Lectures on Motivic Aspects of Hodge Theory: The Hodge Characteristic. Summer School Istanbul 2014

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Introduction

These notes reflect the lectures I have given in the summer school *Algebraic Geometry and Number Theory, 2-13 June 2014*, Galatasaray University Istanbul, Turkey. They are based on [P].

The main goal was to explain why the topological Euler characteristic (for cohomology with compact support) for complex algebraic varieties can be refined to a characteristic using mixed Hodge theory. The intended audience had almost no previous experience with Hodge theory and so I decided to start the lectures with a crash course in Hodge theory.

A full treatment of mixed Hodge theory would require a detailed exposition of Deligne's theory [Del71, Del74]. Time did not allow for that. Instead I illustrated the theory with the simplest non-trivial examples: the cohomology of quasi-projective curves and of singular curves.

For degenerations there is also such a characteristic as explained in [P-S07] and [P, Ch. 7–9] but this requires the theory of limit mixed Hodge structures as developed by Steenbrink and Schmid.

For simplicity I shall only consider $\mathbb{Q}$–mixed Hodge structures; those coming from geometry carry a $\mathbb{Z}$–structure, but this iwill be neglected here. So, whenever I speak of a (mixed) Hodge structure I will always mean a $\mathbb{Q}$–Hodge structure.

1 A Crash Course in Hodge Theory

For background in this section, see for instance [C-M-P, Chapter 2].

1.1 Cohomology

Recall from Loring Tu's lectures that for a sheaf $\mathcal{F}$ of rings on a topological space X, cohomology groups $H^k(X, \mathcal{F})$, $q \geq 0$ have been defined. In particular, for any ring R this gives groups

$$H^k(X, R) := H^k(X, R_X), \quad R_X \text{ the constant sheaf } R.$$

If $f : X \to Y$ is continuous there are induced homomorphisms $f^* : H^k(Y, \mathbb{Q}) \to H^k(X, \mathbb{Q})$ which provide functors from topological spaces to $\mathbb{Q}$–vector spaces.

The standard cohomology groups, such as the *singular* or *Betti*–cohomology groups $H^k_{\mathrm{B}}(X, R)$ and the de Rham cohomology groups $H^k_{\mathrm{dR}}(X, R)$ (for differentiable manifolds X) can be compared with sheaf cohomology.

To state the comparison results let us provisionally introduce:

Definition 1.1. The category of *perfect* topological spaces Top^* has as its objects Hausdorff second countable spaces that are locally compact and locally contractible and for which the total cohomology $\bigoplus_q H^k(X, \mathbb{Q})$ is finite dimensional.

Example 1.2. Recall that a cell is a topological space homeomorphic to an open ball and a cell complex is a Hausdorff second countable space which is the union of cells glued along the boundaries. Cell complexes need not have finite dimensional cohomology and so are not necessarily perfect. However, if only finitely many cells are needed they form perfect topological spaces. For instance differentiable manifolds, as well as complex manifolds or even complex algebraic varieties can be given the structure of a finite cell complex.

Note that if $X \in \mathsf{Top}^*$, also its one-point compactification X^* is a perfect topological space, so that the above finiteness properties also hold for compactly supported cohomology

$$H^k_c(X, \mathbb{Q}) := \tilde{H}^k(X^*, \mathbb{Q}),$$

where $\tilde{H}^k(Y) = \mathrm{coker}(a_Y^* : H^k(\mathrm{pt}) \to H^k(Y))$ is reduced cohomology of Y, ($a_Y : Y \to \mathrm{pt}$ is the constant map). Reduced cohomology differs from ordinary cohomology only in degree 0.

The following comparison results hold:

1. For a perfect topological space X there is a canonical functorial isomorphism $H^k(X, R) \xrightarrow{\sim} H^k_{\mathrm{B}}(X, R)$;
2. For a differentiable manifold integration over singular chains induces a functorial isomorphism (De Rham's isomorphism theorem)

$$H^k_{\mathrm{B}}(X, \mathbb{R}) \xrightarrow{\sim} H^k_{\mathrm{dR}}(X) = \frac{\ker(d : A^p(X) \to A^{p+1}(X))}{\mathrm{Im}(d : A^{p-1}(X) \to A^p(X))},$$

where $A^p(X)$ stands for the real vector space of global p–forms on X.

Suppose next that X is a complex manifold; using complex-valued forms, the De Rham theorem can be stated for these as well:

$$H^k_{\mathrm{B}}(X,\mathbb{C}) \xrightarrow{\sim} H^k_{\mathrm{dR}}(X)_{\mathbb{C}},$$

where the subscript refers to the use of complex-valued forms. Such forms decompose in types: locally in a chart with holomorphic coordinates $\{z_1,\dots,z_n\}$ a form of type (p,q) is a linear combination of forms

$$dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\cdots\wedge d\bar z_{j_q}.$$

whose coefficients are C^∞ functions. Because the gluing functions are holomorphic, a form which is of type (p,q) in one coordinate system is also of the same type (p,q) in any other. Consequently, there is a direct sum decomposition

$$A^k(X;\mathbb{C}) = \bigoplus_{p+q=k} A^{p,q}(X), \tag{1}$$

where $A^{p,q}(X)$ is the vector space of forms of type (p,q) on M and $A^k(X;\mathbb{C}) = A^k(X)\otimes\mathbb{C}$ is the vector space of complex-valued k-forms. Note that

$$\overline{A^{p,q}(X)} = A^{q,p}(X).$$

We now ask whether this decomposition passes to cohomology. To make this precise, we let

$$H^{p,q}(X) := \{\text{classes representable by a closed } (p,q)\text{-form}\} \subset H^{p+q}_{\mathrm{dR}}(X)$$

so that obviously

$$\overline{H^{p,q}(X)} = H^{q,p}(X). \tag{$*$}$$

The decomposition we are after is

$$H^k_{\mathrm{dR}}(X\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

and is called a *Hodge decomposition* for H^k. These might or might not exist. It is customary to employ the notation

$$h^{p,q}(X) = \dim H^{p,q}(X)$$

for the dimension of these spaces. These are the *Hodge numbers*.

Here is a simple example which shows that there need not be a Hodge decomposition. It is based on the fact that if H^1 has a Hodge decomposition, its dimension is equal to $h^{1,0}+h^{0,1} = 2h^{1,0}$, an even number.

Example 1.3. Start with $\mathbb{C}^n - \{0\}$ equipped with an action of the infinite cyclic group with generator g acting as $g(x) = 2x$. A fundamental domain for this action is the annulus $1 \le \|z\| \le 2$ with the inner and outer sphere identified. So the quotient X^n which is a complex manifold, the n–dimensional Hopf manifold, is homeomorphic to $S^1\times S^{2n-1}$ and so, by standard calculations has $H^1_{\mathrm{B}}(X^n;\mathbb{R}) = \mathbb{R}$, an odd-dimensional space.

1.2 Hodge Decomposition

A compact oriented differentiable manifold X can be equipped with a Riemannian metric g and using it, one can define the so-called harmonic forms on X. These exist in all degrees and are closed forms giving a basis for the De Rham cohomology groups. The spaces of harmonic forms $\mathrm{Harm}^k(X)$ are finite dimensional and so gives a nice model for the De Rham cohomology groups.

This applies to compact complex manifolds (they have a natural orientation) and one gets an isomorphism

$$H^k_{\mathrm{dR}}(X)_{\mathbb{C}} = \mathrm{Harm}^k(X)_{\mathbb{C}}.$$

What happens with (p,q)–forms? The problem is that in general the (p,q)–components of a harmonic form are no longer harmonic. Also, the complex conjugate of a harmonic form need not be harmonic.

These problems can be shown not to occur for a special class of metrics:

Definition 1.4. A *Kähler metric* is a hermitian metric which in some holomorphic chart $(z_1,\dots,z_n)$ is euclidean to order 2 at least:

$$h_{i\bar{j}} = \delta_{ij} + O(z^2, (\bar{z})^2)$$

where the local expression of hermitian metric h is $\sum_{i,j} h_{i\bar{j}} dz_i \otimes \bar{z}_j$.

Equivalently, the imaginary part ω_h of the hermitian metric h is a closed form. Recall that ω_h is a global form of type $(1,1)$ which locally in coordinates $(z_1,\dots,z_n)$ is given by

$$\mathrm{i}/2 \cdot \sum_{i,j} h_{i\bar{j}} dz_i \wedge \bar{z}_j.$$

A manifold equipped with a Kähler metric is called a *Kähler manifold.*

As indicated above, one has the following central result in Hodge theory:

Theorem 1.5 (Hodge Decomposition)**.** *Let X be a compact Kähler manifold. Then the complex harmonic forms admit a type decomposition compatible with complex conjugation and hence one has a decomposition*

$$H^k_{\mathrm{dR}}(X)_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}(X), \quad H^{p,q}(X) = \overline{H^{q,p}(X)}.$$

Example 1.6. (1) The *Fubini-Study metric* on $\mathbb{P}^n$ is given by the following form on $\mathbb{C}^{n+1} - \{0\}$:

$$\Omega = \mathrm{i}\partial\overline{\partial} \log \|z\|^2, z = (z_0,\dots,z_n).$$

It is real because in general the conjugate of $\mathrm{i}\partial\overline{\partial}\rho$, where ρ is real-valued, is

$$-\mathrm{i}\overline{\partial}\partial\rho = \mathrm{i}\partial\overline{\partial}\rho,$$

where we have used $\partial\overline{\partial} + \overline{\partial}\partial = 0$. It is closed because in general

$$\begin{aligned} d\partial\overline{\partial} &= (\partial + \overline{\partial})\partial\overline{\partial} \\ &= \partial^2\overline{\partial} - \overline{\partial}^2\partial \\ &= 0. \end{aligned}$$

What is less trivial is the positivity of the form. It is left as an exercise.
(2) If X has a Kähler metric, it restricts to a Kähler metric on every complex submanifold. In particular one sees: *all complex projective manifolds admit a Kähler metric.*
(3) A complex torus is the quotient of $\mathbb{C}^n$ by a lattice of maximal rank. The euclidean metric is invariant under translation and hence gives a metric on the torus. This metric is Kähler. Since there are non-algebraic tori, this shows that the class of Kähler manifolds is strictly larger than the class of complex projective manifolds.

1.3 Pure Hodge Structures

Formalizing the Hodge decomposition, one introduces

Definition 1.7. (1) A weight k Hodge structure consists of a finite dimensional $\mathbb{Q}$–vector space H with a direct sum decomposition, the *Hodge decomposition*

$$H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}, \quad \overline{H^{p,q}} = H^{q,p}.$$

Equivalently, $H_{\mathbb{C}}$ admits a decreasing filtration $F^{\bullet}$, the *Hodge filtration* with

$$H_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}.$$

The relation is given by:

$$F^p = \bigoplus_{a \geq p} H^{a,k-a}, \quad H^{p,q} = F^p \cap \overline{F^q}.$$

(2) A graded pure Hodge structure is a finite direct sum of weight k Hodge structures;
(3) A morphism of Hodge structures is a $\mathbb{Q}$–linear map $f : H_1 \to H_2$ such that for the complex extension of $f_{\mathbb{C}}$ one has $f_{\mathbb{C}} : H_1^{p,q} \to H_2^{p,q}$.
(4) The category of graded pure Hodge structures is denoted $\mathfrak{hs}$.

One can show that the category $\mathfrak{hs}$ is Abelian. For example, if V_1, V_2 are pure of weight k_1, k_2 the space $\mathrm{Hom}(V_1, V_2)$ is pure of weight $k_1 - k_2$ with

$$[\mathrm{Hom}(V_1, V_2)]^{p,q} = \{f : (V_1)_{\mathbb{C}} \to (V_2)_{\mathbb{C}} \mid fV_1^{a,b} \subset V_2^{a+p,b+q}\}.$$

Examples 1.8. (1) For a complex projective manifold X the cohomology $H^k(X)$ is a weight k Hodge structure and if $f : X \to Y$ is a morphism between smooth complex projective varieties, the induced homomorphism $f^* : H^k(Y) \to H^k(X)$ is a morphism of Hodge structures. The total cohomology $H^*(X)$ is a graded pure Hodge structure. This gives a functor

$$H^* : \mathsf{SmProj} \to \mathfrak{hs},$$

where SmProj is the category of complex projective manifolds.
(2) The Hodge-Tate structures $\mathbb{Q}(k)$ are 1-dimensional of pure type $(-k, -k)$ where $\mathbb{Q}(k) = (2\pi \mathrm{i})^k\mathbb{Z} \subset \mathbb{C}$. The Lefschetz structure is $\mathbb{L} = \mathbb{Q}(-1)$.

2 The Hodge Characteristic

2.1 Topological Considerations

For X in the category Top^* the two Euler-characteristics are finite

$$\chi_{\rm top}(X) = \sum(-1)^k b_k(X), \quad b_k(X) = \dim_{\mathbb{Q}} H^k(X;\mathbb{Q});$$
$$\chi^c_{\rm top}(X) = \sum(-1)^k b^c_k(X), \quad b^c_k(X) = \dim_{\mathbb{Q}} H^k_c(X;\mathbb{Q}).$$

In general the first is not additive, while the second is:

Proposition 2.1. *If $Y \subset X$ is a closed subset $Y, X \in \mathsf{Top}^*$ then $\chi^c_{\rm top}(X) = \chi^c_{\rm top}(Y) + \chi^c_{\rm top}(X - Y)$.*

Proof: Set $i : Y \to X$ and $j : U = X - Y \to X$ and consider the exact sequence

$$\cdots \to H^k_c(U) \xrightarrow{j_!} H^k_c(X) \xrightarrow{i^*} H^k_c(Y) \xrightarrow{\delta} H^{k+1}_c(U) \to \cdots,$$

where $j_!$ is the extension by zero map: a cocycle on U having support in $K \subset U$ can be extended to a cocycle on X. □

One can reformulate this K-theoretically as follows. Introduce the free group $\mathbb{Z}[\mathsf{Top}^*]$ on homeomorphism classes in Top^*: an element from this group is a finite sum $n_j\{X_j\}$, $n_j \in \mathbb{Z}$ and $\{X_j\}$ the homeomorphism class of X_j. Introduce an equivalence relation given by the *scissors relation*:

$$\{X\} \sim \{Y\} + \{X - Y\} \text{ whenever } Y \text{ is closed in } X.$$

The equivalence classes will be denoted $[X]$. Hence in the quotient group

$$K_0(\mathsf{Top}^*) = \mathbb{Z}[\mathsf{Top}^*]/\sim$$

one has equality $[X] = [Y] + [X - Y]$. The preceding Proposition is equivalent to the fact that the Euler characteristic with compact support induces a homomorphism

$$\chi^c_{\rm top} : K_0(\mathsf{Top}^*) \to \mathbb{Z}.$$

In fact, the right hand side can be viewed as a K-group as well. Recall that for any abelian category $\mathfrak{A}$ the associated group $K_0(\mathfrak{A})$ consists of the free group $\mathbb{Z}[\mathfrak{A}]$ on isomorphism classes of objects in $\mathfrak{A}$ modulo the subgroup generated by the relations

$$[A] = [B] + [C], \qquad \text{whenever } 0 \to B \to A \to C \to 0 \text{ is an exact sequence.}$$

So the Euler characteristic becomes

$$\chi^c_{\mathrm{top}} : K_0(\mathsf{Top}^*) \to K_0(\mathsf{Vect}) \simeq \mathbb{Z},$$

where Vect denotes the category of finite dimensional $\mathbb{Q}$-vector spaces.

2.2 Hodge Theoretic Version

I want to investigate what happens if we restrict the Euler characteristic to the category SmProj. Since the cohomology groups have a richer structure, namely they carry pure Hodge structures, one expects a homomorphism

$$\chi_{\mathrm{Hg}} : K_0(\mathsf{SmProj}) \to K_0(\mathfrak{hs}).$$

The problem here is that the scissors relation does not preserve the category SmProj: cutting out a closed subvariety yields a quasi-projective variety and such a variety cannot be expected to have cohomology with pure Hodge structures.

However, one can compare the left hand with an a priori different K-group which is built from SmProj: the equivalence relation is given by the blow-up relation:

$$\{X\} - \{Y\} \sim \{Z\} - \{E\}$$

where

$$\begin{array}{ccc} E & \xrightarrow{j} & Z = \mathrm{Bl}_Y X \\ \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{i} & X, \end{array} \tag{2}$$

X is smooth projective and $Y \subset X$ is a smooth subvariety; the map i is the inclusion and $\pi : \mathrm{Bl}_Y X \to X$ is the blow-up of X along Y; finally E is the exceptional divisor included in Z through the inclusion j. The quotient $K_0(\mathsf{SmProj})$ is the free group $\mathbb{Z}[\mathsf{SmProj}]$ on isomorphism classes of smooth projective varieties modulo the equivalence relation given by the blow-up relation $[X]-[Y] = [Z]-[E]$ where X, Y, Z, E are as in the blow-up diagram (2) together with the trivial relation $[\varnothing] = 0$.

The comparison with the K-group built from the scissors relation on the category Var of complex algebraic varieties is given by a result due to Bittner:

Theorem 2.2 ([Bitt]). *The inclusion induces an isomorphism*

$$K_0(\mathsf{SmProj}) \xrightarrow{\simeq} K_0(\mathsf{Var}).$$

Corollary 2.3. *There is a commutative diagram*

$$\begin{array}{ccc} K_0(\mathsf{Top}^*) & \xrightarrow{\chi^c_{\rm top}} & K_0(\mathsf{Vect}) \\ \downarrow & & \downarrow \\ K_0(\mathsf{SmProj}) & \xrightarrow{\chi_{\rm Hg}} & K_0(\mathfrak{hs}). \end{array}$$

Proof: It suffices to show that one has a long exact sequence

$$\cdots \to H^k(X) \xrightarrow{\pi^*+i^*} H^k(Z) \oplus H^k(Y) \xrightarrow{j^*-(\pi|E)^*} H^k(E) \to \cdots$$

Indeed, this sequence is a sequence of pure Hodge structures and the relation $\chi_{\rm Hg}(X) - \chi_{\rm Hg}((Y) = \chi_{\rm Hg}(Z) - \chi_{\rm Hg}(E)$ follows.

The above exact sequence is a consequence of a special type Mayer-Vietoris sequence associated to the topological space $Z \coprod_{\pi|E} Y$ where one glues $E \times [0,1]$ to Z by identifying the "top" $(e,0)$ to $e \in E$ and to Y by identifying the "bottom" $(1,z)$ and $\pi(z) \in Y$. Indeed, the usual Mayer-

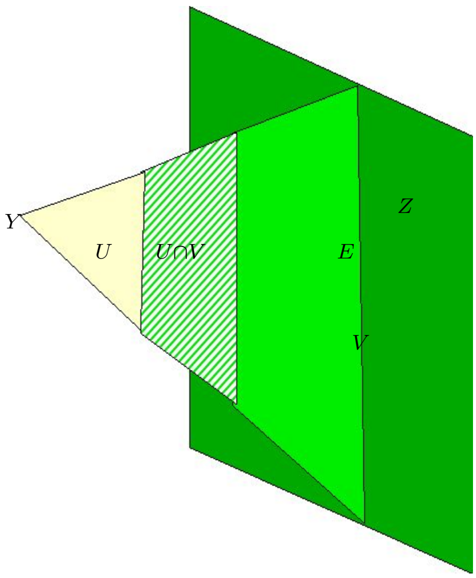

Figure 1: Mayer-Vietoris for a blow-up

Vietoris sequence reads

$$\cdots \to H^k(U \cup V) \to H^k(V) \oplus H^k(U) \to H^k(U \cap V) \to \cdots$$

and in the above situation the picture shows that X is a deformation retract of $U \cup V$, V retracts to Z, U to Y and $U \cap V$ to E. $\square$

The goal of these lectures is to understand the resulting homomorphism $K_0(\mathsf{Var}) \xrightarrow{\chi_{\rm Hg}} K_0(\mathfrak{hs})$ directly in terms of Hodge theory. For this one needs an extension of Hodge structures.

3 Mixed Hodge Structures

3.1 Main Results

Definition 3.1. 1. A *mixed Hodge structure* is a triple (H, W, F) where W is an increasing filtration on H, the *weight filtration* and F a decreasing filtration on $H_{\mathbb{C}}$, the *Hodge filtration*, such that F induces a pure Hodge structure of weight k on $\mathrm{Gr}_k^W := W_k/W_{k-1}$.

2. *Morphisms of mixed Hodge structure* are $\mathbb{Q}$-linear maps preserving weight and Hodge filtrations.

3. The *Hodge numbers* are $h^{p,q}(H) = \dim_{\mathbb{C}} \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^{W^{\mathbb{C}}} H_{\mathbb{C}}$ where $W^{\mathbb{C}} := W \otimes_{\mathbb{Q}} \mathbb{C}$.

The K-group associated to the abelian category of mixed Hodge structures is just $\mathrm{K}_0(\mathfrak{hs})$ since in K_0 there is the relation $[H] = \sum [\mathrm{Gr}_k^W H]$. This suggests to define $\chi_{\mathrm{Hdg}}(X)$ using a mixed Hodge structure on $H_c^k(X)$. That this is indeed possible is a consequence of the following central result:

Theorem 3.2 ([Del71, Del74])**.** *Let U be a complex algebraic variety and $V \subset U$ a closed subvariety (V may be empty). Then $H^k(U, V)$ has a mixed Hodge structure which is functorial in the sense that if $(U, V) \to (U', V')$ is a morphism of pairs, the induced morphism $H^k(V', U') \to H^k(U, V)$ is a morphism of Hodge structures. Moreover, for U smooth projective, the mixed Hodge structure on $H^k(U)$ is the classical one, i.e. the pure weight k Hodge structure coming from the decomposition of forms into types.*

3.2 Examples

Assuming that the mixed Hodge structure behaves functorially, I shall calculate mixed Hodge structures for curves.

Example 1: A punctured curve $C = \bar{C} - \Sigma$ where $\bar{C}$ is a smooth projective curve and Σ is a set of M points. Let $j : C \hookrightarrow \bar{C}$ be the inclusion. For ordinary cohomology one has an exact sequence of mixed Hodge structures

$$0 \to H^1(\bar{C}) \xrightarrow{j^*} H^1(C) \xrightarrow{\text{res}} \bigoplus_{x \in \Sigma} H^0(x)(-1) \to 0.$$

This shows that $H^1(C) = W_2 \subset W_1 = H^1(\bar{C})$ and $\mathrm{Gr}_W^2 = M \cdot \mathbb{L}$.

For compact supported cohomology one has

$$0 \to H^0(\bar{C}) \to H^0(\Sigma) \to H_c^1(C) \to H^1(\bar{C}) \to 0,$$

showing that $H_c^1(C) = W_1 \subset W_0 = \widetilde{H}^0(\Sigma) = (M - 1) \cdot 1$ where $1 = \mathbb{Q}(0)$ while $\mathrm{Gr}_W^1 = H^1(\bar{C})$.

Let $V_g(\bar{C}) = H^1(\bar{C})$ be the usual weight one Hodge structure. It follows that $\chi_{\text{Hodge}}(C) = (1-M)\cdot 1 - V_g(\bar{C}) + \mathbb{L}$. Topological considerations give that $b_0(C) = b_2^c(C) = 1$. Also $H^2(C) = 0$ since C is affine. Then by the above exact sequences Table 1 below can be made up.

From the table one finds $\chi_{\text{Hdg}}(C) = (1-M)\cdot 1 - V_g(\bar{C}) + \mathbb{L}$. Note also that the analogous character for ordinary cohomology reads $1 - V_g(\bar{C}) + (M-1)\cdot\mathbb{L}$ which is different from the Hodge characteristic!

Table 1: Cohomology of the punctured curve C

	H^0	H^1	H^2	H^2_c	H^1_c	H^0_c
weight 0	1	0	0	0	$(M-1)\cdot 1$	0
weight 1	0	$V_g(\bar{C})$	0	0	$V_g(\bar{C})$	0
weight 2	0	$(M-1)\mathbb{L}$	0	$\mathbb{L}$	0	0

Example 2: D a singular curve with N double points forming $\Sigma \subset D$. Consider the normalization $n : \widetilde{D} \to D$, a curve of genus g with $H^1(\widetilde{D}) = V_g(\widetilde{D})$, a Hodge structure of weight one. The exact sequence

$$0 \to \bigoplus H^0(n^{-1}\Sigma) \to H^1(D) \xrightarrow{n^*} H^1(\widetilde{D}) \to 0$$

shows that $H^1(D) = W^1 \supset W_0$ with $\mathrm{Gr}^1_W = H^1(\widetilde{D})$. One gets the following table showing that $\chi_{\text{Hdg}}(D) = (1-N)\cdot 1 - V_g(\widetilde{D}) + \mathbb{L}$.

Table 2: Cohomology of the singular curve D

	H^0	H^1	H^2
weight 0	1	$N\cdot 1$	0
weight 1	0	$V_g(\widetilde{D})$	0
weight 2	0	0	$\mathbb{L}$

3.3 The Mixed Hodge Characteristic

Compactly supported cohomology on U can be calculated with the help of a compactification X of U as relative cohomology:

$$H^k_c(U) = H^k(X, T), \qquad T = X - U,$$

and so, as a consequence of Theorem 3.2, it carries a mixed Hodge structure. This Hodge structure can be shown not to depend on the way one compactifies U. The (mixed) Hodge characteristic can now be defined by setting:

$$\chi_{\mathrm{Hdg}}(U) = \sum_k [\mathrm{Gr}^W_k H^k_c(U)] \in \mathrm{K}_0(\mathfrak{hs}).$$

What has to be shown is that it is additive, i.e. that it respects the scissors relation. This uses the following result.

Theorem 3.3. *Given a triple (U, V, W) of complex algebraic varieties, i.e W is closed in V and V is closed in U. Then the inclusions define a long exact sequence of mixed Hodge structures*

$$\cdots \to H^k(U, V) \to H^k(U, W) \to H^(V, W) \to H^{k+1}(U, V) \to \cdots$$

For a proof see e.g. [P-S07, Ch. 5].

Corollary 3.4. *The Hodge characteristic induces a homomorphism*

$$\chi_{\mathrm{Hg}} : K_0(\mathsf{Var}) \to K_0(\mathfrak{hs}).$$

Proof: Let me apply Theorem 3.3 as follows. Let (X, Y) be a pair in Var and choose a compactification $\bar{X}$ of X and let $\bar{Y}$ be the Zariski closure of Y in $\bar{X}$. Set $T = \bar{X} - X$ and consider the triple $(\bar{X}, \bar{Y} \cup T, T)$ and remark that also $\bar{\cup} T$ is a compactification of Y. So one gets a long exact sequence of mixed Hodge structures

$$\begin{array}{ccccccc} \cdots \to H^k_c(X - Y) & \to & H^k_c(X) & \to & H^k_c(Y) & \to & H^{k+1}_c(X - Y) \to \cdots \\ \| & & \| & & \| & & \| \\ H^k(\bar{X}, \bar{Y} \cup T) & \to & H^k(\bar{X}, T) & \to & H^k(\bar{Y} \cup T) & \to & H^{k+1}(\bar{X}, \bar{Y} \cup T). \end{array}$$

This shows that χ_{Hg} is additive. □

4 Mixed Hodge Complexes

In this Lecture I explain some of the technical ideas behind the construction of a functorial mixed Hodge structure on cohomology. Some general notions having to do with complexes (in an abelian category) are useful. A complex $K^\bullet$ is *bounded below* if $K^p = 0$ for all $p \le p_0$. A *complex with decreasing filtration* $(K^\bullet, F^\bullet)$ consists of subcomplexes $F^p K^\bullet$ for all p such that $F^p K^k \supset F^{p+1} K^k$ for all p, k. The *associated graded complex* $\mathrm{Gr}^p_F K$ has $F^p K^k / F^{p+1} K^k$ at place k.

Likewise, we can speak of increasing filtered complexes, denoted with a lower index $(K^\bullet, W_\bullet)$.

It should be clear what is meant by a bi-filtered complex.

The following examples are useful:

Examples 4.1. (1) For any complex $L^\bullet$ the *trivial decreasing filtration* is given by

$$F^pL = L^{\geq p} = \{0 \to \cdots \to 0 \to L^p \to L^{p+1} \to \cdots\},$$

a complex starting in degree p.
(2) For any complex $K^\bullet$, one has the *canonical filtration* which is the increasing filtration given by

$$\tau_p K = [\cdots \to K^{p-1} \to \ker d^p \to 0 \to 0 \cdots].$$

There are further notions that play a role. We say that a morphism of complexes $f : K \to L$ is a *quasi-isomorphism* if the induced morphisms in cohomology $H^k(f) : H^k(K) \to H^k(L)$ are isomorphisms. A morphism of filtered complexes $f : (K, F) \to (L, G)$ is a *filtered quasi-isomorphism* if the induced morphisms $\mathrm{Gr}^q(f) : \mathrm{Gr}^q_F K \to \mathrm{Gr}^q_G L$ are quasi-isomorphisms. This implies that f itself is a quasi-isomorphism.

4.1 Hodge Theory Revisited

Let X be a complex projective manifold. The proof of the existence of a Hodge structure on cohomology can be analyzed as follows.

1. On the level of *complexes of sheaves* one has the constant sheaves coming with an inclusion $\mathbb{Q}_X \hookrightarrow \mathbb{C}_X$. This puts a rational structure on $H^k(X, \mathbb{C})$.

   The cohomology groups of the latter sheaf is calculated using the de Rham complex $(\mathcal{A}^\bullet_X)_\mathbb{C}$ of sheaves of complex valued differential forms. Alternatively, one may use the holomorphic de Rham complex $\Omega^\bullet_X$. This complex is resolved by $(\mathcal{A}^\bullet_X)_\mathbb{C}$. Note that $(\mathcal{A}^\bullet_X)_\mathbb{C} = s\mathcal{A}^{\bullet,\bullet}_X$, the simplex complex associated to the double complex $\mathcal{A}^{\bullet,\bullet}_X$ of C^∞ forms of type (p, q). Indeed, for any double complex $(L^{\bullet,\bullet}, d', d'')$ one has $sL^k = \bigoplus_{p+q=k} L^{p,q}$ with differential $d = d' + d''$. In our case $d' = \partial$ and $d'' = \overline{\partial}$.

2. One has the trivial filtration

$$F^p\Omega^\bullet_X = \Omega^{\geq p}_X \hookrightarrow F^p\mathcal{A}^{\bullet,\bullet}_X = \mathcal{A}^{\geq p,\bullet}_X.$$

3. Now we pass to cohomology groups. On the level of complexes $K^\bullet$ of sheaves on X this is done by hypercohomology, the so-called derived section-functor $R\Gamma$. This is computed using a fine resolution $K^\bullet \hookrightarrow L^\bullet$ by setting

$$R^k\Gamma(X, K^\bullet) = H^k(\Gamma(X, L^\bullet)).$$

In the above situation, for $K^\bullet = \Omega_X^\bullet$, respectively $K^\bullet = \Omega_X^p$, one has

$$\begin{aligned} H^k(X,\mathbb{C}) &= R^k\Gamma(X,(\mathcal{A}_X^\bullet)_{\mathbb{C}}) = R^k\Gamma(X,\Omega_X^\bullet) \\ H^q(X,\Omega_X^p) &= R^q\Gamma(X,\mathcal{A}^{p,\bullet}). \end{aligned}$$

The first is by de Rham's theorem (or the holomorphic de Rham theorem); the second is Dolbeault's theorem.

4. We do the same for the filtrations:

$$F^p(R^k\Gamma(\Omega_X^\bullet)) = R^k\Gamma(X, s\mathcal{A}_X^{\geq p,\bullet}) = \bigoplus_{b\geq p} H^{k-b}(X,\Omega^b).$$

Indeed the single complex associated to $s\mathcal{A}_X^{\geq p,\bullet}$ is the following complex which starts in degree p:

$$\cdots 0 \to 0 \to \mathcal{A}_X^{p,0} \to \mathcal{A}_X^{p+1,0} \oplus \mathcal{A}_X^{p,1} \to \cdots \to \bigoplus_{k\geq 0} \mathcal{A}_X^{p+r-k,k} \cdots .$$

This is the direct sum of several Dolbeault complexes each placed in a suitable degree, namely the one for Ω_X^b, placed in degree b for $b = p,\dots,n$. This shifts the degrees when we calculate cohomology in degree k; for Ω_X^b it calculates cohomology in degree $k-b$.

5. Next, one considers the spectral sequence for the F filtration on the double complex $A^{\bullet,\bullet}(X)$:

$$\begin{array}{ccccc} E_2^{p,q} & = & R^q\Gamma(X,(\mathcal{A}_X^{p,\bullet},\delta)) & \Longrightarrow & R\Gamma^{p+q}(X,s\mathcal{A}^{\bullet,\bullet}) \\ & & \| & & \| \\ & & H^q(X,\Omega_X^p) & \Longrightarrow & H^{p+q}(X,\mathbb{C}). \end{array}$$

The crucial information from the Hodge decomposition theorem implies that this spectral sequence degenerates at E_2: the Hodge decomposition has the property that the associated filtration $F^pH^k(X,\mathbb{C})$ is same as the filtration induced by F on the "limit" $H^{p+q}(X,\mathbb{C})$. This implies that the dimensions of the terms $E_2^{p,q}$ and the terms $E_\infty^{p,q}$ in the spectral sequence are the same. This in turn is equivalent to degeneration of the spectral sequence.

Alternatively, purely in terms of the F-filtration, one has $\mathrm{Gr}_F^p\,\Omega_X^\bullet = \Omega_X^p[p]$, i.e. the sheaf Ω_X^p placed in degree p. Its hypercohomology in degree $(p+q)$ then is $H^q(X,\Omega_X^p)$. So the spectral sequence for the F–filtration on hypercohomology for $\Omega^\bullet = L^\bullet$ reads

$$E_1^{p,q} = R\Gamma^{p+q}(X,\mathrm{Gr}_F^p\,L^\bullet) \Longrightarrow R^{p+q}\Gamma(X,L^\bullet). \tag{3}$$

6. Since only cohomology groups matter, one better works in a different category, the *derived category of bounded below complexes of sheaves of* $\mathbb{C}$*-vector spaces on* X. Here the objects are bounded below complexes of sheaves on X, but the morphisms are a bit complicated. Without giving details, two morphisms linked by a homotopy are considered the same; their class is called a homotopy class. They induce the same homomorphism on the level of cohomology sheaves. Next, all morphisms that induce isomorphisms on cohomology, i.e. the quasi-isomorphisms, should be considered invertible in this category. This has an important advantage: in $D(X)$ quasi-isomorphic complexes are isomorphic. For instance $[\mathbb{C}_X], [\Omega^\bullet_X]$ and $[s\mathcal{A}^{\bullet,\bullet}_X]$ are all isomorphic in this category.

The above can be formalized by the following concepts:

Definition 4.2. (1) A *Hodge complex of weight* m on X is a triple $(L^\bullet, (L^\bullet_{\mathbb{C}}, F^\bullet), \alpha)$, where

- $L^\bullet$ is a bounded complex of $\mathbb{Q}$–vector spaces,
- $(L^\bullet_C, F^\bullet)$ is a bounded filtered complex of $\mathbb{C}$–vector spaces whose differentials are strict with respect to F (recall: this means that $\mathrm{Im}(d) \cap F^p = \mathrm{Im}(d|F^p)$ for all p),
- $\alpha : L^\bullet \to L^\bullet_{\mathbb{C}}$, the *comparison morphism*, which is a morphism of complexes such that $\alpha \otimes 1 : L^\bullet \otimes \mathbb{C} \to L^\bullet_{\mathbb{C}}$ is a quasi-isomorphism.
- Moreover, F induces a Hodge structure of weight $m + k$ on $H^k(L^\bullet)$.

(2) A *Hodge complex of sheaves* on X is a triple $(\mathcal{L}^\bullet, (\mathcal{L}^\bullet_C, F^\bullet), \alpha)$, where

- $\mathcal{L}^\bullet$ is a bounded complex of sheaves $\mathbb{Q}$–vector spaces,
- $(\mathcal{L}^\bullet_{\mathbb{C}}, F^\bullet)$ is a bounded filtered complex of sheaves of $\mathbb{C}$–vector spaces,
- $\alpha : \mathcal{L}^\bullet \to \mathcal{L}^\bullet_{\mathbb{C}}$, the comparison morphism, is a morphism of complexes such that $\alpha \otimes 1 : \mathcal{L}^\bullet \otimes \mathbb{C} \to \mathcal{L}^\bullet_{\mathbb{C}}$ is a quasi-isomorphism.
- The complex $(R\Gamma(X, \mathcal{L}^\bullet), (R\Gamma(X, \mathcal{L}^\bullet_{\mathbb{C}}), F), R\Gamma\alpha)$ is a Hodge complex of weight m.

Remark. The demand that $(R\Gamma(X, \mathcal{L}^\bullet_{\mathbb{C}}), F^\bullet)$ is a bounded filtered complex of $\mathbb{C}$–vector spaces whose differentials are strict with respect to F is equivalent to degeneration at E_1 of the F-spectral sequence (3). So, indeed

$$\mathsf{Q}^\bullet_X := \{\mathbb{Q}_X, (\Omega^\bullet_X, F), \mathbb{Q}_X \hookrightarrow \mathbb{C}_X \hookrightarrow \Omega^\bullet_X\} \tag{4}$$

is a Hodge complex of sheaves on X of weight 0.

Summarizing: to show that a hypercohomology group $R^k\Gamma(X, \mathcal{L}^\bullet)$ carries a Hodge structure, one should construct a Hodge complex of sheaves out of $\mathcal{L}^\bullet$. One should make full use of the flexibility of the derived category.

4.2 Mixed Hodge Theory Revisited

The above philosophy can be made to work for $H^k(X,\mathbb{C})$ for X a complex algebraic variety, and also for the cohomology of pairs or triples in the category Var provided one throws in an extra ingredient, the *weight filtration.*

To explain how this works on the level of complexes, in the definition of Hodge complexes of sheaves one replaces the complex of sheaves $\mathcal{L}^\bullet$ by a *filtered* complex $(\mathcal{L}^\bullet, W)$ where now W is a decreasing filtration. Next, one replaces $(\mathcal{L}^\bullet_{\mathbb{C}}, F)$ by $(\mathcal{L}^\bullet_{\mathbb{C}}, W_{\mathbb{C}}, F)$ a bi-filtered complex of sheaves with comparison $\alpha : (\mathcal{L}^\bullet, W) \to (\mathcal{L}^\bullet_{\mathbb{C}}, W_{\mathbb{C}})$ such that $\alpha \otimes 1$ becomes a filtered quasi-isomorphism. Moreover, taking the gradeds $\mathrm{Gr}^W_m \mathcal{L}^\bullet$ should give a Hodge complex of sheaves of weight m. The resulting data consisting of the triple

$$\mathsf{L}^\bullet = \{(\mathcal{L}^\bullet, W), (\mathcal{L}^\bullet_{\mathbb{C}}, W_{\mathbb{C}}, F), (\mathcal{L}^\bullet, W) \xrightarrow{\alpha} (\mathcal{L}^\bullet_{\mathbb{C}}, W_{\mathbb{C}})\}$$

is called a *mixed Hodge complex of sheaves* on X.

One passes from mixed Hodge complexes of sheaves on X to *mixed Hodge complexes* upon applying the derived section functor.

The next result gives the main tool for constructing mixed Hodge structures.

Theorem. *Given a mixed Hodge complex of sheaves, the cohomology of the resulting mixed Hodge complex receives a natural functorial mixed Hodge structure.*

Now there is one subtlety. The mixed Hodge structure on $R^k\Gamma(X, \mathcal{L}^\bullet)$ has its weight filtration not induced by W but by the shifted weight filtration $W[k]$. So the m–th graded part comes from $W_{m-k}\mathcal{L}^\bullet$.

Instead of explaining how this yields Deligne's mixed Hodge structure on cohomology, let me do this for one example: the cohomology of a punctured curve $C = \bar{C} - \Sigma$ where $\bar{C}$ is a smooth projective curve and Σ is a set of M points.

To describe the mixed Hodge complex of sheaves, we let $j : C \hookrightarrow \bar{C}$ be the inclusion and introduce the filtered complex in degrees 0 and 1

$$(Rj_*\mathbb{Q}_C, \tau) := (\mathbb{Q}_{\bar{C}} \xrightarrow{0} \mathbb{Q}_\Sigma,\, \tau).$$

Observe that

$$\begin{aligned} \tau_0 &= \mathbb{Q}_{\bar{C}} \text{ in degree } 0 \\ \mathrm{Gr}^\tau_1 &= \mathbb{Q}_\Sigma \text{ in degree } 1. \end{aligned} \tag{5}$$

To calculate $H^1(C,\mathbb{C})$ one uses the complex $\Omega^\bullet_{\bar{C}}(\log \Sigma)$ which differs from $\Omega^\bullet_C$ in that it is generated by dz/z near a puncture where z is a local coordinate. Its weight filtration is given by

$$\underbrace{\Omega^\bullet_{\bar{C}}}_{W_0} \hookrightarrow \underbrace{\Omega^\bullet_{\bar{C}}(\log \Sigma)}_{W_!}$$

Note that its graded pieces are the complexes $\Omega^\bullet_{\bar{C}}$ resolving $\mathbb{C}_{\bar{C}}$ and

$$\mathrm{Gr}^1_W = \mathbb{C}_\Sigma \text{ in degree 1.}$$

So, by (5), the complexes $Rj_*\mathbb{Q}_C \otimes \mathbb{C}$ and $\{\Omega^\bullet_{\bar{C}}(\log \Sigma), W\}$ are filtered quasi-isomorphic. The comparison is given by

$$\alpha : Rj_*\mathbb{Q}_C \to Rj_*\mathbb{C}_C \xrightarrow{\text{qis}} \Omega^\bullet(\log \Sigma),$$

where "qis" means that the morphism a quasi-isomorphism. Using F for the trivial filtration, the triple

$$\mathsf{Q}^\bullet_C := \{(Rj_*\mathbb{Q}_C, \tau), (\Omega^\bullet_{\bar{C}}(\log \Sigma), W, F), \alpha\}$$

then is a mixed Hodge complex of sheaves.

To see that this gives the desired mixed Hodge structure on $H^1(C, \mathbb{Q})$, consider the filtration induced by F on the weight-graded complexes. Then one sees that $\tau_0 j_*\mathbb{Q}_C$ is the Hodge complex $\mathsf{Q}_{\bar{C}}$ from (4) and so calculates the pure weight one Hodge structure $H^1(\bar{C}, \mathbb{Q})$ while $\mathrm{Gr}^1_\tau j_*\mathbb{Q}_C$ calculates the weight 2 Hodge structure coming from the points Σ.

Here, note that in view of the shifts, Gr^1_W being a weight 1 Hodge complex of sheaves, it induces a weight 2 Hodge structure on $\mathrm{Gr}^2_W H^1(C)$ (and similarly of course for W_0 which induces a weight 1 structure on $W_1H^1(C)$.

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