Compact Complex Surfaces

In: *Differential Geometry,*
Global Analysis, and Topology
Proc. Special Session of the Canadian Math. Soc.
Summer Meeting June 1990

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Introduction

In these notes we give an elementary introduction to surface theory, i.e the theory of compact two-dimensional complex manifolds. We refer freely to [Troika] for details. For a more concise treatment and more on related topological questions see [FM 2].

For these notes I have in mind the (differential)-topologist who wants to understand the recent results on the topology of algebraic surfaces, but who is not well versed in algebraic geometry.
I want to present a 'modern' i.e postciteTroika-point of view. Two new ingredients now go into the theory:

1. Mori's theory of the interplay between nef-divisors and effective 1-cycles. In Chapter II, based on Wilson's article [Wi], we show how powerful this point of view is when dealing with the classification of surfaces.

2. Reider's method of constructing rank 2 vector bundles associated to points which have a special role with regards to the geometry of a linear system of divisors (i.e base points). We sketch in Chapter III how this can be used to obtain a nice short and uniform treatment of the topic of pluricanonical maps for surfaces of general type.

In Chapter V on the topology we only look at simply connected algebraic surfaces and gather some of the recent results on the differentiable structures on a given surface oriented homeomorphic to a simply connected algebraic surface. The Dolgachev surfaces introduced in Chapter I play an important role here. Also Barlow surfaces turn out to give interesting examples. These are treated in Chapter IV, where we collect various recent results on the geography of surfaces.

It goes without saying that many interesting and important topics had to be left out. We mention a few of these:

- Commutative algebra methods to describe the finer classification of surfaces with special Chern invariants. We refer to [Cat 1] and [Reid 2] for recent surveys.

- The recent results on obstructed surfaces by Catanese, which show how abundant these are. See [Cat 2].

- Reider's work on irregular surfaces of general type. See [Reider 2].
- Recent bounds on the size of the automorphism group of surfaces of general type. See [Xiao 2, HS, Corti].

I hope this is enough to convey the beauty and wealth of a subject which has kept on developing since the appearance of [Troika].

Chapter 1

Basic material

1 Invariants

We let S be a compact connected complex manifold of dimension 2. Since S is an oriented compact four- manifold we have

$$b_0 = b_4 = 1$$
$$b_1 = b_3.$$

The cup product form

$$H^2(S,\mathbf{Z}) \times H^2(S,\mathbf{Z}) \to \mathbf{Z}$$

defined by $a \cdot b = a \cup b[S], a, b \in H^2(S,\mathbf{Z})$ is unimodular on $H^2(S,\mathbf{Z})$/torsion. With b^+, resp b^-= number of positive, resp. negative eigenvalues of this form, the *signature* is

$$\tau(S) = b^+ - b^-.$$

There are further invariants a priori depending on the almost-complex structure: the Chern classes c_1 and c_2. The Chern numbers $c_1^2[S]$ and $c_2[S]$ turn out to be topological invariants as well. For the second number this is obvious:

$$c_2[S] = e(S) \text{ (the Euler number) } = 2 - 2b_1 + b_2.$$

while the first Chern number can be expressed in the signature and the Euler number using Hirzebruch's signature formula :

$$\tau(S) = 1/3(c_1^2[S] - 2c_2[S]).$$

Let Ω^p be the sheaf of holomorphic p-forms on S and set

$$h^{p,q} = \dim H^q(S,\Omega^p) \text{ (the Hodge numbers) }.$$

Another important formula is Noether's formula, which relates the invariants

$$q = h^{0,1} \text{ (the irregularity)}$$
$$p_g = h^{0,2} = \text{(the geometric genus) }.$$

In fact Noether's formula states:

$$\chi := 1 - h^{0,1} + h^{0,2} \text{ (the arithmetic genus)} = 1/12(c_1^2[S] + c_2[S]).$$

If we eliminate $c_1^2[S]$ from the signature formula and Noether's formula, we find

$$(b^+ - 2p_g) + (2q - b_1) = 1.$$

It is easy to see that both numbers in parenthesis in the above formula are non-negative, so we have two possibilities

$$\begin{aligned} &(i) \quad b^+ = 2p_g + 1, \quad b_1 = 2q, \\ &(ii) \quad b^+ = 2p_g, \quad b_1 = 2q - 1. \end{aligned}$$

From the inequalities $b_1 \geq 2h^{1,0}$ and $h^{0,1} + h^{1,0} \geq b_1$ it follows that $h^{1,0} = \frac{1}{2}b_1$ in the first case and $h^{1,0} = \frac{1}{2}(b_1 - 1)$ in the second case and so

$$b_1 = h^{0,1} + h^{1,0}$$

in both cases. Notice that $e(S) = \sum_{0 \leq p,q \leq 2}(-1)^{p+q} h^{p,q} = 1 - 2b_1 + (h^{2,0} + h^{1,1} + h^{0,2})$ (we used Serre-duality to replace $h^{2,1}$, resp. $h^{1,2}$ by $h^{0,1}$, resp. $h^{1,0}$). It follows that $b_2 = h^{2,0} + h^{1,1} + h^{0,2}$ and so

$$h^{1,1} = b_2 - 2p_g.$$

So we see:

for complex surfaces the Hodge numbers $h^{p,q}$ are topological invariants.

As a side remark, we note that for Kähler manifolds we have of course the first possibility for the invariants. Much less trivial is the converse:

Theorem 1.1. *A compact complex surface is Kähler if and only if the first Betti number is even.*

The proof of this theorem goes far beyond this introduction. It really uses the classification theory of surfaces to reduce to the case of elliptic surfaces and K3-surfaces. In 1974 Miyaoka proved the theorem for elliptic surfaces with elementary means([Mi 1]), while the result for K3-surfaces although conjectured already in the fifties by André Weil, had to wait untill Siu in 1983 using very sophisticated arguments (essentially using Yau's solution of Calabi's conjectures) finally proved that all K3-surfaces are Kähler (see [Siu]).

2 Divisors and pluricanonical bundles

First we recall some generalities about divisors and their associated linear systems.

By 'divisor' we always mean Cartier divisors, i.e objects which are locally on a coordinate patch U_i are given by one equation $f_i = 0$ (f_i meromorphic and nonzero) and such that in overlaps $U_i \cap U_j$ the quotients f_i/f_j are invertible holomorphic functions. A divisor D defines a line bundle $\mathcal{O}(D)$ defined by the transition functions f_i/f_j on intersections $U_i \cap U_j$ and so there is a global section of $\mathcal{O}(D)$, locally on U_i given by f_i, vanishing along D. Sections of this line bundle yield divisors D' which are said to be *linearly equivalent* to D. Since two sections that are the same up to a multiplicative constant define the same divisor, the projective space $\mathbf{P}H^0(\mathcal{O}(D))$ parametrizes *effective divisors* (i.e. non negative combinations of curves) linearly equivalent to D. It is *the linear system* $|D|$ associated to D. It is not true that to any line bundle there corresponds a divisor; indeed, such a line bundle would lead to a meromorphic function having a pole along the divisor, but there are manifolds without any non-constant meromorphic functions. If however S is algebraic, the existence of sufficiently many meromorphic functions shows (after some work) that to any line bundle there is a corresponding divisor.

If $|D|$ is a linear system of divisors, there is a maximal divisor D_{fixed}, the fixed part of $|D|$ with the property that for $D' \in |D|$ we have $D' = D'' + D_{\text{fixed}}$, D'' effective.

If $\mathcal{L}$ is any line bundle, choose a basis $\{s_0, s_1, \ldots, s_N\}$ for the sections of $\mathcal{L}$ and define the meromorphic map

$$\begin{aligned} f_{\mathcal{L}} : S &\to \mathbf{P}^N \\ s &\mapsto (s_0(s) : s_1(s) : \cdots : s_N(s)). \end{aligned}$$

This is not defined at points where all sections of $\mathcal{L}$ vanish. If $\mathcal{L} = \mathcal{O}(D)$ for some divisor D these are the *base points* of the linear system $|D|$.

The *canonical bundle* $\mathcal{K}$ of S is the line bundle associated to the sheaf Ω^2 of holomorphic 2-forms. In case there is a divisor whose line bundle is $\mathcal{K}$ we call it the *canonical divisor* and denote it by $K_S = K$. The m-th tensor power of $\mathcal{K}$ is the m-th *pluricanonical bundle* and we set

$$P_m(S) = h^0(\mathcal{K}^{\otimes m}) \quad \text{the } m\text{-th plurigenus of } S.$$

These bundles play an important role in classification.

Let us next recall that the intersection number of two line bundles $\mathcal{L}$ and $\mathcal{M}$ is defined as

$$\mathcal{L} \cdot \mathcal{M} := c_1(\mathcal{L}) \cdot c_1(\mathcal{M})$$

and hence intersection numbers of divisors are also defined. Very useful is the following formula:

$$\underline{\text{Adjunction formula}}: \quad \mathcal{K} \cdot C + C \cdot C = 2 \cdot (g(\tilde{C}) + \delta - 1), \tag{1.1}$$

where $\tilde{C}$ is a smooth model of a curve C and $\delta \geq 0$ its defect, which vanishes precisely when C is smooth.
Now, look at irreducible curves C with $\mathcal{K} \cdot C < 0$.

Proposition 2.1. *If there exists a curve C on S with $\mathcal{K} \cdot C < 0$ and $C \cdot C \geq 0$, all plurigenera of S are zero. If S is a surface with at least one non-vanishing plurigenus and C is a curve on S with $\mathcal{K} \cdot C < 0$, the curve C is an exceptional curve of the first kind, i.e. C is a smooth rational curve with $C \cdot C = -1$.*

Proof: Let D be a pluricanonical divisor and separate out the possible part of C it contains: $D = aC + R$. Then $m\mathcal{K} \cdot C = D \cdot C = aC \cdot C + R \cdot C \geq aC \cdot C$. Since this is ≥ 0 in the first case, the plurigenera must all vanish. In the second case, if $\mathcal{K} \cdot C \leq -2$ the adjunction formula gives $C \cdot C \geq 0$ and we again have a contradiction. So $\mathcal{K} \cdot C = -1$ and the adjunction formula shows that C is an exceptional curve of the first kind. □

The exceptional curves of the first kind are easy to deal with. Such a curve can be blown down to a point q in a new smooth surface T (Castelnuovo's contractability criterion). Conversely, one can blow up any point in S and the result is again a smooth surface T. Topologically T is the connected sum of S and $\mathbf{P}_2$ with orientation reversed (the neighbourhood of the exceptional line in T looks like the neighbourhood of a line the complex projective plane with the orientation reversed.) It follows that

$$b_2(T) = b_2(S) + 1, \quad e(T) = e(S) + 1, \quad \tau(T) = \tau(S) - 1$$

and hence the arithmetic genus $\chi(S) = 1/4\,(e(S) + \tau(S))$ is invariant. A simple application of Hartog's theorem shows that the plurigenera are invariant as well. One can show that any bimeromorphic map between surfaces can be decomposed as a sequence of such blowings down and their inverses. We conclude:

Proposition 2.2. *The plurigenera and the arithmetic genus are bimeromorphic invariants.*

If we succesively blow down all exceptional curves we reach a surface which by definition is (relatively) minimal. Such a model need not be unique, but in case at least one plurigenus is positive, one can show that it is unique.
Going back to pluricanonical bundles, let us look at the pluricanonical maps f_{mK_S}, defined if $P_m > 0$. The maximal dimension of the image for varying m is called the *Kodaira dimension* $\kappa(S)$. By definition $\kappa(S) = -\infty$ if all plurigenera are 0. We have seen that a surface with a curve C for which $\mathcal{K} \cdot C < 0$ either can be blown down or all the plurigenera of S vanish.

A divisor D with $D \cdot C \geq 0$ for all curves C is called nef.

So we can paraphrase the preceding proposition as follows

Suppose S is a surface whose canonical bundle is not nef. Then either S is not minimal or $\kappa(S) = -\infty$.

Later we need a characterisation of the Kodaira dimension. In fact, we can define the Kodaira dimension for any line bundle $\mathcal{L}$ on a variety X as the maximal dimension of $\dim f_{\mathcal{L}^{\otimes m}}(X)$ for growing m. We denote this number by $\kappa(X,\mathcal{L})$.

Lemma 2.3. *We have bounds*

$$C' m^{\kappa(X,\mathcal{L})} \leq \dim H^0(X,\mathcal{L}^{\otimes m}) \leq C'' m^{\kappa(X,\mathcal{L})}$$

characterizing the Kodaira dimension. Here C' and C'' are positive constants

3 Classification

The Kodaira-dimension is one of the basic invariants used to classify surfaces. To explain the main result, we need some terminology.

A *rational surface* is a surface which is bimeromorphic to $\mathbf{P}^2$. Apart from $\mathbf{P}^2$ itself there are the Hirzebruch surfaces $\mathbf{F}_n$, $n = 0, 1, \ldots$. These are defined as the total space of the projective bundle over $\mathbf{P}^1$ obtained from $\mathcal{O} \oplus \mathcal{O}(n)$. Only $\mathbf{F}_1$ has an exceptional curve. The other surfaces are minimal and there are no other minimal rational surfaces.

A *ruled surface* is a $\mathbf{P}^1$-bundle over a curve.

A *properly elliptic surface* is a surface of Kodaira-dimension 1 admitting an elliptic fibration, i.e. a holomorphic map $f : S \to C$ with general fibre an elliptic curve.

A *bielliptic surface* is a surface with $b_2 = 2$ admitting a holomorphic, locally trivial fibre bundle structure over an elliptic curve with fibre an elliptic curve. These are all of the form $E \times C/G$, with E and C elliptic, $G \subset C$ a finite group of translations acting on E not only by translations. There are only 7 possible groups G (see [Troika, p.148]) and all surfaces are algebraic.

A *Kodaira-surface* is a surface with $b_1 = 3$ and admitting a holomorphic, locally trivial fibre bundle structure over an elliptic curve with fibre an elliptic curve or quotients of these under the action of a finite group which acts freely (for these $b_1 = 1$). Kodaira surfaces are not algebraic.

A *K3-surface* is a simply connected surface with trivial canonical bundle. Examples: (i) Kummer surfaces, i.e. quotients of tori by the natural involution $z \mapsto -z$, or (ii) degree 4 surfaces in $\mathbf{P}^3$.
A K3-surface is not necessarily algebraic.

An *Enriques surface* is a surface with $p_g = b_1 = 0$ and $\mathcal{K}^{\otimes 2}$ trivial. The universal cover is a K3-surface which doubly covers the surface. All Enriques surfaces admit elliptic fibrations $S \to \mathbf{P}^1$ with precisely two double fibres. Examples of Enriques surfaces are degree 6 surfaces in $\mathbf{P}^3$ passing doubly through the edges of a tetrahedron (classical Enriques construction). All of them arise in this way or as degenerations of such surfaces; in particular they are all algebraic.

A *surface of class VII* is a surface with $\kappa = -\infty$ and $b_1 = 1$. Such surfaces are non-algebraic.

A *surface of general type* is by definition a surface with maximal Kodaira dimension 2. Examples: surfaces of degree ≥ 5 in $\mathbf{P}^3$.

Theorem 3.1 (Enriques-Kodaira classification)**.** *Every minimal surface belongs to one of the following types:*

1. *Minimal rational or ruled surfaces* ($\kappa = -\infty$, *algebraic),*
2. *Minimal class VII surfacess* ($\kappa = -\infty$, *non-algebraic),*
3. *Tori, K3-surfaces, Enriques surfaces, bielliptic surfaces* ($\kappa = 0$, *Kähler),*
4. *Kodaira surfaces* ($\kappa = 0$, *non-Kähler),*
5. *Minimal properly elliptic surfaces* ($\kappa = 1$*).*
6. *Surfaces of general type* ($\kappa = 2$.)

When we deal with the topology we need a classification of the simply connected algebraic surfaces. We need:

Corollary 3.2. *Simply connected minimal Kähler surfaces can be classified as follows:*

(a) $\mathbf{P}^2$ *resp.* $\mathbf{F}_n$, $n = 0, 2, 3\ldots$ ($\kappa = -\infty$ *and* $c_1^2 = 9$, $c_2 = 3$ *resp.* $c_1^2 = 8$, $c_2 = 4$*).*

(b) *K3-surfaces* ($\kappa = 0$ *and* $c_1^2 = 0$, $c_2 = 24$*)*

(c) *Simply connected and minimal properly elliptic surfaces.* ($\kappa = 1$, $c_1^2 = 0$*).*

(d) *Simply connected and minimal surfaces of general type.* ($\kappa = 2$, $c_1^2 > 0$*).*

Whereas classes (a) and (d) are always algebraic, surfaces in classes (b) and (c) can always be deformed into algebraic ones. We recall that $S = X_0$ can be deformed into S' if there are finitely many surjective holomorphic maps of maximal rank $f_j : X_j \to T_j$, $j = 1, \ldots n$ of smooth threefolds to connected smooth curves T_j (we say that f_j is a *family of surfaces over a curve*) such that S is a fibre of the first family f_1, S' a fibre of the last family f_n and two families f_j, f_{j+1}, $j = 1, \ldots, n-1$ have at least one fibre in common. Since each family f_j is differentiably a fibre bundle we conclude:

Surfaces which can be deformed into one another are mutually oriented diffeomorphic.

This applies in particular to K3-surfaces (see [Troika, Chapter VIII]) and so

The K3-surfaces form the only one diffeomorphism type of simply connected Kähler surfaces with $\kappa = 0$.

As for the elliptic surfaces, this is certainly not true, since there are many possible Euler numbers.

Example 3.3. Let $S^k \subset \mathbf{P}^1 \times \mathbf{P}^2$ the zero set of a generic bihomogeneous polynomial of bidegree $(1+k, 3), k = 0, 1, \ldots$. This is a simply connected surface and the projection onto the first factor gives it the structure of a minimal elliptic fibration over $\mathbf{P}^1$. For any pair (p, q) of relatively prime integers we can perform the logarithmic transforms of multiplicity p and q over two distinct points over which the fibre is smooth [Troika, p.164]. The resulting simply connected elliptic surface $S^k_{p,q}$ has two multiple fibres pF_p and qF_q. The canonical bundle formula [Troika, p.161] gives (here F is a general fibre):

$$\mathcal{K} \cong (k-1)F + (p-1)F_p + (q-1)F_q.$$

One concludes that for $k = 0, \min(p, q) = 1$ we have a rational surface, that for $k = p = q = 1$ we get a K3-surface and that for all other values the surface is properly elliptic with invariants

$$p_g(S^k_{p,q}) = k, \quad c_2(S^k_{p,q}) = 12(1+k).$$

It is well known that any surface with these invariants can be deformed into one of the surfaces $S^k_{p,q}$ and so we obtain

The diffeomorphism types of minimal simply connected $\kappa = 1$ surfaces are precisely those of $S^k_{p,q}$ except if $k = 0, \min(p, q) = 1$ or $k = p = q = 1$.

The surfaces $S^0_{p,q}$, $p \geq 2$, $q \geq 2$ are elliptic $\kappa = 1$ surfaces with $q = p_g = 0$ and are called *Dolgachev surfaces*. We come back to the topology in Chapter V.

4 Tools: Riemann-Roch, Vanishing Theorems, Algebraic Index Theorem

The Riemann-Roch theorem is an extension of Noether's formula:

Theorem 4.1 (Riemann-Roch)**.** *For a line bundle $\mathcal{L}$ on a surface S we have:*

$$\begin{aligned}\chi(\mathcal{L}) &:= \dim H^0(\mathcal{L}) - \dim H^1(\mathcal{L}) + \dim H^2(\mathcal{L}) \\ &= \frac{1}{2}\mathcal{L} \cdot (\mathcal{L} - \mathcal{K}) + \frac{1}{12}(c_1^2(S) + c_2(S))\end{aligned}$$

For a proof we refer e.g. to [Hir].
Since we are often interested in getting sections for line bundles it is quite helpful to know when higher cohomology groups vanish. Recall that a line bundle is called *ample* if the sections of a positive tensor power embed the surface into projective space. (i.e the map f_{mL} is an embedding for large m). A standard vanishing theorem is:

Theorem 4.2 (Kodaira Vanishing Theorem)**.** *Let $\mathcal{L}$ be an ample line bundle on a compact complex manifold X. Then $H^i(X, \mathcal{K} \otimes \mathcal{L}) = 0$ for $i > 0$.*

Often we use Serre-duality in conjunction with Riemann-Roch:

Theorem 4.3 ((Serre Duality Theorem))**.** *Let X be a compact complex manifold of dimension n. We have:*

$$\dim H^i(X, \mathcal{L}) = \dim H^{n-i}(X, \mathcal{K} \otimes \mathcal{L}^\vee).$$

Example 4.4. Let S be a surface H an ample line bundle and C some curve on S. Then there exist effective divisors in $|mH - C|$ for m large enough. This can be seen as follows: By Serre duality we have $\dim H^2(mH - C) = \dim H^0(C + K - mH) = 0$, since otherwise H would intersect a divisor in $|C + K - mH|$ negatively for large m. Then we can apply Riemann-Roch: $\dim H^0(mH - C) \geq \chi(mH - C) = \frac{1}{2}(mH - C - K) \cdot (mH - C) + \chi > 0$ for m large enough.

A second tool, which is used often, is the so-called Algebraic Index Theorem. To state it, we recall that a **Q**-divisor is just a divisor with rational coefficients and such a divisor is numerically trivial if and only if it has zero intersection product with any divisor. This is the case if and only if its first Chern class is trivial in rational cohomology. Chern classes of line bundles are represented by closed forms of type $(1, 1)$ and since the intersection product has signature $(1, h^{1,1} - 1)$ on the space $H^{1,1}$ spanned by classes of $(1, 1)$-forms we conclude:

Theorem 4.5 (Algebraic Index Theorem)**.** *Let D and E be* **Q***-divisors on an algebraic surface with $D \cdot D > 0$ and $D \cdot E = 0$. Then $E \cdot E \leq 0$ with equality if and only if E is numerically trivial.*

Chapter 2

Mori theory

5 Cone of effective 1-cycles

This aspect of the theory only works well in the case of algebraic surfaces and so in this section we assume that S is algebraic.

On a surface any divisor is at the same time a 1-cycle, but for higher dimensions the notions are dual under intersection pairing. In Mori theory this is essential. It is also essential to work on possibly singular varieties. Two 1-cycles on a (not necessarily smooth) algebraic variety X are said to be numerically equivalent if they have the same intersection product with any divisor on X. Likewise we define numerical equivalence for divisors. This equivalence relation is denoted by $\equiv$. We set

$$\begin{aligned} N_1(X) &= (\{ \text{ 1-cycles on } X\}/\equiv) \otimes \mathbf{R} \\ N^1(X) &= (\{ \text{ divisors on } X\}/\equiv) \otimes \mathbf{R} \\ NE(X) &= \{ \text{ cone generated by effective 1-cycles } \} \\ \overline{NE(X)} &= \text{closure of } NE(X) \text{ in } N_1(X). \end{aligned}$$

Both real vector spaces are finite-dimensional and are dual under the intersection pairing. If X is smooth we have Lefschetz theorem on (1,1)-classes:

$$NS(X) := (\{ \text{ 1-cycles on } X\}/\equiv) \cong H^{1,1} \cap H^2(X,\mathbf{Z}) \text{ (the Néron-Severi group of } S),$$

which shows that the dimensions of the preceding vector spaces are equal to

$$\operatorname{rank} NS(X) \text{ (the Picard number of } X).$$

We can formulate Kleiman's ampleness criterion [Klei] as follows:

Proposition 5.1 (Kleiman's Ampleness Criterion)**.** *Any* $\mathbf{Q}$*–divisor* D *(i.e. a divisor with rational coefficients) defines a linear form on* $N_1(X)$ *with rational coefficients which vanishes on a hyperplane through the origin. Ample divisors are precisely those divisors which have* $\overline{NE(X)} - \{0\}$ *on the strictly positive side of this hyperplane.*

Often one uses the following description of the *nef*-cone:

Corollary 5.2. *The nef-cone is the closure of the ample cone, in particular $L \cdot L \geq 0$ for any nef divisor L.*

In the next section we need:

Lemma 5.3. *Let L be a nef divisor on a surface S and suppose that for some ample divisor H we have $L \cdot H = 0$. Then $L \equiv 0$.*

Proof: By Example 4.4 the system $|mH - C|$ contains a divisor D. Then $0 \leq D \cdot L = (mH - C) \cdot L = -C \cdot L \leq 0$ and so $C \cdot L = 0$ for all curves C. □

6 Rationality theorem and an application

If a divisor D is not nef it defines a hyperplane in $N_1(X)$ for which part of $\overline{NE(X)}$ is strictly on the negative side. So, if H is some ample divisor, in the connecting pencil $H_t = H + tD$ there is a member H_b whose hyperplane just touches the cone $\overline{NE(X)}$.

Theorem 6.1 (Rationality theorem)**.** *If $X = S$ a surface and $D = K_S$ is not nef, there is a rational b such that the hyperplane corresponding to $H + bD$ touches the cone $\overline{NE(X)}$.*

Proof: We follow [Wi, p.13]. For clarity, we recall

$$b := \sup\{t \in \mathbf{R} \mid H_t = H + tK_S \text{ is nef }\}.$$

Set

$$P(v,u) := \chi(vH + uK_S).$$

By Riemann-Roch (Theorem 4.1) this is a quadratic polynomial in v,u. If u and v are positive integers with $(u-1)/v < b$ the divisor $vH + (u-1)K_S$ is ample and so by Kodaira Vanishing (Theorem 4.2) $H^i(vH + uK_S) = 0$ for $i = 1,2$. It follows that $P(v,u) \geq 0$.

Assume now that b is irrational. Number theory tells us that b can be approximated by rational numbers of the form p/q, p and q arbitrarily large integers in such a way that

$$p/q - 1/(3q) < b < p/q.$$

The polynomial $P(kq,kp)$ is quadratic in k. If it is identically zero, $P(v,u)$ must be divisible by $(vp - uq)$. Taking p and q sufficiently large we may assume that this is not the case. For $k = 1,2,3$ the numbers $v = kq$ and $u = kp$ satisfy $(u-1)/v < b$ and hence $P(kq,kp) \geq 0$ for these three values of k. Since a quadric has at most two zeroes, it follows that for at least one pair of positive integers (v,u) with $t_0 := u/v > b$ we have $\dim H^0(vH + uK_S) > 0$. So there is an effective $\mathbf{Q}$-divisor $L := H_{t_0} = \sum a_j\Gamma_j$, $a_j > 0$. Now H_{t_0} is not nef. Since L is effective, it can only be negative on the Γ_j. But then one can subtract off a rational multiple of K_S from H_{t_0} to get H_b and so b would be rational contradicting our assumption. □

As an application we give a sketch of the classification theorem for algebraic surfaces with $\kappa = -\infty$. First, we prove that minimal surfaces with K not nef are rational or ruled and hence have $\kappa = -\infty$:

Proposition 6.2. *A minimal algebraic surface with K not nef is either a ruled surface or $\mathbf{P}^2$.*

Proof: Clearing denominators we have a nef divisor

$$L = vH + uK_S,\ b = u/v = \sup\{t \in \mathbf{R} \mid H_t = H + tK_S \text{ is nef }\}.$$

We have $L \cdot L \geq 0$ (see Coroll. 5.2). We conclude from Kleiman's ampleness criterion 5.1 that $mL - K_S$ is ample for all sufficiently large m. Serre duality (Theorem 4.3) implies that $\dim H^2(mL) = \dim H^0(-(mL - K_S)) = 0$ and so by Riemann-Roch 4.1

$$\dim H^0(mL) \geq \chi(mL) = \chi(S) + mL \cdot (mL - K_S)/2.$$

We can distinguish two cases:
i) $L \cdot L > 0$. Riemann-Roch then shows that $\dim H^0(mL)$ grows like $C \cdot m^2$ and hence the sections of a large multiple of L give a map $f : S \to S'$ onto a surface S' (2.4).
ii) $L \cdot L = 0$. In this case an application of Lemma 5.3 shows that either $L \equiv 0$ or $L \cdot H > 0$ and from $0 = 1/u(L \cdot L) = L \cdot (H + bK_S)$ we infer that $L \cdot K_S < 0$ and so $\dim H^0(mL)$ grows like a linear function of m and we get a map $f : S \to C$ onto a curve C.

In the first case any irreducible curve D which $L \cdot D = 0$ is an exceptional curve of the first kind. Indeed, from the definition of L we see that $K_S \cdot D < 0$, while the algebraic index theorem (Theorem 4.5) applied to L and D shows that $D \cdot D < 0$. In combination with the adjunction formula (1.1) this shows that D has to be an exceptional curve of the first kind. By assumption these don't exist and so $L \cdot D > 0$ for all curves D and L is ample, contradicting the fact that K_S is not nef. It follows that we are in the second case. If $L \equiv 0$ we see that $-K_S$ is ample and it is a nice exercise to show that a minimal surface with $-K_S$ ample is the projective plane.

In the second case we observe that f can also be given by the divisor $L' = L - L_{\text{fixed}}$ so that $|L'|$ does not have fixed components. Since L is nef and $L \cdot L = 0$, we infer that $L' \cdot L = 0$. Since $|L'|$ moves, it must be nef and from $L' \cdot L = 0$ we conclude that $L' \cdot L' = 0$. But then $|L'|$ cannot have fixed points and hence f is a morphism. If F is a general fibre we must have $L' \cdot F = 0$ and one easily concludes from this that $L \cdot F = 0$. So for any component D of F we have $L \cdot D = 0$ and hence $K_S \cdot D < 0$. The adjunction formula shows that D is a smooth rational curve. Taking the Stein factorization of f we see that S is a ruled surface. □

Conversely, we need to show that if K is nef, the Kodaira-dimension is not $-\infty$. First we recall (Cor. 5.2) that $K \cdot K \geq 0$. Riemann-Roch (plus Serre duality) for mK reads:

$$\dim H^0(mK) + \dim H^0(-(m-1)K) \geq \frac{1}{2}m(m-1)K \cdot K + 1 - q + p_g.$$

Fix a very ample H. We distinguish two cases:
a) $K \cdot H > 0$. Then $H^0(-(m-1)K) = 0$ for $m \geq 2$ and so, we get

$$q = 0 \Longrightarrow P_2 > 0, \quad K \cdot K > 0 \Longrightarrow \kappa = 2.$$

b) If $K \cdot H = 0$ the algebraic index theorem implies that $K \cdot K \leq 0$ and hence $K \cdot K = 0$ and the second part of the index theorem shows that $K \equiv 0$. In particular, if $q = 0$ a multiple of K is trivial and so $\kappa = 0$.
This analysis shows that we only have to look at the cases with $K \cdot K = 0$, $q > 0$. Again, if $p_g > 0$ we are done, so we assume that $p_g = 0$. Noether's formula now gives $b_2 = 10 - 8q$ and so $q = 1$. Next, one needs to do a very detailed analysis of the so called Albanese mapping $S \to C$ which in this case maps S onto an elliptic curve with elliptic fibres. This is done in [Beau, Chapter VI]. The result is:

Theorem 6.3. *Any algebraic surface with K nef,* $p_g = 0$, $q = 1$ *is bielliptic and has* $\kappa = 0$.

This now completes the proof of the classification of algebraic surfaces with $\kappa = -\infty$, but it shows much more:

1. A surface if rational if and only $q = 0$ and $P_2 = 0$ (Castelnuovo's Rationality Criterion).

2. A surface with K nef and $K \cdot K > 0$ is of general type.

3. A minimal algebraic surface with $\kappa = 0$ and $q = 1$ is bielliptic.

I hope that this gives enough evidence of the powerfulness of Mori's theory even in the case of surfaces.

Chapter 3

Reider's Method

7 Reider's Theorem

For this entire section we refer to [Reider 1] for the details and references to literature.

Zero cycles Z and rank 2 vector bundles on surfaces S are intimately related. A result due to Griffiths and Harris says that given a line bundle $\mathcal{L} = \mathcal{O}(L)$ on S and a zero cycle Z, there is a rank two vector bundle $\mathcal{E}$ with $\det(\mathcal{E}) = \mathcal{L}$ and a section e of $\mathcal{E}$ vanishing precisely in Z if and only if Z is *special* with respect to the adjoint system $|K_S + L|$. This means that any divisor in this system passing through a subcycle Z' of Z with $\deg Z' = \deg Z - 1$ passes automatically through Z.

Examples 7.1. (i) $Z = p$ is special if and only if it is a basepoint of the adjoint system,
(ii) $Z = p + q$ is special if and only if the adjoint system does not separate p from q.

Let us compute the invariants of $\mathcal{E}$. Since $\det(\mathcal{E}) = \mathcal{L}$ we find $c_1(\mathcal{E}) \cdot c_1(\mathcal{E}) = L \cdot L$. For the second Chern class we get $c_2(\mathcal{E}) = Z$ (indeed $\mathcal{E}$ has a section vanishing exactly in Z). Recall Bogomolov's deep theorem (cf. [**?**] for a short proof) :

If $c_1(\mathcal{E}) \cdot c_1(\mathcal{E}) > 4c_2(\mathcal{E})[S]$, i.e. if

$$L \cdot L > 4 \deg Z, \tag{3.1}$$

the bundle $\mathcal{E}$ is unstable with respect to any ample line bundle H,

i.e. there exists a locally free rank 1 subsheaf ('line bundle') M of $\mathcal{E}$ with

$$\frac{c_1(\mathcal{E})}{2} \cdot H \leq M \cdot H.$$

This line bundle M fits into a 'destabilizing sequence'

$$0 \to M \to \mathcal{E} \to E \otimes \mathcal{I}_A \to 0,$$

where E is another line bundle and $\mathcal{I}_A$ is an ideal sheaf of a zero cycle (points with multiplicity) (see [loc. cit.]. The inequality for M is equivalent to

$$(M - E) \cdot H \geq 0. \tag{3.2}$$

Observe that for the Chern classes of $\mathcal{E}$ this sequence gives the following expressions:

$$c_1(\mathcal{E}) = L = M + E, \quad c_2(\mathcal{E}) = M \cdot E + \deg A \geq M \cdot E. \tag{3.3}$$

Moreover (loc. cit.)

$$(M - E) \cdot (M - E) > 0. \tag{3.4}$$

In our situation Reider makes the following observation (not entirely obvious):

$$\text{There is an effective divisor in } |E| \text{ containing } Z. \tag{3.5}$$

Let us state Reider's main application (see [Reider 1])

Theorem 7.2 (Reider's Theorem). *Let $\mathcal{L} = \mathcal{O}(L)$ be a nef line bundle on an algebraic surface S.*
(i) *If $L \cdot L \geq 5$ and p is a base point of $|K_S + L|$, there exists an effective divisor E passing through p such that*

$$\begin{aligned} &\textit{either} L \cdot E = 0,\, E \cdot E = -1 \\ &\textit{or } L \cdot E = 1,\, E \cdot E = 0. \end{aligned}$$

ii) If $L \cdot L \geq 10$ and p and q are not separated by the adjoint system for L, there is an effective divisor E through p and q such that

$$\begin{aligned} &\textit{either } L \cdot E = 0 \textit{ and } E \cdot E = -1 \textit{ or } -2 \\ &\textit{or } L \cdot E = 1 \textit{ and } E \cdot E = -1 \textit{ or } 0 \\ &\textit{or } L \cdot E = 2 \textit{ and } E \cdot E = 0 \end{aligned}$$

Sketch of proof. Observe that the inequalities for $L \cdot L$ imply (3.2). We take for E the effective divisor whose existence is stated in (3.5).

We have to check the possible intersection numbers. Let us do this for (i). Look at the plane in the Néron-Severi group $NS(S)$ spanned by $L = M + E$ and $M - E$. Since $L \cdot L > 0$, the index theorem implies that the intersection form on this plane has signature $(1, 1)$ and hence

$$\det \begin{pmatrix} M \cdot M & M \cdot E \\ M \cdot E & E \cdot E \end{pmatrix} \leq 0.$$

So $(M \cdot M)(E \cdot E) \leq (M \cdot E)^2 \leq 1$ (by (3.3)) Now $2M = M - E + L$ and so $4M \cdot M = (M - E) \cdot (M - E) + L \cdot L + 2(M - E) \cdot L$ is positive because of (3.1), (3.2), (3.4) and the fact that L, being a nef divisor is in the closure of the ample cone (Corollary 5.2). It follows that $E \cdot E \leq 0$. On the other hand, since L is nef we get $0 \leq L \cdot E = M \cdot E + E \cdot E$ and so $E \cdot E \geq -M \cdot E \geq -1$. So either $E \cdot E = 0$ (and then $L \cdot E = 1$ or 0, but the last possibility is excluded by the algebraic index theorem) or $E \cdot E = -1$ and $L \cdot E = 0$. □

8 An application: pluricanonical maps

We need the notion of C-isomorphism:

Definition 8.1. Let $\mathscr{L}$ be a line bundle on a surface S. We say that a $f_{\mathscr{L}}$ is a C-isomorphism if it is a birational morphism onto a surface with finitely many rational double points, which is an isomorphism away from the inverse images of these rational double points.

This notion arises naturally for surfaces of general type, because any curve with $K \cdot C = 0$ is a rational curve with self intersection -2 (a so called (-2)-curve) and any maximal connected set of (-2)-curves is mapped onto a rational double point by a pluricanonical map, if this map is a morphism.

The following result due to Reider gives an optimal description of pluri-canonical mappings. The proof is substantially simpler than the previous proofs and has the merit hat it treats all special cases with one method. One thing should be stressed: apparently the Bombieri method of producing 1-connected divisors (a technical notion- we don't go into details) seems totally absent in this approach. However, Catanese pointed out that the presence of (-2)-curves forces to use a clever combination of Reider's and Bombieri's methods (see [Cat 3]).

Theorem 8.2. *Let S be a surface of general type. Let $f_m = f_{mK}$ be the m-th pluricanonical map. We have:*
i) *f_m is a C-isomorphism for $m \geq 5$.*
ii) *f_4 is a C-isomorphism except for $K \cdot K = 1$.*
iii) *f_3 is a morphism for $K \cdot K \geq 2$ and an embedding for $K \cdot K \geq 3$.*
iv) *f_2 is a morphism for $K \cdot K \geq 5$ and a birational morphism for $K \cdot K \geq 10$ except when S has a genus two fibration.*

We only give the proof that f_m is a morphism for $K \cdot K$ in the given range. This proof is however typical for the sort of reasoning that is used.
We simply apply the previous theorem to $L := (m-1)K$. One assumes that f_m is not a morphism. The inequality we need is $(m-1)^2 K \cdot K \geq 5$. The conclusion is the existence of effective divisors E with $K \cdot E = 0, E \cdot E = -1$ or $(m-1)K \cdot E = 1$, $E \cdot E = 0$. The adjunction formula excludes both. This gives the required contradiction.
The proof of the other assertions goes similarly, but there are technical complications due to the presence of -2-curves. See [Cat 3] for complete details.

Another important application of Reider's method is the study of the adjoint mapping for very ample line bundles. It gives back and completes results previously obtained by Sommese and Van de Ven.

Chapter 4

Surface geography

9 Overview of surface geography

There are several restrictions on a pair (c_1^2, c_2) of positive natural numbers to occur as a pair of Chern numbers for a minimal surface of general type (it is well known that these numbers must be strictly positive [Troika, p.208–209]. In fact:

$$c_1^2 + c_2 \text{ is divisible by } 12 \text{ (Noether condition)}$$
$$c_1^2 - 1/5c_2 + 36/5 \geq 0$$
$$c_1^2 \leq 3c_2.$$

The first follows from Noether's formula, the second is a consequence of Noether's inequality $p_g \leq \frac{1}{2}c_1^2+2$, while the third is the Bogomolov-Miyaoka-Yau inequality. See [Troika, Chapter VII] for details and historical remarks. In fig. 9 one can see the region in the (c_1^2, c_2)-plane given by these inequalities.
We shall use the self-evident terminology of *Noether-line* and *BMY-line* for the borderlines. Note another interesting line: the line with $c_1^2 = 2c_2$ is the line with signature 0. Above it surfaces have positive signature and below it the signature is negative. Let us call these regions *positive signature region*, resp. *negative signature region*. Surface geography consists of mapping out subregions where surfaces have specific properties. There are the following geographic results:

Theorem 9.1. i) *In the region*

$$\frac{1}{5}(c_2 - 36) \leq c_1^2 < 3c_2 - 1388$$

every Chern number satisfying the Noether condition can be realized by a minimal surface.
ii) *In the region given by (recall that $\chi = 1/12(c_1^2 + c_2)$)*

$$\frac{88}{179}c_2 + 209.125\chi^{2/3} < c_1^2 < \frac{4661}{1726}c_2 - 1353.26\chi^{2/3}$$
$$c_2 > \text{large constant},$$

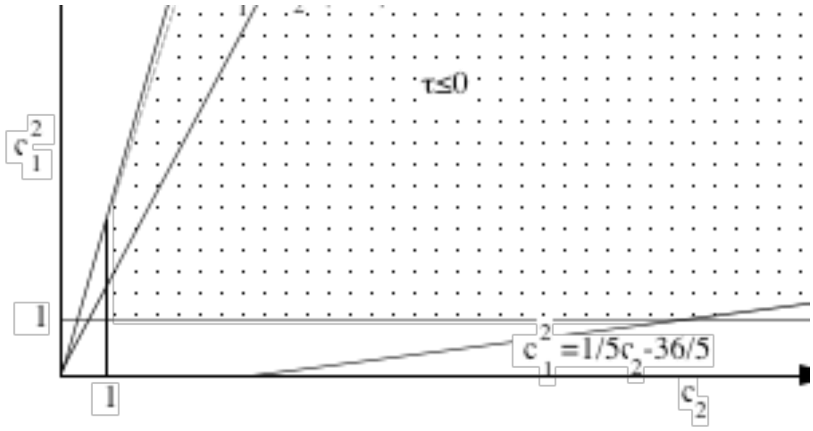

Figure 4.1: The inhabitated region

we can realize all pairs (c_1^2, c_2) *satisfying the Noether condition by a simply connected surface.*
A similar assertion holds without a further condition on c_2 *for the part of the negative signature region given by*

$$\frac{1}{5}(c_2 - 36) \leq c_1^2 \leq 2c_2 - \frac{3}{4}\chi^{2/3}.$$

In particular all rational slopes $c_1^2/c_2 \in [1/5, 4661/1726)$ *occur and can be realized by simply connected surfaces (observe that* $88/179 = 0.49..$ *and* $4661/1726 = 2.70..$).
iii) *Surfaces on the Noether line and on the line* $c_1^2 - 1/5c_2 + 30/5 = 0$ *immediately above it are all double covers of Hirzebruch surfaces and the possible branch-loci are classified.*
iv) *All rational slopes* $c_2/c_1^2 \in [1/5, 3]$ *occur and can be realized by irregular surfaces.*

A proof of (i) can be found in [Chen 1], where a slightly more precise statement is proved. These results supersede earlier work by Persson [Per 1, Xiao 1, Chen 1]. For (ii) we refer to [Chen 1, Per 2]. Item (iii) is work of Horikawa and is reported on in [Troika]. Item (iv) follows from (i) plus the fact that surfaces on the borderline $c_1^2 = 3c_2$ exist. There is a much simpler proof due to Sommese, which we will give below.

Apart from these general results there are many scattered results for which we refer the reader to [Per 2]. Below we shall review two spectacular but simple constructions, one giving a surface on the BMY-line, the other giving a simply connected surface with $p_g = 0$ in which differential topologists have become interested lately (Barlow's surface).

10 Hirzebruch's Example with $c_1^2 = 3c_2$

We refer to [BHH, Kap. 3] for details on what follows.

Start with k homogeneous linear forms $l_1, \ldots, l_k$ in three variables z_0, z_1, z_2 defining k lines in $\mathbf{P}^2$ and consider the abelian field extension

$$K\left(\sqrt[n]{(l_2/l_1)}, \ldots, \sqrt[n]{(l_k/l_1)}\right) \supset K := \mathbb{C}(z_1/z_0, z_2/z_0).$$

The group of this extension is $G := (\mathbf{Z}/n\mathbf{Z})^{k-1}$ and it corresponds to a covering $S' \to \mathbf{P}^2$ ramified in the k given lines. The surface S' has singularities above points where three or more of these lines meet. Blowing up $\mathbf{P}^2$ at these points we get $\sigma : \tilde{\mathbf{P}}^2 \to \mathbf{P}^2$ and we can form the covering

$$f : S \to \tilde{\mathbf{P}}^2$$

with group G ramified in the proper transform of these curves. Then S is smooth. If p is a point where r lines meet, there are n^{k-1-r} isomorphic curves C_p over this point in S. Each of these is mapped to the corresponding exceptional curve E_p in $\tilde{\mathbf{P}}^2$ with degree n^{r-1}. There are r branch points on E_p over each of which one has n^{r-2} points. The Riemann-Hurwitz formula gives the Euler-number:

$$e(C_p) = n^{r-1}(2 - r) + rn^{r-2}.$$

Above each point of a line where no other lines intersect there are $n^{k-1}/n = n^{k-2}$ points. Above the points were only two of the lines meet there are exactly n^{k-3} points. So, if we let L be the union of the k lines, $\operatorname{sing} L$ be the singular locus of L and

$$t_r := \text{number of points where } r \text{ lines meet}$$
$$f_1 := \sum r t_r, \quad f_0 := \sum t_r$$

we get

$$\begin{aligned}
&e(S' - \operatorname{sing} S') \\
&\quad = n^{k-1} e(\mathbf{P}^2 - L) + n^{k-2} e(L - \operatorname{sing} L) + n^{k-3}(t_2) \\
&\quad = n^{k-1}(3 - 2k + \sum (r-1)t_r) + n^{k-2}(2k - \sum r t_r) + n^{k-3} t_2, \\
e(S) &= e(S' - \operatorname{sing} S') + \sum_{r \geq 3} n^{k-1-r} t_r \left(n^{r-1}(2 - r) + r n^{r-2}\right) \\
&= n^{k-1}(3 - 2k + f_1 - f_0) + 2n^{k-2}(k - f_1 + f_0) + n^{k-3}(f_1 - t_2).
\end{aligned}$$

We now specify to the following arrangement: take three distinct non-concurrent lines l, m, n and add the lines through the three points $l \cap m$, $l \cap n$, $m \cap n$ and a fourth point not on $l \cap m \cap n$. So we have $t_2 = 3$, $t_3 = 4$ and the other t_r vanish. Observe that the arrangement is essentially unique. We let $n = 5$ and we abuse notation by letting S be the surface obtained by the preceding construction. We find

$$e(S) = 15 \cdot 5^3.$$

To calculate $K \cdot K$ we use the formula for the canonical divisor on S (H is a line on $\mathbf{P}^2$):

$$\begin{aligned} K &= f^* \left(\sigma^* K_{\mathbf{P}^2} + \sum E_p + 4/5(\sum E_p + \sigma^* H - 3 \sum E_p)\right) \\ &= f^*(\sigma^*(9/5H) - 3/5 \sum E_p). \end{aligned}$$

and we get

$$K \cdot K = 5^2(81 \cdot 5^2 - 36 \cdot 5^2) = 45 \cdot 5^3.$$

Comparing the two numbers we see indeed that S is on the line $c_1^2 = 3c_2$.

Remark 10.1. i) Ishida [I] calculated the irregularity: $q = 30$.
ii) He also found 4 groups of order 5^2 which operate freely on S giving quotients with

$$c_1^2 = 3c_2 = 225; \quad q = 10, 6, 4, 0$$

and a group of order 5^3 operating freely with quotient having invariants

$$c_1^2 = 3c_2 = 45; \quad q = 2.$$

iii) With arrangements one can find at least two more examples on the borderline:

- L= the twelve lines in a Hesse pencil with $c_1^2 = 3c_2 = 48 \cdot 3^{10}$,
- L= dual of the configuration of the 9 inflection points on a cubic and the connecting lines with $c_1^2 = 3c_2 = 333 \cdot 5^6$.

iv) There is an obvious fibration $Y \to C_p$, which can be used to construct surfaces with given slopes $c_1^2/3c_2$ in the next section.

11 Surfaces with given slope

The following theorem is due to Sommese, see [Sommese].

Theorem 11.1. *Every rational point in* $[1/5, 3]$ *occurs as the slope of some irregular algebraic surface of general type.*

Proof: We only consider the interval $[2, 3]$, the remaining interval can be treated similarly. See e.g. [BHH, 156–157],
Let

$$s = c_1^2/c_2 \text{ (the Chern-slope)} .$$

The construction we use is based on the existence of a fibration $S \to C$ with connected fibres F over a curve C of genus ≥ 1. Then S is irregular (pull back 1-forms from the curve to the surface). If we pull back this fibration by a ramified covering $C' \to C$ of degree d such that the ramification divisor on C' has degree ρ we get a new irregular surface S'. Its Chern-slope is easily calculated:

$$\begin{aligned} s(S') &= \frac{d \cdot c_1^2(S) - 2\rho \cdot (e(F)}{d \cdot e(S) - \rho \cdot e(F)} \\ &= s(S) + (2 - s(S)) \cdot \frac{-\rho \cdot e(F)}{d \cdot e(S) - \rho \cdot e(F)}. \end{aligned}$$

If now $g(C) \geq 1$ and $s(S) \geq 2$ every slope between 2 and $s(S)$ can be realized as follows: take

$$C' \xrightarrow{\alpha} C'' \xrightarrow{\beta} C$$

with α a double cover with $2y$ branchpoints and β an unramified cover of degree x. Note these do exist! Now $d = 2x$ and $\rho = 2y$ and we find

$$s(S') = s(S) + (2 - s(S)) \cdot \frac{-y \cdot e(F)}{x \cdot e(S) - y \cdot e(F)},$$

So if we are given the rational number p/q, $0 \leq p < q$ we can take $x = -(q-p)e(F)$ and $y = pe(S)$ and we see that the coefficient of $2 - s(S)$ is exactly p/q.
We apply this to Hirzebruch's surface, which has a fibration over a curve of genus 6. □

12 Barlow's simply connected surface

There is only one known example of a simply connected surface of general type with $p_g = 0$. It has been constructed by Barlow and recently has attracted the attention of the differential-topologists. See [Barlow] for the construction of the surface and [OV 2] for the relation with differential topology .
Let us outline the construction of this surface. The basic observation is as follows.

Suppose a finite group G acts on a simply connected variety X and let $H :=$ group generated by $\{g \in G \mid g \text{ has a fixed point}\}$, then $\pi_1(X/G) \cong G/H$. In particular, if G is generated by elements having a fixed point, X/G is simply connected.

We want to apply this to a simply connected non-singular surface X in a particularly simple case, where each $g \in G - \{1\}$ has a finite number t of fixed points with isotropy group $\mathbf{Z}/2\mathbf{Z}$. The quotient X/G has t ordinary double points. We resolve them and call the resulting surface S . If G has order n we easily find:

$$\begin{aligned} c_1^2(S) &= 1/n \cdot c_1^2(X) \\ \chi(S) &= 1/n \cdot (\chi(X) + 1/4t). \end{aligned}$$

If X has no exceptional curves, K_X is nef and one can easily show that K_S then is also nef and hence S must be minimal.

Example 12.1. Suppose we have a minimal simply connected X with $p_g = 4$, $c_1^2 = 10$ which admits an action of the dihedral group $\mathbf{D}_{10} = \langle \alpha, \beta \rangle$ (α is an involution and β has order 5) such that
1) β acts freely,
2) α has finite fixed locus then S is a simply connected surface with $p_g = 0$, $c_1^2 = 1$.

Indeed, $\mathbf{D}_{10}$ is generated by 5 conjugate involutions and so S is simply connected. Now we form the quotient in two steps:

$$X \to X/\langle \beta \rangle \to X/\mathbf{D}_{10}.$$

The first map is a five-fold unramified covering and so is smooth with invariants $\chi = 1$, $c_1^2 = 2$, $q = 0$ and it follows that $p_g = 0$ and a fortiori this holds for S. Since $\chi(S) > 0$ as for any surface of general type, we find from the preceding formulas that $\chi(X/\mathbf{D}_{10}) = 1 = \frac{1}{2}(1 + 1/4t)$ and hence $t = 4$ and $c_1^2(S) = 1$.
To construct X Barlow proceeds as follows: look at the quintic surface Y in $\mathbf{P}^4$ with homogeneous coordinates $z_0, \ldots, z_4$ given by

$$\sum_j z_j = 0$$
$$\sum_j z_j^5 - 5/4 \sum_j z_j^2 \sum_j z_j^3 = 0.$$

This surface has an obvious action of the group $\mathfrak{A}_5$ and there are 20 nodes on this surface which is the $\mathfrak{A}_5$-orbit of $(2, 2, 2, -3 - \sqrt{-7}, -3 + \sqrt{-7})$. This surface is the canonical model of the Hilbert modular surface X for the 2-congruence subgroup Γ of $SL_2(\mathcal{O}_K)$, $K = \mathbf{Q}(\sqrt{21})$. This surface X is known to be simply connected and has the correct invariants. The group $SL_2(\mathcal{O}_K)/\Gamma$ is isomorphic to $\mathfrak{A}_5$ and acts on X. The surface X is the double cover of Y branched in the 20 nodes and the covering involution σ can be identified as the one coming from $\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}$ where $\epsilon = 55 + 12\sqrt{21}$ is a positive unit in K. The involution commutes with the $\mathfrak{A}_5$-action and this explains the action on Y. Now take the dihedral group generated by the 5-cycle in $\mathfrak{A}_5$ and the involution $(14)(23)\sigma$. One checks easily that the 5-cycle acts freely on Y and hence on X. The two-cycle fixes a line $L := \{(0, \lambda, -\lambda, \mu, -\mu) \mid \lambda, \mu \in \mathbb{C}\}$ on the surface and the set S consisting of the five intersection points of the line $z_1 - z_4 = 0$, $z_2 - z_3 = 0, \sum z_j = 0$ with the surface Y. The line L splits on X into two curves which are interchanged by the involution σ. These curves are fixed by $a := (14)(23)$ or by $a \cdot \sigma$. In the first case the involution $a \cdot \sigma$ has fixed locus contained in the finite set above S and in the second case this holds for a itself. A closer inspection shows that $a \cdot \sigma$ has a finite fixed point set (but for our example it is only necessary to know that some involution exists!).

Chapter 5

Topology of simply connected algebraic surfaces

13 Severi's question

A famous question posed by Severi (1954) is :

> Is every algebraic surface homeomorphic to the projective plane biholomorphic to it?

Since p_g is a topological invariant, one can pose the related question:

> Is every surface with $p_g = q = 0$ rational?

This question was already solved negatively by Castelnuovo in a 1896-paper [Cast] in which he presented what is now called Castelnuovo's Rationality Criterion (see the end of Chapter 6: a surface is rational if and only if $P_2 = q = 0$) plus –a couple of properly elliptic non-rational surfaces with $p_g = q = 0$. Also Enriques' examples of Enriques surfaces were shown to belong to this class.
It is an amusing exercise to show that Castelnuovo's Criterion is optimal in the sense that any surface with $P_2 = q = 0, \quad b_2 = 1$ must be isomorphic to the plane [Troika, Chapter V, Theorem 1.1]. Unfortunately it is not clear a priori whether the vanishing of P_2 is a topological property or not and we need further technology if we want to find an answer to Severi's question. It turns out that Yau's solution to the Calabi Conjectures (1977) finally provided an affirmative answer to Severi's question. We refer to loc.cit. for details.

14 Topological classification

By Freedman's fundamental work ([Freedman]) it follows that the oriented homeomorphism type of a simply connected four-manifold having at least one smooth structure is completely determined by the integral equivalence class of the unimodular intersection form on $H^2(S, \mathbf{Z})$. Intersection forms of simply connected algebraic surfaces are always indefinite with the only exception of $\mathbf{P}^2$. The positive answer to Severi's problem deals with this case (see the previous section) and so we can assume that the form is indefinite. Such forms are uniquely classified by their rank($= b_2$), their signature τ and their parity (a form is even if all self-intersections are even and odd otherwise). See [Serre] for details. With the usual notation, any such form is isometric to

$$p\langle 1\rangle \oplus q\langle -1\rangle, \quad \tau = p - q \quad \text{(odd forms)}$$
$$rpH \oplus q(\pm E_8), \quad \tau = \pm 8q \quad \text{(even forms)}.$$

There is an easy way of determining the parity by Wu's formula [MS, p.132]:

The intersection is even precisely when K is 2-divisible in in the Néron-Severi group. (5.1)

Examples 14.1. *1.* All surfaces with odd intersection form must be homeomorphic to connected sums of copies of the projective plane and of the projective plane with the orientation reversed.
2. Any K3-surface has even intersection form of signature -16 and since $b_2 = 22$ the intersection form must be

$$L := H \oplus H \oplus H \oplus \oplus(-E_8) \oplus -E_8).$$

3. Any quadric has H as intersection form. Since any Hirzebruch surface $\mathbf{F}_n$ with n odd is a deformation of a quadric [Troika, p.202.] H is the intersection form also for those. All $\mathbf{F}_n$ with n odd are deformations of $\mathbf{F}_1$, i.e. of the projective plane once blown up and hence diffeomorphic to to $\mathbf{P}^2\#\overline{\mathbf{P}^2}$ with intersection form $\langle 1\rangle \oplus \langle -1\rangle$.
4. From the previous examples we see that the direct sum of $3a + b$ copies of H and $2a$ copies of $\pm E_8$ can be realized as the connected sum of b quadrics and a K3-surfaces or K3's with reversed orientation.

> **Question**: Are all simply connected oriented differentiable 4-manifolds with even intersection form oriented homeomorphic to connected sums of quadrics and K3's or K3's with orientation reversed?

This would follow from

Conjecture 14.2 (The 11/8 Conjecture)**.** If M is a simply connected oriented differentiable four-manifold with even intersection form we have the inequality

$$b_2 \geq \frac{11}{8}|\tau|.$$

Indeed, for differentiable four-manifolds Rochlin's theorem says that the signature for even forms is divisible by 16 and we can take $a := |\tau|/16$ copies of a K3-surface (if the signature is negative) or of a K3 with orientation reversed (if

$\tau \geq 0$) and of $b := 1/16(8b_2 - 11|\tau|)$ copies of a quadric. The fact that b is an integer follows from the fact that the unimodular intersection form is isometric to a direct sum of copies of H and $\pm E_8$ so that b_2 is even. For almost complex surfaces we have $b_2 = c_2 - 2$ and $\tau = 1/3(c_1^2 - 2c_2)$ and we find that the 11/8-conjecture in this case boils down to

$$\begin{cases} 11c_1^2 + 2c_2 - 48(= 48b) \geq 0 & \text{if } \tau \leq 0 \\ -11c_1^2 + 46c_2 - 48(= 48b) \geq 0 & \text{if } \tau \geq 0. \end{cases}$$

For algebraic surface the intersection form being even implies that the surface is minimal and by the classification results (see Corollary 3.2) we have $c_1^2 \geq 0$ and $c_2 \geq 3$ and so if the signature is negative the conjecture follows. If the signature is positive, we need the far from trivial Bogomolov-Miyaoka-Yau inequality (see chapter 4) to show that in this case the inequality holds. So we have

> Any simply connected algebraic surface with even intersection form is homeomorphic to a connected sum of quadrics and K3's or K3's with orientation reversed.

Let us go back to the surfaces S^k_{pq} from section 3. From the canonical bundle given there we find that the intersection form is even if and only if k, p, q are all odd. The other invariants show that

$$\begin{array}{lll} S^k_{p,q} \overset{\text{homeo}}{\sim} \overset{\frac{1}{2}(k+1)}{\#} (\text{K3-surface}) \overset{\frac{1}{2}(k-1)}{\#} (\mathbf{P}^1 \times \mathbf{P}^1) & & \text{if } k, p, q \text{ all odd} \\ S^k_{p,q} \overset{\text{homeo}}{\sim} \overset{2k+1}{\#} (\mathbf{P}^2) \overset{10k+9}{\#} \overline{\mathbf{P}^2} & & \text{otherwise.} \end{array}$$

In particular, the Dolgachev surfaces $S^0_{p,q}$ are all homeomorphic to $\mathbf{P}^2$ blown up in 9 points.

5. As a final example, consider a question posed to me by Ronny Lee, Ron Stern and Bob Gompf:

> If we reverse the orientation of a (simply connected) complex surface, does there exist a compatible complex structure on the new manifold?

There are obvious examples where the answer is 'no', like $\mathbf{P}^2$ and others, where the answer is 'yes', like $\mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbf{P}^2$ blown up in a point. To put this question into perspective, we shall recall [Troika, p. 130] the criterion for an oriented four-manifold to have an almost complex structure:

Theorem 14.3. *An oriented four-manifold M admits an almost complex structure with given $c_1 = h \in H^2(M, \mathbf{Z})$ if and only if*
i) *h is equal to the second Stiefel- Whitney class in mod 2-cohomology,*
ii) $h^2 = 3\tau(M) + 2e(M)$.

We look first at simply connected four-manifolds with odd intersection form, i.e the connected sums of a projective planes and b planes with orientation reversed. A little number theory shows that almost complex structures are only possible if and only if a is odd and positive. This shows :

> $S := \overset{a}{\#} (\mathbf{P}^2) \overset{b}{\#} \overline{\mathbf{P}^2}$ as well as $\overline{S}$ have almost complex structures if and only if both a and b are positive and odd.

We now look at complex structures. From the classification (Corollary 3.2) we see that $3a - b = c_1^2 \geq 0$ and so a necessary condition for a positive answer to the preceding question is

$$1/3 \cdot b \leq a \leq b. \tag{5.2}$$

In particular we see that the answer is no for the surfaces $S^k_{p,q}$ with at least one of k, p, q even. For surfaces of general type on the other hand the answer is often 'yes' as we see from the following

Theorem 14.4. *Let S be a simply connected complex surface.*
i) If

$$c_1^2 < c_2$$

the surface $\overline{S}$ does not admit a complex structure.
ii) Assume that S has odd intersection form. Then $\overline{S}$ does not admit an almost complex structure if $e(S) - \tau(S) \not\equiv 0 \bmod 4$. On the other hand, if $e(S) - \tau(S) = 20\chi(S) - 2c_1^2(S)$ is divisible by 4, i.e. if $c_1^2(S)$ is even and if moreover the Chern numbers of S satisfy the inequalities

$$\frac{2935}{1726} c_2 + 1353.26(5c_2 - c_1^2)^{2/3} < c_1^2 \leq 3c_2$$
$$c_2 > \textit{large constant},$$

or if we have

$$2c_2 + \frac{3}{4}(5c_2 - c_1^2)^{2/3} \leq c_1^2 \leq 3c_2$$

(without further restrictions on c_2), the oppositely-oriented surface $\overline{S}$ admits a complex structure.

Proof: (i) For any S in the mentioned region, the surface $\overline{S}$ lies on the wrong side of the BMY-line.
(ii) Since b must be odd (this follows from the observations about almost complex structures) we have that $e(S) - \tau(S) = 2(b + 1)$ is divisible by four.
The potential new first Chern number equals $-c_1^2 + 4c_2$, while c_2 remains unchanged. We easily check that the stated inequalities just mean that the potential new Chern numbers are in the range where we can apply Theorem 9.(ii) to conclude existence of a simply connected complex surface with these new Chern numbers.

The surfaces constructed by Chen all have odd intersection form and correspond to the first region. Persson's constructions give odd forms in the region

$$\frac{1}{5}(c_2 - 30) \leq c_1^2 \leq 2c_2 - \frac{3}{4}\chi^{2/3},$$

which corresponds to the second region. These facts can be checked, using the fact that the non-negligible singularities used by the aforementioned authors give rize to odd forms (use (5.1)). We omit the (tedious) details. □

It is much harder to obtain results for surfaces of general type in the case of even forms since there are not many surfaces known in the allowed region with even intersection form, but clearly a small modification of the preceding argument would give a positive answer to the preceding question if existence of such surfaces would be established.

As to the existence of almost complex structures, it is not difficult to show:

> If the 11/8-conjecture is true, any simply connected oriented differentiable four-manifold with even intersection form has an almost complex structure and if we reverse the orientation, also the new oriented manifold has an almost complex structure.

The inequalities from surface geography tells us that we have restrictions on the pairs of numbers (a, b) with a = number of K3-'s (or reversed K3-'s) and b = numbers of quadrics in the decomposition of such an almost complex surface in order that it be complex. One finds:

$$b \geq \frac{2}{3}a - 1 \text{ (if } \tau \leq 0),$$
$$b \geq 6a \text{ (if } \tau \leq 0).$$

As to 'special' surfaces $S^k_{p,q}$ with even form, this shows immediately that $\overline{S^k_{p,q}}$ can have no complex structure. We already noticed that for a simply connected surface S of general type with $c_1^2 < c_2$ the surface with orientation reversed has no complex structure, but -as remarked previously-it is hard to get positive results in the remaining region.

15 Some results on the differentiable structure

The most striking application of Donaldson-theory is probably to the h-cobordism conjecture

> If two simply connected differentiable manifolds are h-cobordant they are diffeomorphic.

Smale's work [Smale] shows that this is true for dimensions ≥ 5 and Freedman's theorem implies that h-cobordant simply connected smooth four-manifolds

are homeomorphic, so in dimension four one only needs to see whether a given homeomorphism-class can contain different diffeomorphism types. Looking for examples in our classification, we see that the Dolgachev surfaces $S^0_{p,q}$ have the same topological structure as $\mathbf{P}^2$ blown up in 9 points. Donaldson showed [Donaldson] that this rational surface is not diffeomorphic to $S^0_{2,3}$ and so the h-cobordism conjecture is not true.
Several authors[FM 1, OV 1] showed that we get countably many distinct differentiable structures when we let p and q vary while $k = 0$ and so there are infinitely many differentiable structures on $S_m := \mathbf{P}^2 \overset{m}{\#} \overline{\mathbf{P}^2}$ when $m = 9$, a result extended to any $m \geq 9$ by Friedman and Morgan (see [FM 1]).
As to $m \leq 8$ there is only one known simply connected surface of general type homeomorphic to S_m, namely the Barlow surface (see Chapter IV). It has $p_g = 0$, $c_1^2 = 1$ and hence $m = 8$. This surface does not have the differentiable structure of S_8: if we blow up Barlow's surface once the resulting surface is neither diffeomorphic to S_9 nor to $S^0_{p,q}$. See [OV 2, Ko].

Using the Donaldson polynomials plus some information on the monodromy-group of big families of differentiable isomorphic surfaces Ebeling showed (see [Ebe]) that complete intersection surfaces of general type which have p_g even and which are homeomorphic (e.g. with the same c_1^2 and with $K = a\,H$ a same parity mod 2), but whose canonical class has different divisibility properties cannot be diffeomorphic. An example of two such surfaces: multidegrees $(10, 7, 7, 6, 3, 3)$ in $\mathbf{P}^8$ and $(9, 5, 3, 3, 3, 3, 3, 2, 2)$ in $\mathbf{P}^{11}$. Here $c_1^2 = 2^2 \cdot 3^9 \cdot 5 \cdot 7^2$, $K = 28H$, resp. $22H$.
Salvetti proved that given any natural number n there are n surfaces of general type which are all homeomorphic, but mutually not diffeomorphic. See [Sal]. His surfaces are not complete intersections, but repeated double covers of $\mathbf{P}^2$.

As to the other surfaces $S^k_{p,q}$ with $k \geq 1$, there is a nice result ([FM 1]:

> Let $k \geq 1$. If a surface $S^k_{p,q}$ blown up in r points is diffeomorphic to a surface $S^{k'}_{p',q'}$ blown up in r' points then $r = r'$ and $pq = p'q'$.

This gives a negative answer to the differentiable analog of Kodaira's question (which has a positive answer because of Freedman's results):

> Is any surface with the same homotopy-type of a K3-surface homeomorphic to a K3-surface?

Indeed, take p and q both odd, but > 1, then $S^1_{p,q}$ has the same topology as a K3-surface, while the preceding theorem says that it cannot be diffeomorphic to the genuine K3-surface $S^1_{1,1}$. The K3-surfaces have $\kappa = 0$, while the other surfaces $S^1_{p,q}$ have $\kappa = 1$. This gives evidence for one of the challenging conjectures that remains in this field:

Conjecture 15.1. The Kodaira-dimension of an algebraic surface is a differentiable invariant.

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