})\). The new set of equations doesn’t look promising to handle.

In the case \( k = 2 \), however, a simple argument can be given to exclude solutions. Here the analogs of equations \((\ref{eq:10}), (\ref{eq:12}), (\ref{eq:13})\) lead to the system of equations

\[
\begin{pmatrix}
1 & 2 & 1 \\
a & a + b & b \\
a^2 & 2ab & b^2
\end{pmatrix}
\begin{pmatrix}
b_1b_2 \\
a_1b_2 + a_2b_1 \\
a_1a_2
\end{pmatrix}
= \begin{pmatrix}
2^6 \\
3 \cdot 2^6 \\
3^2 \cdot 2^6
\end{pmatrix},
\]

\((\ref{eq:19})\)
where \( a = a_1 + a_2 - 3, b = b_1 + b_2 - 3 \). Consider the second column as variables \( b, c, d \) to be solved for. Then one gets the following cases for the solutions.

1. Case \( a = b \). Then \( a = b = 3 \) from the first and second equation in (19), and \( b + 2c + d = 2^6 \). This has no solutions (case by case analysis).

2. Case \( a \neq b \). Then, setting \( u = \frac{3-a}{b-a} \) (and hence \( 1 - u = \frac{b-3}{b-a} \)), we obtain

\[
\begin{align*}
\text{b} &= b_1b_2 = (u-1)^2 \cdot 2^6, \\
\text{c} &= a_1b_2 + a_2b_1 = u(1-u) \cdot 2^6, \\
\text{d} &= a_1a_2 = u^2 \cdot 2^6.
\end{align*}
\]

In this case we have \( u \neq 0 \) and \( 1 - u \neq 0 \) (from the expressions for \( b, c, d \)).

Now

\[
\begin{align*}
\text{d} &= a_1a_2 = \frac{u^2}{(u-1)^2}, \\
\text{b} &= b_1b_2 = \frac{u}{1-u}, \\
\text{c} &= a_1b_2 + a_2b_1 = \frac{u(1-u) \cdot 2^6}{(u-1)^2}.
\end{align*}
\]

This means that \( x_{1,2} = \frac{a_{1,2}}{b_{1,2}} \) are solutions of the quadratic equation

\[
X^2 + \frac{u}{u-1}X + \frac{u^2}{(u-1)^2} = 0.
\]

But this equation has discriminant \(-3\frac{u^2}{(u-1)^2} < 0\) and so has no real solutions.

## 5 Moduli

### 5.1 Basic deformation theory

Let us recall (see e.g. [16, 14, 6.2]) some basic deformation theory for compact complex manifolds. A deformation of a compact complex manifold \( X_0 \) is a proper submersive morphism \( p : (X,p^{-1}0) \to (S,0) \) between pairs of analytic spaces, where \( X_0 \approx p^{-1}(0) \). Non-triviality of a deformation can be measured by means of a linear map\(^3\)

\[
\kappa : T_0S \longrightarrow H^1(\Theta_{X_0}),
\]

its Kodaira–Spencer map. If surjective, the deformation is complete and if injective, one says that it is effective. A deformation which is both complete and effective is called semi-universal. It always exists and since Kuranishi gave a first construction, they are called Kuranishi or semi-universal local deformation. The base space \( \mathcal{K}_{X_0} \) is called the

\(^3\)The source space \( T_0 \) denotes the Zariski tangent space.
**Kuranishi space** of $X_0$. If $h^0(\Theta_{X_0}) = 0$ it is known that the semi-universal deformation is unique up to isomorphism and we speak of a *universal local deformation*. So in this case, it coincides with the Kuranishi deformation. Due to the functorial nature of the latter, the automorphism group of $X_0$ acts on $\mathcal{K}_{X_0}$ and the quotient under this action, if it exists as an analytic space, is called the *local moduli space* for $X_0$. This is the case for instance if the automorphism group of $X_0$ is finite.

The Kuranishi space need not be smooth; it may even be non-reduced. However, if it is smooth, it has dimension $h^1(\Theta_{X_0})$. In fact, there is an "obstruction" map

$$\text{ob} : H^1(\Theta_{X_0}) \to H^2(\Theta_{X_0}), \quad \text{ob}(0) = 0,$$

which starts off with terms of order $\geq 2$ in the Taylor expansion and the Kuranishi space is locally isomorphic to the analytic subspace of the complex vector space $H^1(\Theta_{X_0})$ given as the zeros of the obstruction map. The obstruction map can described by means of the graded Lie-algebra structure on $H^*(\Theta_{X_0})$ induced by the usual Lie-bracket $[-,-]$ on the tangent bundle. In particular, if the brackets are always zero, the obstruction map vanishes. From this description we find:

$$T_0\mathcal{K}_{X_0} = H^1(\Theta_{X_0}) \iff \mathcal{K}_{X_0} \text{ is smooth at } 0$$
$$\iff [-,-]|H^1(\Theta_{X_0}) = 0$$
$$\iff \dim_0 \mathcal{K}_{X_0} = h^1(\Theta_{X_0}).$$

In favorable cases the Kuranishi deformation also gives the Kuranishi deformation for nearby fibres, for instance if this deformation is regular in the following sense.

**Definition 5.1.** A family $\{X_t\}_{t \in S}$ is *regular* $\iff h^1(\Theta_{X_t}) = h^1(\Theta_{X_0}) = \mu$ for all $t \in S$.

If the Kuranishi deformation $\{X_t\}_{t \in \mathcal{K}_{X_0}}$ is regular and a moduli space for the $X_t$ exists, its dimension then equals $\mu$.

**Example 5.2.** In the introduction we mentioned Horikawa’s example of a specialization of a quintic surface which is no longer a quintic surface. It is illustrative to look at their respective Kuranishi spaces. For the quintic surface $h^1(T) = 40$ is the number of moduli and the Kuranishi space is smooth. This also holds for the double covers of the quadric having the same invariants as the quintic. However, for the specialization which is the double cover of the Hirzebruch surface, one has $h^1(T) = 41$ and the Kuranishi space is a union of two divisors in an open 41-dimensional ball meeting transversally in a divisor through the origin. We say that this Kuranishi family is “obstructed”. For a quintic surface we have $h^2(\Theta_X) = 0$, while for the specialization $h^2(\Theta_X) = 1$. This is in accordance with above description of the Kuranishi space: if $h^2(\Theta_X) = 0$ there are no obstructions to smoothness, while if $h^2(\Theta_X) = 1$ there is at most one equation which incorporates obstructions. Horikawa has shown this this equation is non-zero and starts off with a quadratic term.
In general, for a surface \( S \) of general type the numbers \( h^1(\Theta_S) \) and \( h^2(\Theta_S) \) are related by means of the Riemann-Roch formula

\[
h^1(\Theta_S) - h^2(\Theta_S) = \frac{1}{6}(7c_1^2 - 5c_2),
\]

(20)

where we use \([17]\) that the group of biholomorphism of \( S \) is finite and hence that \( h^0(\Theta_S) = 0 \). This confirms what we said for quintic surfaces, since for them \( c_1^2 = 5 \), \( c_2 = 55 \) so that the right hand side equals 40.

If \( X_0 \) is a submanifold of some manifold \( Y \), the deformations within \( Y \) can be measured as follows. Consider the long exact sequence associated to the normal bundle sequence

\[0 \to \Theta_{X_0} \to \Theta_Y|X_0 \to N_{X_0/Y} \to 0.\]

If \( h^0(\Theta_{X_0}) = 0 \), we find

\[0 \to H^0(\Theta_{X_0}) \to H^0(N_{X_0/Y}) \xrightarrow{\delta} H^1(\Theta_{X_0}) \to H^1(\Theta_Y|X_0) \to H^1(N_{X_0/Y}),\]

(21)

and the image of \( \delta \) measures deformations of \( X_0 \) within \( Y \). In fact, the Kodaira-Spencer map of a deformation \( \{X_t\}_{t \in S} \) factors through \( H^0(N_{X_0/Y}) \) as follows

\[
\begin{array}{ccc}
T_0S & \xrightarrow{\sigma} & H^0(N_{X_0/Y}) \\
\downarrow & \sigma & \downarrow \rho \\
& \xrightarrow{\delta} & H^1(\Theta_{X_0}),
\end{array}
\]

where \( \sigma \) is called the characteristic map. The following, central, result is in essence due to Kodaira and Spencer \([16]\) Chapter V,VI:

**Proposition 5.3.** Let \( X \) be a compact complex submanifold of the compact complex manifold \( Y \). Suppose that \( h^0(\Theta_X) = 0 \), that the restriction map \( H^0(Y, \Theta_Y) \to H^0(X, \Theta_Y|X) \) is an isomorphism, and that there exists a regular deformation of \( X \) within \( Y \) with smooth base such that the characteristic map is onto. Then there is a subdeformation for which the Kodaira-Spencer map is an isomorphism onto the image of \( \delta \). This deformation is locally universal for deformations of \( X_0 \) within \( Y \).

To understand the conditions of this proposition, note that \( h^0(\Theta_X) = 0 \) implies that the Kuranishi family is locally universal, not merely versal. The second condition states that the “infinitesimal automorphisms” of \( Y \) act faithfully on \( X \). The proof consists in showing that this implies that a small slice transversal to the orbit of \( X \) under this action is the base of a locally universal family of deformations of \( X \) within \( Y \).

Since the image of \( \delta \) has codimension \( h^1(\Theta_Y|X_0) \) within \( H^1(\Theta_X) \) we conclude:

**Corollary 5.4.** If in addition \( h^1(\Theta_Y|X_0) = 0 \), the deformation of Proposition 5.3 is the Kuranishi deformation and the number of moduli for \( X \) equals

\[
h^1(\Theta_X) = h^0(N_{X/Y}) - h^0(\Theta_Y|X).
\]
Example 5.5. For this example we refer to [9]. Let $Y$ be a smooth projective variety. The Hilbert space parametrizing subschemes of $Y$ with the same Hilbert polynomial $P = P(X)$ as the subvariety $X \subset Y$ exist as a scheme $H^{(P)}$. Moreover, one has a tautological family over it, i.e. the fiber of this family over $[X'] \in H^{(P)}$ is the variety $X'$. Let $H_{X/Y}$ be the component to which $X$ belongs. One has

$$T_{[X]}(H_{X/Y}) = H^0(X, N_{X/Y}) \text{ and } h^0(X, N_{X/Y}) - h^1(X, N_{X/Y}) \leq \dim H_{X/Y} \leq h^0(X, N_{X/Y}).$$

Moreover,

$$h^1(X, N_{X/Y}) = 0 \implies H_X \text{ is smooth at } [X] \text{ and } \dim H_{X/Y} = h^0(X, N_{X/Y}).$$

This implies that locally at $[X]$ the universal family is regular. Conversely, if we have a regular deformation of $X \subset Y$ with smooth base $S$ and surjective characteristic map $\sigma : T_{[X]}S \to H^0(N_{X/Y})$, then the same conclusion holds. Note that these conditions are exactly the ones one needs to be able to apply Proposition 5.3.

Our situation concerns complete intersection surfaces $S$ in products $P$ of projective spaces, say of codimension $c$. Then $N_{S/P}$ is the restriction to $S$ of a vector bundle $N$ on $P$. We make use of the Koszul resolution for $O_S$ given by

$$0 \to N^*_c \to N^*_{c-1} \to \cdots \to N^*_1 \to O_P \to O_S \to 0, \quad N^*_j = \Lambda^j N^*.$$

The Koszul sequence gives a resolution of the ideal sheaf $J_S$.

Lemma 5.6. Let $\mathcal{F}$ be a locally free sheaf on $P$. Set $N^*_0 = O_P$.

1. If $h^j(\mathcal{F} \otimes N^*_j) = 0$ for $j = 0, \ldots, c - 1$, then $h^0(\mathcal{F} \otimes J_S) = 0$ and if $h^j(\mathcal{F} \otimes N^*_j) = 0$ for $j = 1, \ldots, c$ then $h^1(\mathcal{F} \otimes J_S) = 0$.

2. If $h^{j+1}(\mathcal{F} \otimes N^*_j) = 0$ for $j = 0, \cdots, c$, then $h^1(\mathcal{F}|S) = 0$.

3. If $h^j(\mathcal{F} \otimes N^*_j) = 0$ for $j = 1, \cdots, c$, then $h^0(\mathcal{F}|S) = \sum_{j=0}^c (-1)^j h^0(\mathcal{F} \otimes N^*_j)$.

Proof: Tensor the long exact sequence (22) with $\mathcal{F}$ and break up the resulting sequence in short sequences, the first of which reads

$$0 \to \mathcal{F} \otimes J_S \to \mathcal{F} \to \mathcal{F}|S \to 0,$$

the last is of the form

$$0 \to \mathcal{F} \otimes N^*_c \to \mathcal{F} \otimes N^*_{c-1} \to K_{c-1} \to 0,$$

and the intermediate steps $j = 1, \ldots, c - 2$ are of the form

$$0 \to K_{j+1} \to \mathcal{F} \otimes N^*_j \to K_j \to 0.$$

Now use descending induction.

\footnote{For the projective space see also [21, 23].}
The vanishing results we need are as follows:

**Proposition 5.7.** (a) (Cf. [II, III, Theorem 5.1]) \( h^i(P^k, O(\lambda)) = 0 \) for all \( \lambda \) if \( i \neq 0, k; h^0(P^k, O(\lambda)) = 0 \) if \( \lambda < 0 \) and \( h^k(P^k, O(\lambda)) = 0 \) if \( \lambda > -k - 1 \).

(b) (Cf. [5]) We have
- \( h^\ell(P^k, \Theta_{P^k}(\lambda)) = 0 \) if \( \lambda \leq -2 \).
- \( h^q(P^k, \Theta_{P^k}(\lambda)) = 0 \) for all \( \lambda \) and for \( 1 \leq q \leq k - 2 \),
- \( h^{k-1}(P^k, \Theta_{P^k}(\lambda)) = 0 \) if \( \lambda \neq -k - 1 \).
- \( h^k(P^k, \Theta_{P^k}(\lambda)) = 0 \) for \( \lambda \geq -k - 2 \).

### 5.2 Deformations of complete intersections of quadrics

We use:

**Corollary 5.8.** Let \( S \subset P^{2k+1} \) be a complete intersection of \((2k-1)\) quadrics with \( k \geq 3 \). We have

1. The restriction map \( H^0(\Theta_{P^{2k+1}}) \to H^0(\Theta_{P^{2k+1}}|S) \) is an isomorphism.
2. \( h^0(N_{S/P^{2k+1}}) = (2k-1) \cdot \left( \binom{2k+3}{2} - (2k-1) \right) \) and \( h^0(\Theta_{P^{2k+1}}|S) = (2k+2)^2 - 1 \).
3. \( h^1(N_{S/P^{2k+1}}) = h^1(\Theta_{P^{2k+1}}|S) = 0 \).
4. The Kuranishi space is smooth of dimension \( h^1(\Theta_S) = 4k^3 - 3k - 7 \). In particular, for \( k = 3 \) we find \( h^1(\Theta_S) = 92 \).

**Proof:** In (22) take \( N = \bigoplus^{2k-1} O_{P^{2k+1}}(2) \).

1. In Lemma 5.6, take \( \mathcal{F} = \Theta_{P^7} \). The required vanishing conditions follow from Proposition 5.7.
2. Take \( \mathcal{F} = O_{P^{2k+1}}(2) \), respectively \( \mathcal{F} = \Theta_{P^7} \). The assertion for \( h^0(N_{S/P^{2k+1}}) \) follows from Lemma 5.6 with \( \mathcal{F} = N = \bigoplus^{2k-1} O_{P^{2k+1}}(2) \) since then \( h^0(\mathcal{F} \otimes N^*) = (2k-1)^2 \). To calculate \( h^0(\Theta_{P^{2k+1}}|S) \), restrict the Euler sequence

\[
0 \to O_{P^{2k+1}} \to \bigoplus^{2k+2} O_{P^{2k+1}}(1) \to \Theta_{P^{2k+1}} \to 0
\]

1. First observe that the family of smooth complete intersection surfaces of fixed type is regular. Since \( S \) is a surface of general type, as noticed before, one has \( h^0(\Theta_S) = 0 \). Now apply Proposition 5.3 and our previous calculation.

**Remark 5.9.** 1. By [4] the relevant component of the moduli space is an affine variety of the given dimension.
2. The exception \( k = 2 \) in Corollary 5.8 covers the complete intersections \( S \) of 3 quadrics in \( P^5 \). One has \( h^1(\Theta_{P^5}|S) = 1 \).
5.3 Deformations of the complete intersection surface \( T \subset P \)

Recall that \( P = P^1 \times P^3 \). With \( p : P^1 \times P^3 \to P^1 \), \( q : P^1 \times P^3 \to P^3 \) the two projections, for any coherent sheaf \( \mathcal{F} \) on \( P \), write

\[
\mathcal{F}(a, b) = \mathcal{F} \otimes (p^* \mathcal{O}(a) \boxtimes q^* \mathcal{O}(b)).
\]

Note that \( \Theta_P(a, b) = (a + 2, b) \oplus q^* \Theta_{P^3}(a, b) \).

We shall be needing the following \( \chi \)-characteristics.

\[
\chi(a, b) = \chi(a)\chi(b) = (a + 1) \cdot \left(\frac{b + 3}{3}\right),
\]

\[
\chi(\Theta_P(a, b)) = (2ab + 8a + 3b + 9) \cdot \frac{(b + 3)(b + 2)}{3}.
\]

In our case for \( N = (4, 2) \oplus (0, 4) \) one gets

\[
\chi(N) = 85, \quad \chi(N \otimes N^*) = -28, \quad \chi(N^*) = -1
\]

\[
\chi(\Theta_P) = 18, \quad \chi(\Theta_P \otimes N^*) = -2, \quad \chi(\Theta_P \otimes \Lambda^2 N^*) = 28.
\]

Let us collect some vanishing results deduced from Proposition 5.7 together with the Künneth formula for a sheaf \( \mathcal{F} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \) on \( P \), which, we recall, reads

\[
h^{i+1}(\mathcal{F}(a, b)) = h^0(\mathcal{F}_1(a)h^{i+1}(\mathcal{F}_2b) + h^1(\mathcal{F}_1a))h^i(\mathcal{F}_2b)).
\]

One gets:

<table>
<thead>
<tr>
<th>( h^i(a, b) = 0 )</th>
<th>conditions</th>
<th>( h^i(\Theta_P(a, b)) = 0 )</th>
<th>conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j = 0 )</td>
<td>( a &lt; 0 ) or ( b &lt; 0 )</td>
<td>( j = 0 )</td>
<td>( a &lt; -2 ) or ( b &lt; -1 )</td>
</tr>
<tr>
<td>( j = 1 )</td>
<td>( a &gt; -2 ) or ( b &lt; 0 )</td>
<td>( j = 1 )</td>
<td>( a &gt; 0 ) or ( b &lt; -1 )</td>
</tr>
<tr>
<td>( j = 2 )</td>
<td>always</td>
<td>( j = 2 )</td>
<td>( a &gt; 0 ) or ( b = -4 )</td>
</tr>
<tr>
<td>( j = 3 )</td>
<td>( a &lt; 0 )</td>
<td>( j = 3 )</td>
<td>( a &lt; -2 ) or ( b = -4 )</td>
</tr>
<tr>
<td>( j = 4 )</td>
<td>( a &gt; -2 ) or ( b &gt; -4 )</td>
<td>( j = 4 )</td>
<td>( a &gt; 0 ) or ( b &gt; -5 )</td>
</tr>
</tbody>
</table>

Using this and the \( \chi \)-characteristic we find the data in Tables 2 and 3 below.

**Corollary 5.10.** We have \( h^1(\Theta_T) = 96 \) and \( h^2(\Theta_T) = 32 \).

**Proof:** The long exact sequence for the tangent bundle sequence \( 0 \to \Theta_T \to \Theta_P|T \to N_{T/P} \to 0 \) reads as follows

\[
0 \to H^0(\Theta_P|T) \to H^0(N_{T/P}) \to H^1(\Theta_T) \to H^1(\Theta_P|T) \xrightarrow{\alpha} H^1(N_{T/P}) \to \cdots,
\]
Table 2: Data for the normal bundle $N_{T/P}$.

| $h^i$ | $N$ | $N \otimes N^*$ | $N \otimes \Lambda^2 N^*$ | $N \otimes J_T$ | $N|T$ |
|-------|-----|-----------------|---------------------------|----------------|-------|
| 0     | 85  | 2               | 0                         | 2              | 113   |
| 1     | 0   | 30              | 0                         | 30             | 1     |
| 2     | 0   | 0               | 0                         | 1              | 0     |
| 3     | 0   | 0               | 1                         | 0              | 0     |
| 4     | 0   | 0               | 0                         | 0              | 0     |

Table 3: Data for $\Theta_p|T$.

| $h^i$ | $\Theta_p$ | $\Theta_p \otimes N^*$ | $\Theta_p \otimes \Lambda^2 N^*$ | $\Theta_p \otimes J_T$ | $\Theta_p|T$ |
|-------|-------------|------------------------|--------------------------------|------------------------|------------|
| 0     | 18          | 0                      | 0                              | 0                      | 18         |
| 1     | 0           | 0                      | 0                              | 0                      | 1          |
| 2     | 0           | 1                      | 0                              | 1                      | 31         |
| 3     | 0           | 3                      | 0                              | 31                     | 0          |
| 4     | 0           | 0                      | 28                             | 0                      | 0          |

and from Tables 2 and 3 we see that $\alpha$ is a homomorphism between 1-dimensional vector spaces and hence is an isomorphism or the zero map. It is indeed the zero map as follows from the commutative diagram

\[
\begin{array}{ccc}
H^1(\Theta_p|T) & \xrightarrow{\alpha} & H^1(N|T) \\
\approx & & \approx \\
H^2(\Theta_p \otimes J_T) & \xrightarrow{=} & H^2(N \otimes J_T) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{=} & H^3(\Theta_p \otimes N_2^*) \\
& & \approx \\
& & H^3(N \otimes N_2^*).
\end{array}
\]

Then the result follows immediately from the above table and the fact that $\chi(\Theta_T) = -26$ by (20).

Let us consider the Kuranishi space of $T$, the surfaces of Theorem 4.2. Since these surfaces heuristically depend on 83 parameters we get $83 - (3 + 15) = 65$ ”embedded” moduli.

However, from Table 2 we see that

\[ h^0(T,N|T) = h^0(N) + h^1(N \otimes J_T) = 83 + 30 \]

So these parameters account only for a part of the moduli, the so-called ”natural” moduli which come from varying the global equations. The second term in the above ex-
pression shows that there are 30 “hidden” deformation parameters. Indeed, the main result is as follows.

**Proposition 5.11.** The Kuranishi space $\mathcal{M}_2$ for complete intersection $T$ of type $(0,4), (4,2)$ in $\mathbb{P}^1 \times \mathbb{P}^3$ is a smooth variety of dimension 96.

**Proof:** Let us study the deformations of the bundle $N$ on $P$ restricting to the normal bundle of $T$ in $P$. Since

$$h^1(\text{End}(N)) = h^1(N \otimes N^*) = 30$$

$$h^2(\text{End}(N)) = h^2(N \otimes N^*) = 0,$$

by [15, Theorem 3.23] the local moduli space for the vector bundle $N$ is smooth of dimension 30. To see that the sections of $N$ extend to all small deformations $N_t$ we argue as follows. Since $H^1(N,N_t) = 0$, by [16, Thm. 2.3], for any small deformation $N_t$ of $N$ the dimension of $H^0(P,N_t)$ is independent of $t$ and hence, by [16, Thm. 2.2] the claim that sections of $N$ extend to all small deformations $N_t$ follows. In particular, the two sections of $N$ whose common zero locus corresponds to the surface $T$ can be deformed together with $N$ yielding a (supplementary) 30-dimensional family of deformations of $T$. Combining this with the 65-dimensional family we already had we get a complete family of embedded deformations of $T$. From Theorem 5.3 it follows that the Kuranishi space for the embedded deformations of $T$ within $P$ is smooth of dimension 95.

The remaining question is whether $T$ can be deformed in a direction projecting onto a generator of $H^1(\Theta|T)$? Consider the canonical embedding $\kappa : T \to \mathbb{P}^{30}$ for which $\kappa^*\Theta(1) = (2,2)$. The Euler sequence gives

$$0 \to \mathcal{O}_T \to \bigoplus_{i=0}^{31} \mathcal{O}_T(2,2) \to \Theta_{\mathbb{P}^{30}}|T \to 0$$

Since $h^1(K_T) = h^1(\Theta_T(2,2)) = 0$ and $h^2(K_T) = 1$, this yields an exact sequence

$$0 \to H^1(\Theta_{\mathbb{P}^{30}}|T) \to H^2(\Theta_T) \xrightarrow{\alpha} \bigoplus_{i=0}^{31} H^2(K_T),$$

where $\alpha$ is a homomorphism between two vector spaces of dimension 31. This map is induced by the Euler map $f \mapsto f(X_0, \ldots, X_{30})$, where the $X_j$ are homogeneous coordinates in $\mathbb{P}^{30}$. The map dual to $\alpha$ is given by

$$\bigoplus_{i=0}^{31} H^0(\Theta_T) \to H^0(K_T)$$

$$(a_0, \ldots, a_{30}) \mapsto \sum_{j=0}^{30} a_j X_j$$

and hence is an isomorphism. It follows that $H^1(\Theta_{\mathbb{P}^{30}}|T) = 0$ and so in the canonical embedding all deformations are embedded deformations.

We finally need to show that any non-embedded deformation of $T \subset P$ is unobstructed. This is best seen with a spectral sequence argument. Write $T^{-2} := \Theta_p \otimes N_2^*$, $T^{-1} := \Theta_p \otimes N_1^*$, $T^0 := \Theta_p$. This yields the complex $T^\bullet$ quasi-isomorphic to $\Theta_p|Y$. Give
it the structure of a complex of differential graded Lie algebras where we use the usual bracket on $\Theta_P$ on $T^0$ and between $T^0$ and $T^{-1}$, while on local sections $\theta_\alpha \cdot \nu_\alpha$, $\alpha = 1, 2$ of $T^{-1} = \Theta_P \otimes N^*$ we define the Lie-bracket $[\theta_1 \cdot \nu_1, \theta_2 \cdot \nu_2]$ as $[\theta_1, \theta_2] \cdot \nu_1 \wedge \nu_2$ which is indeed a local section of $T^{-2}$. We can interpret the data in the first 3 columns of Table 2 as the $E_2$-terms for the spectral sequence for the hypercohomology groups $H^*(T^*)$. Since $d_r = 0$ for $r \geq 2$ we see that this spectral sequence degenerates at $E_2$ and gives

$$
\begin{align*}
H^1(T^*) &= E_\infty^{-1,2} = H^2(T^{-1}) \\
H^2(T^*) &= E_\infty^2 \supset E_\infty^{-1,3}, \\
E_\infty^2/E_\infty^{-1,3} &= E_\infty^{-2,4} = H^4(T^{-2}).
\end{align*}
$$

The (symmetric) bracket on the one-dimensional space $H^1(T^*)$ then descends to a bracket

$$
\begin{array}{ccc}
H^2(T^{-1}) \otimes H^2(T^{-1}) & \longrightarrow & H^4(T^{-2}) = H^1(\Theta_{P^1}(-4)) \otimes H^3(\Theta_{P^3}(-6)) \\
\uparrow & & \uparrow \\
H^2(\Theta_{P^3}(-4)) \otimes H^2(\Theta_{P^3}(-4)) & \longrightarrow & H^4(\Theta_{P^3}(-8)) = 0
\end{array}
$$

and so the image of the bracket lands in the subspace $E_\infty^{-1,3} \subset H^2(T^*)$, and therefore comes from $[E_\infty^{-1,2}, E_\infty^{0,1}]$. But since $E_\infty^{0,1} = 0$, this image intersects the latter subspace only in (0). It follows that the bracket is zero and so the obstructions are all zero. \[\square\]

### 5.4 Summary of the local moduli calculations

**Theorem 5.12.** Let $\mathcal{M}$ be the moduli space of simply connected minimal surfaces of general type with $c_1^2 = 2^7$ and $c_2 = 2^8$.

(i) The Kuranishi space for intersections $S$ of five quadrics in $P^7$ is smooth of dimension $h^1(\Theta_S) = 92$. Also, $h^2(\Theta_S) = 28$. The corresponding component $\mathcal{M}_1$ of $\mathcal{M}$ is smooth of dimension 92.

(ii) The Kuranishi space $\mathcal{M}_2$ for complete intersection $T$ of type $(0, 4), (4, 2)$ in $P^1 \times P^3$ is smooth of dimension $h^1(\Theta_T) = 96$, and $h^2(\Theta_T) = 32$.

(iii) The closure of the component $\mathcal{M}_1$ in $\mathcal{M}$ does not meet the component $\mathcal{M}_1$.

**Proof:** We only have to show the last part. To see this we may use upper semicontinuity of $h^1(\Theta)$: in a neighborhood of a given point $h^1(\Theta)$ can only decrease. \[\square\]

**Remark 5.13.** In Example 5.5 we recalled some facts about the dimension of the Hilbert scheme $H_{X/P^n}$ of $X \hookrightarrow P^n$. Applying this to the example of the complete intersection $S$ of 5 quadrics in $P^7$ one gets $h^0(N_{X/P^7}) = 155$ for this dimension, which confirms the calculation $\dim \mathcal{H}_S = 92$ for the dimension of the Kuranishi space of $S$, since $155 = 92 + 63$.

As to $T$, we have constructed a family of deformations of $T$ within $P$ whose characteristic map is is an isomorphism. Then the dimension of the component of the
Hilbert scheme $H_{T/P}$ at $[T]$ equals $h^0(T,N_{T/P}) = 113$. If we Segre-embed this family in $\mathbb{P}^7$ and let the group of automorphisms act we get a family with smooth base and dimension $113 + (63 - 18) = 158$ and the characteristic map is an injection onto a subspace of $H^0(T,N_{T/P})$ of dimension 158. On the other hand, we can show that $h^0(N_{T/P}) = 158$ by considering the long exact sequence of

$$0 \to \Theta_T \to \Theta_{\mathbb{P}^7}|T \to N_{T/P} \to 0.$$

The crucial point here is that $h^1(\Theta_{\mathbb{P}^7}|T) = 1$ while the map $H^1(\Theta_{\mathbb{P}^7}|T) \to H^1(N_{T/P})$ is the zero map. This can be shown as in the proof of Corollary [5.10]. It follows that the Hilbert scheme $H_{T/P}$ is smooth at $[T]$ and has dimension 158. So the two corresponding components of the Hilbert scheme $H_S$ of surfaces $S$ with deg $S = 2^5$, $K_S \cdot H = 2^6$, $K_S^2 = 2^7$, $c_2(S) = 2^8$ in $\mathbb{P}^7$ have different dimensions.

References


