Complete intersection quadrics that degenerate into complete intersections on Segre varieties

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Abstract

We investigate whether surfaces that are complete intersections of quadrics can degenerate in complete intersection surfaces in Segre embedded products of projective spaces. The first such example is a K3 surface in projective 5-space degenerating into a hypersurface on the Segre threefold. We show that there is only one more example.

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1 Introduction

This note is motivated by exercise 11, Ch. VIII in [1], where two types of surfaces of degree 8 in $\mathbb{P}^5$ are compared: those that are complete intersections of three quadrics and those that arise as smooth hypersurfaces of bidegree $(2, 3)$ in the Segre embedded $\mathbb{P}^1 \times \mathbb{P}^2$. The exercise asks to show that the latter arise as limits of some well-chosen complete intersection of quadrics. The limit surfaces in the example form a divisor in the boundary of the 19-dimensional family consisting of complete intersections of three quadric hypersurfaces in $\mathbb{P}^5$ and the problem is to make this explicit. We have included a construction in Section 3 where it appears as Theorem 3.1.

This phenomenon seems very special. First of all, if we want to construct higher dimensional examples in a similar fashion, we are doomed to fail since by the Lefschetz hyperplane theorem complete intersections of dimension $\geq 3$ have the same second Betti number as the surrounding variety in which they are embedded (see [3]) and so cannot live in a non-trivial product of projective spaces.

In this note we investigate the simplest generalization for surfaces, namely we compare complete intersection quadrics in $\mathbb{P}^{2k+1}$ with complete intersection surfaces lying on $\mathbb{P}^1 \times \mathbb{P}^k$ for $k \geq 3$ using the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^k \hookrightarrow \mathbb{P}^{2k+1}$.

\footnote{We remark that Beauville’s exercise is closely related to Saint-Donat’s work on projective models of K3-surfaces [7]; see also [6 Chap. 7].}
That indeed examples are hard to come by is already apparent within our modest search area: we show that there is exactly one other example where the above phenomenon also occurs: a complete intersection of 5 quadrics in $\mathbb{P}^7$ which degenerates in a complete intersection of type $(0, 4), (4, 2)$ in a Segre embedded $\mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7$. Such surfaces are of general type. This is the main new result of this paper. See Section 4 for details.

We would like to observe that this construction suggests that moduli spaces for surfaces with the invariants of complete intersection surfaces may consist of several components. In any case, the examples are shown to belong to the same Hilbert scheme and turn out to be oriented homeomorphic. If the Hilbert scheme would be irreducible, they would even be diffeomorphic. However, it seems unlikely to us that this is the case, which would indeed imply that there are two (or more) irreducible components.

Even if this is the case, surfaces corresponding to the two components could still be diffeomorphic. This is hard to decide. The main reason is that where moduli theory for $K_3$-surfaces is governed by powerful transcendental theory (e.g. the Torelli theorem, see [2, Chap. VIII]) which makes it a priori clear that there are no two distinct components for the Hilbert scheme, such a theory is lacking for surfaces of general type.

The structure of the paper is as follows. Since some standard properties of numerical invariants of surfaces play a central role in the proofs, we start by recalling these in Section 2.1 and use them in Section 2.2 to calculate the basic invariants for the two types of surfaces. Then we solve Beauville’s exercise, and finally we compare the basic invariants and prove the main result. This comparison is quite delicate since the formulas are involved. The reader will be quickly convinced that a general comparison with complete intersection surfaces in products of projective spaces having 3 or more factors, is out of reach with the present approach. The best one could hope for is to find similar examples with a structured computer search. In a remark at the end of the paper we briefly address the case $\mathbb{P}^2 \times \mathbb{P}^k$, and provide some details for $k = 2$.

\section{Calculation of some numerical invariants}

\subsection{Surface invariants}

If $S$ is a complex projective surface, the following topological invariants

\begin{align*}
    b_1(S) &= b_3(S), b_2(S), \quad \text{the Betti numbers,} \\
    e(S) &= 2 - 2b_1(S) + b_2(S), \quad \text{the Euler number,}
\end{align*}

are related to the Chern classes $c_1(S), c_2(S)$ and the complex invariants through the formulas (see e.g. [23, Chapter I.5]):

\begin{align*}
    c_2(S) &= e(S), \\
    b_1(S) &= 2q(S), \\
    \chi(\mathcal{O}_S) &= p_2(S) - q(S) + 1 = \frac{1}{12}(c_1^2(S) + c_2(S)) \quad \text{(Noether’s formula).} \quad (1)
\end{align*}
So, the information given by the basic triple invariant \( \{b_1(S), c_1^2(S), c_2(S)\} \) completely determines the complex invariants \( K_S^2 = c_1^2(S), p_g(S) \) and \( q(S) \). In particular, if \( S \) is simply connected (and hence \( b_1(S) = 0 \)), the Chern classes suffice for that. However, to determine the topological type, one also needs the intersection form on \( H^2(S) \) which is determined by the signature and the parity. The signature is given by the Chern classes, but the parity is determined by the parity of the first Chern class in \( H^2(S) \).

Suppose that \( S \) comes with a preferred embedding \( S \hookrightarrow \mathbb{P}^{n+2} \) as a codimension \( n \) submanifold, and \( H = \mathcal{O}_{\mathbb{P}^{n+2}}(1) \) is the hyperplane bundle, then the embedding yields two more invariants:

\[
\deg(S) := [S] \cdot H^2 \in H^{2n+4}(\mathbb{P}^{n+2}) = \mathbb{Z}, \quad K_S \cdot H|_S.
\]

**Lemma 2.1.** The invariants \( \{2\} \) together with the basic triple invariant determine the Hilbert scheme of \( S \).

**Proof:** By Hartshorne [5], p. 366, Exercise 1.2, the Hilbert polynomial for a surface \( S \) is

\[
P_S(z) = \frac{1}{2}az^2 + bz + c, \quad a = \deg S, b = \frac{1}{2}\deg S + 1 - \pi, c = \chi(\mathcal{O}_S) - 1,
\]

\[
\pi = \text{genus of the curve } (S \cap H) = \frac{1}{2}(K_S \cdot H + \deg S + 2). \quad \square
\]

**Remark 2.2.**

1. We shall only consider simply connected surfaces. For these it is well known that the invariants \( c_1^2, c_2 \) together with the parity of \( c_1 \) completely determine the topological type of the surface. See e.g., [2], Chap. VIII, Lemma 3.1, Chap. IX.1.

2. By [4] the Hilbert scheme \( H_S \) of \( S \subset \mathbb{P}^{n+2} \) is connected. This implies that if \( S' \in H_S \), the surface \( S \) can be deformed into \( S' \). In fact Hartshorne in loc. cit. proves that this deformation can be done via a linear deformation. Suppose that the resulting family is through smooth surfaces, then by [3] they would be diffeomorphic. In general, all one can say is that \( S \) and \( S' \) deform to the same surface which may or may not be singular.

### 2.2 Complete intersection surfaces

The general situation is as follows: \( P \) is a smooth projective manifold of dimension \( n + 2 \) and \( X \subset P \) is a complete intersection surface cut out by hypersurfaces \( f_1, \ldots, f_n \). The Whitney product relation between the total Chern class \( c = 1 + c_1 + c_2 + \cdots \) of \( X \) and \( P \):

\[
c(X) \cdot (1 + F_1) \cdots (1 + F_n) = c(P), \quad F_j = c_1(f_j)
\]

allows to calculate the invariants \( c_1(X) \) and \( c_2(X) \).

Suppose that the Chern classes of \( P \) are in the subring \( A(P) \subset H^*(P) \) generated by the Picard group \( \text{Pic}(P) \). Then, the Chern classes of \( X \) belong to the subring \( j^*A(P) \subset H^*(X) \), where \( j : X \hookrightarrow P \) is the inclusion, which leads to simple formulas for them. Below this is applied to complete intersections in products of projective spaces.
For the calculation of the Betti numbers $b_j(X)$ the following consequence of Lefschetz' hyperplane theorem will play a role:

$$b_j(Y) = b_j(P), \quad j \leq \dim Y - 1, \quad Y \text{ a complete intersection in } P. \quad (3)$$

For instance, if $P = \mathbb{P}^{n+2}$, one has $b_1(X) = 0$. In fact, a sharper version of the Lefschetz hyperplane theorem implies that $X$ is simply connected.

**Example 2.3** (Surface complete intersections of quadrics). Let $S \subset P = \mathbb{P}^{2k+1}$ be a smooth complete intersection of $2k - 1$ quadrics. Its degree is

$$\deg S = 2^{2k-1}.$$  

Since $\text{Pic}(P) = \mathbb{Z}H$, where $H$ is the class of a hyperplane in $H^*(P, \mathbb{Z})$, and $c(P) = (1 + H)^{2k+2}$, setting

$$h := j^*H, \quad S \hookrightarrow P \text{ the embedding},$$

one finds from the Whitney product relation:

$$c_1(S) = -2(k-2)h \quad (4)$$

$$c_1^2(S) = 4(k-2)^2 \cdot 2^{2k-1} \quad (5)$$

$$c_2(S) = (2k^2 - 5k + 5)h^2 = (2k^2 - 5k + 5) \cdot 2^{2k-1} \quad (6)$$

$$-c_1(S) \cdot h = 2(k-2) \cdot 2^{2k-1} \quad (7).$$

**Example 2.4** (Surface complete intersections in $\mathbb{P}^1 \times \mathbb{P}^k$). If $P = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, the Picard group is given by $\text{Pic}(P) = \mathbb{Z}H_1 + \mathbb{Z}H_2$ where $H_j$ is the pull back of the generator of $\text{Pic}(\mathbb{P}^{n_j})$, $j = 1, 2$. For any complete intersection surface $T \subset P$ we write

$$h_k = j^*H_k, \quad k = 1, 2, \quad j : T \hookrightarrow P \text{ the embedding}.$$  

From now on, assume that $n_1 = 1$ and $n_2 = k$ and suppose that $T \subset P$ is a complete intersection of $k - 1$ hypersurfaces of bidegrees $(a_1, b_1), \ldots, (a_{k-1}, b_{k-1})$:

$$[T] = \underbrace{(a_1H_1 + b_1H_2)}_{F_1} \cdots \underbrace{(a_{k-1}H_1 + b_{k-1}H_2)}_{F_{k-1}} \in H^*(P, \mathbb{Z}).$$

One has the intersection table in $H^*(T, \mathbb{Z})$:

Now $h_1 + h_2$ comes from an ample class, so $0 < h_1 \cdot (h_1 + h_2) = b$ implies that all $b_i \geq 1$. Since we wish to compare with a surface of general type (a complete intersection of quadrics), the case $a_i = 0$ ($i = 1, \ldots, k - 1$) is not of interest to us. Indeed, in our computations later on we will assume and use that at least one of the $a_i$ is positive. In summary:

$$b \neq 0, \quad c \neq 0. \quad (8)$$

Let $s : \mathbb{P}^1 \times \mathbb{P}^k \hookrightarrow \mathbb{P}^{2k+1}$ be the Segre embedding and let

$$h := s^*H = h_1 + h_2, \quad \text{Pic}(\mathbb{P}^{2k+1}) = \mathbb{Z} \cdot H.$$
Table 1:

<table>
<thead>
<tr>
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<th>$h_1$</th>
<th>$h_2$</th>
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<tbody>
<tr>
<td>$h_1$</td>
<td>$b$</td>
<td>$c$</td>
</tr>
<tr>
<td>$h_2$</td>
<td>$b$</td>
<td>$c$</td>
</tr>
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$b := b_1 \cdots b_{k-1}$, $c := \sum_{j=1}^{k-1} a_j \cdot (b_1 \cdots \hat{b}_j \cdots b_{k-1})$

Setting

\[
\alpha := \sum_{j=1}^{k-1} a_j, \quad \beta := \sum_{j=1}^{k-1} b_j, \\
\gamma := \sum_{i < j} a_i b_j, \quad \delta := \sum_{i < j} b_i b_j, \\
x := (\alpha - 1)(2\beta - k - 1) + (k + 1) - \gamma, \quad y := \beta^2 + (k + 1)(\frac{k}{2} \beta - \delta), \\
u := 2(\alpha - 2)(\beta - (k + 1)), \quad v := (\beta - (k + 1))^2,
\]

one finds first of all

\[
j^*c(P) = (1 + 2h_1)(1 + (k + 1)h_2 + \frac{1}{2}k(k + 1)h_2^2) \\
= 1 + 2h_1 + (k + 1)h_2 + 2(k + 1)h_1 h_2 + \frac{1}{2}k(k + 1)h_2^2,
\]

\[
j^*(1 + F_1) \cdots j^*(1 + F_{k-1}) = 1 + \alpha h_1 + \beta h_2 + \gamma h_1 h_2 + \delta h_2^2,
\]

and thus, by the Whitney formula:

\[
c_1(T) = (-\alpha + 2)h_1 + (-\beta + (k + 1))h_2 \tag{9} \\
c_1^2(T) = ub + vc \tag{10} \\
c_2(T) = xb + yc \tag{11} \\
\deg T = h_2^2 = 2b + c \tag{12} \\
-c_1(T) \cdot h = (\alpha + \beta - (k + 3))b + (\beta - (k + 1))c. \tag{13}
\]

Equation (10), (11) together with the parity of $\alpha$ and $\beta$ determine the topological type. By Lemma 2.1, together with the remaining two equations they determine the component of the Hilbert scheme of subvarieties of $\mathbb{P}^{2k+1}$.

3 The case $k = 2$: degeneration of degree 8 $K_3$ surfaces in $\mathbb{P}^5$

A smooth complete intersection $S$ of three quadrics in $\mathbb{P}^5$ is simply connected (by the Lefschetz hyperplane theorem) and has trivial canonical bundle (by (4)). So it is a $K_3$ surface. As an intersection of 3 quadrics it has degree 8.

We next consider a smooth hypersurface $T$ of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Again, it is simply connected by Lefschetz, and by (6) it follows that $K_T$ is trivial: $T$ is also a $K_3$-surface. Consider its Segre image $s(T)$ in $\mathbb{P}^5$. From from (12) and Table 19 we see that
deg s(T) = 8. The equations describing the image of $\mathbf{P}^1 \times \mathbf{P}^2$ in $\mathbf{P}^5$ will appear below after analyzing bihomogeneous polynomials of bidegree $(2, 3)$. The surfaces $S$ and $T$ have the same Hilbert polynomial and so by Lemma 2.1 they belong to the same connected Hilbert scheme. The component to which $T$ belongs has dimension $3 \cdot 10 - 1 = 29$ with the bi-projective group of dimension $18 \cdot 3 = 54$. To explain this, we remark that the ideal generated by three quadrics $Q_k, k = 1, 2, 3$ can be normalized: we may assume that there exist three monomials exactly one of which occurs in one of the $Q_j$ but not the other two. The projective group of dimension $6^2 - 1 = 35$ then acts on the Hilbert scheme with 19-dimensional quotient. This calculation shows that in moduli, the surfaces $T$ give a divisor on the $19$-dimensional moduli space of those projective $K_3$ surfaces that have a genus $5$ hyperplane section. So there is only one component of the Hilbert scheme and the following theorem is a consequence. We want however to give a constructive proof.

**Theorem 3.1.** There exists a one parameter family $\{S_t\}$ whose fibers $S_t$ with $t \neq 0$ are complete intersections of three quadrics and whose special fiber $S_0$ is the given Segre embedded surface $s(T)$.

**Proof:** Let $R = \mathbf{C}[u, v] \otimes \mathbf{C}[x_1, x_2, x_3]$, the homogeneous coordinate ring of $\mathbf{P}^1 \times \mathbf{P}^2$. Consider a bihomogeneous polynomial of bidegree $(2, 3)$ defining the surface $T$:

$$F = u^2 C_{11} + uv C_{12} + v^2 C_{22},$$

where the $C_{ij} \in \mathbf{C}[x_1, x_2, x_3]$ are homogeneous cubics. Our first goal is to rewrite $F$ in terms of the products $ux_i$ and $vx_j$ so that the link with the coordinates of $\mathbf{P}^5$ can be exploited. Write

$$C_{ij} = \sum_a Q_{ij}^a x_a,$$

where the $Q_{ij}^a \in \mathbf{C}[x_1, x_2, x_3]$ are quadratic polynomials. Note that the $Q_{ij}^a$ are not uniquely determined by the $C_{ij}$.

Now every homogeneous quadratic polynomial $Q$ corresponds to a bilinear form $q$ such that $Q = q(x, x)$, $x = (x_1, x_2, x_3)$. If $q_{ij}^a$ denotes the bilinear form corresponding to $Q_{ij}^a$, then

$$u^2 C_{11} = \sum_a x_a q_{11}^a (ux, ux)$$
$$v^2 C_{22} = \sum_a x_a q_{22}^a (vx, vx)$$
$$uv C_{12} = \sum_a x_a q_{12}^a (ux, vx),$$

so that $F$ can be written in the form

$$F = \sum_a x_a \left[ q_{11}^a (ux, ux) + q_{22}^a (vx, vx) + q_{12}^a (ux, vx) \right].$$
Next consider $R' = \mathbf{C}[X_1, X_2, X_3, X_1', X_2', X_3']$, the homogeneous coordinate ring of $\mathbf{P}^5$, and the homomorphism $h : R' \to R$ induced by the Segre embedding, and which is determined by

$$(X_1, X_2, X_3, X_1', X_2', X_3') \mapsto (x_1u, x_2u, x_3u, x_1v, x_2v, x_3v).$$

If

$$A := \begin{pmatrix} X_1 & X_2 & X_3 \\ X_1' & X_2' & X_3' \end{pmatrix}$$

and if $A_1, A_2, A_3$ are the subdeterminants of $A$ obtained by omitting the first, second and third column, respectively, then the three quadratic equations $A_3 = 0$ describe the Segre embedded $\mathbf{P}^1 \times \mathbf{P}^2$ in $\mathbf{P}^5$ (and $h$ induces an injective map $R'/\langle A_1, A_2, A_3 \rangle \to R$). Now the polynomial $F$ gives rise to two polynomials in the coordinate ring of $\mathbf{P}^5$, namely, $uF = \sum X_a Q^a$, which we denote by $C$, and $vF = \sum X_a' Q^a$, which we denote by $C'$. Note furthermore that the quadrics $A_1, A_2, A_3$ obey the two relations $\sum(-1)^a X_a A_a = 0$ and $\sum(-1)^a X_a' A_a = 0$.

**Claim.** It is possible to choose the quadrics $Q^a (\alpha = 1, 2, 3)$ in such a way that for small non-zero values of $t$ the 3 quadrics $Q^1_t = tQ^a + (-1)^a A_\alpha (\alpha = 1, 2, 3)$ define a complete intersection $S_t$ as mentioned in the theorem.

Before proving this claim, first note that $\sum X_a Q^a_t = t \sum X_a Q^a = tC$, and similarly for $C'$. So the zero set of the ideal $I_t = (Q^1_t, Q^2_t, Q^3_t, C, C')$ is the intersection $Q^1_t = Q^2_t = Q^3_t = 0$ of quadrics in $\mathbf{P}^5$ for $t \neq 0$, whilst for $t = 0$ the zero set of $I_0$ gives the ‘degenerate’ $K3$ surface $s(T)$.

**Proof of the Claim.** The preceding construction regarding polynomials of bidegree $(2, 3)$ can be rephrased as follows. Let $U$ and $V$ be complex vector spaces of dimensions 2 and 3, respectively. These ‘model’ the degree 1 parts of $C[u, v]$ and $\mathbf{C}[x_1, x_2, x_3]$. Now put

$$W = U \otimes V.$$ 

Then the obvious bilinear map $U \times V \to U \otimes V = W$ induces the Segre embedding. In these terms, we view our bidegree $(2, 3)$ polynomial $F$ as an element of $S^2 U \otimes S^3 V$. The maps

$$S^2 V \otimes V \to S^3 V,$$

induced by $Q \otimes L \mapsto Q \cdot L,$

and

$$S^2 (U \otimes V) \to S^2 U \otimes S^2 V,$$

induced by $(x \otimes v) \cdot (x' \otimes v') \mapsto x \cdot x' \otimes v \cdot v',$

are surjective. These fit in the following commutative diagram:

$$
\begin{array}{ccc}
S^2 U \otimes S^2 V \otimes V & \longrightarrow & S^2 U \otimes S^3 V \\
\downarrow & & \downarrow \\
S^2 (U \otimes V) \otimes V & \cong & \bigoplus^3 S^2 (U \otimes V) = \bigoplus^3 S^2 W
\end{array}
$$
In this diagram the map \( q \) represents the explicit construction described above. Indeed,

\[
q(Q^1(ux, vx), Q^2(ux, vx), Q^3(ux, vx)) = \sum_{\alpha=1}^{3} x_\alpha Q^\alpha(ux, vx) = \left( \sum_{\alpha=1}^{3} x_\alpha q^\alpha_1(x, x) \right) uv + \left( \sum_{\alpha=1}^{3} x_\alpha q^\alpha_2(x, x) \right) u^2 + \left( \sum_{\alpha=1}^{3} x_\alpha q^\alpha_2(x, x) \right) v^2.
\]

Now \( \mathbf{P}(S^2 U \otimes S^3 V) \) contains a Zariski open subset corresponding to smooth surfaces, including the one defined by \( F \). Its preimage under \( q \) gives rise to a Zariski open subset in \( \mathbf{P}(\bigoplus^3 S^2 W) \). This latter open subset meets the Zariski open part of \( \mathbf{P}(\bigoplus^3 S^2 W) \) corresponding to smooth intersections of three quadrics in a Zariski open set. In particular, in our construction with \( F \), a triple \((Q^1, Q^2, Q^3)\) of quadrics in \( \bigoplus^3 S^2 W \) can be chosen defining a complete intersection. This proves the above claim and hence the Theorem.

\[\square\]

4 Comparison for \( k \geq 3 \)

The following integers come up in the relevant formulae below; for clarity, we also recall some of the previous notation:

\[
K_T = ah_1 + bh_2, \text{ where } a = \alpha - 2 = \sum a_j - 2, \quad b = \beta - (k + 1) = \sum b_j - (k + 1),
\]

\[
b = \prod_{j=1}^{k-1} b_j, \quad c = \sum_{j=1}^{k-1} a_j \cdot (b_1 \cdots \hat{b}_j \cdots b_{k-1}), \quad x = 2ab + a(k + 1) + 2b + 2(k + 1) - \gamma, \quad \gamma = \sum_{i \neq j} a_i b_j, \quad y = b^2 + b(k + 1) + \frac{1}{2} k(k + 1) - \delta, \quad \delta = \sum_{i < j} b_i b_j.
\]

The calculations in Example 2.4 together with Remark 2.2 imply:

**Lemma 4.1.** 1. The topological invariants \( c^2_1(T), c^2_2(T) \) equal those of a smooth complete intersection \( S \) of \((2k - 1)\) quadrics in \( \mathbf{P}^{2k+1} \) precisely if

\[
2ab b + b^2 c = 2^{2k-1}(2(k - 2))^2, \quad x b + y c = 2^{2k-1}(2k^2 - 5k + 5).
\]
Suppose that $K_T$ is ample. Then $a \geq 0$ and $b \geq 1$. If such a $T$ exists with even $a$ and $b$ it is oriented homeomorphic to a smooth complete intersection of $(2k-1)$ quadrics in $P^{2k+1}$.

2. The surfaces $S$ and $T$ belong to the same Hilbert scheme if, moreover,

$$2b + c = 2^{2k-1}, \quad (a + b)b + bc = 2^{2k-1}(2(k-2)).$$

(16) \hspace{1cm} (17)

We first consider the case $k = 3$ and then we have:

$$b = b_1b_2 = \delta, \quad \gamma = a_1b_2 + a_2b_1 = c.$$ \hspace{1cm} (18)

**Theorem 4.2.** A smooth complete intersection $T$ of two hypersurfaces of type $(4,2)$ and $(0,4)$ in $P^1 \times P^3$ is oriented homeomorphic to a smooth complete intersection $S$ of 5 quadrics in $P^7$.

This is the only possibility among complete intersections of $P^1 \times P^3$.

The two surfaces belong to the same Hilbert scheme of $P^7$ when we consider $T$ as embedded in $P^7$ through the Segre embedding $P^1 \times P^3 \hookrightarrow P^7$. In particular they deform to the same, possibly singular, surface. ²

**Proof:** Because of (18), our system of equations reduces to

$$2ab\delta + b^2\gamma = 2^7, \quad (2ab + 4a + 2b + 8 - \gamma)\delta + (b^2 + 4b + 6 - \delta)\gamma = 2^8.$$ By (8), $\gamma = c \neq 0$. Rewriting the first equation as $b(2a\delta + b\gamma) = 2^7$, we see that $b$ is a power of 2 and that $b^2 \leq b(2a\delta + b\gamma) = 2^7$ so we conclude that $b = 2^\ell$ with $\ell = 0, 1, 2, 3$. Hence

$$2a\delta + b\gamma = 2^{7-\ell}.$$ Subtracting this twice from the second equation, after some rewriting, yields,

$$(\gamma - (2^\ell + 4))(\delta - (2^\ell + 3)) = (2^\ell + 4)(2^\ell + 3) + 2^{7-\ell} - 2^6.$$ The right-hand side equals 84, 30, 24, respectively, for $\ell = 0, 1, 2$, respectively.

- **Case** $\ell = 0$, i.e. $b = 1$. Then $(\gamma - 5)(\delta - 4) = 84 = 7 \cdot 4 \cdot 3$. Now $0 \leq \delta = b_1b_2 = b_1(b + 4 - b_1) = b_1(5 - b_1) \leq 6$, and $\gamma \geq 0$, so both factors $\gamma - 5$ and $\delta - 4$ must be positive. But then $\delta$ must be 6 and $\gamma$ must be 47. But the equation $2ab\delta + b^2\gamma = 2^7$ reduces to $12a + 47 = 128$ which has no integer solutions.

- **Case** $\ell = 1$, i.e. $b = 2$. Then $(\gamma - 6)(\delta - 5) = 30 = 2 \cdot 3 \cdot 5$. The solution $\gamma = 0$ and $\delta = 0$ is ruled out, since we saw that $\gamma \neq 0$.

²If it can be chosen to be smooth, the two types of surfaces are diffeomorphic.
Otherwise \(1 \leq \delta = b_1(6-b_1) \leq 9\) so that \(-4 \leq \delta - 5 \leq 4\). For divisibility reasons, the only possibility for \(\delta\) is 8 and thus \(\gamma = 16\). Then the equation \(2ab\delta + b^2\gamma = 2^7\) reduces to \(a = 2\). We get
\[
\begin{align*}
\delta &= b_1b_2 = 8, \\
\gamma &= a_1b_2 + a_2b_1 = 16, \\
a + 2 &= a_1 + a_2 = 4.
\end{align*}
\]
The first equation has solutions \((b_1, b_2) = (1, 8), (2, 4)\). The first is incompatible with the other two equations. The second leads to the only solution \((a_1, a_2) = (4, 0), (b_1, b_2) = (2, 4)\) compatible with the three equations.

- **Case** \(\ell = 2\), i.e. \(b = 4\). Then \((\gamma - 8)(\delta - 7) = 24 = 2^3 \cdot 3\). The equation \(2ab\delta + b^2\gamma = 2^7\) reduces to \(a\delta + 2\gamma = 16\). Now \(\delta = b_1(8-b_1)\) can only assume the values 0, 1·7, 2·6, 3·5 and 4·4. From divisibility the only possibility left for \(\delta\) is 15. But then \(a\delta + 2\gamma = 16\) implies \(a = 0\) and \(\gamma = 8\). But \(\gamma \neq 8\) because the factor \(\gamma - 8\) must be nonzero.

- **Case** \(\ell = 3\). Here we have \(8(2a\delta + 8\gamma) = 2^7\) so that \(a\delta + 4\gamma = 8\). Since \(a \geq 0, \gamma \geq 1\), the only possibilities for \(\gamma\) are 1 and 2, but that conflicts with \((\gamma - 12)(\delta - 11) = 84\).

Concluding, we have shown that the only solution to the first two equations is as stated. However, for this solution, \(a = b = 2\) the remaining equations are identical to the first equation and so \(T\) and \(S\) belong to the same Hilbert scheme.

**Remark 4.3.** An intersection of five quadrics in \(\mathbb{P}^7\) depends on \(5 \cdot 31 = 155\) parameters and gives \(155 - 63 = 92\) moduli. Surfaces \(T\) depend on 83 parameters, giving \(83 - 11 = 72\) moduli, which is less and such surfaces could correspond to points in the closure of the first component. However, since there is no obvious way to generalize the proof of Theorem 3.1, this suggests that, unlike what happens for \(K_3\)-surfaces, the surfaces \(T\) belong to a different irreducible component of the Hilbert scheme.

If this is indeed the case, it would show that moduli spaces for complete intersection surfaces of general type may consist of more than one component, unlike what happens for hypersurfaces. Note that, even if this is true, the two types of surfaces might still be diffeomorphic.

We complete the above result by showing that the phenomenon of Theorem 4.2 does not occur for \(k \geq 4\):

**Proposition 4.4.** If \(k \geq 4\) there cannot exist two surfaces \(S\) and \(T\) of the above type which belong to the same Hilbert scheme.

**Proof:** The idea here is to consider the three equations \((14), (16), (17)\) as a system of equations for \(b, c\) with coefficients involving \(a\) and \(b\). By \((8)\), if an integer solution exists the rank of the coefficient matrix has to be at most 1. This means that \(ab = b^2, a + b = 2b\) and so \(a = b\). But then the equations imply that \(a = b = 2(k - 2)\).
To exclude this solution, argue as follows:

\[
\sum_{j=1}^{k-1} a_j = a + 2 = 2(k - 1)
\]
\[
\sum_{j=1}^{k-1} b_j = b + (k + 1) = 3(k - 1).
\]

The maximal value of \( b \) can be computed with the methods of Lagrange multipliers: the maximum for \( b \) occurs for \( b_j = 3 \) and equals \( 3^{k-1} \). Note that there is an extremal value for \( c \) when \( a_j = 2, b_j = 3 \) but this is not a maximum. But we may use that \( \sum_{j \neq i} b_j \leq 3(k - 1) - 1 \) since \( b_i \geq 1 \). We then use the Lagrange multiplier method for the product of \( (k - 2) \) different \( b_j \). This gives:

\[
\prod_{j \neq i} b_j \leq \left( 3 + \frac{2}{k - 2} \right)^{k-2}
\]

and hence

\[
c \leq \left( 3 + \frac{2}{k - 2} \right)^{k-2} \cdot \left( \sum_{j=1}^{k-1} a_j \right) = \left( 3 + \frac{2}{k - 2} \right)^{k-2} \cdot (2(k - 1)).
\]

But this would imply

\[
4^{k-1} = \frac{1}{2} (2b + c) \leq 3^{k-2} \left( 3 + \frac{2}{3(k - 2)} \right)^{k-2} \cdot (k - 1)
\]

which is false as soon as \( k \geq 6 \).

To exclude \( k = 4, 5 \) we have to use that the \( b_j \) are positive integers summing up to \( 3(k - 1) \). For \( k = 5 \), writing down all possibilities for the quadruple \((b_1, b_2, b_3, b_4)\), we see that the product of three among them can be 48 for \((1, 3, 4, 4)\), 45 for \((1, 3, 3, 5)\) and at most 40 for all other quadruples. The first quadruple gives, using that \( a_1 + a_2 + a_3 + a_4 = 8 \),

\[
4^4 = 256 = \frac{1}{2} c + b = 24a_1 + 8a_2 + 6(a_3 + a_4) + 48 = 18a_1 + 2a_2 + 6 \cdot 8 + 48
\]

and so \( 80 = 9a_1 + a_2 \) which has no solutions since \( a_1 + a_2 \leq 8 \). For \((1, 3, 3, 5)\) we find

\[
256 = \frac{1}{2} (45a_1 + 15a_2 + 15a_3 + 9a_4) + 45 = \frac{1}{2} (36a_1 + 6a_2 + 6a_3 + 9 \cdot 8) + 45 = 18a_1 + 3(a_2 + a_3) + 81,
\]

which gives a contradiction modulo 3.

In the other cases, we have

\[
256 = \frac{1}{2} c + b \leq \frac{1}{2} 40 \cdot 8 + 3^4 = 241,
\]
and hence no solution either.

For \( k = 4 \) there is a solution to \( b + \frac{1}{2}c = 4^3 \), namely \((a_1, a_2, a_3) = (0, 0, 6)\), and \((b_1, b_2, b_3) = (4, 4, 1)\). This can be seen to be the only one: we only have to test whether for each of the values of the triples \((b_1, b_2, b_3) = (1, 1, 7), (1, 2, 6), (1, 3, 5), (1, 4, 4), (2, 2, 5), (2, 3, 4), (3, 3, 3)\), i.e., the positive integral solutions of \( b_1 + b_2 + b_3 = 9 \), one can find a triple \((a_1, a_2, a_3)\) with \( a_1 + a_2 + a_3 = 6 \) such that

\[
\frac{1}{2}(a_1b_2b_3 + a_2b_1b_3 + a_3b_1b_2) + b_1b_2b_3 = 64.
\]

This gives for \((1, 4, 4)\) one solution only, which is the one we had (up to renumbering). For the other triples the argument resembles the one for \( k = 5 \). To test for instance \((b_1, b_2, b_3) = (1, 2, 6)\), one gets

\[
6a_1 + 3a_2 + a_3 + 12 = 5a_1 + 2a_2 + 18 \leq 5(a_1 + a_2) + 18 \leq 48 < 64.
\]

It remains to exclude the solution we found, \((a_1, a_2, a_3) = (0, 0, 6)\), \((b_1, b_2, b_3) = (4, 4, 1)\). For this we observe that it does not satisfy the remaining equation \((15)\) since \( x = 22, y = 10 \) while \( b = 16, c = 96 \) and thus \((15)\) would give

\[
22 \cdot 16 + 10 \cdot 96 = 17 \cdot 128,
\]

which is false.

\[\square\]

**Remark 4.5.** For \( k \geq 4 \), there could still be solutions to the "topological" equations \((14), (15)\). We have not tested this since these equations become unwieldy. Some experimentation suggest that existence of solutions is very unlikely.

Referring to Remark 2.2 if these do exist, they would give other examples of complete intersection surfaces in \( \mathbb{P}^1 \times \mathbb{P}^k \) oriented homeomorphic to complete intersections of quadrics.

**Remark 4.6.** If we replace \( \mathbb{P}^1 \times \mathbb{P}^k \) by \( \mathbb{P}^2 \times \mathbb{P}^k \), and try to compare complete intersection surfaces in the latter space with complete intersections of quadrics in \( \mathbb{P}^{3k+2} \), we are led to introduce a new variable, since the intersection table corresponding to Table \((19)\) no longer contains a 0. In this case we need to bring more equations into play than the ones corresponding to \((16)\) and \((17)\). The new set of equations doesn’t look promising to handle.

In the case \( k = 2 \), however, a simple argument can be given to exclude solutions. Here the analogs of equations \((10), (12), (13)\) lead to the system of equations

\[
\begin{pmatrix}
1 & 2 & 1 \\
\frac{a}{a} & a + b & b \\
\frac{a^2}{2ab} & 2ab & b^2
\end{pmatrix}
\begin{pmatrix}
\frac{b_1b_2}{a_1b_2 + a_2b_1} \\
\frac{b_1b_2}{a_1b_2 + a_2b_1}
\end{pmatrix}
= \begin{pmatrix}
2^6 \\
3 \cdot 2^6 \\
3^2 \cdot 2^6
\end{pmatrix},
\]

where \( a = a_1 + a_2 - 3, b = b_1 + b_2 - 3 \). Consider the second column as variables \( b, c, d \) to be solved for. Then one gets the following cases for the solutions.
1. Case $a = b$. Then $a = b = 3$ from the first and second equation in \([19]\), and $b + 2c + d = 2^6$. This has no solutions (case by case analysis).

2. Case $a \neq b$. Then, setting $u = \frac{3-a}{b-a}$ (and hence $1 - u = \frac{b-3}{b-a}$), we obtain

$$b = b_1 b_2 = (u - 1)^2 \cdot 2^6,$$
$$c = a_1 b_2 + a_2 b_1 = u(1 - u) \cdot 2^6,$$
$$d = a_1 a_2 = u^2 \cdot 2^6.$$

In this case we have $u \neq 0$ and $1 - u \neq 0$ (from the expressions for $b, c, d$).

Now

$$d = \frac{a_1 a_2}{b_1 b_2} = \frac{u^2}{(u - 1)^2},$$
$$c = \frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{u}{1 - u}.$$

This means that $x_{1,2} = \frac{a_{1,2}}{b_{1,2}}$ are solutions of the quadratic equation

$$X^2 + \frac{u}{u - 1} X + \frac{u^2}{(u - 1)^2} = 0.$$

But this equation has discriminant $-3 \frac{u^2}{(u-1)^2} < 0$ and so has no real solutions.

References


