Tata Lectures on Motivic Aspects of Hodge Theory

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Contents

Introductory Remarks

These notes are based on a series of lectures given at the Tata Institute of Fundamental Research at Mumbai during the months of november and december 2007. In fact only Lectures 1–8 have been delivered at that time. These concern the absolute case. The remaining Lectures deal with Hodge-theoretic aspects of families of varieties.

The main theme of the notes is the Hodge theoretic motive associated to various geometric objects. Starting with the topological setting (Lecture 1) I pass to Hodge theory and mixed Hodge theory on the cohomology of varieties (Lectures 2–6). Next comes degenerations and the limiting mixed Hodge structure and the relation to singularities (Lectures 7–9).

Bittner's theorem plays an important role; the (original) proof is presented as an appendix to Lecture 2. It also contains some applications of this theorem, the most important of which is the relation to Chow motives.

The main theme continues by generalizing to the situation of relative varieties. Here the machinery of mixed Hodge modules come into play. See Lectures 10–13.

In Lecture 14 I consider an important application to the topology of singular varieties. To be more specific, the Riemann-Roch formalism inspires a way to consider Chern classes for singular varieties which can be explained in the motivic setting using Bittner's approach. The full functorial meaning however becomes only apparent using mixed Hodge modules.

As a final remark, I want to point out that the Hodge characteristic originally appeared in relation with motivic integration and string theory, a subject that will be explained in Appendix B. The treatment of this Appendix is much inspired by [**Craw**] and [**Blick**].

The lectures were partially meant to popularize certain major parts of the monograph [**P-S**]. For this reason, in the lectures I merely explained and motivated the main results from loc. cit. with only a hint at proofs. To counterbalance this, all results are amply illustrated by examples, some of which are useful in themselves. For instance I give a modern treatment of the Hodge theoretic part of Persson's thesis [**Per**] extending his results to the motivic framework. Also, several new examples are given related to the motivic nearby and vanishing cycle functors.

I clearly owe a lot to my co-author of [**P-S**], Joseph Steenbrink. This is equally true for the motivic interpretation of the nearby cycles; indeed Lectures 7–9 contain a simplified version of [**P-S07**]. The short outline [**Sr**] containing relevant background for the lectures has also been extremely useful in preparing the lectures.

Special thanks go to the members and visitors of the Tata Institute who participated in my lectures. Their pertinent questions and remarks owe a lot to this presentation.

LECTURE 1

Motives and Topology

1.1. Introduction

Let me first introduce the motivic viewpoint in the topological category. To make this work, one has to restrict to a certain category of "good" topological spaces X which include those that are coming from complex algebraic geometry. More about this a in little while; roughly, one wants compactly supported cohomology[1] to behave nicely. In particular:

i) The *Betti numbers* $b_k^c(X) = \dim H_c^k(X;\mathbb{Q})$ are finite;
ii) the Betti numbers are zero for $i \gg 0$ so that the *topological Euler characteristic* $\chi_{\text{top}}^c(X) := \sum(-1)^k b_k^c(X)$ is well defined.

Moreover, there should be a long exact sequence for cohomology with compact cohomology for pairs (X,Y) of topological spaces where $Y \subset X$ is closed:

$$\cdots \to H_c^k(U) \to H_c^k(X) \to H_c^k(Y) \to H_c^{k+1}(U) \to \cdots, \quad U := X - Y. \tag{1}$$

This last condition is known to be satisfied for locally compact and locally contractible Hausdorff topological spaces which have a countable basis for the topology. Call such topological spaces *perfect*.

So one could take for the category $\mathcal{C}$ of topological spaces those that are perfect and satisfy (i) and (ii) above. Complex algebraic varieties as well as complex analytic spaces are examples of perfect topological spaces. They also satisfy (i); the first also satisfies (ii) and for the second one needs the space to have at most finitely many irreducible components.

The point of (1) is that is implies the *scissor relation*:

$$\chi_{\text{top}}^c(X) = \chi_{\text{top}}^c(U) + \chi_{\text{top}}^c(Y) \text{ whenever } X, Y \in \mathcal{C}. \tag{2}$$

Note that this makes sense: if X is perfect, automatically U is. Hence (1) can be used. Then condition (i) and (ii) for X and Y are true for U as is seen by induction using (1).

This can be reformulated as follows. Consider the free group generated by homeomorphism classes $\{X\}$ of spaces $X \in \mathcal{C}$. Divide out by the equivalence relation generated by the scissor relations: $\{X\} = \{U\} + \{Y\}$. The resulting factor group is the group $\mathrm{K}_0(\mathcal{C})$.

On the target side the character χ_{top}^c can be refined: take the alternate sum of the $\mathbb{Q}$-vector spaces $H_c^k(X)$ inside the K-group $\mathrm{K}_0(\mathrm{Vect}_{\mathbb{Q}}^f)$ for the category $\mathrm{Vect}_{\mathbb{Q}}^f$ of finite dimensional $\mathbb{Q}$-vector spaces. By definition this is

[1] I always take (co)homology with $\mathbb{Q}$-coefficients

the free group on finite dimensional $\mathbb{Q}$-vector spaces modulo the equivalence relation generated by

$$V \sim U + W \iff 0 \to U \to V \to W \to 0 \text{ is an exact sequence of } \mathbb{Q}\text{-vector spaces.}$$

The topological Euler characteristic can thus be seen as the composition

$$\chi^c_{\rm top} : \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathrm{Vect}^f_{\mathbb{Q}}) \xrightarrow{\sim} \mathbb{Z}$$

where the last isomorphism is induced by taking $\dim_{\mathbb{Q}} V$.

The word "motivic" relates to everything which respects the scissor relations. Often the motivic point of view allows to make deductions about Betti numbers.

Examples 1.1.1. (1) Write the complex projective plane $\mathbb{P}^2$ as the disjoint union of the affine plane and the line at infinity which itself further breaks up into a the affine line and the point at infinity:

$$\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{C} \cup \{p\} \implies \chi^c_{\rm top}(\mathbb{P}^2) = \chi^c_{\rm top}(\mathbb{C}^2) + \chi^c_{\rm top}(\mathbb{C}) + \chi^c_{\rm top}(p) = 3.$$

This alone together with the fact that there are no odd Betti-numbers together with the standard fact that $b_0 = b_4 = 1$ implies that $b_2 = 1$.

(2) Let me consider a smooth degree d curve C in $\mathbb{P}^2$. Calculate $\chi^c_{\rm top}(C)$ by projecting onto $\mathbb{P}^1$ from a general point p in the plane. There are $d(d-1)$ tangents to C from P each having a simple tangency. Hence $\chi^c_{\rm top}(C) - d(d-1)^2 = d(\chi^c_{\rm top}(\mathbb{P}^1) - d(d-1))$ so that $\chi^c_{\rm top}(C) = -d^2 + 3d$ in accord with $g(C) = \frac{1}{2}(d-1)(d-2)$. v Similarly, one can compute the Betti numbers of a smooth hypersurface S in $\mathbb{P}^n$ of degree d. Let me just explain this for $n = 3$. Project the surface onto $\mathbb{P}^2$. If the centre is sufficiently general, the tangents from P form a cone of degree $d(d-1)$ and again, they will be simply tangent along a smooth curve C of degree $d(d-1)$ which projects isomorphically to a plane curve whose topological Euler characteristic just has been computed. One then gets $\chi^c_{\rm top}(S) = d(1-4d+6d^2)$ and since S is known to be simply connected $b_1(S) = b_3(S) = 0$ so that $b_2(S) = d(6-4d+d^2) - 2$.

(3) To see that additivity does not work for the Euler characteristic e, look at S^1 with $e(S^1) = 0$. Remove a point and you get the interval which is contractible and hence has the same Euler characteristic as a point. Hence $e(S^1) \neq e(S^1 - p) + e(p)$.

Suppose now that one restricts to a subcategory $\mathcal{D} \subset \mathcal{C}$ for which the H^q_c have extra structure, say they all belong to some abelian category $\mathfrak{A}$. Then there is an additive refinement

$$\chi^c_{\rm top} : \mathrm{K}_0(\mathcal{D}) \xrightarrow{\chi_{\mathfrak{A}}} \mathrm{K}_0(\mathfrak{A}) \xrightarrow{F} \mathrm{K}_0(\mathrm{Vect}^f_{\mathbb{Q}}) \xrightarrow{\sim} \mathbb{Z},$$

where F is the forgetful map. In good cases this works also multiplicatively. The idea is to apply this to the category $\mathcal{D}$ of complex algebraic varieties and $\mathfrak{A}$ the category of Hodge structures. This point of view makes it possible to extract information about the Hodge theoretic invariants by doing calculations on simpler locally closed subsets of a given variety, as we shall see.

1.2. Pure Hodge Structures

Here are the relevant definitions:

Definition 1.2.1. (1) A pure (rational) *Hodge structure of weight* k is a couple $(V, \{V^{p,q}\}_{p+q=k})$ where V is a finite dimensional rational vector space and $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$ with $\overline{V^{p,q}} = V^{q,p}$ (the *reality constraint*). The conjugation is coming from conjugation on $\mathbb{C}$. The numbers $h^{p,q}(V) := \dim V^{p,q}$ are the *Hodge numbers* of the Hodge structure.

(2) A *graded pure Hodge structure* is a direct sum of Hodge structures, possibly of different weights.

(3) A *morphism of Hodge structures* $f : V \to W$ is a $\mathbb{Q}$-linear map which induces $f_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ having the property that it preserves types, i.e. $f_{\mathbb{C}}(V^{p,q}) \subset W^{p,q}$ for all p and q.

Examples 1.2.2. (1) The standard example of a weight k Hodge structure is $H^k(X)$ for X a smooth complex projective variety. To explain this, consider $H^k(X)_{\mathbb{C}}$ as the De Rham group of closed *complex* valued k-forms modulo exact ones. Each such form has a decomposition into types: a form of type (p,q) in any coordinate patch with coordinates $(z_1, \ldots, z_n)$ is a linear combination $\sum f_{i_1 \ldots i_n} \cdot dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{i_{p+1}} \wedge \cdots \wedge d\bar{z}_{i_n}$. where $f_{i_1 \ldots i_n}$ is a C^∞function. For general complex varieties this type decomposition does not descend to cohomology, but for Kähler varieties it does. Since any smooth complex projective variety is a Kähler variety (use the Fubini-Study metric) this holds for X. The reality constraint is obvious. Hence the entire cohomology $H(X) = \bigoplus H^k(X)$ is a graded pure Hodge structure.

(2) If $f : X \to Y$ is a morphism between smooth complex projective varieties the induced map $f^* : H^k(Y) \to H^k(X)$ is a morphism of Hodge structures since f induces on forms a map which preserves the type decomposition because f is holomorphic.

(3) The *Hodge structure of Tate type* $\mathbb{Q}(k)$ is the pair $\{\mathbb{Q}(2\pi i)^k, \mathbb{C} = \mathbb{C}^{-k,-k}\}$. A special case is $\mathbb{Q}(-1) = H^2(\mathbb{P}^1) = \mathbb{L}$, the *homological Lefschetz motive.*

Various linear algebra constructions can be applied to Hodge structures, like the direct sum, tensor product and Hom. The first operation is clear. For the last two assume $(V, \{V^{p,q}\}_{p+q=k})$ and $(W, \{W^{p,q}\}_{p+q=\ell})$ then set

$$
\begin{aligned}
[V \otimes W]^{p,q} &= \bigoplus_{a+c=p;b+d=q} V^{a,b} \otimes W^{c,d} \\
\mathrm{Hom}^{p,q}(V,W) &= \{f : V_{\mathbb{C}} \to W_{\mathbb{C}} \mid f(V^{a,b}) \subset W^{a+p,b+q} \text{ for all } a, b \in \mathbb{Z}\}.
\end{aligned}
$$

Clearly, $V \otimes W$ has weight $k + \ell$ and $\mathrm{Hom}(V,W)$ has weight $-k + \ell$. As a special case the dual V^* has weight $-k$.

Examples 1.2.3. (1) The homology group $H_k(X)$ of a smooth complex projective variety X, as the dual of $H^k(X)$ has a weight $(-k)$

Hodge structure and for a morphism of smooth complex varieties the induced maps in homology are morphisms of Hodge structures.

(2) $\mathbb{Q}(1) = H_2(\mathbb{P}^2)$.

(3) The Künneth decomposition $H^k(X \times Y) = \bigoplus_{a+b=k} H^a(X) \otimes H^b(Y)$ (remember: we are working with rational coefficients here) for smooth projective X and Y is an isomorphism of Hodge structures. This is not entirely trivial and I give a proof in Lecture 4.

It is not hard to show that the above linear algebra constructions imply that the category of Hodge structures forms an abelian category $\mathfrak{hs}$ with tensor-products so that $\mathrm{K}_0(\mathfrak{hs})$ is a ring. One has:

Lemma 1.2.4. *The* Hodge number polynomial *for a graded pure Hodge structure* $(V, \{V^{p,q}\})$

$$P(V) := \sum h^{p,q} u^p v^q \in \mathbb{Z}[u, v, u^{-1}, v^{-1}]$$

defines a ring homomorphism from $\mathrm{K}_0(\mathfrak{hs}) \to \mathbb{Z}[u, v, u^{-1}, v^{-1}]$

Proof: The map P is easily seen to be well defined and additive. That it is also multiplicative follows from Example 1.2.3. (3).

□

LECTURE 2

The Hodge Characteristic Makes its Appearance

2.1. The Hodge Characteristic

For a smooth complex projective variety X the *Hodge characteristic* is defined by

$$\chi_{\mathrm{Hdg}}(X) := \sum_k (-1)^k [H^k(X)] \in \mathrm{K}_0(\mathfrak{hs}). \tag{3}$$

My aim is to show that this leads to a character from the Grothendieck group on the category of complex algebraic varieties to $\mathrm{K}_0(\mathfrak{hs})$. But first I need to study *the blow-up diagram*

$$\begin{array}{ccc} E & \xrightarrow{j} & Z = \mathrm{Bl}_Y X \\ \downarrow{\scriptstyle \pi_{X_0}} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{i} & X, \end{array} \tag{4}$$

where X is smooth projective and $Y \subset X$ is a smooth subvariety; the map i is the inclusion and $\pi : \mathrm{Bl}_Y X \to X$ is the blow-up of X along Y; finally E is the exceptional divisor included in Z through the inclusion j.

Claim. There is a long exact sequence of Hodge structures, the *Mayer-Vietoris sequence* for blow-up diagrams:

$$0 \to H^k(X) \xrightarrow{\pi^* \oplus i^*} H^k(Z) \oplus H^k(Y) \xrightarrow{j^* - (\pi|E)^*} H^k(E) \to 0. \tag{5}$$

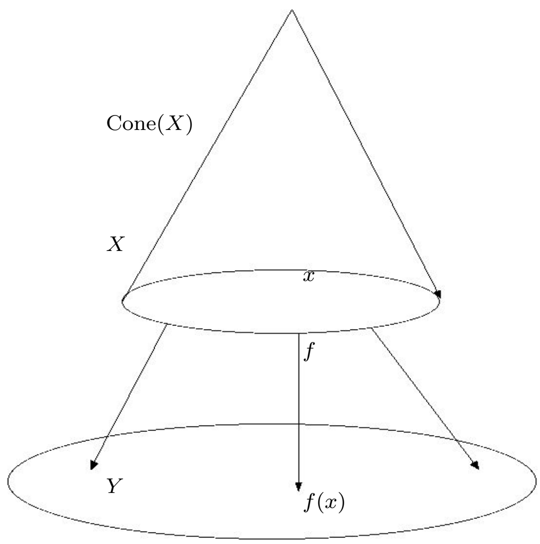


FIGURE 1. The mapping cone of $f : X \to Y$

I shall give a proof of the claim which is adapted to later needs. In this proof the topological *mapping cone* plays an important role. It is constructed as follows. Let $f : X \to Y$ be any continuous map. Then Cone(f) is obtained from the disjoint union $X \times [0,1] \coprod Y$ by collapsing $X \times \{0\}$ to a point (this gives Cone(X) and identifying a point $(x,1)$ with $f(x) \in Y$. In other words, take a Cone(X) and glue any point of the base of onto its image in Y. See Fig 1.

Example 2.1.1. Note that Cone(X) always contracts to a its vertex, so if $j : Z \hookrightarrow X$ is an inclusion of a closed set, its cone consists of X with Z contracted to a point. This topological space is denoted X/Z. One has $H^q(X,Z) = \widetilde{H}^q(X/Z)$, i.e the *reduced cohomology* of $Y = X/Z$ defined by taking the cokernel of $a_Y^* : H^q(\mathrm{pt}) \to H^q(Y)$. So, and this is important for what follows, relative cohomology can be viewed as the cohomology of a topological cone. It will be explained later that this cohomology also can be realized using a cone construction on the level of complexes. See Remark 4.2.2. Looking ahead, I want to point out that this will be the crucial construction making it possible to put mixed Hodge structures on relative cohomology.

Proof of the Claim. The mapping cone Cone(π) is constructed by gluing the mapping cone Cone$(\pi|E)$ to Z along the base $E \times \{1\}$ of Cone(π). Let U the open set coming from $Y \times [0, \frac{3}{4}[$ and V be the set coming from $Z \sqcup Y \times]\frac{1}{4}, 1]$. Clearly U retracts to Y, V to Z and $U \cap V$ to E. Moreover $U \cup V = \mathrm{Cone}(\pi)$ retracts to X. So one can use the usual Mayer-Vietoris sequence.

What remains to be checked "by hand" is that the first map in (5) is into and the last map is onto. Since both fit into a long exact sequence, it is sufficient to prove that the first map is into. This in turn is a consequence of the fact that $\pi^* : H^k(X) \to H^k(Z)$ is into, or equivalently that $\pi_* : H_k(Z) \to H_k(X)$ is onto which is clear: every k-cycle in X can be moved to a cycle γ which meets Y transversally and the closure in Z of $\gamma - \gamma \cap Y$ is cycle whose class maps onto the class of γ. □

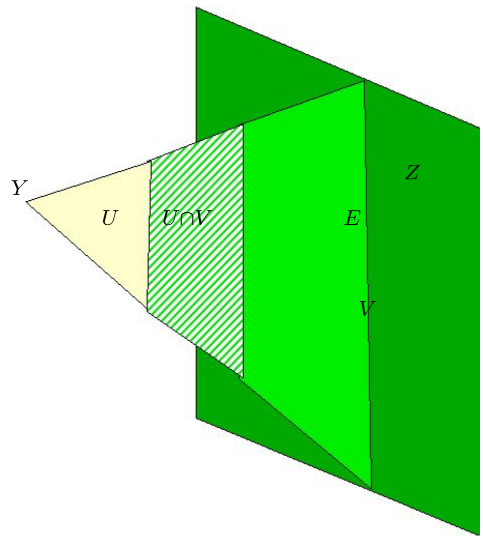


FIGURE 2. Mayer-Vietoris for a blow-up

***Remark* 2.1.2.** Setting $X_0 = Y$, $X_{01} = E$, $X_1 = Z$, $X_\varnothing = X$ and $d_{0,\{01\}} = \pi_{X_0}$, $d_{1,\{0,1\}} = j$, $d_{\varnothing,0} = i$, $d_{\varnothing,1} = \pi$ one obtains a 2-cubical variety $X_\bullet$. It gives an augmented 1-simplicial variety whose geometric realization is just the natural map $\mathrm{Cone}(\pi) \to X$. This will be explained in Lecture 5.

First I recall the notion $K_0(\mathcal{C})$ from Lecture 1, where $\mathcal{C}$ was some suitable category of topological spaces. For $\mathcal{C} = \underline{\mathrm{Var}}$ this formally becomes:

Definition 2.1.3. Let $\underline{\mathrm{Var}}$ be the category of complex algebraic varieties. Define $\mathrm{K}_0(\underline{\mathrm{Var}})$ to be the quotient ring $\mathbb{Z}[\underline{\mathrm{Var}}]/J$ of finite formal sums $\sum n_V\{V\}$, $n_V \in \mathbb{Z}$ of isomorphism classes $\{V\}$ of varieties V modulo the ideal J generated by$\{X\} - \{Y\} - \{X - Y\}$ for any closed subvariety $Y \subset X$. This means that in $\mathrm{K}_0(\underline{\mathrm{Var}})$ the *scissor-relations* $[X] = [Y] + [X - Y]$ hold. Here $[X]$ denotes the class of $\{X\}$ in $\mathrm{K}_0(\underline{\mathrm{Var}})$.

Next I shall prove:

Theorem 2.1.4. *There is a unique homomorphism of rings*

$$\chi_{\mathrm{Hdg}} : \mathrm{K}_0(\underline{\mathrm{Var}}) \to \mathrm{K}_0(\mathfrak{hs})$$

such that

(1) $F \circ \chi_{\mathrm{Hdg}} = \chi_{\mathrm{top}}$, *where* $F : K_0(\mathfrak{hs}) \to K_0(\mathrm{Vect}^f_{\mathbb{Q}})$ *is the forgetful morphism,*
(2) *For a smooth and projective* X *one has* $\chi_{\mathrm{Hdg}}(X) = \sum(-1)^k H^k(X)$.

Here is an “easy proof” of the theorem: Use the fact that by (5) the Hodge characteristic (3) is compatible with the blow-up relation and then apply the following theorem:

Theorem 2.1.5 ([**Bitt1**])**.** $\mathrm{K}_0(\underline{\mathrm{Var}})$ *is generated by classes* $[X]$ *of smooth projective varieties subject to the following two relations*

(1) $[\varnothing] = 0$.
(2) *the blow-up relation* $[X] - [Y] = [Z] - [E]$ *where* X, Y, Z, E *are as in the blow-up diagram* (4).

The proof of this theorem is not hard provided you admit another deep theorem, the *weak factorization theorem* [**A-K-M-W**] which describes how birational maps can be composed in blowups along smooth centers and their inverses. This is carried out in the Appendix to this Lecture.

***Remark* 2.1.6.** Bittner's theorem has an interesting consequence: on the category of complex algebraic varieties χ^c_{top} equals the ordinary topological Euler characteristic e. This is because on compact projective varieties they are the same. Note that for a smooth but not complete variety X, Poincaré duality implies that $\chi^c_{\mathrm{top}}(X) = e(X)$, but this uses that $\dim X$ is even. This already hints at the non-triviality of this consequence. Note also that as a corollary, on the category $\underline{\mathrm{Var}}$ the scissor relations are respected by the ordinary Euler characteristic.

A more elementary proof of this runs as follows. Recall that complex algebraic varieties admit stratifications that satisfy the two Whitney conditions (see e.g. [**P-S**, Appendix C]) so that in particular the local topology near open strata does not change. This implies that each closed stratum is a strong deformation retract of a tubular neighbourhood. In particular, if Y is a closed stratum, there is a Mayer-Vietoris sequence for the decomposition $X = Y \cup (X - Y)$ so that $e(X) = e(Y) + e(X - Y)$. If $X = \cup_{D \in S} D$ is a Whitney stratification in smooth strata, for each stratum one has $e(D) = \chi^c_{\text{top}}(D)$. Let Y be a stratum of minimal dimension (and hence closed). By induction on the number of strata, one may assume that $e(X - Y) = \chi^c_{\text{top}}(X - Y)$. Hence $e(X) = \chi^c_{\text{top}}(Y) + \chi^c_{\text{top}}(X - Y) = \chi^c_{\text{top}}(X)$ by additivity for χ^c_{top}.

Another, similar approach uses the stratification of varieties into smooth and singular loci and resolution of singularities to reduce to smooth varieties. See the appendix of [**K-P**] where this is explained in detail.

Examples 2.1.7. (1) Let $E \to Y$ be a $\mathbb{P}^k$-bundle over Y, a projective variety. Since E is locally trivial in the Zariski topology, the scissor relations imply

$$[E] = [Y] \cdot [\mathbb{P}^k] = [Y] \cdot (1 + \mathbb{L} + \cdots \mathbb{L}^k), \quad \mathbb{L} = \mathbf{A}^1 \tag{6}$$

where $\mathbf{A}^1$ is the affine line, the geometric incarnation of the Lefschetz motive.

(2) Using the preceding, for a blow up $\mathrm{Bl}_Y X$ where $\mathrm{codim}_Y X = c$ one gets

$$[\mathrm{Bl}_Y X] - [X] = [Y] \cdot ([\mathbb{P}^{c-1}] - 1). \tag{7}$$

In the following lectures a proof of Theorem 2.1.4 is sketched based on the existence of a functorial mixed Hodge structure on the cohomology of complex algebraic varieties. For the moment I would like to draw attention to the following results, which uses that cohomology groups of smooth projective varieties have weights ≥ 0:

Corollary 2.1.8. *The Hodge number polynomial*

$$P_{\text{Hdg}} := P \circ \chi_{\text{Hdg}} : \mathrm{K}_0(\underline{\mathrm{Var}}) \to \mathbb{Z}[u, v] \tag{8}$$

is a morphism of rings.

The existence of such a polynomial does not say anything about the actual values of the coefficients in front of $u^p v^q$. It will be shown that these can be calculated with the help of mixed Hodge theory.

2.2. Mixed Hodge Theory

Before giving the definition let me rephrase the concept of a Hodge structure in terms of filtrations rather than decompositions. This goes as follows. Let $(H, \{H^{p,q}\}_{p+q=k})$ be a Hodge structure of weight k. Put $F^p = \bigoplus_{p' \geq p} H^{p',q'}$. Then one verifies that $F^p \oplus \overline{F^{k-p+1}} = H_{\mathbb{C}}$. Conversely, given a decreasing filtration with this property one gets a Hodge structure of weight k by putting $H^{p,q} = F^p \cap \overline{F^q}$. The filtration F is called the *Hodge filtration.*

Now the definition of a mixed Hodge structure can be stated:

Definition 2.2.1.
(1) A *mixed Hodge structure* is a triple (H, W, F) where W is an increasing filtration on H, the *weight filtration* and F a decreasing filtration on $H_{\mathbb{C}}$, the *Hodge filtration*, such that F induces a pure Hodge structure of weight k on $\mathrm{Gr}^W_k := W_k / W_{k-1}$.
(2) *Morphisms of mixed Hodge structure* are $\mathbb{Q}$-linear maps preserving weight and Hodge filtrations.

(3) The *Hodge numbers* are $h^{p,q}(H) = \dim_{\mathbb{C}} \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^{W^{\mathbb{C}}} H_{\mathbb{C}}$ where $W^{\mathbb{C}} := W \otimes_{\mathbb{Q}} \mathbb{C}$.

One word of explanation: if there are two (decreasing) filtrations F, G on the same vector space V, one of these induces a filtration on the gradeds of the other as follows. The natural maps $G^k \cap F^p/G^k \cap F^{p+1} \to F^p \cap G^k/F^{p+1} \to \mathrm{Gr}_F^p = F^p/F^{p+1}$ are injective and hence if one puts $G^k(\mathrm{Gr}_F^p) = G^k \cap F^p/G^k \cap F^{p+1}$ this gives a filtration on Gr_F^p. Similar remarks apply to the situation where one of the filtration is increasing, which is the situation of the above definition.

The same linear algebra constructions as for the pure case apply: one has internal direct sums, tensor products and "Homs". This makes the category of mixed Hodge structures abelian. The only non-trivial point here is the fact that, given a morphism $f : H \to H'$ of mixed Hodge structures, the natural morphism $H/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ is an isomorphism. This fact is equivalent to strictness which I prove now:

Lemma 2.2.2. *A morphism $f : H \to H'$ of mixed Hodge structures is* strict *for both the weight and Hodge filtrations, i.e. $f[W_k(H)] = W_k(H') \cap \mathrm{Im}(f)$and similarly for the F-filtration.*

Here I only give a *sketch of proof.* See also [**C-K-S86**, Theorem 2.13].

First note that for a pure Hodge structure this poses no problem because the Hodge decomposition is completely canonically associated to the Hodge filtration. For the mixed situation there is something similar, the *Deligne decomposition*

$$H_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} I^{p,q}, \quad I^{p,q} = F^p \cap W^{\mathbb{C}}_{p+q} \cap (\overline{F^q} \cap W^{\mathbb{C}}_{p+q} + \sum_{j \geq 2} \overline{F^{q-j+1}} \cap W^{\mathbb{C}}_{p+q-j})$$

with the property that $W_k^{\mathbb{C}} = \bigoplus_{p+q \leq k} I^{p,q}$ and $F^p = \bigoplus_{p' \geq p} I^{p',q'}$. Such a splitting is completely canonical and hence is preserved under morphisms; as in the pure case this guarantees strictness. $\square$

***Remark* 2.2.3.** Projection gives an isomorphism of complex vector spaces $I^{p,q} \xrightarrow{\sim} \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^{W^{\mathbb{C}}} H_{\mathbb{C}}$ so that $\dim_{\mathbb{C}} I^{p,q} = h^{p,q}(H)$.

A first important consequence of strictness is the following test for isomorphisms of mixed Hodge structures:

Corollary 2.2.4. *A morphism of mixed Hodge structures is an isomorphism if and only if it is a vector space isomorphism.*

The K-group associated to the abelian category of mixed Hodge structures is just $\mathrm{K}_0(\mathfrak{hs})$ since in K_0 there is the relation $[H] = \sum [\mathrm{Gr}_k^W H]$. This suggests to define $\chi_{\mathrm{Hdg}}(X)$ using a mixed Hodge structure on $H_c^k(X)$. That this is indeed possible is a consequence of the following central result:

Theorem 2.2.5 ([**Del71, Del74**])**.** *Let U be a complex algebraic variety and $V \subset U$ a closed subvariety (V may be empty). Then $H^k(U,V)$ has a mixed Hodge structure which is functorial in the sense that if $f : (U,V) \to (U',V')$ is a morphism of pairs, the induced morphism $H^k(V',U') \to H^k(U,V)$ is a morphism of Hodge structures. Moreover*

(1) *For U smooth projective, the mixed Hodge structure on $H^k(U)$ is the classical one, i.e. the pure weight k Hodge structure coming from the decomposition of forms into types;*
(2) *If U is smooth with smooth compactification X, then $H^k(U)$ has weights $\geq k$ and $\mathrm{Im}[H^k(X) \to H^k(U)] = W_k H^k(U)$;*
(3) *If U is projective $H^k(U)$ has weights $\leq k$ and in the interval $[2k - 2n, k]$ if $k \geq n$;*
(4) *If U, V are smooth then the weights of $H^k(U, V)$ are in the range $[k-1, 2k]$; if U, V are complete, then the weights of $H^k(U, V)$ are in the range $[0, k]$;*
(5) *Duality: cap product with the orientation class of U in Borel-Moore homology induces a morphism of mixed Hodge structures*
$$H^k(U)(n) \to H^{\mathrm{BM}}_{2n-k}(U) = [H_c^{2n-k}(U)]^* \quad (n = \dim U)$$
which is an isomorphism if U is smooth.

Since compactly supported cohomology on U can be calculated with the help of a compactification X of U as relative cohomology: $H_c^k(U) = H^k(X, T)$ where $T = X - U$, for the weights one has:

Corollary 2.2.6. *$H_c^k(U)$ has weights in the interval $[0, k]$.*

The Hodge characteristic can now be defined by setting:
$$\chi_{\mathrm{Hdg}}(U) = \sum_k [\mathrm{Gr}_k^W H_c^k(U)] \in \mathrm{K}_0(\mathfrak{hs}),$$
and turns out to respect the scissor relations and to be multiplicative.

The fact that the scissor relations are respected uses the long exact sequence for pairs. That this long exact sequence is compatible with the mixed Hodge structures follows from the functoriality statement in Theorem 2.2.5; the trick here is to describe the connecting homomorphism as coming from a morphism of complexes so that functoriality also applies to this morphism. How this is done will be explained in Lecture 6. In that Lecture also the multiplicative structure will be treated. This will be preceded by a rough sketch of the proof of Theorem 2.2.5; first, in Lectures 4-5 this is done for pairs (U, V) where V is empty, and then the general case is explained after having given the cone construction in Lecture 6.

Appendix A: A Proof of Bittner's Theorem and Some Applications

A-1. Statement of the Weak Factorization Theorem

This theorem is actually is true over any field k of characteristic zero (not necessarily algebraically closed).

Theorem (Weak Factorization [**A-K-M-W**]). *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between complete smooth connected varieties over k, let $U \subset X_1$ be an open set where ϕ is an isomorphism. Then ϕ can be factored into a sequence of blow-ups and blow-downs with smooth centers disjoint from U, i.e. there exists a sequence of birational maps*

$$X_1 = V_0 \overset{\phi_1}{\dashrightarrow} V_1 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_i}{\dashrightarrow} V_i \overset{\phi_{i+1}}{\dashrightarrow} V_{i+1} \overset{\phi_{i+2}}{\dashrightarrow} \cdots \overset{\phi_{l-1}}{\dashrightarrow} V_{l-1} \overset{\phi_l}{\dashrightarrow} V_l = X_2$$

where $\phi = \phi_l \circ \phi_{l-1} \circ \cdots \phi_2 \circ \phi_1$. Moreover, each factor ϕ_i is an isomorphism over U, and $\phi_i : V_i \to V_{i+1}$ or $\phi_i^{-1} : V_{i+1} \to V_i$ is a morphism obtained by blowing up a smooth center disjoint from U (here U is identified with an open subset of V_i).

Moreover, there is an index i_0 such that for all $i \le i_0$ the map $V_i \dashrightarrow X_1$ is defined everywhere and projective, and for all $i \ge i_0$ the map $V_i \dashrightarrow X_2$ is defined everywhere and projective.

If $X_1 - U$ (respectively, $X_2 - U$) is a simple normal crossings divisor, then the factorization can be chosen such that the inverse images of this divisor under $V_i \to X_1$ (respectively, $V_i \to X_2$) are also simple normal crossing divisors, and the centers of blowing up have normal crossings with these divisors.

A-2. A Proof of Bittner's Theorem

Step 1: reduction to *smooth* varieties – modulo the scissor relations.

Introduce the group $\mathrm{K}_0(\underline{\mathrm{Var}})^{\mathrm{sm}}$, by definition the free abelian group on isomorphism classes of smooth varieties modulo the scissor relations with respect to *smooth* closed subvarieties Y of smooth varieties X. It is a commutative ring with respect to the product of varieties. The class of X in $\mathrm{K}_0(\underline{\mathrm{Var}})^{\mathrm{sm}}$ will be denoted by $[X]_{\mathrm{sm}}$

Claim. The ring homomorphism $\mathrm{K}_0(\underline{\mathrm{Var}})^{\mathrm{sm}} \longrightarrow \mathrm{K}_0(\underline{\mathrm{Var}})$, $[X]_{\mathrm{sm}} \mapsto [X]$ is an isomorphism.

The group $\mathrm{K}_0(\underline{\mathrm{Var}})$ can in fact be generated by classes of smooth quasi-projective varieties subject to the scissor relations (restricted to smooth varieties).

Proof: Construct an inverse as follows For any variety X stratify $X = \bigsqcup_{N\in S} N$ in such a way that all $N \in S$ are smooth and equidimensional, and such that the closure of N is a union of strata. Consider the expression $\sum_{N\in S}[N]_{\mathrm{sm}}$ in $\mathrm{K}_0(\underline{\mathrm{Var}})^{\mathrm{sm}}$. If X is itself smooth, then $\sum_{N\in S}[N]_{\mathrm{sm}}$ equals $[X]_{\mathrm{sm}}$ as can be seen by induction on the number of elements of S: let $N \in S$ be an element of minimal dimension, then $[X]_{\mathrm{sm}} = [X - N]_{\mathrm{sm}} + [N]_{\mathrm{sm}}$, and by the induction hypothesis $[X - N]_{\mathrm{sm}} = \sum_{N'\in I-\{N\}}[N']_{\mathrm{sm}}$.

Two stratifications S and S' of X have a common refinement T. The above argument shows that $[N]_{\mathrm{sm}} = \sum [L]_{\mathrm{sm}}$ where L runs over the strata from T which make up N. Hence $\sum_{L\in T}[L]_{\mathrm{sm}}$ is equal to $\sum_{N\in S}[N]_{\mathrm{sm}}$ and, in the same way, it equals $\sum_{N\in S'}[N]_{\mathrm{sm}}$, therefore $\sum_{N\in S}[N]_{\mathrm{sm}}$ is independent of the choice of the stratification and one can set

$$e(X) := \sum_{N\in S}[N]_{\mathrm{sm}}.$$

If $Y \subset X$ is a closed subvariety there is a stratification for which Y is a union of strata. Hence $e(X) = e(X-Y)+e(Y)$, i.e. e respects the scissor relations and hence factors through $\mathrm{K}_0(\underline{\mathrm{Var}})$ and the induced map on $\mathrm{K}_0(\underline{\mathrm{Var}})$ is an inverse for $\mathrm{K}_0(\underline{\mathrm{Var}})^{\mathrm{sm}} \longrightarrow \mathrm{K}_0(\underline{\mathrm{Var}})$.

Decomposing into connected components and noting that instead of cutting a smooth closed subvariety Y out of a smooth connected variety X one can also take out the connected components of Y one by one, shows that one can restrict to smooth *connected* varieties. Stratifying by smooth quasi-projective varieties shows that one can even restrict to smooth (connected) quasi-projective varieties. □

Step 2: passing to the blow-up relations.

Introduce the auxiliary group $\mathrm{K}_0(\underline{\mathrm{Var}})^{\mathrm{bl}}$, the free abelian group on isomorphism classes $[X]_{\mathrm{bl}}$ of smooth complete varieties X modulo the blow-up relations for blow-ups of smooth complete varieties X along smooth closed subvarieties Y and the relation $[\varnothing]_{\mathrm{bl}} = 0$ (then $[X \sqcup Y]_{\mathrm{bl}} = [X]_{\mathrm{bl}} + [Y]_{\mathrm{bl}}$, which can be seen by blowing up along Y).

Decomposing into connected components and noting that the blow-up along a disjoint union is the successive blow-up along the connected components one sees that this can also be described as the free abelian group on isomorphism classes $[X]_{\mathrm{bl}}$ of connected smooth complete varieties with imposed relations $[\varnothing]_{\mathrm{bl}} = 0$ and $[\mathrm{Bl}_Y X]_{\mathrm{bl}} - [E]_{\mathrm{bl}} = [X]_{\mathrm{bl}} - [Y]_{\mathrm{bl}}$, where $Y \subset X$ is a connected closed smooth subvariety.

Also $\mathrm{K}_0(\underline{\mathrm{Var}})^{\mathrm{bl}}$ carries a commutative ring structure induced by the product of varieties.

Claim. The ring homomorphism $\mathrm{K}_0(\underline{\mathrm{Var}})^{\mathrm{bl}} \longrightarrow \mathrm{K}_0(\underline{\mathrm{Var}})$ which sends $[X]_{\mathrm{bl}}$ to $[X]$ is an isomorphism.

Proof: Again one constructs an inverse. Using the Claim of Step 1, one may restrict to smooth connected varieties. Let X be a smooth connected variety, let $X \subset \overline{X}$ be a smooth completion with $D = \overline{X} - X$ a simple normal crossing divisor. Let $D^{(l)}$ be the normalization of the l-fold intersections of D, where $D^{(0)}$ is understood to be $\overline{X}$ (so $D^{(l)}$ is the disjoint union of the l-fold intersections of the irreducible components of D). Consider the expression $\sum(-1)^l[D^{(l)}]_{\mathrm{bl}}$ in $\mathrm{K}_0(\underline{\mathrm{Var}})^{\mathrm{bl}}$.

This expression is independent of the choice of the completion: Let $X \subset \overline{X}'$ and $X \subset \overline{X}$ be two smooth completions of X with $\overline{X} - X = D$ and $\overline{X}' - X = D'$ simple normal crossings divisors. Due to the weak factorization theorem the birational map $\overline{X}' \dashrightarrow \overline{X}$ can be factored into a sequence of blow-ups and blow-downs with smooth centers disjoint from X which have normal crossings with the complement of X. Hence we may assume that $\overline{X}' = \mathrm{Bl}_Z(\overline{X})$ with $Z \subset D$ smooth and connected such that Z has normal crossings with D.

Let D_0 be the irreducible component of D containing Z and let $\{D_i \mid i \in I\}$ be the remaining irreducible components. Then the irreducible components of D' are $D'_i = \mathrm{Bl}_{Z \cap D_i}(D_i)$ (where $i \in \{0\} \cup I$) and the exceptional divisor E of the blow-up. For $K \subset \{0\} \cup I$ we put $D_K := \bigcap_{j \in K} D_j$ (where $D_\varnothing$ is understood to be $\overline{X}$), $D'_K := \bigcap_{j \in K} D'_j$, $Z_K := Z \cap D_K$ and $E_K := E \cap D'_K$. As Z has normal crossings with D we get $D'_K = \mathrm{Bl}_{Z_K}(D_K)$ with exceptional divisor E_K, hence

$$[D'_K]_{\mathrm{bl}} - [E_K]_{\mathrm{bl}} = [D_K]_{\mathrm{bl}} - [Z_K]_{\mathrm{bl}}.$$

Denote by $E^{(l)}$ the preimage of E in $D'^{(l)}$ and by $Z^{(l)}$ the preimage of Z in $D^{(l)}$. Then for $l = 0, \ldots, n$ the preceding identity yields

$$\begin{aligned}
[D'^{(l)}]_{\mathrm{bl}} &= \sum_{|K|=l} [D'_K]_{\mathrm{bl}} + \sum_{|K|=l-1} [E_K]_{\mathrm{bl}} \\
&= \sum_{|K|=l} ([D_K]_{\mathrm{bl}} + [E_K]_{\mathrm{bl}} - [Z_K]_{\mathrm{bl}}) + \sum_{|K|=l-1} [E_K]_{\mathrm{bl}} \\
&= [D^{(l)}]_{\mathrm{bl}} + [E^{(l)}]_{\mathrm{bl}} - [Z^{(l)}]_{\mathrm{bl}} + [E^{(l-1)}]_{\mathrm{bl}},
\end{aligned}$$

(for $l = 0$ the last term is zero). As $Z \subset D_0$ we get $Z_{\{0\} \cup K} = Z_K$ for $K \subset I$, thus $\sum(-1)^l[Z^{(l)}]_{\mathrm{bl}} = 0$. Taking the alternating sum hence yields

$$\sum(-1)^l[D'^{(l)}]_{\mathrm{bl}} = \sum(-1)^l[D^{(l)}]_{\mathrm{bl}}.$$

Therefore one can put

$$e(X) := \sum(-1)^l[D^{(l)}]_{\mathrm{bl}}.$$

One has to check that $e(X) = e(X - Y) + e(Y)$ for $Y \subset X$ a connected closed smooth subvariety of a connected smooth variety X. Choose $\overline{X} \supset X$ smooth and complete such that $D = \overline{X} - X$ is a simple normal crossings divisor and such that the closure $\overline{Y}$ of Y in $\overline{X}$ is also smooth and has normal crossings with D (one can take first a smooth completion of X with boundary a simple normal crossing divisor and then an embedded resolution of the closure of Y compatible with this divisor – compare e.g. [**A-K-M-W**, Section 1.2]). In particular $D \cap \overline{Y}$ is a simple normal crossings divisor in $\overline{Y}$. Denote the irreducible components of D by $\{D_i\}_{i \in I}$, for $K \subset I$ let D_K be defined as above, let $Y_K := \overline{Y} \cap D_K$ and $Y^{(l)} = \bigsqcup_{|K|=l} Y_K$. Then $e(Y) = \sum(-1)^l[Y^{(l)}]_{\mathrm{bl}}$.

Let $\widetilde{X} := \mathrm{Bl}_{\overline{Y}}(\overline{X})$ and denote the exceptional divisor by E. Denote the proper transform of D_i by $\widetilde{D}_i$. The complement $\widetilde{D}$ of $X - Y$ in $\widetilde{X}$ is the simple normal crossings divisor $\bigcup \widetilde{D}_i \cup E$. If $E_K := E \cap \widetilde{D}_K$ then as above $\widetilde{D}_K$ is the blow-up of D_K along Y_K with exceptional divisor E_K and hence

$$[\widetilde{D}_K]_{\mathrm{bl}} - [E_K]_{\mathrm{bl}} = [D_K]_{\mathrm{bl}} - [Y_K]_{\mathrm{bl}}.$$

Thus

$$\begin{aligned}[\widetilde{D}^{(l)}]_{\mathrm{bl}} &= \sum_{|K|=l} [\widetilde{D}_K]_{\mathrm{bl}} + \sum_{|K|=l-1} [E_K]_{\mathrm{bl}} \\ &= \sum_{|K|=l} ([D_K]_{\mathrm{bl}} - [Y_K]_{\mathrm{bl}} + [E_K]_{\mathrm{bl}}) + \sum_{|K|=l-1} [E_K]_{\mathrm{bl}} \\ &= [D^{(l)}]_{\mathrm{bl}} - [Y^{(l)}]_{\mathrm{bl}} + [E^{(l)}]_{\mathrm{bl}} + [E^{(l-1)}]_{\mathrm{bl}},\end{aligned}$$

where $E^{(l)}$ denotes the preimage of E in $\widetilde{D}^{(l)}$. Taking the alternating sum yields $e(X - Y) = e(X) - e(Y)$. Hence e induces a morphism $\mathrm{K}_0(\underline{\mathrm{Var}}) \to \mathrm{K}_0(\underline{\mathrm{Var}})^{\mathrm{bl}}$ which clearly is an inverse for the mapping $\mathrm{K}_0(\underline{\mathrm{Var}})^{\mathrm{bl}} \to \mathrm{K}_0(\underline{\mathrm{Var}})$. Using the fact that we can restrict to quasi-projective generators in the smooth presentation given in Step 1, and that a connected smooth quasi-projective variety has a smooth projective simple normal crossings completion one sees that one can restrict to projective generators. □

A-3. Applications

Duality. There are various reasons to enlarge the Grothendieck ring $\mathrm{K}_0(\underline{\mathrm{Var}})$ by inverting the Lefschetz motive $\mathbb{L}$. The new ring is called the *naive motivic ring*

$$\mathcal{M} := \mathrm{K}_0(\underline{\mathrm{Var}})[\mathbb{L}^{-1}]. \tag{9}$$

There is a *duality involution* D on $\mathcal{M}$ characterized by the property

$$\mathsf{D}[X] = \mathbb{L}^{-d_X}[X], \quad X \text{ smooth }, d_X = \dim X.$$

Indeed, using Bittner's theorem, one needs to show that this operator respects the blow-up relation. So let $Z = \mathrm{Bl}_Y(X)$ be the blow up and E the exceptional divisor. Since E is a projective bundle over Y it is locally trivial in the Zariski-topology and hence $[E] = [Y]\cdot[\mathbb{P}^{c-1}] = [Y]\cdot(1+\mathbb{L}+\cdots+\mathbb{L}^{c-1})$ with c the codimension of Y in X, the blow-up relation can be rewritten as

$$[Z] - \mathbb{L}\cdot[E] = [X] - \mathbb{L}^c\cdot[Y]. \tag{10}$$

and hence $\mathsf{D}[Z] - \mathsf{D}[E] = \mathsf{D}[X] - \mathsf{D}[Y]$. This proves the claim.

Chow Motives. The idea is to enlarge the category of smooth complex projective varieties to include projectors, i.e. idempotent correspondences. Recall that a correspondence from X to Y of degree s is an element of

$$\mathrm{Corr}^s(X,Y) := \mathrm{Chow}^{d_X+s}(X\times Y)\otimes\mathbb{Q}$$

where $d_X = \dim X$ and $\mathrm{Chow}^k(X)$ denotes the Chow group of cycles of codimension k on X. Correspondences can be composed (using the morphisms induced by the various projections): if $f \in \mathrm{Corr}^s(X,Y)$, $g \in \mathrm{Corr}^t(Y,Z)$, then $g\circ f \in \mathrm{Corr}^{s+t}(X,Z)$. More precisely, if one denotes the projection from $X\times Y\times Z$ to $X\times Z$ by p_{XZ}, then $g\circ f = pr_{XZ}(\{(f\times Z)\cap(X\times g)\}$, where $\cap$ is the intersection product of algebraic cycle classes on $X \times Y \times Z$. One speaks of a projector $p \in \mathrm{Corr}^0(X, X)$ if $p \circ p = p$. A pair (X, p) consisting of a smooth variety and a projector is called an *effective motive*. A morphism $(X, p) \to (Y, q)$ is a correspondence which is of the form $q \circ f \circ p$ where $f \in \mathrm{Corr}^0(X, Y)$. Pure motives are triples (X, p, n) where (X, p) is effective and $n \in \mathbb{Z}$. A morphism $(X, p, n) \to (Y, q, m)$ is a correspondence of the form $q \circ f \circ p$ with $f \in \mathrm{Corr}^{m-n}(X, Y)$. The category of pure motives is denoted $\underline{\mathrm{Mot}}$.

A correspondence $f \in \mathrm{Corr}^r(X, Y)$ operates on cohomology:

$$f_* : H^q(X) \to H^{q+r}(Y), \quad x \mapsto f_*(x) := (pr_Y)_*\{f \cap (pr_X)^*(x)\}, x \in H^q(X).$$

Here p_X and p_Y are the obvious projections from $X \times Y$ onto X, respectively Y.

Any smooth projective variety X defines an effective motive

$$h(X) := (X, \mathrm{id}).$$

One can extend this functorially by letting a morphisms $f : X \to Y$ correspond to the *transpose* of the graph ${}^{\mathsf{T}}\Gamma_f \in \mathrm{Corr}^0(Y, X)$: the functor goes from the *opposite* of the category of varieties to Grothendieck motives.

There are other motives that play a crucial role: the *Lefschetz motive* $\mathbb{L} := (\mathrm{point}, \mathrm{id}, -1)$ and the *Tate motive* $\mathbb{T} = (\mathrm{point}, \mathrm{id}, 1)$.

The disjoint sum of varieties induces a direct sum on the level of motives: set $h(X) \oplus h(Y) = h(X \coprod Y)$. More generally, if $M = (X, p, m)$, $N = (X', p', m')$ with $m \le m'$ one first rewrites

$$M \otimes \mathbb{L}^{m'-m} = (X \times \mathbb{P}^{m'-m}, q, m') = (Y, q, m')$$

for a suitable projector q and then

$$M \oplus N := (Y \coprod X', q \coprod p', m').$$

In any case, $h(\mathbb{P}^1) = 1 \oplus \mathbb{L}$ as motives. There is also a tensor product:

$$(X, p, m) \otimes (X', p', m') := (X \times X', p \times p', m + m').$$

With this product, $\mathbb{L} \otimes \mathbb{T} = 1$, i.e. $\mathbb{T} = \mathbb{L}^{-1}$.

Example A-3.1. Manin [**Manin**] has shown that for a projective bundle $E = \mathbb{P}(V)$ of a vector bundle over Y of rank k one has:

$$h(E) = \sum_{j=0}^{k-1} h(Y) \otimes \mathbb{L}^j \tag{11}$$

This can be used to calculate the motive of a blow-up $Z = \mathrm{Bl}_Y X$ of X in the codimension c subvariety Y from the split exact sequence (loc. cit.)

$$0 \to h(Y) \otimes \mathbb{L}^c \to h(X) \oplus (h(E) \otimes \mathbb{L}) \to h(Z) \to 0.$$

Indeed, this sequence implies that in the Grothendieck ring $\mathrm{K}_0(\underline{\mathrm{Mot}})$ one has $[h(Y) \otimes \mathbb{L}^c] + [h(Z)] = [h(X)] + [h(E) \otimes \mathbb{L}]$ and then (11) shows

$$[h(Z)] - [h(E)] = [h(X)] - [h(Y)] \tag{12}$$

in accordance with (10).

One has the following comparison between $\mathrm{K}_0(\underline{\mathrm{Var}})$ and $\mathrm{K}_0(\underline{\mathrm{Mot}})$:

Theorem A-3.2. *The map $X \to h(X)$ induces a ring homomorphism*

$$\chi^c_{\mathrm{mot}} : \mathcal{M} = \mathrm{K}_0(\underline{\mathrm{Var}})[\mathbb{L}]^{-1} \to \mathrm{K}_0(\underline{\mathrm{Mot}}).$$

It sends the class of the affine line $\mathbf{A}$ *to the motive* $\mathbb{L} = h(\mathbb{P}^1) - 1$.

Proof: This follows directly from (12). □

The Grothendieck group of pairs. For a good duality theory compatible with Grothendieck motives one needs the Grothendieck group $\widetilde{\mathrm{K}}_0(\underline{\mathrm{Var}})$ for pairs (X, Y) of varieties where $Y \subset X$ is a closed subvariety. It is the free group on those pairs modulo certain relations generated by *excision*, *Gysin maps* and *exactness* as explained below. The isomorphism class of (X, Y) in this ring is denoted $\{(X, Y)\}$. The class of $(X, \varnothing)$ is also denoted $\{X\}$.

- *Excision*: If $f : X' \to X$ is proper and $Y \subset X$ is a closed subvariety such that f induces an isomorphism $X' - f^{-1}Y \cong X - Y$, then $\{(X', f^{-1}Y)\} = (X, Y)$.
- *Gysin maps*: If X is smooth and connected and $D \subset X$ is a smooth divisor, then $\{X - D\} = \{X\} - \{(\mathbb{P}^1 \times D, \{\infty\} \times D)\}$.
- *Exactness*: If $X \supset Y \supset Z$, Y closed in X and Z closed in Y, then $\{(X, Z)\} = \{(X, Y)\} + \{(Y, Z)\}$.

Excision implies that $\{(X, Y)\}$ only depends on the isomorphism class of the pair (X, Y). Using exactness one gets $\{(X, Y)\} = \{X\} - \{Y\}$. Exactness and excision yield $\{(X \sqcup Y)\} = \{X \sqcup Y, Y\} + \{Y\} = \{X\} + \{Y\}$ and $\{(\varnothing)\} = 0$. Set

$$\chi^c(X) := \{(W, W - X)\}, \quad X \subset W,\ X \text{ open and } W \text{ complete}.$$

The excision property implies that $\{(W, W - X)\}$ is independent of the choice of the open embedding. For $Y \subset X$ closed we have

$$\chi^c(X) = \{(W, W - X)\} = \{(W, \overline{Y} \cup (W - X))\} + \\ + \{(\overline{Y} \cup (W - X), W - X)\} = \chi^c(X - Y) + \chi^c(Y),$$

where $\overline{Y}$ denotes the closure of Y in W. Hence χ^c factors through $\mathrm{K}_0(\underline{\mathrm{Var}})$, i.e. there is a group homomorphism

$$\chi^c : \mathrm{K}_0(\underline{\mathrm{Var}}) \to \widetilde{\mathrm{K}}_0(\underline{\mathrm{Var}}), \quad [X] \mapsto \{X\}.$$

One can show that this extends to a ring homomorphism where $\widetilde{\mathrm{K}}_0(\underline{\mathrm{Var}})$ gets a ring structure from

$$\{(X, Y)\} \cdot \{(X', Y')\} := \{(X \times X', X \times Y' \cup Y \times X')\}.$$

One has:

Theorem A-3.3 ([**Bitt1**, Theorem 4.2])**.** *The ring homomorphism* χ^c *is an isomorphism.*

The proof is of the same level of difficulty as Bittner's main theorem 2.1.5. The main point consists in showing that an inverse of χ^c can be explicitly given in terms of a smooth simple normal completion $(\overline{X}, D)$ of a given smooth variety X: assign to $\{X\}$ the element $\sum(-\mathbb{L})^l[D^{(l)}]$ where $D^{(l)}$ is the disjoint union of l-fold intersections of the irreducible components of D.

Note that one can also define

$$\chi_{\text{mot}} : \widetilde{\mathrm{K}}_0(\underline{\mathrm{Var}}) \to \mathrm{K}_0(\underline{\mathrm{Mot}}), \quad \{(X,Y)\} \mapsto h(X) - h(Y).$$

If one sets $\widetilde{\mathcal{M}} := \widetilde{\mathrm{K}}_0(\underline{\mathrm{Var}})[\mathbb{L}]^{-1}$, then χ^c extends to an isomorphism

$$\chi^c : \mathcal{M} \to \widetilde{\mathcal{M}}.$$

The dual motive for $h(X)$ is

$$[h(X)]^* := \mathbb{L}^{-d_X} \otimes [h(X)], \quad d_X = \dim X.$$

Then the duality operator D extends to $\widetilde{\mathcal{M}}$ and it exchanges the two motivic characters:

$$[\chi_{\text{mot}}\{X\}]^* = \mathbb{L}^{-d_X} \otimes \chi^c_{\text{mot}}[X] = \chi^c_{\text{mot}}\mathsf{D}([X]).$$

The obvious diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\chi^c_{\text{mot}}} & \\ \downarrow \chi^c & & \mathrm{K}_0(\underline{\mathrm{Mot}}) \\ \widetilde{\mathcal{M}} & \xrightarrow{\chi_{\text{mot}}} & \end{array}$$

is *not* commutative. This is shown by the following simple example: let Y be a smooth projective variety and let X be an affine cone on Y. Then, since X is contractible we have $\chi_{\text{mot}}(X) = 1$, but since $X - \{\text{vertex } v\} = Y \times (\mathbf{A} - v)$, we have $\chi^c_{\text{mot}}(X) - \chi_{\text{mot}} \circ \chi^c(X) = \chi^c_{\text{mot}}(X) - 1 = \chi^c_{\text{mot}}(Y)(\mathbb{L} - 1)$. This proves in fact that the difference of the two motives belongs to the ideal generated by $1 - \mathbb{L} \in K_0\underline{\mathrm{Mot}}(k)$.

Lifting the Hodge characteristic to Chow motives. For motives cohomology can be defined as follows:

$$H^k(M) := \mathrm{Im}(p^* : H^{k+2n}(X) \to H^{k+2n}(X)), \quad M = (X, p, n).$$

Here p^* denotes the induced action of the correspondence on cohomology. This action preserves the Hodge structures. Note however that $H^k(M)$ is pure of weight $k + 2n$ reflecting the weight n of the motive $M = (X, p, n)$. Nevertheless, one clearly has:

Theorem. *The Hodge characteristic "lifts to motives", i.e. one has a commutative diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\chi^c_{\text{mot}}} & \mathrm{K}_0(\underline{\mathrm{Mot}}) \\ & {\scriptstyle \chi_{\text{Hdg}}} \searrow \quad \swarrow {\scriptstyle \chi_{\text{Hdg;mot}}} & \\ & \mathrm{K}_0(\mathfrak{hs}) & \end{array}$$

where for a motive M one sets $\chi_{\text{Hdg;mot}}(M) = \sum (-1)^k H^k(M)$.

There is strong refinement of this result due to Gillet and Soulé and, independently, to Guillen and Navarro Aznar. [1] First of all one needs to replace the category $\underline{\mathrm{Var}}$ by $\mathbb{Z}\underline{\mathrm{Var}}$ whose objects are the same but where now

[1] To explain this result I freely make use of some concepts and results that will be treated in later chapters.

a morphism $X \to Y$ is a finite formal sum $\sum n_f f$, $n_f \in \mathbb{Z}$ of morphisms $f : X \to Y$. Chain complexes in this new category exist:

$$\cdots C_n \to C_{n-1} \to \cdots \to C_0, \quad C_j \in \underline{\mathrm{Var}}$$

and form an abelian category $C(\mathbb{Z}\underline{\mathrm{Var}})$. These arise naturally if you consider simplicial varieties (see Lecture 5)

$$X_\bullet = \quad \{X_0 \xleftarrow{d_0} X_1 \; \substack{\xleftarrow{d_1} \\ \xleftarrow{d_0}} \; X_2 \; \substack{\xleftarrow{d_2} \\ \xleftarrow{d_1} \\ \xleftarrow{d_0}} \; X_3 \qquad \cdots\}$$

Indeed its *associated simple complex*

$$\mathbf{s}X_\bullet := \left\{X_0 \xleftarrow{d_0} X_1 \xleftarrow{d_0 - d_1} X_2 \xleftarrow{d_0 - d_1 + d_2} X_3 \cdots\right\}$$

belongs to this category $C(\mathbb{Z}\underline{\mathrm{Var}})$. The functor $X \mapsto h(X)$ which maps $X \in \underline{\mathrm{Var}}$ to the motive (X, id_X) now extends to a functor

$$C(h) : C(\mathbb{Z}\underline{\mathrm{Var}}) \to \mathbf{H}\underline{\mathrm{Mot}},$$

where for any abelian category $\mathfrak{A}$ one lets $\mathbf{H}(\mathfrak{A})$ be the homotopy category of *cochain complexes* in $\mathfrak{A}$.

In Lecture 5 a mixed Hodge structure will be put on the cohomology of any variety U. A basic step is Theorem 5.2.6 which asserts that for any sufficiently nice compactification X of U the pair $(X, D := X - U)$ admits a so called cubical hyperresolution whose associated augmented semi-simplicial variety is denoted

$$\epsilon : (X_\bullet, D_\bullet) \to (X, D).$$

Here we don't need to know the precise definition. It is sufficient to know that each of the X_i are smooth varieties and the $D_i \subset X_i$ are normal crossing divisors and that together they form a simplicial object in the category of such pairs. One associates to the augmentation ϵ the inclusion of semi-simplicial varieties

$$i_\bullet : D_\bullet \to X_\bullet$$

which induces a map of complexes $\mathbf{s}i_\bullet : \mathbf{s}D_\bullet \to \mathbf{s}X_\bullet$ and yields the main new invariant from [**G-S**], the *weight complex*

$$W^\bullet(U) := \left[\mathrm{Cone}^{\bullet-1}(C(h)\mathbf{s}i_\bullet)\right] \in \mathbf{H}(\underline{\mathrm{Mot}}).$$

There is a spectral sequence converging[2] to compactly supported cohomology for U:

$$E_2^{j,n} = H^j\left(\bullet \mapsto H^n(W^\bullet(U)\right) \implies H^{j+n}(X, D) = H_c^{j+n}(U) \qquad (13)$$

Let me explain the left hand side. The individual varieties $V_k := D_k \sqcup X_{k-1}$ which occur in the cone complex assemble into a chain complex. Cohomology behaving contravariantly, this then gives the cochain complex

$$\cdots \to H^n(V_k) \to H^n(V_{k+1}) \to \cdots$$

and $E_2^{j,n}$ is the j-th cohomology of this cochain complex.

The result alluded to is:

[2]It degenerates at E_2 for $\mathbb{Q}$-coefficients

Theorem A-3.4 ([**G-S, G-N**])**.** *The weight complex $W(U)$ up to canonical isomorphism depends only on the original variety U. The construction is categorical in the sense that*

- *a proper map $f : X \to Y$ induces $f^* : W(Y) \to W(X)$ and composable proper maps f, g satisfy $(g \circ f)^* = f^* \circ g^*$;*
- *open immersions $i : U \hookrightarrow X$ induce $i_* : W(U) \to W(X)$; composable open immersion behave likewise functorial;*

It obeys the product rule $W(X \times Y) = W(X) \otimes W(Y)$ and it has a motivic property: If $i : Y \hookrightarrow X$ is a closed immersion with complement $j : U = X - Y \hookrightarrow X$ one has a distinguished triangle (see § 4.1)

$$W(U) \xrightarrow{i_*} W(X) \xrightarrow{j^*} W(U) \to [1].$$

As an application, if $\Gamma : \underline{\mathrm{Mot}} \to \mathfrak{A}$ is any covariant functor to an abelian category $\mathfrak{A}$, for any variety X one gets a complex $\Gamma(W(X))$ in $\mathfrak{A}$ for which one calculates the cohomology $H^i\Gamma(W(X))$. The theorem implies for instance that this assigment is contravariantly functorial in X and one has long exact sequences associated to pairs (X, Y) where $Y \subset X$ is a closed subvariety. This can first of all be used to put a weight filtration on *integral* cohomology with compact support which induces the one on rational cohomology: just take $\Gamma = H^n$ and apply (13). In [**G-S**] it is shown, using the product rule, that the torsion group $H^3(S \times S')$ where S is a (singular) Kummer surface (i.e. the quotient of an abelian surface by the standard involution) and S' an Enriques surface admits a weight filtration with at least 2 non-trivial steps so that this gives new motivic invariants which do not follow from the mixed Hodge structure!

Furthermore, the motivic property can be used to prove directly without any Hodge theory that there exists a Hodge characteristic which lifts to motives. I refer to [**G-S, G-N**] for details.

LECTURE 3

The Hodge Characteristic, Examples

Assuming that the Hodge characteristic is additive and multiplicative I shall now illustrate how in some cases the Hodge structures on the graded parts of the weight filtration can be found.

Example 1: A punctured curve $C = \bar{C} - \Sigma$ where $\bar{C}$ is a smooth projective curve and Σ is a set of M points. Additivity implies that $\chi_{\text{Hdg}}(C) = \chi_{\text{Hdg}}(\bar{C}) - N \cdot 1$. Let $V_g = H^1(\bar{C})$ be the usual weight one Hodge structure. it follows that $\chi_{\text{Hodge}}(C) = (1 - M) \cdot 1 - V_g + \mathbb{L}$. Topological considerations give that $b_0(C) = b_2^c(C) = 1$. Also $H^2(C) = 0$ since C is affine. Then Theorem 2.2.5 gives that $H^1(C)(1) = [H_c^1(C)]^*$ and $W_1 H^1(C) = H^1(\bar{C}) = V_g$. Again using restriction on the weights given by Theorem 2.2.5 and Corr. 2.2.6 one calculates the graded Hodge structure $\text{Gr}_W H^*(C)$ and $\text{Gr}_W H_c^*(C)$: From the table one finds $\chi_{\text{Hdg}}(C) = (1 - M) \cdot 1 - V_g^*(-1) + \mathbb{L}$.

TABLE 3.1. Cohomology of the punctured curve C

	H^0	H^1	H^2	H_c^2	H_c^1	H_c^0
weight 0	1	0	0	0	$M-1$	0
weight 1	0	V_g	0	0	$V_g^*(-1)$	0
weight 2	0	$(M-1)\mathbb{L}$	0	$\mathbb{L}$	0	0

That $V_g^*(-1) = V_g$ follows from the existence of a non-degenerate skew pairing on V_g, the cup-product pairing. It gives a polarization for the Hodge structure. See Remark 12.1.2. Note also that the analogous character for ordinary cohomology reads $1 - V_g + (M-1) \cdot \mathbb{L}$ which is different from the Hodge characteristic!

Example 2: U is the complement inside $\mathbb{P}^2$ of d lines in general position, meaning that at most two lines pass through a point. The scissor relations give

$$\chi_{\text{Hdg}}(U) = g - (d-1)\mathbb{L} + \mathbb{L}^2,$$

where $g = 1 + \frac{d(d-3)}{2}$, the genus of a smoothing of the d lines.

To find the actual Hodge structures, one needs some topological results: since U is affine of dimension 2, there is no cohomology beyond degree 2. Moreover, the fundamental group of U is known to be the free abelian group on $(d-1)$ generators and so $b_1(U) = b_c^3(U) = (d-1)$. Then one finds for $\text{Gr}_W H(U)$:

Example 3: D a singular curve with N double points forming $\Sigma \subset D$. Consider the normalization $n : \widetilde{D} \to D$, a curve of genus g with $H^1(\widetilde{D}) = V_g$, a Hodge structure of weight one. Since $n^{-1}\Sigma$ consists of $2N$ points, one has

TABLE 3.2. Cohomology of the complement of d lines

	H^0	H^1	H^2	H_c^2	H_c^3	H_c^4
0	1	0	0	g		0
2	0	$(d-1)\mathbb{L}$	0	0	$(d-1)\mathbb{L}$	0
4	0	0	$g\mathbb{L}^2$	0	0	$\mathbb{L}^2$

$\chi_{\text{Hdg}}(C) - N\cdot 1 = \chi_{\text{Hdg}}(\widetilde{D}) - 2N\cdot 1$ and hence $\chi_{\text{Hdg}}(D) = (1-N) - V_g + \mathbb{L}$.

TABLE 3.3. Cohomology of the singular curve D

	H^0	H^1	H^2
weight 0	1	N	0
weight 1	0	V_g	0
weight 2	0	0	$\mathbb{L}$

Example 4: Consider a singular connected surface X consisting of k smooth surfaces $X_1,\dots,X_k$ meeting transversally in ℓ smooth curves $C_1,\dots,C_\ell$. Finally suppose that X has m ordinary triple points. The intersection configuration is described by a dual graph Γ, a triangulated real surface. For any smooth projective variety Y write

$$\boxed{V_{g_Y} = H^1(Y), \quad g_Y = h^{1,0}(Y),}$$

a Hodge structure of weight 1 with Hodge numbers $h^{0,1} = h^{0,1} = g_Y$, and

$$\boxed{W_{p_Y,q_Y}(Y) = H^2(Y), \quad p_Y = h^{2,0}(Y),\ q = h^{1,1}(Y),}$$

a Hodge structure of weight 2 with Hodge numbers $h^{2,0} = h^{0,2} = p_Y$, $h^{1,1} = q_Y$. By the Lefschetz decomposition, which is recalled in a later Lecture (see (43)), one has $H^3 = \mathbb{L}V_{g_Y}$ and hence

$$\begin{aligned}\chi_{\text{Hdg}}(X) &= \sum_i \chi_{\text{Hdg}}(X_i) - \sum_j \chi_{\text{Hdg}}(C_j) + m\cdot 1\\ &= \chi_{\text{top}}(\Gamma) - \Big(\sum_i V_{g_{X_i}}(X_i)(1+\mathbb{L}) + \sum_j V_{g_{C_j}}(C_j)\Big)\\ &\qquad\qquad + \sum_i W_{p_{X_i},q_{X_i}}(X_i) + k\mathbb{L}^2\end{aligned}$$

Example 5: Let $S_d \subset \mathbb{P}^3$ be a smooth surface of degree d. The Betti-numbers have been calculated earlier (Example 1.1.1 (3)): $b_1 = b_3 = 0$ and $b_2 = d(6-4d+d^2)-2$. The Hodge structure $H^2(S_d)$ splits into $\mathbb{L}\oplus H^2_{\text{prim}}(S_d)$ and

$$h^{2,0}(S_d) = p_d := \binom{d-1}{3} \tag{14}$$

$$h^{1,1}(S_d) = q_d := \frac{1}{3}d(2d^3 - 6d + 7). \tag{15}$$

Then one has $\chi_{\mathrm{Hdg}}(S_d) = (1 + \mathbb{L} + \mathbb{L}^2) + W_{p_d,q_d-1}(S_d)$.

Example 6: The singular fibre $\bar{X}_0$ of the family $tF + F_1F_2 = 0$ inside $\mathbb{P}^3 \times \mathbb{C}$ where $\deg F = d, \deg F_1 = d_1, \deg F_2 = d_2$. The surfaces $F = 0$, $\bar{D}_1 = \{F_1 = 0\}$ and $\bar{D}_2 = \{F_2 = 0\}$ are supposed to be smooth and the total space $\bar{X}$ of the family is supposed to have only ordinary double point singularities at the $M = d_1d_2d$ points $\bar{T}_j$ of intersection of $F = 0$ with $\bar{D}_1$ and $\bar{D}_2$. Blow up $\mathbb{P}^3 \times \Delta$ at these points. The proper transform X of $\bar{X}$ is smooth and the new fibre at 0 consists of two components, the proper transforms D_1 of $\bar{D}_1$ and D_2, the proper transform of $\bar{D}_2$. These are isomorphic to hypersurfaces of $\mathbb{P}^3$ of degrees d_1, d_2 respectively blown up in the T_j. The exceptional $\mathbb{P}^3$ meets X in M quadrics E_j. The original double curve $\bar{C} = \{F_1 = F_2 = 0\}$ stays isomorphic to its proper transform C, but the point $\bar{T}_j$, originally on the curve C gets replaced by the quadric E_j which meets the C in a new point T_j, a triple point on the new fibre $X_0 = D_1 \cup D_2 \cup \bigcup_j E_j$. The new double locus consists of $C \cup \bigcup_j L_j^{(1)} \cup L_j^{(2)}$ where $L_j^{(1)}$ and $L_j^{(2)}$ are two rulings of E_j in which this quadric meets D_1 and D_2.

Note that there is another possible degeneration obtained by blowing up the indeterminacy locus $F = 0 = F_1F_2$ of the meromorphic function $\mathbb{P}^3 \to \Delta$ given by F_1F_2/F which consists of two curves $C_1 = \{F = F_1 = 0\}$ and $C_2 = \{F = F_2 = 0\}$. One either first blows up $\mathbb{P}^3$ in C_1 and then in the proper transform of C_2 or the other way around. This gives a morphism $\mathrm{Bl}(\mathbb{P}^3) \to \Delta$. The resulting degeneration is different: one gets a two-component degeneration. Depending on the order of blowing up, the first component is isomorphic to the hypersurface $F_1 = 0$ and the second component isomorphic to $F_2 = 0$ blown up in d_1d_2d points, or the other way around. These meet transversally in a curve isomorphic to C. See [**Per**, p. 125] for details. This degeneration is obtained if one blows down the exceptional quadrics of the first degeneration along either one of the rulings. This makes the quadrics disappear and only one of the remaining components is a blown up hypersurface.

Let me calculate the Hodge characteristic in the two cases using the calculations of Example 5. In the first case one gets

$$\begin{aligned}\chi_{\mathrm{Hdg}} &= 1 + (2M + 1)) \cdot \mathbb{L} + M \cdot \mathbb{L}^2 + V_{\frac{1}{2}(d_1d_2(d_1+d_2-4)+1} \\ &\quad + W_{p_{d_1},q_{d_1}-1} + W_{p_{d_2},q_{d_2}-1}, \quad M = d_1d_2(d_1 + d_2).\end{aligned}$$

while for the second case one has

$$\begin{aligned}\chi_{\mathrm{Hdg}} &= 1 + (M + 1) \cdot \mathbb{L} + 2\mathbb{L}^2 + V_{\frac{1}{2}(d_1d_2(d_1+d_2-4)+1} \\ &\quad + W_{p_{d_1},q_{d_1}-1} + W_{p_{d_2},q_{d_2}-1}.\end{aligned}$$

These two characters are completely different!
To find the actual Hodge structures more information is needed which will be explained later. See § 6.3 and more specifically f Table 6.1.

Example 7: The singular fibre X_0 of the family $\{Q_1Q_2 + tF_4 = 0\} \subset \mathbb{P}^3 \times \mathbb{C}$, where Q_1, Q_2 are homogeneous quadric forms with zero locus smooth quadrics and F_4 a quartic form whose zero locus is a smooth surface. The total space of this family has 16 isolated singularities at $\{t = Q_1 = Q_2 =$

$F_4 = 0\} \subset \mathbb{P}^3$ and hence one blows up once at each of these points to get a family $p : X \to \mathbb{C}$. The original double curve $C = \{t = Q_1 = Q_2 = 0\}$, a smooth elliptic curve passes through these points p_j and after blowing up the exceptional surface E_j meets the proper transform of C at a new point q_j which becomes a triple point for the fibre $S = p^{-1}(0)$ over the origin. The surface E_j is a quadric and the two rulings through q_j are the two curves along which the proper transforms of the two quadrics $\{Q_1 = 0\}$ and $\{Q_2 = 0\}$ meet. So the double locus of S consists of C and 32 exceptional curves isomorphic to $\mathbb{P}^1$ and there are 16 triple points q_j. It follows that

$$\chi_{\text{Hdg}}(X_0) = 1 + (V_1 + 35\mathbb{L}) + 18\mathbb{L}^2.$$

For topological reasons (or using a Mayer-Vietoris argument) one has $H^1(S) = H^3(S) = 0$. With $H^1(C) = V_1$, a weight one Hodge structure, one thus finds for the even cohomology:

Table 3.4. Cohomology of the singular surface X_0

	H^0	H^2	H^4
weight 0	1	0	0
weight 1	0	V_1	0
weight 2	0	$35\mathbb{L}$	0
weight 3	0	0	0
weight 4	0	0	$18\mathbb{L}^2$

The entry $18\mathbb{L}^2$ reflects the fact that S has 18 components.

LECTURE 4

Hodge Theory Revisited

4.1. A Digression: Cones in the Derived Category

I shall give a brief summary of what is needed from the theory cones in the derived categories. See for instance [**P-S**, Appendix A].

Start from any abelian category $\mathfrak{A}$. In the homotopy category $\mathbf{H}(\mathfrak{A})$ of complexes in $\mathfrak{A}$ the objects are the complexes of objects in $\mathfrak{A}$ while the morphisms are the equivalence classes $[f]$ of morphisms between complexes up to homotopy. In the derived category $D(\mathfrak{A})$ more flexibility is allowed in that the quasi-isomorphism become invertible. Recall that a *quasi-isomorphism*, denoted

$$s : K \xrightarrow[\sim]{\text{qis}} L$$

is a morphism between complexes such that the induced morphisms $H^q(s) : H^q(K) \to H^q(L)$ are all isomorphisms. Hence, if $f : K \to M$ is a morphism of complexes $K \xrightarrow{f} L \xleftarrow{s} K$ defines a morphism the derived category, usually denoted $[f]/[s]$ because it resembles fractions of homotopy classes of maps. One of the problems with derived categories is that exact sequences don't make sense. These should be replaced by distinguished triangles. By definition these come from the exact sequence of the cone as I now explain.

Definition 4.1.1. Let K, L be two complexes in an additive category and let $f : K \to L$ be a morphism of complexes. The *cone* $\mathrm{Cone}(f)$ *over* f is the complex

$$\begin{aligned} \mathrm{Cone}^k(f) = &\quad K^{k+1} \oplus L^k \\ \text{differential} = &\quad \begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}. \end{aligned}$$

Suppose now that there is a short exact sequence of complexes

$$0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0. \tag{16}$$

Then the map $\mathrm{Cone}(f) \to M$ given by $\begin{pmatrix} x \\ z \end{pmatrix} \mapsto g(z)$ is a quasi-isomorphism and so in $D(\mathfrak{A})$ one may replace M by $\mathrm{Cone}(f)$. Note that there also is a projection morphism

$$p_1 : \mathrm{Cone}(f) \to K[1], \quad \begin{pmatrix} x \\ z \end{pmatrix} \mapsto -x \tag{17}$$

where the $(-)$-sign is needed since one wants a morphism of complexes. The important point here is that this morphism gives the coboundary for the long exact sequence in cohomology associated to (16). Indeed, this map fits into another short exact sequence. the *sequence of the cone*:

$$0 \to L \to \mathrm{Cone}(f) \to K[1] \to 0. \tag{18}$$

If you consider(16) in $D(\mathfrak{A})$ you get the standard example of a *distinguished triangle* which is usually written as

$$\begin{array}{ccc} K & \xrightarrow{\;f\;} & L \\ & {}_{p_1}\nwarrow^{[1]} \quad \swarrow & \\ & \mathrm{Cone}(f). & \end{array} \tag{19}$$

or, equivalently, as

$$K \xrightarrow{f} L \to \mathrm{Cone}(f) \to [1].$$

The meaning of this is as follows. The arrows are morphisms in the derived category which are represented by the morphism f, the injection $L \to \mathrm{Cone}(f)$ and the projection $p_1 : \mathrm{Cone}(f) \to K[1]$ from (17). In the derived category the first two maps can also be represented by the two maps in the short exact sequence (16). Its associated long exact sequence involves induced morphisms and hence do not depend on choices within $D(\mathfrak{A})$. So already two of the maps in the above triangle induce the same maps as those from (16). As for the connecting morphism, one can easily prove that it is induced by the map p_1 from (17). Hence, and this is the crucial advantage of working with cones in the derived category, **all morphisms in the long exact sequence**

$$\cdots H^q(K) \to H^q(L) \to H^q(M) = H^q(\mathrm{Cone}(f)) \to H^{q+1}(K) = H^q(K[1]) \cdots$$

are induced by morphisms of complexes.

4.2. Classical Hodge Theory via Hodge Complexes

The flexibility of the derived category is needed to change between different complexes computing cohomology in order to construct compatible weight and Hodge filtrations: in one incarnation the weight filtration is more natural, in another the Hodge filtration is easier to define.

At the base of this lies a functorial construction at the level of sheaves and which is due to Godement. For any sheaf of abelian groups $\mathcal{F}$ on a topological space X, the Godement resolution $C(\mathcal{F})$ is inductively defined as follows. Put $C^0(\mathcal{F})(U) = \coprod_{x \in U} \mathcal{F}_x$ which generates a presheaf with associated sheaf $C^0(\mathcal{F})$; next put $\mathcal{Z}^0(\mathcal{F}) = \mathcal{F}$, $C^k(\mathcal{F}) = C^0(\mathcal{Z}^{k-1})$ and $\mathcal{Z}^k(\mathcal{F}) = C^k(\mathcal{F})/Z^{k-1}(\mathcal{F})$. There are natural morphisms $d : C^k(\mathcal{F}) \to C^{k+1}(\mathcal{F})$ obtained as follows. By definition $Z^{k+1}(\mathcal{F})$ is a quotient sheaf of $C^k(\mathcal{F})$ while it is a subsheaf of $C^{k+1}(\mathcal{F})$ and d is the composition $C^k(\mathcal{F}) \twoheadrightarrow Z^{k+1}(\mathcal{F}) \hookrightarrow C^{k+1}(\mathcal{F})$ of the obvious two natural morphisms. One has $d \circ d = 0$ since $Z^{k+1}(\mathcal{F})$ gets killed in forming $Z^{k+2}(\mathcal{F}) = C^{k+1}(\mathcal{F})/Z^k(\mathcal{F})$ and hence $\{C(\mathcal{F}), d\}$ is a complex.

Clearly this construction is functorial and can be extended also to complexes of sheaves $\mathcal{F}$: the $C^p(\mathcal{F}^q)$ form a double complex and one takes $sC(\mathcal{F})$, the associated single complex. One way of defining the sheaf cohomology is to take the cohomology of the complex of global sections for the Godement resolution:

$$H^k(X, \mathcal{F}) := H^k(\Gamma(X, C(\mathcal{F})).$$

For a complex $\mathcal{F}$ it is convenient to use the language of the derived section functor $R\Gamma$ which from a complex of sheaves of abelian groups produces a complex of abelian groups :

$$R\Gamma(\mathcal{F}) = \Gamma(X, sC\mathcal{F}).$$

Its cohomology gives *hypercohomology*:

$$\mathbb{H}^k(X, \mathcal{F}) = R\Gamma^k(\mathcal{F}) := H^k\left[\Gamma(sC(\mathcal{F}))\right].$$

If $f : \mathcal{F} \to \mathcal{G}$ is a quasi-isomorphism (of complexes of sheaves) the induced morphisms $\mathbb{H}^k(X, \mathcal{F}) \to \mathbb{H}^k(X, \mathcal{G})$ turn out to be isomorphisms.

***Remark* 4.2.1.** The following constructions employing the Godement resolution are constantly used:

(1) If a complex $\mathcal{F}$ comes with a filtration F, then it induces one on $sC(\mathcal{F})$ and one writes $R\Gamma(\mathcal{F}, F)$ for the derived section complex equipped with this filtration.
(2) If $f : X \to Y$ is a continuous map, $\mathcal{F}$ a complex of sheaves on X, the *derived direct image* $Rf_*\mathcal{F}$ is a complex of sheaves on Y given by $sf_*(C(\mathcal{F}))$. Almost by definition one then has
$$\mathbb{H}^k(Y, Rf_*\mathcal{F}) \simeq \mathbb{H}^k(X, \mathcal{F}).$$
(3) If $\mathcal{G}$ is a complex of sheaves on Y there is a natural *adjunction morphism*
$$f^\# : \mathcal{G} \to Rf_*(f^{-1}\mathcal{G}).$$
Combing this with the previous remark one thus has induced maps
$$\mathbb{H}^k(f^\#) : \mathbb{H}^k(Y, \mathcal{G}) \to \mathbb{H}^k(X, f^*\mathcal{G}).$$

***Remark* 4.2.2.** Let $f : X \to Y$ be any continuous map. The relation of the cone-complex with the topological cone $\operatorname{Cone}(f)$ can now be explained. The adjunction map for $\underline{\mathbb{Q}}_Y$ reads

$$f^\# : \underline{\mathbb{Q}}_Y \to Rf_*(\underline{\mathbb{Q}}_X)$$

and I **claim** that

$$\mathbb{H}^q(Y, \operatorname{Cone}(f^\#)) = \widetilde{H}^{q+1}(\operatorname{Cone}(f)).$$

The proof goes as follows. The exact sequence of the cone (18) yields

$$\cdots \to \mathbb{H}^q(Y, \underline{\mathbb{Q}}_Y) \to \mathbb{H}^q(Y, Rf_*\mathbb{Q}_X) \to \mathbb{H}^q(\operatorname{Cone}(f^\#)) \to \mathbb{H}^{q+1}(Y, \underline{\mathbb{Q}}_Y) \to \ldots$$

and hence

$$\cdots \to H^q(Y) \xrightarrow{f^*} H^q(X) \to \mathbb{H}^q(\operatorname{Cone}(f^\#)) \to H^{q+1}(Y) \to \ldots.$$

On the topological side there is the exact sequence

$$\cdots \widetilde{H}^q(Y) \xrightarrow{f^*} \widetilde{H}^q(X) \to \widetilde{H}^{q+1}(\operatorname{Cone}(f)) \xrightarrow{j^*} \widetilde{H}^q(X) \cdots \qquad (20)$$

as will be explained shortly. From this it follows that indeed

$$\mathbb{H}^q(Y, \operatorname{Cone}(f^\#)) \simeq \widetilde{H}^{q+1}(\operatorname{Cone}(f)).$$

To explain (20), one argues as follows. If $f : X \to Y$ is a continuous map, the *cylinder* $\operatorname{Cyl}(f)$ *over* f is obtained by gluing the usual cylinder $X \times I$ to Y upon identifying a bottom point $(x, 1) \in X \times I$ with $f(x) \in Y$. The map $i : x \mapsto (x, 0)$ identifies X as a subspace of $\operatorname{Cyl}(f)$. The inclusion

$Y \to \mathrm{Cyl}(f)$ of Y as the bottom of this cylinder is a homotopy equivalence since the cylinder retracts onto Y. Under this retraction the inclusion $X \to \mathrm{Cyl}(f)$ (as the top) deforms into the map $f : X \to Y$. The (topological) mapping cone $\mathrm{Cone}(f)$ is obtained by collapsing the top of the mapping cylinder to a single point v. The quotient space $\mathrm{Cyl}(f)/X$ is canonically homeomorphic to $\mathrm{Cone}(f)$. Since for any pair (X, A) with A closed, we have $H^*(X, A) = \widetilde{H}^*(X/A)$ the long exact sequence associated to the pair $(\mathrm{Cyl}(f), X)$ therefore can be identified with (20), whereby completing the proof of the claim.

A particular case of the above is relative cohomology which comes from the cone on the inclusion map as explained before (Example 2.1.1).

If X is a perfect topological space (see the first lecture), the (singular) cohomology group $H^k(X)$ coincides with $H^k(X, \underline{\mathbb{Q}}_X)$, the cohomology for the constant sheaf $\underline{\mathbb{Q}}_X$ and so it is possible to calculate $H^k(X)$ using any complex quasi-isomorphic to the constant sheaf. For example, if X is a complex manifold, complex cohomology $H^k(X; \mathbb{C}) = H^k(X) \otimes \mathbb{C}$ can be calculated as the hypercohomology of the holomorphic De Rham complex $\Omega^\bullet_X$ since the inclusion $\underline{\mathbb{C}}_X \to \Omega^\bullet_X$ is a quasi-isomorphism (the holomorpic Poincaré lemma). The trivial filtration

$$F^p(\Omega^\bullet_X) = [0 \to \cdots \to 0 \to \Omega^p_X \to \cdots \Omega^n_X \to 0], \quad n = \dim X$$

passes to hypercohomology and it follows from classical Hodge theory that for smooth projective manifolds this F-filtration on complex cohomology coincides with the Hodge filtration. The morphism of complexes $\alpha : \underline{\mathbb{Q}}_X \to \underline{\mathbb{C}}_X \to \Omega^\bullet_X$ where the first morphism is inclusion, is called the *comparison morphism*. The triple

$$\boxed{\mathbb{Q}^{\mathrm{Hdg}}_X = (\underline{\mathbb{Q}}_X, (\Omega^\bullet_X, F), \alpha) \quad \text{(the \textit{Hodge-De Rham complex of sheaves on } X\text{)}}}$$

is the basic example of a so-called *Hodge complex of sheaves* of weight 0.

In order to define this concept one needs to enlarge the collection of allowed “maps”.

Definition 4.2.3. Let K, L two bounded below complexes in an abelian category. A *pseudo-morphism* between K and L is a chain of morphisms of complexes

$$K \xrightarrow{f} K_1 \xleftarrow[\sim]{\text{qis}} K_2 \xrightarrow[\sim]{\text{qis}} \cdots \xrightarrow[\sim]{\text{qis}} K_{n+1} = L.$$

It induces a morphism in the derived category. I shall denote such a pseudo-morphism by

$$K \dashrightarrow L.$$

If also f is a quasi-isomorphism I shall call it a *pseudo-isomorphism*. It becomes invertible in the derived category. These will be denoted by

$$K \overset{\text{qis}}{\underset{\sim}{\dashrightarrow}} L.$$

A *morphism* between two pseudo-morphisms $K \xrightarrow{f} K_1 \xleftarrow[\sim]{\text{qis}} \cdots \xrightarrow[\sim]{\text{qis}} K_m$ and $L \xrightarrow{g} L_1 \xleftarrow[\sim]{\text{qis}} \cdots \xrightarrow[\sim]{\text{qis}} L_m$ consists of a sequence of morphism $K^j \to L^j$, $j = 1, \ldots, m$ such that the obvious diagrams commute. Note that such morphisms are only possible between chains of equal length.

Definition 4.2.4. Let X be a topological space. A *Hodge complex of sheaves* of weight m on X consists of the following data

- A bounded below complex of sheaves of $\mathbb{Q}$-vector spaces $\mathcal{K}$ such that $\dim \mathbb{H}^k(X, \mathcal{K}) < \infty$,
- A filtered complex of sheaves of complex vector spaces $(\mathcal{K}_{\mathbb{C}}, F)$ and a *comparison morphism*: a pseudo-morphism $\alpha : \mathcal{K} \dashrightarrow \mathcal{K}_{\mathbb{C}}$ in the category of sheaves of $\mathbb{Q}$-vector spaces on X inducing a pseudo-isomorphism (of sheaves of $\mathbb{C}$-vector spaces)
$$\alpha \otimes \mathrm{id} : \mathcal{K} \otimes \mathbb{C} \overset{\text{qis}}{\underset{\sim}{\dashrightarrow}} \mathcal{K}_{\mathbb{C}},$$
and such that the $\mathbb{Q}$-structure on $\mathbb{H}^k(\mathcal{K}_{\mathbb{C}})$ induced by α and the filtration induced by F determine a Hodge structure of weight $k+m$ for all k. Moreover, there is a technical condition: one requires that the differentials of the derived complex $R\Gamma(X, \mathcal{K}_{\mathbb{C}})$ strictly preserve the F-filtration.

The notion of a morphism $(f, f_{\mathbb{C}}, \kappa) : (\mathcal{K}, (\mathcal{K}_{\mathbb{C}}, W), \alpha_{\mathcal{K}}) \to (\mathcal{L}, (\mathcal{L}_{\mathbb{C}}, W), \alpha_{\mathcal{L}})$ between sheaves of Hodge complexes of weight m is what you think it is; for instance $\kappa : \alpha_{\mathcal{K}} \to \alpha_{\mathcal{L}}$ is a morphism of pseudo-morphisms.

If one passes to the derived section functor, morphisms of Hodge complexes of sheaves clearly give morphisms between the associated Hodge complexes and hence, on the level of hypercohomology there are induced morphisms of Hodge structures. Note also that for the relevant Hodge structures morphisms of Hodge complexes of sheaves that are quasi-isomorphisms give isomorphisms so that one can employ the derived category

$D^+F\underline{\mathbb{C}}_X$:	bounded below filtered complexes of sheaves of $\mathbb{C}$-vector spaces on X.

Note that the category of filtered complexes of sheaves of $\mathbb{C}$-vector spaces is *not* abelian. Indeed this would mean that the derivatives are strict, which is not automatic. However, there still is a canonical way to form the derived category. In this category *filtered quasi-isomorphisms* $f : (K, F) \to (K', F')$ are inverted. By definition these are morphism of filtered complexes which on the graded parts induces quasi-isomorphisms. Note that using this notion one can also speak of pseudo-(iso)morphisms in the category of *filtered* complexes.

As explained in [**P-S**, Chap 2.3], using this language, classical Hodge theory can be summarized as follows:

Theorem 4.2.5. *Let X be a smooth projective variety. The Hodge-De Rham complex $\mathbb{Q}_X^{\mathrm{Hdg}}$ which was introduced just above Defiition 4.2.3 induces the classical Hodge structure on $H^k(X)$; this structure only depends on the class of $\mathbb{Q}_X^{\mathrm{Hdg}}$ in $(D^{\mathrm{b}}\underline{\mathbb{Q}}_X, D^+F\underline{\mathbb{C}}_X)$. Indeed, one has $(H^k(X), F, \alpha)) = R\Gamma(\mathbb{Q}_X^{\mathrm{Hdg}})$, where $\alpha : H^k(X) \hookrightarrow H^k(X; \mathbb{C})$ is the coefficient homomorphism.*

***Remark* 4.2.6.** Given a Hodge complex of sheaves (of weight m) L, the triple $R\Gamma\mathsf{L}$ is called a *Hodge complex* of weight m. Its cohomology groups $H^k(R\Gamma\mathsf{L})$ are weight $k+m$ Hodge structures.

For what follows it should be remarked that several algebraic constructions can now be done: direct sums, tensor products, Tate twists and shifts. The direct sum construction is obvious, the other ones require an explanation:

1) **Tensor products**. This uses a universal construction for filtered complex (K, F), (L, G):

$$\begin{aligned}(K \otimes L)^n &= \bigoplus_{i+j=n} K^i \otimes L^j, \quad d(x \otimes y) = dx + (-1)^{\deg x} x \otimes dy \\ (F \otimes G)^m &= \sum_{i+j=m} F^i K \otimes G^j L.\end{aligned}$$

For Hodge complexes of sheaves of weight m and m' put

$$(K, (K_{\mathbb{C}}, F), \alpha) \otimes (L, (L_{\mathbb{C}}, G), \beta) = (K \otimes L, (K_{\mathbb{C}}, F) \otimes (L_{\mathbb{C}}, G), \alpha \otimes \beta).$$

It gives a Hodge complexes of sheaves of weight $m + m'$.

2) **Tate twists**

$$(K, (K_{\mathbb{C}}, F), \alpha)(k) = (K \otimes (2\pi \mathrm{i})^k, (K_{\mathbb{C}}, F[k]), \alpha(k)).$$

where $\alpha(k)$ is induced by α followed by multiplication by $(2\pi\mathrm{i})^k$. This is a Hodge complex of sheaves of weight $m - 2k$.

3) **Shifts**

$$(K, (K_{\mathbb{C}}, F), \alpha)[k] = (K[k], (K_{\mathbb{C}}[k], F[k]), \alpha[k]),$$

a Hodge complex of sheaves of weight $m + k$.

Let me illustrate these constructions by giving a proof of the fact that the Künneth decomposition for a product $X \times Y$ of smooth projective varieties X and Y is an isomorphism of Hodge structures. Consider the natural map

$$\underline{\mathbb{Q}}_X^{\mathrm{Hdg}} \boxtimes \underline{\mathbb{Q}}_Y^{\mathrm{Hdg}} \to \underline{\mathbb{Q}}_{X \times Y}^{\mathrm{Hdg}}$$

Both complexes are Hodge complexes of sheaves on $X \times Y$ and on the level of hypercohomology they thus give a morphism of Hodge structures

$$\bigoplus_{i+j=k} H^i(X) \boxtimes H^j(X) \to H^k(X \times Y).$$

This is exactly the map which gives the Künneth isomorphism. Now apply Cor. 2.2.4 which implies that the above morphism is indeed an isomorphism of mixed Hodge structures.

One important construction, that of the cone cannot be done in this framework because by its very nature it will mix up the weights and so can only be performed if one enlarges the scope so as to include mixed complexes.

LECTURE 5

Mixed Hodge Theory

5.1. Mixed Hodge Complexes

Definition 5.1.1. A *mixed Hodge complex of sheaves* on X

$$\mathsf{K} = ((\mathcal{K}, W), (\mathcal{K}_{\mathbb{C}}, W, F), \alpha)$$

consists of the following data

- A bounded complex of sheaves of $\mathbb{Q}$-vector spaces $\mathcal{K}$ equipped with an *increasing* filtration W;
- a complex of sheaves of complex vector spaces $\mathcal{K}_{\mathbb{C}}$ equipped with an increasing filtration W and a decreasing filtration F, together with a pseudo-morphism in the category of sheaves of *filtered* $\mathbb{Q}$-vector spaces on X, the comparison pseudo-morphism

$$\alpha : (\mathcal{K}, W) \dashrightarrow (\mathcal{K}_{\mathbb{C}}, W).$$

The latter is required to induce a pseudo-isomorphism

$$\alpha \otimes \mathrm{id} : (\mathcal{K} \otimes \mathbb{C}, W) \overset{\text{qis}}{\underset{\sim}{\dashrightarrow}} (\mathcal{K}_{\mathbb{C}}, W)$$

such that the following axiom is satisfied:

for all $m \in \mathbb{Z}$ the triple $\mathrm{Gr}^W_m \mathsf{K} = \big(\mathrm{Gr}^W_m \mathcal{K}, (\mathrm{Gr}^W_m \mathcal{K}_{\mathbb{C}}, F), \mathrm{Gr}^W_m(\alpha)\big)$ *is a Hodge complex of sheaves of weight m.*

The following theorem is a basic observation due to Deligne (see e.g. [**P-S**, Theorem 3.18] for a proof).

Proposition 5.1.2. *Let $(\mathcal{K}, (\mathcal{K}_{\mathbb{C}}, W), \alpha)$ be a mixed Hodge complex of sheaves on X. Let W and F stand for the filtrations they induce on the hypercohomology groups $\mathbb{H}^k(X, \mathcal{F})$. Then $(\mathbb{H}^k(X, \mathcal{K}), W[k], F)$ is a mixed Hodge structure.*

***Remark* 5.1.3.** Given a mixed Hodge complex of sheaves L, the triple $R\Gamma\mathsf{L}$ is called a *mixed Hodge complex*. So its cohomology groups $H^k(R\Gamma\mathsf{L})$ have natural mixed Hodge structures.

One can form direct sums, tensor products, Tate twists and shifts as in the pure case. Likewise, the definition of a morphism of mixed Hodge complexes of sheaves resembles the one in the pure case and should be obvious. Since after tensoring with $\mathbb{C}$ the pseudo-morphisms become pseudo-isomorphisms it is natural to work in the derived category

$D^+FW\underline{\mathbb{C}}_X$:	bounded below bi-filtered complexes of sheaves of $\mathbb{C}$-vector spaces on X.

Again the category of bi-filtered complexes in an abelian category in general is not abelian, but one can still form the derived category.

5.2. Cohomology of Varieties Have a Mixed Hodge Structure

Let me first say a few words about the cohomology of a smooth complex variety U. Details are to be found in [**P-S**, Ch 4]. One chooses a compactification X such that $D = X - U$ is a simple normal crossing divisor. The log-complex $\Omega^\bullet_X(\log D)$ of rational forms on X, regular on U and with at most logarithmic poles along D can be defined as follows.

Consider a local chart with coordinates $(z_1, \dots, z_n)$ in which D has the equation $z_1 \cdots z_k = 0$. Then $\Omega^1(\log D)$ by definition is freely generated over $\mathcal{O}_X$ by $\{dz_1/z_1, \dots, dz_k/z_k, dz_{k+1}, \dots, dz_n\}$ and $\Omega^k_X(\log D) = \bigwedge^k \Omega^1_X(\log D)$. The usual derivative on forms makes this into a complex. It comes with a weight filtration which counts the number of dz_j/z_j in local coordinates $(z_1, \dots, z_n)$ adapted to the normal crossing divisor. In other words

$$W_m(\Omega^\bullet_X(\log D) = \left[\begin{array}{l} 0 \to \mathcal{O}_X \cdots \to \Omega^{m-1}_X(\log D) \to \\ \qquad\qquad \to \Omega^m_X(\log D) \wedge [\mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{n-m}_X] \end{array} \right].$$

Let me explain why this can be defined over the rationals. First note that the logarithmic complex computes *complex* cohomology on U. For rational cohomology one uses the complex $Rj_*\underline{\mathbb{Q}}_U$, a complex on X which by definition consists of the direct image of the Godement resolution, i.e. $Rj_*\underline{\mathbb{Q}}_U = j_*(C\underline{\mathbb{Q}}_U)$. There is a filtration which can be put on any complex K, the so called *canonical filtration*:

$$[\tau(K)]_p = \left[\cdots \to K^{p-1} \to \mathrm{Ker}(d) \to 0 \cdots 0\right].$$

One then proves that the inclusion $(\Omega^\bullet_X(\log D), W) \to (\Omega^\bullet_X(\log D), \tau)$ is a filtered quasi-isomorphism. So one can put the canonical filtration on $Rj_*\underline{\mathbb{Q}}_U$ which provides the rational component of a Hodge complex of sheaves and a comparison pseudo-morphism

$$\alpha : (Rj_*\underline{\mathbb{Q}}_U, \tau)) \to (\Omega^\bullet_X(\log D), \tau) \overset{\text{qis}}{\underset{\sim}{\longleftarrow}} (\Omega^\bullet_X(\log D), W).$$

The Hodge filtration is the same as in the smooth projective case: you take the trivial filtration F on the log-complex. This then gives a canonical way to associate to the pair (X, D) the mixed Hodge complex denoted

$$\mathbb{Q}^{\mathrm{Hdg}}_{(X,D)} := ((Rj_*\underline{\mathbb{Q}}_U, \tau), (\Omega^\bullet_X(\log D), W, F), \alpha).$$

One can show that moreover, the resulting mixed Hodge structure on $H^k(U)$ does not depend on choices.

To handle singular varieties one uses the concepts of simplicial and cubical varieties. For details see [**P-S**, Ch. 5].

Definition 5.2.1. (1) The *simplicial category* Δ is the category with objects the ordered sets $[n] := \{0, \dots, n\}$, $n \in \mathbb{Z}_{\geq 0}$, and with morphisms non-decreasing maps. If one only considers the strictly increasing maps one speaks of the *semi-simplicial category* $\triangle$. The *cubical category* is the category $\square$ whose objects are the finite subsets of $\mathbb{N}$ and for which $\mathrm{Hom}(I, J)$ consists of a single element if $I \subset J$ and else is empty. In this category the *face maps* $\delta^j : [n-1] \to [n]$ are non-decreasing maps defined by $\delta^j(k) = k$ if $k < j$ and $= (k+1)$ if $k \geq j$. The n-truncated simplicial, semi-simplicial category, respectively cubical category is the full sub-category of the category Δ_n, $\triangle_n$, respectively $\square$ whose objects are the $[k]$ with $k \in [n-1]$.

(2) A *simplicial object* in a category $\mathfrak{C}$ is a contravariant functor $K_\bullet : \Delta \to \mathfrak{C}$. A morphism between such objects is to be understood as a morphism of corresponding functors. Similarly one speaks of a *semi-simplicial objects* and *cubical objects*. One obtains an n-*simplicial object* by replacing Δ by Δ_n and similarly for n-*semi-simplicial objects*. Set

$$K_n := K_\bullet[n] \text{ (the set of } n\text{-simplices)}, \quad d_j = K(\delta^j).$$

Moreover, for a cubical object X and $I \subset \mathbb{N}$ finite write

$$\begin{aligned} X_I &:= X(I) \\ d_{IJ} &:= X(I \hookrightarrow J) : X_J \to X_I, \quad I \subset J. \end{aligned}$$

So, a simplicial object $K_\bullet$ in $\mathfrak{C}$ consists of objects $K_n \in \mathfrak{C}$, $n = 0, \ldots,$ and for each non-decreasing map $\alpha : [n] \to [m]$, there are morphisms $d_\alpha : K_m \to K_n$.

Every $(n+1)$-cubical variety (X_I) gives rise to an augmented n-semi-simplicial variety $X_\bullet \to Y$ in the following way. Put

$$X_k = \coprod_{|I|=k+1} X_I, \quad k = 0, \ldots, n$$

and for each inclusion $\beta : [s] \to [r]$ and $I \subset [n]$ with $|I| = r+1$ writing $I = \{i_0, \ldots, i_r\}$, $i_0 < \ldots < i_r$, one lets

$$X(\beta)|_{X_I} = d_{IJ}, \quad J = \beta(I) = \{i_{\beta(0)}, \ldots, i_{\beta(s)}\}.$$

For all $I \subset [n]$ there is a well-defined map $d_{\varnothing I} : X_I \to X_\varnothing = Y$. This is the desired augmentation. Note that this correspondence is functorial.

Examples 5.2.2. (1) The blow-up diagram in § 2.1 is an example of a 2-cubical variety. See Remark 2.1.2.

(2) The normalization $n : \widetilde{C} \to C$ of a singular curve C can be viewed as a 2-cubical variety: let $X_\varnothing = C$, $X_0 = \widetilde{C}$, X_1 = singular points $\Sigma \subset C$, $X_{01} = n^{-1}\Sigma$ with the obvious maps:

$$\begin{array}{ccc} X_{01} & \to & X_1 \\ \downarrow & & \downarrow \\ X_0 & \to & X_\varnothing. \end{array}$$

(3) Let X be a **normal crossing variety**, i.e. $X = \bigcup_{i=1}^k X_k$ where X_k are smooth projective varieties of the same dimension d meeting like the coordinate hyperplanes in $\mathbb{C}^{d+1}$. Set

$$\begin{aligned} X_I &= X_{i_1} \cap \cdots \cap X_{i_m}, \quad I = \{i_1, \ldots, i_m\} \\ a_I &: X_I \hookrightarrow X \\ X(m) &= \coprod_{|I|=m} X_I \\ a_m &= \coprod_{|I|=m} a_I : X(m) \hookrightarrow X. \end{aligned}$$

The $X(m)$ are smooth projective. For $I = (i_1, \ldots, i_m)$ put

$$I_j = (i_1, \ldots, \widehat{i_j}, \ldots, i_m).$$

There are m natural inclusions $a_I^j : X_I \hookrightarrow X_{I_j} \subset X(m+1)$ which assemble to give a an augmented semi-simplicial space

$$X \xleftarrow{\ \epsilon\ } X(1) \;\substack{\longleftarrow \\ \longleftarrow}\; X(2) \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; X(3) \qquad \cdots$$

Suppose that $\epsilon : X_\bullet \to Y$ is an augmentation. A sheaf $\mathcal{F}$ on $X_\bullet$ is a collection of sheaves $\mathcal{F}^{(k)}$ on the X_k compatible with the maps induced by the face maps. More precisely, a sheaf on a simplicial space is a simplicial object in the category of pairs $(X, \mathcal{F})$ of topological spaces and sheaves on them whose morphisms are pairs $(X \xrightarrow{f} Y, f^\# : \mathcal{G} \to f_*\mathcal{F})$.

The vector spaces $\Gamma(X_q, C^p(\mathcal{F}^{(q)}))$ form a double complex $\Gamma C(\mathcal{F})$. Now define

$$H^k(X_\bullet, \mathcal{F}) := H^k(s\Gamma C(\mathcal{F})).$$

The sheaves $\epsilon_* C^p(\mathcal{F}^{(q)})$ form a double complex $\epsilon_* C(\mathcal{F})$ on Y and one then sets

$$R\epsilon_*\mathcal{F} := s[\epsilon_* C(\mathcal{F})], \tag{21}$$

a complex of sheaves on Y whose k-th hypercohomology equals $H^k(X_\bullet, \mathcal{F})$:

$$H^k(X_\bullet, \mathcal{F}) = \mathbb{H}^k(Y, R\epsilon_*\mathcal{F}).$$

Examples are the sheaves $\Omega^k_{X_\bullet}$ and the constant sheaf $\underline{\mathbb{Q}}_{X_\bullet}$.

The natural map

$$\epsilon^\# : \underline{\mathbb{Q}}_Y \to R\epsilon \underline{\mathbb{Q}}_{X_\bullet}$$

plays an important role for calculating the cohomology of Y: the kind of cubical varieties one needs are such that this map induces an isomorphism $H^k(X_\bullet, \underline{\mathbb{Q}}_{X_\bullet}) \simeq H^k(Y)$. This is guaranteed provided $\epsilon^\#$ is a quasi-isomorphism. Such augmentations are called *of cohomological descent*.

There is a simple test for cohomological descent which uses the construction of the *geometric realization* $|X_\bullet|$ which is defined as follows. Let $K_\bullet$ be a simplicial space. Every non-decreasing map $f : [q] \to [p]$ has a geometric realization $|f| : \Delta_q \to \Delta_p$ where Δ_k is the standard affine k-simplex. Set

$$|K_\bullet| = \coprod_{p=0}^{\infty} \Delta_p \times K_p / R,$$

where the equivalence relation R is generated by identifying $(s, x) \in \Delta_q \times K_q$ and $(|f|(s), y) \in \Delta_p \times K_p$ if $x = K(f)y$ for all non-decreasing maps $f : [q] \to [p]$. The topology on $|K|$ is the quotient topology under R obtained from the direct product topology (note that the K_p are topological spaces by assumption). If $\epsilon : X_\bullet \to Y$ is an augmented semi-simplicial complex variety, there is an induced continuous map $|\epsilon| : |X_\bullet| \to Y$. I quote without proof:

Proposition 5.2.3. *Let $\epsilon : X_\bullet \to Y$ be an augmented semi-simplicial complex variety. If $|\epsilon| : |X_\bullet| \to Y$ is proper and has contractible fibres, the augmented semi-simplicial complex variety is of cohomological descent.*

Examples 5.2.4. (1) The geometric realization of the blow-up diagram has been described in Lecture 2. From it one sees that the fibre of $|\epsilon|$ over $x \in X$ is a point if $x \notin Y$ and the cone over $\pi^{-1}x$ if $x \in Y$. It follows that the blow-up diagram is of cohomological descent.
(2) Consider example 5.2.2. 2), the normalization of the curve C. The geometric realization of the 2-cubical variety of the example consists of the normalization $\widetilde{C}$ to which for each singular point p of C with counter-images $p_1, \dots, p_r \in \widetilde{C}$ you glue a graph which is a "star" with vertex p and edges from p to the points p_i. The map $|\epsilon|$ contracts these stars to the point $p \in C$ and it is the identity on $C - \Sigma$. Hence the fibre of $|\epsilon|$ over $x \in C$ is either a point or a contractible star. It follows that the augmented semi-simplicial variety is of cohomological descent.
(3 Continue with Example 5.2.2.3). Each variety X_I with $|I| = m + 1$ meets exactly m of the X_J for which $|J| = m$. So the associated geometric realisation $|\epsilon|$ has as fibres exactly one m-simplex over each point lying on X_I. It follows that ϵ is of cohomological descent.

Note that being of cohomological descent is a purely local property and so if for some locally closed subvariety $U \subset Y$ with $\epsilon^{-1}U = U_\bullet$, one also has $H^k(U_\bullet, \underline{\mathbb{Q}}_{U_\bullet}) = H^k(U)$, a remark to be used for open varieties.

But let me first suppose that X is proper. One wants of course all the X_I be smooth and proper so that the Hodge complexes $\underline{\mathbb{Q}}_{X_I^{\mathrm{Hdg}}}$ can be used which push down to X. Indeed, by (21) all of its constituents push down to X giving $R\epsilon_*\underline{\mathbb{Q}}_{X_\bullet^{\mathrm{Hdg}}}$ which is a mixed Hodge complex of sheaves on X. Since the dimensions of the X_I in general vary, this complex is usually not pure. To calculate cohomology of X one uses a *cubical hyperresolution*, i.e. a cubical variety such that the X_I, $I \neq \varnothing$ are smooth and proper and the associated augmented simplicial variety is of cohomological descent. It can be shown that these exist (see Theorem 5.2.6 for the full statement) and that the resulting mixed Hodge structure on the $H^k(X)$ no longer depends on choices.

The case of an open variety is slightly more complicated, but the idea is the same. One needs the concept of a *log-pair* (X, D). To say what this is, first suppose that X is irreducible. Then by definition $U = X - D$ is smooth and D is a simple normal crossing divisor. If X is reducible, possibly with components of different dimensions, this definition should be modified: (X, D) is a log-pair if a given component of X *either*
— does not meet D,
— *or* is entirely contained in D,
— *or* is cut by D in a normal crossing divisor.

Definition 5.2.5. Let X be a compactification U. Put $T = X - U$. A *cubical hyperresolution of* (X, T) is a cubical variety $\{X_I\}$ whose associated augmented variety $\epsilon : X_\bullet \to X$ has the following properties:

(1) the augmentation maps are proper;
(2) the X_k are smooth and the inverse images $\epsilon^{-1}T$ on each component C of the X_k are either empty, or C in its entirety, or a normal

crossing divisor on C. Set

$$D_\bullet = \epsilon^{-1}T.$$

It follows that $X_\bullet, D_\bullet) \to (X,T)$ is an *augmental* simplicial log-pair.

(3) the augmentation $X_\bullet \to X$ is of cohomological descent.

The full result reads as follows:

Theorem 5.2.6. *Let U be a complex algebraic variety. There exists a cubical hyperresolution of (X,T), where X is a compactification of U and $T = X - U$. Then $R\epsilon_*\underline{\mathbb{Q}}^{\mathrm{Hdg}}_{(X_\bullet,D_\bullet)}$ is a mixed Hodge complex of sheaves on U. The resulting mixed Hodge structure on $H^k(U)$ does not depend on choices.*

LECTURE 6

Motivic Hodge Theory

6.1. The Hodge Characteristic Revisited

A first goal is to show that given a pair of complex algebraic varieties (U, V) the resulting exact sequence in cohomology is a sequence of mixed Hodge structures. But first one has to reinterpret the relative cohomology group $H^k(U, V)$.

Lemma 6.1.1. *Let $j : V \hookrightarrow U$ be the inclusion of a closed set. and let $j^\# : C(\underline{\mathbb{Q}}_U) \to C(j_* \underline{\mathbb{Q}}_V$ be the induced map. Then*

$$H^k(U, V) = \mathbb{H}^{k-1}(\text{Cone}(j^\#)).$$

Proof: This is the conjunction of two facts proven earlier. See Example 2.1.1 and Remark 4.2.2: $H^k(U, V) = \widetilde{H}^k(\text{Cone}(j)) = \mathbb{H}^{k-1}(\text{Cone}(j^\#))$. □

Next, replace $\underline{\mathbb{Q}}$ by the mixed Hodge complex of sheaves $\underline{\mathbb{Q}}^{\text{Hdg}}$ constructed from suitable compactifications $\bar{U}$, $\bar{V}$ of U and V respectively.

The third step uses a construction which provides a mixed Hodge complex of sheaves on the cone of a morphism $\mathsf{f} : \mathsf{K} = (\mathcal{K}, (\mathcal{K}_{\mathbb{C}}, F), \alpha) \to \mathsf{L} = (\mathcal{L}, (\mathcal{L}_{\mathbb{C}}, F), \beta)$ between such complexes. This construction, the *mixed cone* $(\text{Cone}(\mathsf{f}), W), (\alpha, \beta))$ goes as follows:
Put

$$W_m \text{Cone}(\mathsf{f})^p = W_{m-1}\mathcal{K}^{p+1} \oplus W_m \mathcal{L}^p, \quad F^r \text{Cone}(\mathsf{f})^p_{\mathbb{C}} = F^r \mathcal{K}^{p+1}_{\mathbb{C}} \oplus F^r \mathcal{L}^p_{\mathbb{C}},$$

together with the comparison morphism given by

$$(\alpha, \beta) : \text{Cone}(\mathsf{W}) \dashrightarrow \text{Cone}(\mathsf{W}, F)_{\mathbb{C}}.$$

Lemma 6.1.2. *The mixed cone is a mixed Hodge complex of sheaves.*

Proof: A morphism of pseudo-morphisms consists of morphisms between the constituents of the chains which make up a pseudo-morphisms, and such that the obvious diagrams commute. This implies that each such diagram defines a morphism of cones or a quasi-isomorphism of cones or an inverse of such. In this way one gets the pseudo-morphism defining the comparison morphism for a cone.

The map f maps $W_m(\mathcal{K})$ to $W_m(\mathcal{L})$ and so on the graded mixed cone

$$\text{Gr}^W_m(\text{Cone}(\mathsf{f}) = \text{Gr}^W_{m-1} \mathcal{K}[1] \oplus \text{Gr}^W_m(\mathcal{L})$$

the contribution of f to the differential $\begin{pmatrix} -d & 0 \\ \mathsf{f} & d \end{pmatrix}$ vanishes. Since both $\text{Gr}^W_{m-1} \mathcal{K}[1]$ and $\text{Gr}^W_m(\mathcal{L})$ are pure Hodge complexes of sheaves of weight m the results follows. □

***Remark* 6.1.3.** If one would work with comparison morphisms in the derived category, as Deligne does, one cannot define such cones since a morphism in this set-up involves diagrams which commute only *up to homotopy*. Indeed, there is no guarantee that the comparison maps for the cone as defined above commute with the derivative (remember that the derivative of the cone of a map involves the map itself). They might for one choice, but they might not commute for another choice.

For this reason Deligne's set-up has to be modified. In the geometric setting one has explicit representatives, as I have explained in previous Lectures. It follows that one then automatically gets morphisms of mixed complexes (of sheaves) and hence the above cone construction can be applied.

To achieve the goal formulated at the start of this section, the crucial point now is to apply the cone construction to a suitable derived category associated to mixed Hodge complexes of sheaves on $\bar{U}$ where. Indeed, one can consider the "fibre product" of $D^+W\underline{\mathbb{Q}}_{\bar{U}}$ and $D^+FW\underline{\mathbb{C}}_{\bar{U}}$ in the sense that one only considers pairs $([\mathcal{L}],[\mathcal{L}_{\mathbb{C}}]) \in D^+W\underline{\mathbb{Q}}_{\bar{U}} \times D^+FW\underline{\mathbb{C}}_{\bar{U}}$ related by some morphism $\mathcal{L} \to \mathcal{L}_{\mathbb{C}}$ in the *derived category* which becomes an isomorphism after tensoring with $\mathbb{C}$. In this category there is the triangle of the cone and its associated long exact sequence automatically gives a long exact sequence of mixed Hodge structures for the pair (U, V):

$$\cdots H^k(U,V) \to H^k(U) \to H^k(V) \to H^{k+1}(U,V)\cdots .$$

Looking at triples (U, V, W) of algebraic varieties one gets an exact sequence of mixed Hodge structures

$$\cdots H^k(U,V) \to H^k(U,W) \to H^k(V,W) \to H^{k+1}(U,V)\cdots .$$

To see this, look at the coboundary map $H^k(V,W) \to H^{k+1}(U,V)$. This is the composition $H^k(V,W) \to H^k(V) \to H^{k+1}(U,V)$ where the first is restriction and the second is the coboundary for the pair (U, V).

After all this preparation, the promised proof for the scissor relations can be given:

Proposition 6.1.4. *Let X be an algebraic variety and $Y \subset X$ a closed subvariety. Then $\chi_{\mathrm{Hdg}}(X) = \chi_{\mathrm{Hdg}}(Y) + \chi_{\mathrm{hdg}}(X - Y)$.*

Proof: Let $\bar{X}$ be a compactification of X, $T = \bar{X} - X$, and $\bar{Y} \subset \bar{X}$ the closure of Y in $\bar{X}$. Consider the triple $(\bar{X}, \bar{Y} \cup T, T)$. Since $H^k_c(X - Y) = H^k(\bar{X}, \bar{Y} \cup T)$ and since $\bar{Y} \cup T$ is also a compactification of Y, the resulting exact sequence becomes

$$\cdots H^k_c(X-Y) \to H^k_c(X) \to H^k_c(Y) \to H^{k+1}_c(X-Y)\cdots \quad \square$$

6.2. Products

The fact that the Hodge characteristic preserves products can be shown in two steps as follows.

Step 1: external products. Let (X, Y) and $X', Y')$ be two pairs of complex algebraic varieties. The external product

$$H^i(X,Y) \otimes H^j(X',Y') \to H^{i+j}(X \times X', (Y \times X' \cup X \times Y'))$$

is a morphism of Hodge structures. Roughly this goes as follows: start from cubical hyperresolutions for (X,Y) and (X',Y'). Then, using the first barycenter-construction one obtains a cubical hyperresolution $((X\times Y)_\bullet,(X\times D'\cup D\times Y')_\bullet))$ of the pair $(X\times X',(Y\times X'\cup X\times Y'))$. Let $\epsilon:(X_\bullet,D_\bullet)\to(X,Y)$, $\epsilon':(X'_\bullet,D'_\bullet)\to(X',Y')$ and $\epsilon'':((X\times Y)_\bullet,(X\times D'\cup D\times Y')_\bullet))$ be the associated augmented simplicial varieties. The natural morphism of mixed Hodge complexes of sheaves

$$R\epsilon_*\underline{\mathbb{Q}}^{\mathrm{Hdg}}_{(X_\bullet,D_\bullet)}\boxtimes R\epsilon'_*\underline{\mathbb{Q}}^{\mathrm{Hdg}}_{(X'_\bullet,D'_\bullet)}\to R\epsilon''_*\underline{\mathbb{Q}}^{\mathrm{Hdg}}_{((X\times Y)_\bullet,(X\times D'\cup D\times Y')_\bullet)}$$

does the job. See [**P-S**, pp 134–135] for the details of the proof in a similar case.

Step 2: completion of the proof. Apply the previous step to a compactification $(\bar X,X)$ of X and a compactification $(\bar Y,Y)$ of Y. For better visibility, write $\partial X=\bar X-X$ and similarly for ∂Y. Then $\partial(X\times Y)=X\times\partial Y\cup\partial X\times Y$. Since $H^i_c(X)=H^i(\bar X,\partial X)$ and likewise for Y the previous step gives a morphism

$$\bigoplus_{i+j=k}H^i_c(X)\otimes H^j_c(X)\to H^k_c(X\times Y)$$

of mixed Hodge structures which at the same time is a $\mathbb{Q}$-vector space isomorphism. Now apply Corr. 2.2.4. It follows that $\chi_{\mathrm{Hdg}}(X\times Y)=\chi_{\mathrm{Hdg}}(X)\chi_{\mathrm{Hd}}(Y)$.

6.3. Further Examples

Since this is the first time spectral sequences come up in these Lectures I digress a bit to recall the salient facts needed about these. A *spectral sequence* in an abelian category $\mathfrak{A}$ consists of terms (E_r,d_r) $r=0,1\dots,$ each of which is a complex in $\mathfrak{A}$ and whose cohomology gives the next term E_{r+1}. The terms E_r are bigraded, say $E_r=\bigoplus_{p,q\in\mathbb{Z}}E_r^{p,q}$ and d_r has bidegree $(r,-r+1)$.

A filtered complex (K,F) in an abelian category gives the standard example of such a spectral sequence. Let me assume that the filtration is *decreasing* and, in addition, obeys a certain finiteness condition: on each K^m it has finite length. The spectral sequence associated to such a filtration is defined by

$$\begin{aligned}Z_r^{p,q}=&\ \ \mathrm{Ker}\left(d:F^pK^{p+q}\to K^{p+q+1}/(F^{p+r}K^{p+q+1})\right)\\ B_r^{p,q}=&\ \ F^{p+1}K^{p+q}+d(F^{p-r+1}K^{p+q-1})\\ E_r^{p,q}=&\ \ Z_r^{p,q}/(B_r^{p,q}\cap Z_r^{p,q}).\end{aligned}$$

This makes also sense for $r=\infty$. The finiteness condition implies for p and q fixed, from a certain index r on one has

$$\begin{aligned}Z_r^{p,q}=Z_{r+1}^{p,q}=\cdots=&\ \ Z_\infty^{p,q}:=\mathrm{Ker}(d:F^pK^{p+q}\to K^{p+q+1})\\ B_r^{p,q}=B_{r+1}^{p,q}=\cdots=&\ \ B_\infty^{p,q}:=F^{p+1}K^{p+q}+dK^{p+q-1}\end{aligned}$$

and so the $E_r^{p,q}=E_\infty^{p,q}$ from a certain index r on. For the first terms of the spectral sequence one has

$$\left.\begin{aligned}E_0^{p,q}=&\ \ \mathrm{Gr}_F^p(K^{p+q})\\ E_1^{p,q}=&\ \ H^{p+q}(\mathrm{Gr}_F^p(K))\end{aligned}\right\}\tag{22}$$

An easy calculation shows that the differential of the complex K induces the differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ of a spectral sequence; indeed $d_r \circ d_r = 0$ and E_{r+1} is the cohomology of the resulting complex. Next, a computation shows that

$$E_\infty^{p,q} = \mathrm{Gr}_F^p H^{p+q}(K),$$

where F is the filtration induced on cohomology:

$$\begin{aligned} F^p(H^n(K)) = \mathrm{Im}(H^n(F^p(K)) &\xrightarrow{H^n(i)} H^n(K)), \\ \text{with } i : F^p(K) &\hookrightarrow K \text{ the inclusion.} \end{aligned}$$

One summarises this by saying that *the spectral sequence converges to the filtered cohomology of the complex* or that *the spectral sequence abuts to* $H^{p+q}(K)$ or that $H^{p+q}(K)$ is the *abutment* of the spectral sequence, and one writes

$$H^{p+q}(\mathrm{Gr}_F^p(K)) \Longrightarrow H^{p+q}(K).$$

Let me now come back to the example of a normal crossing variety (Example 5.2.2 (3)). Recall that $X = \bigcup X_i$ where the X_i are smooth projective and meet as the coordinate hyperplanes in $\mathbb{C}^{d+1}$, $d = \dim X$. The disjoint union of the m-fold intersections was denoted $X(m)$. The cohomology of the $X(m)$ carry Hodge structures and

$$\chi_{\mathrm{Hdg}}(X) = \sum (-1)^m \chi_{\mathrm{Hdg}}(X(m)).$$

To calculate the actual graded parts of the weight filtration one proceeds as follows. For $I = (i_1, \ldots, i_m)$ and $I_j = (i_1, \ldots, \widehat{i_j}, \ldots, i_m)$ the natural inclusions $a_I^j : X_I \hookrightarrow X_{I_j}$ combine into $a_m^j = \oplus_{|I|=m} a_I^j : X(m) \hookrightarrow X(m-1)$ which induce the Mayer-Vietoris maps

$$\alpha_m^{(k)} = \oplus_j (-1)^j (a_m^j)^* : E_1^{m-1,k} = H^k(X(m-1)) \to H^k(X(m)) = E_1^{m,k}.$$

which are the differentials of the *Mayer-Vietoris spectral sequence*. This spectral sequence abuts to the cohomology of X (since the associated augmented semi-simplicial variety is of cohomological descent). The associated filtration is known to coincide with the weight filtration of the the mixed Hodge structure. Moreover, from the general theory of mixed Hodge structures it is known that the Mayer-Vietoris spectral sequence, which coincides with the weight spectral sequence, degenerates at the E_2-term. Each of the terms in this spectral sequence has a Hodge structure and this makes it possible to give explicit expressions for the Hodge structure on the graded parts of the weight filtration. Indeed, $E_2^{m,k}$ is a subquotient of $H^k(X(m))$ and is isomorphic as a Hodge structure to $\mathrm{Gr}_k^W H^{k+m-1}(X)$. I explain how this works in the following example.

Example 6.3.1. Normal surface degenerations. In this case the Mayer-Vietoris spectral sequence is rather simple. There are two maps which are relevant:

$$\bigoplus_i H^k(X_i) \xrightarrow{\alpha_k} \bigoplus_j H^k(C_j), \quad \alpha_k = a_2^{(k)},\ k = 1, 2.$$

Recall that Γ is the dual graph for the configuration of the intersecting surfaces. One finds:

$H^4(X) = W_4H^4(X) = \bigoplus_i H^4(X_i)$

$\mathrm{Gr}_3^W H^3(X) = \bigoplus_j H^3(X_i)$ $\qquad \mathrm{Gr}_2^W H^3(X) = \mathrm{Coker}(\alpha_2)$

$\mathrm{Gr}_2^W H^2(X) = \mathrm{Ker}(\alpha_2)$ $\qquad \mathrm{Gr}_1^W H^2(X) = \mathrm{Coker}(\alpha_1)$ $\quad W_0H^2(X) = H^2(\Gamma)$

$\mathrm{Gr}_1^W H^1(X) = \mathrm{Ker}(\alpha_1)$ $\qquad W_0H^1(X) = H^1(\Gamma)$.

From this the following table can be constructed.

TABLE 6.1. Cohomology of the singular surface X

	H^0	H^1	H^2	H^3	H^4
weight 0	1	$b_1(\Gamma)$	$b_2(\Gamma)$	0	0
weight 1	0	$\mathrm{Ker}(\alpha_1)$	$\mathrm{Coker}(\alpha_1)$	0	0
weight 2	0	0	$\mathrm{Ker}(\alpha_2)$	$\mathrm{Coker}(\alpha_2)$	0
weight 3	0	0	0	$\bigoplus_i \mathbb{L} \otimes V_{g_{X_i}}(X_i)$	0
weight 4	0	0	0	0	$k\mathbb{L}^2$

The reader can verify that this agrees with the value for the Hodge characteristic as given in Example 2 of Lecture 3.

LECTURE 7

Motivic Aspects of Degenerations

7.1. The Nearby Cycle Complex

Let me summarise some results from [**P-S**, Ch. 11]. Consider a relative situation; X is a complex manifold, $\Delta \subset \mathbb{C}$ the unit disk and $f : X \to \Delta$ a holomorphic map which is smooth over the punctured disk Δ^*. Let me say that f is a *one-parameter degeneration*. Let $X_t = f^{-1}t$ be the fibre over t and $x \in X_0$ a point on the singular fibre.

In this setting one has the notion of the *Milnor fibre* of f at x, $\mathrm{Mil}_{f,x}$: take t very close to 0, say at a distance $\eta > 0$, and form $X_t \cap B(x, r)$ where $0 < \eta \ll r \ll 1$. For small enough $\eta \ll r$ the diffeomorphism type of this manifold does not depend on η and r and any representative is called the Milnor fibre. It is well known that there is a fibre preserving retraction $r : X \to X_0$ so that there are induced maps $r_t : X_t \to X_0$. The complex of sheaves $(Rr_t)_* \underline{\mathbb{Q}}_{\mathbb{Q}_t}$ lives on the singular fibre X_0 and is called the (topological) *complex of sheaves of nearby cycles*. The stalk at x of its cohomology sheaves give the cohomology of the Milnor fibre. Observe at this point that

$$\mathbb{H}^k(X_0, (Rr_t)_* \underline{\mathbb{Q}}_{\mathbb{Q}_t}) = \mathbb{H}^k(X_t, \underline{\mathbb{Q}}_{X_t}) = H^k(X_t). \tag{23}$$

This description is however not complex-analytic and one has to replace it with one which is. Consider the *specialisation diagram*

$$\begin{array}{ccccc} X_\infty & \xrightarrow{k} & X & \xleftarrow{i} & X_0 \\ \big\downarrow \tilde{f} & & \big\downarrow f & & \big\downarrow \\ \mathfrak{h} & \xrightarrow{\mathrm{e}} & \Delta & \leftarrow & \{0\} \end{array}$$

where $\mathfrak{h}$ is the complex upper half plane, $\mathrm{e}(z) := \exp(2\pi \mathrm{i} z)$ and where

$$X_\infty := X \times_{\Delta^*} \mathfrak{h}.$$

Note that this manifold retracts onto any of the smooth fibres X_z and can be seen as an object in the homotopy category which is canonically associated to the smooth part of the family. It can be used to define the *monodromy action*: the translation $z \mapsto z + 1$ in the upper half-plane can be lifted (non-canonically) to a diffeomorphism $X_z \to X_{z+1}$, whence an action on X_t, $t = \mathrm{e}(z)$, the so-called geometric monodromy. Geometric monodromy is well-defined up to homotopy and hence there is a well-defined map

$$T : H^*(X_t) \to H^*(X_t), \quad t \in \Delta^*,$$

the *monodromy operator*. This gives an action of the fundamental group $\pi_1(\Delta^*)$.

For t real and positive the map $z \mapsto (z, \frac{\log t}{2\pi \mathrm{i}})$ embeds X_t in X_∞ and setting $V_{r,\eta} = B(x,r) \cap f^{-1}B(0,\eta)$ the Milnor fibre embeds in $k^{-1}V$ and this embedding can be proven to be a homotopy equivalence. Shrinking r one can see that

$$H^k(\mathrm{Mil}_{f,x}) \simeq \lim_{r\to 0} H^k(k^{-1}V_{r,\eta}) = \Big((R^k k)_* \underline{\mathbb{Q}}_{X_\infty})\Big)_x \tag{24}$$

So the previous construction can be replaced by a canonical object, the (analytic) *nearby cycle complex* on X_0:

$$\psi_f \underline{\mathbb{Q}}_{X_0} = i^* R k_* (k^* \underline{\mathbb{Q}}_X).$$

From (23) one concludes that its hypercohomology computes the cohomology of X_t:

$$\mathbb{H}^k(X_0, \psi_f \underline{\mathbb{Q}}_{X_0}) = H^k(X_t).$$

Now apply this to the algebraic situation. Assume from now on that one has a one-parameter projective degeneration, i.e. **all fibres of f are projective**. Let me also make a simplifying assumption which will be justified later on (see § 9.1):

$$\boxed{X_0 = \bigcup_{i\in I} E_i \text{ is a divisor with strict normal crossings on } X.}$$

An involved construction shows that one can enlarge the nearby cycle complex to give it the structure of a mixed Hodge complex of sheaves

$$\psi_f^{\mathrm{Hdg}} = ((\psi_f \underline{\mathbb{Q}}_{X_0}, W), (\psi_f \underline{\mathbb{C}_{X_0}}, W, F), \alpha),$$

the *Hodge-theoretic nearby cycle complex*. To define the Hodge filtration is relatively easy by finding a different representative of $\psi_f \underline{\mathbb{C}}_{X_0}$ in $D^+(\underline{\mathbb{C}}_{X_0})$. Indeed, the relative log-complex $\Omega^\bullet_{X/\Delta}(\log E)$ when restricted to E gives such a representative and for the Hodge filtration F just take the trivial filtration on this complex. It is called the *limit Hodge filtration* on $\psi_f \underline{\mathbb{C}}_X$. The weight filtration is more complicated to define. I won't describe it on the level of sheaves, but only on the level of cohomology groups. It uses the action of the *monodromy* T on $\psi_f \underline{\mathbb{Q}}_{X_0}$. The point here is that a nilpotent endomorphism comes with a natural filtration. So one has to construct such a morphism out of T. This is possible thanks to the following result.

Lemma 7.1.1. *Suppose that the multiplicities of f along E_i are all* 1*. Then the eigenvalues of the monodromy operator T are all* 1*, i.e. T is unipotent. In the general case a power of T is unipotent; one says that T is* quasi-unipotent*.*

Remark. This result is a weak form of the so-called *monodromy theorem* which, in addition, gives more information on the size of the Jordan blocks: it states that the size is at most equal to the *level* of the Hodge structure, i.e. the largest difference $|p-q|$ for which there is a non-zero Hodge number $h^{p,q}$. See [**P-S**, Corr.11.42].

Let me for simplicity assume that the multiplicities of f along E_i are all 1 so that the monodromy action is unipotent. One then puts

$$N = \log T.$$

It is a nilpotent and, as promised, comes with an intrinsic filtration:

Lemma 7.1.2. *Let N be a nilpotent endomorphism of a finite dimensional rational vector space H. There is a unique increasing filtration $W = W(N)$ on H, the such that $N(W_j) \subset W_{j-2}$ and $N^j : \mathrm{Gr}_j^W \to \mathrm{Gr}_{-j}^W$ is an isomorphism for all $j \geq 0$. This filtration is called the* weight filtration of N.

If $N = \log T$ the shifted filtration $W[k]$ on $H^k(X_\infty)$ is called the *monodromy weight filtration*.

This can also be used to give a more geometric interpretation of the limit mixed Hodge structure. For $t = \mathrm{e}(z)$ fix a choice of logarithm $z = 2\pi \mathrm{i} \log(t)$ and let $F^p(z) = F^p H^k(X_z) \subset H^k(X_\infty) \otimes \mathbb{C}$. Now consider $\exp(-zN)F(z)$. Since $\exp(-(z+1)N) = \exp(-zN)T^{-1}$ one has $\exp(-(z+1)N)F(z+1) = \exp(-z)F(z)$ as a subset of $H^k(X_\infty)$. So $\exp(-zN)F^p(z)$ defines a subspace of $H^k(X_\infty) \otimes \mathbb{C}$ which simply can be written $F^p(t)$. One of Schmid's results [**Schm**] is that this subspace converges (in the sense of points in a Grassmannian) to a limit F^p_∞ when t approaches 0 along radii. This limit, for different p indeed gives the limit mixed Hodge structure.

The basic result is:

Theorem 7.1.3 (Steenbrink, Schmid)**.** *Assume that f is a projective one-parameter degeneration. The Hodge-theoretic nearby cycle complex puts a mixed Hodge structure on the cohomology groups $H^k(X_\infty)$, the* limit mixed Hodge structure. *One has:*

1. *The weight filtration is the monodromy weight filtration;*
2. *the limit Hodge filtration on $H^k(X_\infty)$, coincides with the above defined limit F_∞ of the classical Hodge filtration $FH^k(X_t)$. In particular for all p, k one has $\dim F^p H^k(X_\infty) = \dim F^p H^k(X_t)$.*

Corollary 7.1.4. *One has $h^{p,q}(X_s) = \sum_{s \geq 0} h^{p,s}(H^{p+q}(X_\infty))$.*

A strengthening of Lemma 7.1.2 is needed which holds in the above setting and which describes which types of Jordan blocs occur in terms of a canonical decomposition:

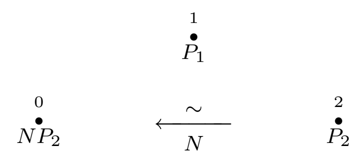
Lemma 7.1.5 ([**Schm**, Lemma 6.4])**.** *There is a Lefschetz-type decomposition*

$$\mathrm{Gr}^W H^k(X_\infty) = \bigoplus_{\ell=0}^{k} \bigoplus_{r=0}^{\ell} N^r P_{k+\ell},$$

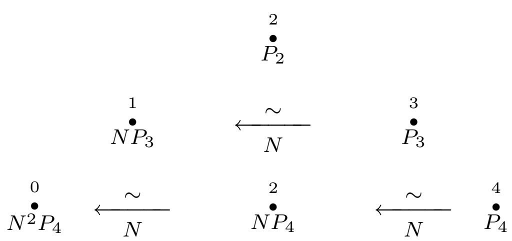
where $P_{k+\ell}$ is pure of weight $k+\ell$. The endomorphism N has $\dim P_{k+m-1}$ Jordan blocs of size m.

Example 7.1.6.

For H^1 one has:



and for H^2:



You see that in the first case there are $\dim \mathrm{Gr}_2^W$ Jordan blocs of size 2 and $\dim \mathrm{Gr}_1^W$ ones of size 1. In the second case the primitive part in weight 2 is needed which gives the Jordan blocs of size 1. Then there are $\dim \mathrm{Gr}_3^W$ blocs of size 2 and Gr_4^W blocs of size 3.

There is an important result, which will be used later on in the examples:

Theorem 7.1.7 (Local Invariant Cycle Theorem)**.** *The invariants in $H^k(X_\infty)$ under local monodromy form a subspace whose mixed Hodge structure is naturally isomorphic to the mixed Hodge structure of $H^k(X_0)$.*

Example 7.1.8. For $k = 1$ the above diagram shows that $\mathrm{Gr}_W \operatorname{Ker} N = NP_2 \oplus P_1$ and hence $W_0 \cap \operatorname{Ker} N = NP_2 = W_0 H^1(X_\infty) \simeq W_0 H^1(X_0)$. The last result stays true for all k, as one easily sees.

For $k = 2$ one has $\mathrm{Gr}\,{}_1^W H^2(X_\infty) = NP_3 \oplus NP_4$ so that $\mathrm{Gr}\,{}_1^W \operatorname{Ker} N = NP_3 = \mathrm{Gr}\,{}_1^W H^2(X_0)$.

7.2. The Motivic Nearby Cycle: Unipotent Monodromy

The goal of this section is to calculate $\chi_{\mathrm{Hdg}}(X_\infty)$. It uses the weight filtration and the spectral sequence related to it. I have treated this in Lecture 6.3. But in the situation at hand the weight filtration W is increasing and one needs to reindex the terms the spectral sequence so as to obtain

$$E_1^{-s,q+s} = H^{-s,q+s}(\mathrm{Gr}_s^W(\psi_f^{\mathrm{Hdg}})) \implies H^q(X_f^{\mathrm{Hdg}})$$

This a spectral sequence of mixed Hodge structures; it can be shown that $\mathrm{Gr}_p^W(\psi_f^{\mathrm{Hdg}})$ is a complex of Hodge structures on the smooth components of the normalizations of partial intersection of the components of $X_0 = f^{-1}(0)$. Indeed, put

$$E_J = \bigcap_{i\in J} E_j,\ E(m) = \coprod_{|J|=m} E_J$$

so that the $E(m)$ are all smooth projective. One shows that up to quasi-isomorphisms

$$\mathrm{Gr}_s^W(\psi_f^{\mathrm{Hdg}}) = \bigoplus_k \underline{\mathbb{Q}}_{E(2k+s+1)}[s+2k](-s-k).$$

so that the E_1-term is the graded Hodge structure

$$\bigoplus_{k\geq 0,s} H^{q-s-2k}(E(2k+r+1))(-s-k) \implies H^q(X_\infty).$$

Let me now step back to the definition of the Hodge characteristic for a variety. It is defined in two steps: first one associates to a variety a certain mixed Hodge structure and secondly one takes the associated class in $\mathrm{K}_0(\mathfrak{hs})$. This last step can be performed abstractly: to any mixed Hodge structure H one associates its class $[H] \in \mathrm{K}_0(\mathfrak{hs})$, its *Hodge characteristic*. It behaves well with respect to spectral sequences. To see this, let (E_r, d_r) be the r-th term of a spectral sequence of mixed Hodge structures converging to E_∞. By definition E_r is a complex with cohomology E_{r+1}. Hence $[E_k] = [E_{k+1}] = \cdots = [E_\infty]$.

In our case this implies that it suffices to calculate the Hodge characteristic $[E_2]$. Let me simplify this calculation as follows. Set $s + k = a$,

$2k+s+1=b$, $q-s-2k=c$. If one goes back to the construction of the mixed Hodge complex of sheaves X_f^{Hdg} one can show that $a,c\geq 0$, $b\geq 1$. Since $k=b-a-1$, one has the restriction $0\leq a\leq b-1$. One finds:

$$\begin{aligned}\chi_{\mathrm{Hdg}}(X_\infty) &= \sum_{b\geq 1, c\geq 0}\sum_{a=0}^{b-1}(-1)^{c+b+1}[H^c(E(b))(-a)] \\ &= \sum_{b\geq 1}(-1)^{b+1}\chi_{\mathrm{Hdg}}(E(b))\cdot\left[\sum_{a=0}^{b-1}\mathbb{L}^a\right].\end{aligned}$$

One obtains:

$$\left.\begin{aligned}\chi_{\mathrm{Hdg}}(X_\infty) &= \textstyle\sum_{b\geq 1}(-1)^{b-1}\chi_{\mathrm{Hdg}}(E(b))\cdot\left[\sum_{a=0}^{b-1}\mathbb{L}^a\right] \\ &= \textstyle\sum_{b\geq 1}(-1)^{b-1}\chi_{\mathrm{Hdg}}(E(b)\times\mathbb{P}^{b-1}).\end{aligned}\right\} \quad (25)$$

It suggest the following definition:

Definition 7.2.1. Suppose that the fibres of f are projective varieties. Following [**Bitt2**, Ch. 2] let me define the *motivic nearby fibre* of f by

$$\psi_f^{\mathrm{mot}} := \sum_{m\geq 1}(-1)^{m-1}[E(m)\times\mathbb{P}^{m-1}]\in \mathrm{K}_0(\underline{\mathrm{Var}}).$$

The motivic fibre is indeed an invariant of the nearby smooth fibre:

Lemma 7.2.2. *Suppose that $\sigma: Y\to X$ is a bimeromorphic proper map which is an isomorphism over $X-E$. Put $g=f\circ\sigma$ Assume that $g^{-1}(0)$ is a divisor with strict normal crossings. Then*

$$\psi_f=\psi_g.$$

Proof: In [**Bitt2**], the proof relies on the theory of motivic integration [**D-L99b**]. I sketch a simplified version of the proof from [**P-S07**] which is based on the weak factorization theorem [**A-K-M-W**]. See Appendix 1 to Lecture 2. This theorem reduces the problem to the following situation: σ is the blowing-up of X in a connected submanifold $Z\subset E$ with the following property: with $A\subset I$ the set of indices i for which $Z\subset E_i$ the manifold Z intersects the divisor $\bigcup_{i\notin A}E_i$ transversely so that in particular $Z\cap\bigcup_{i\notin A}E_i$ is a divisor with normal crossings in Z.

Suppose for simplicity that $|A|=1$ so that Z is contained in just one divisor, say $Z\subset E_1$ and that E_2 is the only component of E meeting Z. This guarantees that the components of $g^{-1}(0)$ all have multiplicity one. Let $c=\operatorname{codim}_Z X$ so that $\operatorname{codim}_Z E_1=\operatorname{codim}_{Z_2}E_{12}=c-1$ and $\operatorname{codim}_{Z_2}E_2=c$. The special fibre $g^{-1}(0)$ has one extra component, namely the exceptional divisor which is denoted E_0'. The proper transforms of the E_j are called E_j'. There are two new 2-fold intersections E_{01}' and E_{02}' and one new triple intersection E_{012}'. It follows that

$$\begin{aligned}\psi_g-\psi_f &= ([E_1']-[E_1])+([E_2']-[E_2])+[E_0']+ \\ &\quad -([E_{12}']-[E_{12}])+[E_{01}']+[E_{02}'])\times[\mathbb{P}^1]+ \\ &\quad +[E_{012}']\times[\mathbb{P}^2].\end{aligned}$$

Now use that E_1' is the blow-up of E_1 along Z, and that E_2' is the blow-up of E_2 along $Z_2:=Z\cap E_2$. For these use the formula (6). Now the full exceptional divisor E'_0 is a $\mathbb{P}^{c-1}$-bundle over Z, E'_{01} is a $\mathbb{P}^{c-2}$-bundle over Z, E'_{02} is a $\mathbb{P}^{c-1}$-bundle over Z_2 while E'_{012} is a $\mathbb{P}^{c-2}$-bundle over Z_2. For these apply (7).

The coefficient of $[Z]$ is found to be equal to

$$[\mathbb{P}^{c-1}] + ([\mathbb{P}^{c-2}] - 1) - [\mathbb{P}^{c-2}] \cdot [\mathbb{P}^1] = 0$$

and the coefficient of $[Z_2]$ equals

$$([\mathbb{P}^{c-1}] - 1) - ([\mathbb{P}^{c-2}] - 1 + \mathbb{P}^{c-1}) \cdot [\mathbb{P}^1] + [\mathbb{P}^{c-2}] \cdot [\mathbb{P}^2] = 0.$$

□

As a consequence:

Corollary 7.2.3.

$$\chi_{\mathrm{Hdg}}(\psi_f^{\mathrm{mot}}) = \chi_{\mathrm{Hdg}}(X_\infty).$$

Motivic Nearby Fibre, Examples

Example 1: Let $F, L_1, \ldots, L_d \in \mathbb{C}[X_0, X_1, X_2]$ be homogeneous forms with $\deg F = d$ and $\deg L_i = 1$ for $i = 1, \ldots, d$, such that $F \cdot L_1 \cdots L_d = 0$ defines a reduced divisor with normal crossings on $\mathbb{P}^2(\mathbb{C})$. Consider the space

$$X = \{([x_0, x_1, x_2], t) \in \mathbb{P}^2 \times \Delta \mid \prod_{i=1}^{d} L_i(x_0, x_1, x_2) + tF(x_0, x_1, x_2) = 0\}$$

where Δ is a small disk around $0 \in \mathbb{C}$. Then X is smooth and the map $f : X \to \Delta$ given by the projection to the second factor has as its zero fibre the union $E_1 \cup \cdots \cup E_d$ of the lines $E_i : L_i = 0$. These lines are in general position. The formula (25) gives

$$\chi_{\mathrm{Hdg}}(\psi_f^{\mathrm{mot}}) = (1 - g)(1 + \mathbb{L})$$

where $g = \binom{d-1}{2}$ is the genus of the general fibre X_t, a smooth projective curve of degree d. The table is

	H^0	H^1	H^2
weight 0	1	g	0
weight 2	0	$g(\mathbb{L})$	$\mathbb{L}$

One sees that there are only even weight terms for $H^1(X_\infty)$ and its only primitive subspace has weight 2 and dimension g (since $\dim H^1(X_\infty) = 2g$) and in particular, by Lemma 7.1.5 N has g Jordan blocs of size 2, i.e. is "maximally unipotent". Indeed the monodromy diagram is

$$\begin{array}{ccc} \overset{0}{\bullet} & \xleftarrow[N]{\sim} & \overset{2}{\bullet} \\ g \cdot 1 & & g\mathbb{L} \end{array}$$

Example 2: In the same example, replace $\mathbb{P}^2$ by $\mathbb{P}^3$ and curves by surfaces, lines by planes. Then the space X will not be smooth but has ordinary double points at the points of the zero fibre where two of the planes meet the surface $F = 0$. There are $d\binom{d}{2}$ of such points, d on each line of intersection. Blow these up to obtain a family $f : X_\infty \to \Delta$ whose zero fibre $D = E \cup F$ is the union of components E_i, $i = 1, \ldots, d$ which are copies of $\mathbb{P}^2$ blown up in $d(d-1)$ points, and components $F_j, j = 1, \ldots, d\binom{d}{2}$ which are copies of $\mathbb{P}^1 \times \mathbb{P}^1$. Thus

$$\chi_{\mathrm{Hdg}}(D(1)) = d(1 + (d^2 - d + 1)\mathbb{L} + \mathbb{L}^2) + d\binom{d}{2}(1 + \mathbb{L})^2.$$

The double point locus $D(2)$ consists of the $\binom{d}{2}$ lines of intersections of the E_i together with the $d^2(d-1)$ exceptional lines in the E_i. So

$$\chi_{\mathrm{Hdg}}(D(2)) = d(d-1)(d + \frac{1}{2})(1 + \mathbb{L}).$$

Finally $D(3)$ consists of the $\binom{d}{3}$ intersection points of the E_i together with one point on each component F_j, so

$$\chi_{\mathrm{Hdg}}(D(3)) = \binom{d}{3} + d\binom{d}{2} = \frac{1}{3}d(d-1)(2d-1).$$

Using the Hodge numbers for a smooth degree d surface are

$$h^{2,0} = h^{0,2} = \binom{d-1}{3}, \quad h^{1,1} = \frac{1}{3}d(2d^2 - 6d + 7)$$

one finds

$$\chi_{\mathrm{Hdg}}(\psi_f^{\mathrm{mot}}) = \left(h^{2,0}+1\right)(1+\mathbb{L}^2) + h^{1,1}\mathbb{L}.$$

The table is

	H^0	H^2	H^4
weight 0	1	$h^{2,0}$	0
weight 2	0	$h^{1,1}(\mathbb{L})$	0
weight 4	0	$h^{2,0}(\mathbb{L}^2)$	$\mathbb{L}^2$

There are only weight 4 and 2 primitive spaces, that $\dim \mathrm{Gr}_2^W = h^{1,1}$ and one has $\dim W_4 = h^{2,0}$. The monodromy diagram simplifies to

$$\overset{2}{\bullet}\ [h^{1,1} - h^{2,0}]\mathbb{L}$$

$$\overset{0}{\bullet}\ h^{2,0}\cdot 1 \xleftarrow[N]{\sim} \overset{2}{\bullet}\ h^{2,0}\mathbb{L} \xleftarrow[N]{\sim} \overset{4}{\bullet}\ h^{2,0}\mathbb{L}^2$$

and hence there are $h^{2,0}$ Jordan blocs of size 3 and $h^{1,1} - h^{2,0}$ blocks of size 1.

Interlude: An *arbitrary* degeneration into a normal crossing surface. I use the same notation as in Example 6.3.1 from Lecture 6. The Hodge characteristic for ψ^{mot} is immediately found from the degeneration:

$$\left.\begin{aligned} \chi_{\mathrm{Hdg}}(\psi_f^{\mathrm{mot}}) \;=\; & \chi(\Gamma)(1+\mathbb{L}^2) + (m-2\ell)\mathbb{L} \\ & +\left(\textstyle\sum_j V_{g_{C_j}} - \sum_i V_{g_{X_i}}\right)(1+\mathbb{L}) \\ & + \textstyle\sum_i W_{p_{X_i}, q_{X_i}}. \end{aligned}\right\} \tag{26}$$

Let me now calculate the limit mixed Hodge structures using the Hodge numbers $g = h^{1,0}(X_t) = h^{0,1}(X_t)$, $p = h^{2,0}(X_t)$ and $q = h^{1,1}(X_t)$ on the nearby smooth surface X_t.

First look at the mixed Hodge structure $H^1(X_\infty)$. Example 7.1.6 exhibits the gradeds of the weight filtration. Let

$$V_{g-c} = \mathrm{Gr}_1^W H^1(X_\infty), \quad g - c := h^{1,0}(\mathrm{Gr}_1^W H^1(X_\infty)).$$

Then the diagram reads:

$$\overset{1}{\bullet}\ V_{g-c}$$

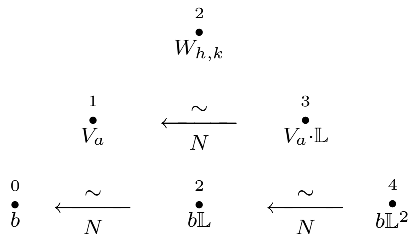
$$\overset{0}{\bullet}\ c \xleftarrow[N]{\sim} \overset{2}{\bullet}\ c\cdot\mathbb{L}$$

By Example 7.1.8 one has $c = h^1(\Gamma) = \dim W_0 H^1(X_0)$.

Next, consider $H^2(X_\infty)$. Let $V_a = \mathrm{Gr}_1^W H^2(X_\infty)$ and let $W_{h,k}$ be the primitive of part of $\mathrm{Gr}_2^W H^2(X_\infty)$. The relevant numbers are

$$a = h^{1,0}(H^2),\ h = h^{2,0}(H^2),\ k = h^{1,1}(H^2),\ b = h^{0,0}(H^2).$$

The diagram becomes



It follows that b is the number of Jordan blocks of size 3, a the number of Jordan blocks of size 2.

By Example 7.1.8 and by inspecting Table 6.1 one finds

$$b = W_0 H^2(X_0) = h^2(\Gamma), \quad V_a = \mathrm{Gr}_1^W H^2(X_0) = \mathrm{Coker}(\alpha_2).$$

Note also that

$$p = h^{2,0}(X_t) = h + a + b, \quad q = h^{1,1}(X_t) = k + 2a + b.$$

This follows from Cor. 7.1.4. So, finally, the relevant information can be summarized as follows.

$g = h^{1,0}(X_t)$	$p = h^{2,0}(X_t)$	$q = h^{1,1}(X_t)$ (Hodge numbers of nearby fibre)
$h^1(\Gamma) = c$	$h^2(\Gamma) = b$	(invariants of local monodromy T)
$a = \dim \mathrm{Coker}(\alpha_2)$	b	(number of Jordan blocks of $T\vert H^2(X_t)$ of size $2,\ 3$).

The table for the graded parts of the limit mixed Hodge structure is:

	H^0	H^1	H^2	H^3	H^4
weight 0	1	$h^1(\Gamma)$	$h^2(\Gamma)$	0	0
weight 1	0	V_{g-c}	V_a	0	0
weight 2	0	$h^1(\Gamma)\cdot\mathbb{L}$	$b\mathbb{L} + W_{h,k}$	$h^1(\Gamma)\cdot\mathbb{L}$	0
weight 3	0	0	$V_a \cdot \mathbb{L}$	$V_{g-c}\cdot\mathbb{L}$	0
weight 4	0	0	$h^1(\Gamma)\cdot\mathbb{L}^2$	$h^1(\Gamma)\cdot\mathbb{L}^2$	$\mathbb{L}^2$

Compare these results with (26):

Proposition 8.1.4. *One has the following equalities in* $\mathrm{K}_0(\mathfrak{hs})$*:*

$$\begin{array}{rcl} V_a - V_{g-c} & = & \sum V_{g_{C_j}} - \sum_i V_{g_{X_i}} \\ (m-2\ell)\cdot\mathbb{L} + \sum_i W_{p_{X_i}, q_{X_i}} & = & (-2c+b)\cdot\mathbb{L} + W_{h,k}. \end{array}$$

In particular

$$\begin{array}{rcl} g & = & \sum_i g_{X_i} - \sum_j g_{C_j} + a + c \\ p & = & \sum_i p_{X_i} + a + b \\ q & = & \sum_i q_{X_i} + m + 2a + 2c - 2\ell. \end{array}$$

Remark. These results make the results from [**Per**, Ch. II] more explicit.

Let me illustrate the above with some concrete examples.

Example 3: I come back to Example 6 from Lecture 3, the degeneration $tF + F_1F_2 = 0$ inside $\mathbb{P}^3 \times \mathbb{C}$ where $\deg F = d, \deg F_1 = d_1, \deg F_2 = d_2$. I explained in Lecture 3 that there are two possible degenerations. It is a nice exercise to calculate that in both cases

$$\begin{aligned}\chi_{\mathrm{Hdg}}(\psi_f^{\mathrm{mot}}) &= 1 + d_1d_2d\cdot\mathbb{L} + \mathbb{L}^2 + W_{p_{d_1},q_{d_1}-1} + W_{p_{d_2},q_{d_2}-1}\\ &\qquad + V_{\frac{1}{2}d_1d_2(d_1+d_2-4)+1}(1+\mathbb{L}).\end{aligned}$$

The table thus becomes

	H^0	H^2	H^4
weight 0	1	0	0
weight 1	0	$V_{\frac{1}{2}d_1d_2(d_1+d_2-4)+1}$	0
weight 2	0	$d_1d_2d\cdot\mathbb{L} + W_{p_{d_1},q_{d_1}-1} + W_{p_{d_2},q_{d_2}-1}$	0
weight 3	0	$V_{\frac{1}{2}d_1d_2(d_1+d_2-4)+1}\cdot\mathbb{L}$	0
weight 4	0	0	$\mathbb{L}^2$

Hence the weight filtration is $W_4 = W_3 \supset W_2 \supset W_1 \supset W_0 = 0$ which implies that $N^2 = 0$, and the monodromy diagram simplifies to

$$\overset{2}{\bullet}\quad d_1d_2d\mathbb{L}+W_{p_{d_1},q_{d_1}-1}+W_{p_{d_2},q_{d_2}-1}$$

$$\underset{V_{\frac{1}{2}(d_1d_2(d_1+d_2-4)+1}}{\overset{0}{\bullet}} \quad \xleftarrow[N]{\sim} \quad \underset{V_{\frac{1}{2}(d_1d_2(d_1+d_2-4)+1}\cdot\mathbb{L}}{\overset{4}{\bullet}}$$

Hence the monodromy has $d_1d_2(d_1+d_2-4)+2$ Jordan blocs of size 2 and $2N + 2 + 2p_{d_1} + 2p_{d_2} + q_{d_1} + q_{d_2}$ blocs of size 1.

Example 4: Now I go back to Example 4 from Lecture 3, the degeneration given by $\{Q_1Q_2 + tF_4 = 0\} \subset \mathbb{P}^3 \times \mathbb{C}$, where Q_1, Q_2 are homogeneous quadric forms with zero locus smooth quadrics and F_4 a quartic form whose zero locus is a smooth surface. After blowing up the 16 double points $\{t = Q_1 = Q_2 = F_4 = 0\}$ of the total space of the family (a hypersurface inside $\Delta \times \mathbb{P}^3$); the special fibre consists of eighteen smooth components which intersect transversally according to the pattern described in loc. cit. One gets

$$\chi_{\mathrm{Hdg}}(\psi_f^{\mathrm{mot}}) = 1 + (18\mathbb{L} + V_1) + (\mathbb{L}^2 + V_1\cdot\mathbb{L}).$$

where $V_1 := H^1(C)$; note that $C = \{t = Q_1 = Q_2 = 0\}$ is an elliptic curve. The table becomes

	H^0	H^2	H^4
weight 0	1	0	0
weight 1	0	V_1	0
weight 2	0	$18\mathbb{L}$	0
weight 3	0	$V_1\cdot\mathbb{L}$	0
weight 4	0	0	$\mathbb{L}^2$

Hence the monodromy has two Jordan blocs of size 2 and 18 blocs of size 1.

LECTURE 9

Motivic Aspects of Degenerations, Applications

9.1. The Motivic Nearby Cycle: the General Case

Now I no longer assume that $\mathrm{ord}_{E_i} f = e_i$, the multiplicity of f along E_i is one. Let $\widetilde{f} : \widetilde{X} \to \Delta$ denote the normalization of the pull-back of X under the map $\mu_e : \Delta \to \Delta$ given by $\tau \mapsto \tau^e = t$. It fits into a commutative diagram describing the *e-th root* of f :

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\rho} & X \\ {\scriptstyle \widetilde{f}}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ \Delta & \xrightarrow{\mu_e} & \Delta. \end{array}$$

Note that $\widetilde{X}$ is in general not a smooth variety so that one has to perform blowings up introducing possibly new components of higher multiplicity. So one should continue extracting roots. One can show [**P-S**, Theorem 11.11] that the process terminates, say at

$$f' : X' \to \Delta, \quad (f')^{-1}(0) = D = D_1 \cup \cdots \cup D_{N'}$$

where the base change $\Delta \to \Delta$ has order a multiple of e which is still called e.

Example 9.1.1. The simplest case is an ordinary multiple point, say of order e. Extracting the e-th root produces a total space $\widetilde{X}$ which has a unique singularity which is an ordinary multiple point, and blowing it up gives a smooth variety X with new fibre $D_1 \cup D_2$, a union of the exceptional component D_1 which is a hypersurface of degree e in $\mathbb{P}^{n+1}$ and D_2 the strict transform of the original singular fibre. The intersection $D_1 \cap D_2$ is a hyperplane section of D_1. Indeed, locally inside $(t, x_1, \ldots, x_{n+1})$-space one has an equation $f = t + g(x_1, \ldots, x_{n+1}) = 0$ where g starts off with a homogeneous polynomial g_e of degree e. Extracting the e-th root means replacing t by t^e. Blowing up in the origin can be done by setting $x_k = ty_k$, $k = 1, \ldots, n+1$ and then f pulls back to

$$\widetilde{f} = t^e(1 + g_e(y_1, \ldots, y_{n+1}) + th(y_1, \ldots, y_{n+1})) = t^e \cdot f_{\mathrm{new}}.$$

The proper transform of the total space becomes $f_{\mathrm{new}} = 0$ with exceptional divisor $t = 0$. The exceptional component of the zero fibre is the homogeneous hypersurface inside $\mathbb{P}^{n+1}$ given by $y_0^e + g_e(y_1, \ldots, y_{n+1}) = 0$. In the given $(t, y_1, \ldots, y_{n+1})$-chart the equation is $t = 0 = 1 + g_e(y_1, \ldots, y_{n+1})$ which in this chart does not meets the proper transform of the old fibre. In the $(u, u_1, \ldots, u_{n+1})$-chart given by $x_1 = u$, $t = uu_1$, $x_k = uu_k$, $k = 2, \ldots, n+1$ one finds the equation

$$f_{\mathrm{new}} = u_1^e + g_e(1, u_2, \ldots, u_{n+1}) + uh(u_2, \ldots, u_{n+1}) = 0$$

with exceptional locus given by $u = 0 = \{u_1^e + g_e(1, u_2, \ldots, u_{n+1}) = 0\}$ and the proper transform of the old fibre by $u_1 = 0 = \{g_e(1, u_2, \ldots, u_{n+1}) + uh(u_2, \ldots, u_{n+1}) = 0\}$ so that the intersection of D_1 and D_2 in this chart is the smooth hypersurface in $(u_2, \ldots, u_{n+1})$-space with equation

$$g_e(1, u_2, \ldots, u_{n+1}) = 0.$$

The cyclic group μ_e acts on the component

$$D_1 = V_{e,n+1} := \{(y_0, y_1, \ldots, y_{n+1}) \in \mathbb{P}^{n+1} \mid y_0^e + g_e(y_1, \ldots, y_{n+1}) = 0\}.$$

as follows

$$(y_0, y_1, y_2, \ldots, y_{n+1}) \mapsto (\epsilon y_0, \epsilon^{-1} y_1, y_2, \ldots, y_{n+1}), \quad \epsilon = \mathrm{e}^{\frac{2\pi \mathrm{i}}{e}}.$$

So on the intersection $D_1 \cap D_2$ it also acts non-trivially. In fact the same action is reproduced on a hypersurface of the same degree but one dimension less.

So I have reduced the general situation to the case treated in the previous Lecture. Here the first N components correspond to the 'old' components D_i while the others come from possible blow-ups. The special fibre $E' = f'^{-1}(0)$ is now a complex variety equipped with the action of the cyclic group of order a multiple of e. Let me introduce the associated Grothendieck-group:

Definition 9.1.2. Let $\mathrm{K}_0^{\hat{\mu}}(\underline{\mathrm{Var}})$ denote the Grothendieck group $\mathrm{K}_0(\underline{\mathrm{Var}}^{\hat{\mu}})$ of complex algebraic varieties with an action of a finite cyclic order automorphism modulo the subgroup generated by expressions $[\mathbb{P}(V)] - [\mathbb{P}^n \times X]$ where V is a vector bundle of rank $n+1$ over X with action which is linear over the action on X. See [**Bitt2**, Sect. 2.2] for details.

To explain why one should divide out by the relations $[\mathbb{P}(V)] - [\mathbb{P}^n \times X]$, recall (Example. 2.1.7 1) that in the ordinary Grothendieck group the relation $[\mathbb{P}(V)] = [\mathbb{P}^n \times X]$ holds. These relations extend to the case where one has a group action.

Still assume that the fibres of f are projective varieties. Following [**Bitt2**, Ch. 2] define the *motivic nearby fibre* of f in this setting by

$$\psi_f^{\mathrm{mot}} := \sum_{m \geq 1} (-1)^{m-1} [D(m) \times \mathbb{P}^{m-1}] \in \mathrm{K}_0^{\hat{\mu}}(\underline{\mathrm{Var}})$$

Actually, all constructions give varieties with natural morphisms to the original fibre $X_0 = f^{-1}(0)$. So it is natural to use a relative version of the Grothendieck group $\mathrm{K}_0(\underline{\mathrm{Var}})$ which came up in § 2.1:

Definition 9.1.3. Let S be a complex algebraic variety. Then $\underline{\mathrm{Var}}_S$ denotes the category of varieties over S to be thought of as morphisms $X \to S$. Let $\mathrm{K}_0(\underline{\mathrm{Var}}_{/S})$ be the free abelian group on isomorphism classes of complex algebraic varieties over S modulo the *scissor relations* where the class $[X]$ of X gets identified with $[X - Y] + [Y]$ whenever $Y \subset X$ is a closed subvariety over S. Identify $\mathrm{K}_0(\underline{\mathrm{Var}}_{/\mathrm{pt}})$ with $\mathrm{K}_0(\underline{\mathrm{Var}})$.

The direct product between a variety over S and a variety over T gives a variety over $S \times T$. This is compatible with the scissor relations and defines an "exterior" product $\mathrm{K}_0(\underline{\mathrm{Var}}_{/S}) \times \mathrm{K}_0(\underline{\mathrm{Var}}_{/T}) \to \mathrm{K}_0(\underline{\mathrm{Var}}_{/S\times T})$. When $S = T$, taking instead the fibred product, defines a ring structure on $\mathrm{K}_0(\underline{\mathrm{Var}}_S)$

with unit the class $[S]$ of the identity morphism $S \xrightarrow{\text{id}} S$. Taking $T = \text{pt}$, the exterior product makes $\mathrm{K}_0(\underline{\mathrm{Var}}_{/S})$ into a $\mathrm{K}_0(\underline{\mathrm{Var}})$-module.

For a morphism $\varphi : S \to T$, composition defines a push forward morphism $\varphi_! : \mathrm{K}_0(\underline{\mathrm{Var}}_{/S}) \to \mathrm{K}_0(\underline{\mathrm{Var}}/T)$ and the fibre product construction gives a pull back $\varphi^{-1} : \mathrm{K}_0(\underline{\mathrm{Var}}_{/T}) \to \mathrm{K}_0(\underline{\mathrm{Var}}_{/S})$ which are $\mathrm{K}_0(\underline{\mathrm{Var}})$-linear.

***Remark* 9.1.4.** One has indeed that $\psi_f^{\mathrm{mot}} \in \mathrm{K}_0^{\hat{\mu}}(\underline{\mathrm{Var}}_{X_0})$.

Example 9.1.5 (Continuation of Example 9.1.1)**.** The cyclic group $G = \mu_e$ acts. The component $D_1 = X_{e,n+1}$ is a smooth hypersurface in $\mathbb{P}^{n+1}$ of degree e, and $D_1 \cap D_2$ can be identified with a smooth hypersurface $X_{e,n}$ inside $\mathbb{P}^n$. The Hodge characteristic becomes

$$\chi_{\mathrm{Hdg}}(D_2) + (-1)^n[H^n_{\mathrm{prim}}(X_{e,n+1}) - H^{n-1}_{\mathrm{prim}}(X_{e,n})(1+\mathbb{L})] - (\mathbb{L} + \cdots + \mathbb{L}^{n-1})$$

The G-invariant part can be found by noting that G acts on $X_{e,n+1}$ with $\mathbb{P}^n$ as quotient; indeed the projection along the y_0-axis exhibits $X_{e,n+1}$ as a cyclic quotient of $\mathbb{P}^n$ ramified along $X_{e,n}$ and this implies that the splitting $\chi_{\mathrm{Hdg}}(X_{e,n+1}) = \chi_{\mathrm{Hdg}}(\mathbb{P}^n) + (-1)^n H^n_{\mathrm{prim}}(X_{e,n+1})$ reflects the splitting into the G-invariant part and the part on which G acts non-trivially. The same holds for $\chi_{\mathrm{Hdg}}(X_{e,n})$.

Let me continue with the general theory. As in the case where all the e_i are one, one can show that

(1) ψ_f^{mot} only depends on the nearby fibre;
(2) $\chi_{\mathrm{Hdg}}(\psi_f^{\mathrm{mot}}) = \chi_{\mathrm{Hdg}}(X_\infty)$

Note however that the individual hypercohomology groups $\mathbb{H}^k(\psi_f^{\mathrm{mot}})$ still have an action of a finite group and so there is finer information floating around of which I want to take advantage. Let me do this by introducing the category of *graded real Hodge structures with finite automorphisms* $\mathfrak{hs}^{\hat{\mu}}$. Its objects are pairs (H, γ) consisting of a graded Hodge structure (i.e. direct sum of pure Hodge structures of possibly different weights) H and an automorphism γ of finite order of this Hodge structure.

I am going to consider a kind of tensor product of two such objects, which is called *convolution* (see [**SchS**], where this operation was defined for mixed Hodge structures and called *join*). I shall explain this by settling an equivalence of categories between $\mathfrak{hs}^{\hat{\mu}}$ and a category $\mathfrak{fhs}$ of so-called *fractional Hodge structures*. Note that the weights are not fractional, but the indices of the Hodge filtration!

Definition 9.1.6 (see [**L**])**.** A *fractional Hodge structure* of weight k is a rational vector space H of finite dimension, equipped with a decomposition

$$H_{\mathbb{C}} = \bigoplus_{a+b=k} H^{a,b}$$

where $a, b \in \mathbb{Q}$, such that $H^{b,a} = \overline{H^{a,b}}$. A *fractional Hodge structure* is defined as a direct sum of pure fractional Hodge structures of possibly different weights.

Lemma 9.1.7. *There is an equivalence of categories* $G : \mathfrak{hs}^{\hat{\mu}} \to \mathfrak{fhs}$*.*

Proof: Let (H,γ) be an object of $\mathfrak{hs}^{\hat\mu}$ pure of weight k. Define $H_a = \mathrm{Ker}(\gamma - \exp(2\pi \mathrm{i} a); H_{\mathbb{C}})$ for $0 \le a < 1$ and for $0 < a < 1$ put

$$\widetilde{H}^{p+a,k-p+[1-a]} = H_a^{p,k-p}, \quad \widetilde{H}^{p,k-p} = H_0^{p,k-p}$$

This transforms (H,γ) into a direct sum $\widetilde{H} =: G(H,\gamma)$ of fractional Hodge structures of weights $k+1$ and k respectively. Conversely, for a fractional Hodge structure $\widetilde{H}$ of weight k one has a unique automorphism γ of finite order which is multiplication by $\exp(2\pi \mathrm{i} b)$ on $\widetilde{H}^{b,k-b}$.
Note that this equivalence of categories does not preserve tensor products! Hence it makes sense to make the following definition:

Definition 9.1.8. The *convolution* $(H',\gamma') * (H'',\gamma'')$ of two objects in $\mathfrak{hs}^{\hat\mu}$ is the object corresponding to the tensor product of their images in $\mathfrak{fhs}$:

$$G\left((H',\gamma') * (H'',\gamma'')\right) = G(H',\gamma') \otimes G(H'',\gamma'').$$

Note that the Hodge character refines to

$$\chi^{\hat\mu}_{\mathrm{Hdg}} : \mathrm{K}_0^{\hat\mu}(\underline{\mathrm{Var}}) \to \mathrm{K}_0(\mathfrak{hs}^{\hat\mu}) = \mathrm{K}_0(\mathfrak{fhs}).$$

Now the Hodge number polynomial P introduced in Lemma 1.2.4 must be redefined slightly as to accommodate rational exponents:

$$P^{\hat\mu} : \mathrm{K}_0(\mathfrak{fhs}) \to \lim_{\leftarrow} \mathbb{Z}[u^{\frac{1}{n}}, v^{\frac{1}{n}}, u^{-1}, v^{-1}].$$

Example 9.1.9. Let μ_n be the cyclic group of order n. For each divisor m of n there is a unique irreducible rational representation $W_m := \mathbb{Q}[t]/(\Phi_m(t))$ where Φ_m is the m-th cyclotomic polynomial. It is of degree $\varphi(m)$. The sum of these W_m over all divisors m of n is precisely the regular representation.

Over the complex numbers one has the one-dimensional representations $U_{\frac{k}{n}} = \mathbb{C}[t]/(t - \mathrm{e}^{\frac{2k\pi}{n}})$ and

$$W_m \otimes \mathbb{C} = \bigoplus_{(k,m)=1} U_{\frac{k}{m}}.$$

Suppose now that W_m carries a Hodge structure of weight w. Then the $H^{p,q}$ split into such irreducible modules. Since $H^{p,q}$ is the complex conjugate of $H^{q,p}$, if $H^{p,q}$ contains $U_{\frac{k}{m}}$, $H^{q,p}$ has to contain $U_{\frac{m-k}{m}}$. The sum $U_{\frac{k}{n}} \oplus U_{\frac{n-k}{n}}$ can then be considered as a fractional *real* Hodge structure with Hodge numbers $h^{p+\frac{k}{m},q+\frac{m-k}{m}} = h^{q+\frac{m-k}{n},p+\frac{k}{m}} = 1$. One can consider the corresponding fractional (rational) Hodge structure on W_m

$$W_m \otimes \mathbb{C} = \widetilde{V}_{\frac{p\|w}{m}} := \bigoplus_{(k,m)=1} W_m^{p+\frac{k}{m},q+\frac{m-k}{m}}$$

with all non-zero fractional Hodge numbers equal to 1 and W_m then underlies a weight $w+1$ fractional Hodge structure.

- For $w = 0$ this fractional Hodge structure will be denoted $\widetilde{V}_{\frac{0}{m}} := \widetilde{V}_{\frac{0\|0}{m}}$.
- If $w = 1$, there is no choice: one must have that $W_m \otimes \mathbb{C} = \widetilde{V}_{\frac{0\|1}{m}}$.
- If $w = 2$ one has more possibilities:
  (1) W_m is pure of type $(1,1)$ and $W_m = \widetilde{V}_{\frac{1\|2}{m}}$.

(2) $W_m^{1,1} = 0$ and $W_n = \widetilde{V}_{\frac{0\|2}{m}}$.

(3) A mixed situation. Let $I_m = \{i_1, \dots, i_{\varphi(m)}\}$ the increasing set of integers between 1 and m coprime with m and let σ be a permutation of I_m.

$$\widetilde{W}_{\frac{k\|\sigma}{m}} := \begin{cases} W_m^{2,0} &= U_{\frac{\sigma(i_1)}{m}} \oplus \cdots \oplus U_{\frac{\sigma(i_k)}{n}}, \quad k < \frac{1}{2}\varphi(m) \\ W_m^{1,1} &= U_{\frac{\sigma(i_{k+1})}{m}} \oplus \cdots U_{\frac{\sigma(i_{\varphi(m)-k-1})}{m}} \\ W_m^{0,2} &= U_{\frac{\sigma(i_{\varphi(m)})}{m}} \oplus \cdots \oplus U_{\frac{\sigma(i_{\varphi(m)-k})}{m}}. \end{cases}$$

Suppose that $\sigma = \mathrm{id}$ and m an odd prime; then the corresponding nonzero fractional Hodge numbers are

$$\begin{array}{ccc} h^{1-\frac{k}{m},2+\frac{k}{m}} & \dots & h^{1-\frac{1}{m},2+\frac{1}{m}}, \\ h^{1+\frac{k+1}{m},2-\frac{k+1}{m}} & \dots & h^{2-\frac{k+1}{m},1+\frac{k+1}{m}} \\ h^{2+\frac{1}{m},1-\frac{1}{m}} & \dots & h^{2+\frac{k}{m},1-\frac{k}{m}} \end{array}$$

Intermezzo: cohomology of smooth hypersurfaces. See for instance [**Grif69**] or [**C-M-P**, Ch. 3.2] for what follows. Let $X_F \subset \mathbb{P}^{n+1}$ be a smooth hypersurface given by a homogeneous equation $F = 0$ of degree d. Let $\{\xi_0, \dots, \xi_{n+1}\}$ be homogeneous coordinates and set

$$\Omega := \sum_j (-1)^j \xi_j d\xi_0 \wedge \cdots \widehat{d\xi_j} \wedge \cdots \wedge d\xi_{n+1}.$$

Then the primitive cohomology group $H^n_{\mathrm{prim}}(X_F)$ can be represented by certain closed rational forms on $\mathbb{P}^{n+1}$ having poles along X_F of order at most $n+1$:

$$\frac{A}{F^{n+1-\ell}}\Omega, \quad \deg A = t(\ell) = d(n-\ell+1) - n - 2, \quad \ell = 0, \dots, n$$

For fixed ℓ one gets F^ℓ where A represents an element in $F^{\ell+1}$ if and only if A belongs to the Jacobian ideal of F. In particular, for $\ell = n$ a basis for the homogeneous polynomials of degree $d - n - 2$ gives a basis for $H^{n,0}(X_F)$.

Example 9.1.10 (Continuation of Example 9.1.1)**.** Apply the above considerations to the hypersurface $X_{e,n+1}$ which has an action of G. The description for $H^{n,0}$ as given in the intermezzo shows that $H^{n,0}(X_{e,n+1})$ splits into the following eigenspaces:

$$\begin{aligned} H^{n,0}(X_{e,n+1}) &= U_{1,e} \otimes V^{[e-n-2]} \oplus U_{2,e} \otimes V^{[e-n-3]} \oplus \cdots \oplus U_{e-n-3,e} \\ V &= \mathbb{C}[y_1, \dots, y_{n+1}], \quad V^{[d]} = \text{ degree } d\text{-part of } V. \end{aligned}$$

This should help to determine the splitting of $H^n_{\mathrm{prim}}(X_{e,n+1})$ into rational G-modules carrying a Hodge structure. I shall treat two special cases

(1) An ordinary curve singularity of order e. One must have

$$H^1(X_{e,2}) = \bigoplus_{d|e,\, d\neq 1,e} \widetilde{V}_{\frac{0\|1}{d}}.$$

The intersection $X_{e,1}$ consists of e points which G permutes cyclically. So this gives the direct sum as representation space the regular representation $\bigoplus_{d|e} W_d$, or as a Hodge structure, $\bigoplus_{d|e} W_{\frac{0}{d}}$.

This implies that the Hodge theoretic nearby fibre is

$$(1 - \sum_{d|e,\, d\neq 1} W_{\frac{0}{d}}) \cdot (1 + \mathbb{L}) - H^1(D_2) - \sum_{d|e,\, d\neq 1} \widetilde{V}_{\frac{0\|1}{d}}.$$

(2) An ordinary surface singularity of multiplicity 5. The exceptional component is a smooth surface of degree 5 in $\mathbb{P}^3$ meeting the proper transform of the original fibre in a smooth plane curve of degree 5, hence of genus 6.

One has $h^{2,0} = 4$, $h^{1,1}_{\text{prim}} = 4 \cdot 13$. Here $H^{2,0}(X_{5,3}) = U_{1,5} + 3U_{2,5}$ so that $H^2_{\text{prim}}(X_{5,3}) = \widetilde{V}_{\frac{0\|5}{5}} \oplus 2\widetilde{W}_{\frac{1\|(2,1,4,3)}{5}} \oplus 10\widetilde{V}_{\frac{1\|2}{5}}$. The Hodge theoretic nearby fibre is

$$\chi_{\text{Hdg}}(D_2) - \widetilde{V}_{\frac{0\|1}{5}} \cdot (1 + \mathbb{L}) - \mathbb{L} + \widetilde{V}_{\frac{0\|5}{5}} + 2\widetilde{W}_{\frac{1\|(2,1,4,3)}{5}} + 10\widetilde{V}_{\frac{1\|2}{5}}.$$

9.2. Vanishing Cycle Sheaf and Applications to Singularities

I shall first explain the construction of the vanishing cycle sheaf $\phi_f(K)$ for a complex K on the total space. It is based on the cone construction on the adjunction morphism $k^\# : K \to (Rk)_* k^{-1}K$:

$$\phi_f(K) := \text{Cone}(i^* k^\# : i^*K \to i^*(Rk)_* k^{-1}K = \psi_f(K)).$$

The *Hodge theoretic vanishing cycle complex*

$$\phi_f^{\text{Hdg}} := \text{Cone}(\underline{\mathbb{Q}}^{\text{Hdg}}_{X_0} \to \psi_f^{\text{Hdg}})$$

as a cone over a morphism of Hodge complexes of sheaves on D has a canonical structure of a Hodge complexes of sheaves on D. Its stalk at $x \in D$ computes the reduced cohomology of the Milnor fibre $F = \text{Mil}_{f,x}$. This is a direct consequence of the comparison result explained in Remark 4.2.2.

The long exact sequence of the cone shows that

$$\begin{aligned}\sum (-1)^k \mathbb{H}^k(D, \phi_f^{\text{Hdg}}) &= \Big(\sum (-1)^k \mathbb{H}^k(D, \psi_f^{\text{Hdg}}))\Big) - \chi_{\text{Hdg}}(E) \\ &= \chi_{\text{Hdg}}(\psi_f^{\text{mot}} - [D]),\end{aligned}$$

so that it makes sense to define

$$\phi_f^{\text{mot}} := \sum (-1)^{j+1} [D(j) \times \mathbb{P}^{j-2} \times \mathbb{A}^1] = \psi_f^{\text{mot}} - [D] \in \mathrm{K}_0^{\hat{\mu}}(\underline{\text{Var}}).$$

Examples 9.2.1 (Continuation of Example 9.1.1)**.** (1) Note that in the curve case the above expression simplifies to $\phi_f^{\text{mot}} = -D(2) \times \mathbb{A}^1$ and that $D(1)$ consists of points permuted by the action of μ_e. It follows that the Hodge theoretic vanishing cycle is just

$$- \sum_{d|e,\, d\neq 1} W_{\frac{0}{d}} \cdot \mathbb{L}$$

(2) In the surface case the expression for the motivic vanishing cycle simplifies to $-D(2) \times \mathbb{A}^1 + D(3) \times \mathbb{A}^1 \times \mathbb{P}^1$. For an isolated ordinary surface singularity of prime multiplicity e one then gets

$$-(1 - \widetilde{V}_{\frac{0\|1}{e}} + \mathbb{L}) \cdot \mathbb{L} = -(\mathbb{L} + 1) \cdot \mathbb{L} + \widetilde{V}_{\frac{0\|1}{e}} \cdot \mathbb{L}.$$

Suppose that now $u : W \to \mathbb{C}$ is a projective morphism where W is smooth, of relative dimension n and that there is a single isolated critical point x such that $u(x) = 0$. Construct $f : X \to \Delta$ by replacing the zero fibre $u^{-1}(0)$ by a divisor with normal crossings E as above. In particular, there is a complex analytic map $\iota : X \to W$. The Milnor fibre of F of u at x corresponds to the Milnor fibre of f at any of the points $y \in \iota^{-1}x$. Since

$$\widetilde{H}^k(F) = H^k(\phi_{f,y}^{\mathrm{Hdg}})$$

a spectral sequence argument which will be omitted shows that $\widetilde{H}^k(F) = \mathbb{H}^k(E, \phi_f^{\mathrm{Hdg}})$, the reduced cohomology of F has a mixed Hodge structure. It is known that F has the homotopy type of a wedge of spheres of dimension $n = \dim F$ so that only $\widetilde{H}^n(F)$ is possibly non-zero. Hence

$$\chi_{\mathrm{Hdg}}^{\hat{\mu}}(F) = \chi_{\mathrm{Hdg}}^{\hat{\mu}}(\phi_f) = (-1)^n[\widetilde{H}^n(F)].$$

Write

$$e^{\hat{\mu}}(F) := P^{\hat{\mu}} \circ \chi_{\mathrm{Hdg}}^{\hat{\mu}}(F) = \sum_{\alpha \in \mathbb{Q},\ w \in \mathbb{Z}} m(\alpha, w) u^{\alpha} v^{w-\alpha}.$$

In the literature several numerical invariants have been attached to the singularity $f : (X, x) \to (\mathbb{C}, 0)$. These are all related to the numbers $m(\alpha, w)$ as follows:

(1) The *characteristic pairs* [**Ste77**, Sect. 5].

$$\mathrm{Chp}(f, x) = \sum_{\alpha, w} m(\alpha, w) \cdot (n - \alpha, w).$$

(2) The *spectral pairs* [**N-S**]:

$$\mathrm{Spp}(f, x) = \sum_{\alpha \notin \mathbb{Z}, w} m(\alpha, w) \cdot (\alpha, w) + \sum_{\alpha \in \mathbb{Z}, w} m(\alpha, w) \cdot (\alpha, w + 1).$$

(3) The singularity spectrum in Saito's sense [**Sa**]:

$$\mathrm{Sp}_{\mathrm{Sa}}(f, x) = e^{\hat{\mu}}(F)(t, 1).$$

(4) The singularity spectrum in Varchenko's sense [**Var**]:

$$\mathrm{Sp}_{\mathrm{V}}(f, x) = t^{-1} e^{\hat{\mu}}(F))(t, 1).$$

As a consequence, calculation of the motivic vanishing cycle gives a way to find all of these invariants.

Examples 9.2.2 (Continuation of Example 9.1.1)**.** (1) For an ordinary curve singularity of prime order e one finds

$$e^{\hat{\mu}}(F) = -\sum_{k=1}^{e-1} u^{\frac{k}{e}} v^{\frac{e-k}{e}}$$

and hence, for example, $\mathrm{Chp}(f, x) = -\sum_{k=1}^{e-1}(1 - \frac{k}{e}, 1)$.

(2) For an ordinary surface singularity of prime order e one finds

$$e^{\hat{\mu}}(F) = -uv - u^2v^2 + uv \left(\sum_{k=1}^{e-1} u^{\frac{k}{e}} v^{2-\frac{k}{e}} \right)$$

which gives for instance $\mathrm{Spp}(f, x) = \sum_{k=1}^{e-1}(\frac{k}{e}, 2) - (1, 3) - (2, 4)$.

As a last illustration of a motivic link I shall rephrase the original Thom-Sebastiani theorem (i.e. for the case of isolated singularities):

Theorem. *Consider holomorphic germs $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ with isolated singularity. Then the germ*

$$f \oplus g : (\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}, (0,0)) \to \mathbb{C}, \quad (f \oplus g)(x, y) := f(x) + g(y), \tag{27}$$

the join *of f and g, has also an isolated singularity, and*

$$\chi^{\hat{\mu}}_{\mathrm{Hdg}}(\phi_{f\oplus g}) = -\chi^{\hat{\mu}}_{\mathrm{Hdg}}(\phi_f) * \chi^{\hat{\mu}}_{\mathrm{Hdg}}(\phi_g)$$

so

$$e^{\hat{\mu}}(\phi_{f\oplus g}) = e^{\hat{\mu}}(\phi_f) \cdot e^{\hat{\mu}}(\phi_g) \in \varprojlim \mathbb{Z}[u^{\frac{1}{n}}, v^{\frac{1}{n}}, u^{-1}, v^{-1}]..$$

This theorem states that the Hodge theoretic vanishing cycle of the join of f and g is up to a sign equal to the convolution-product of the Hodge theoretic vanishing cycles of f and g. It makes you wonder whether there is a general construction on the level of $\mathrm{K}_0^{\hat{\mu}}(\underline{\mathrm{Var}})$. Looijenga [**L**] gave such a construction, but only after making the Lefschetz motive invertible, e.g. in

$$\mathcal{M}^{\hat{\mu}} := \mathrm{K}_0^{\hat{\mu}}(\underline{\mathrm{Var}})[L^{-1}].$$

He shows that the Thom-Sebastiani property holds already in this ring.

In fact, in view of Remark 9.1.4, it is more natural to work with the relative versions (see Definition 9.1.3)

$$\mathcal{M}^{\hat{\mu}}_S := \mathrm{K}_0^{\hat{\mu}}(\underline{\mathrm{Var}}_S)[L^{-1}].$$

Indeed, starting from a function $f : X \to \mathbb{C}$ constructions leading up to the motive associated to the join (27) take place in the relative motives over the fibre over 0. This explained in the next section.

9.3. Motivic Convolution

Here is an excerpt from [**L**] explaining the crucial construction of joins and convolutions in the motivic context.

Instead of considering varieties admitting an action of some finite cyclic group μ_n it is more efficient to consider them all at once by assembling these in the pro-cyclic group μ, thereby explaining the notation $\underline{\mathrm{Var}}^{\hat{\mu}}$. An element of μ can be thought of an infinite sequence $(a_1, a_2, \cdots)$ where only those $a_j \in \mu_j$ are different from 1 for which $j = kn$, a multiple of some integer n and such that for those the compatibility restrictions $a_{kn} = a_n^k$ hold. Clearly, if a variety admits a μ-action it admits a μ_n-action for some n and conversely. If X and Y admit a μ-action, also $X \times Y$ admit a (diagonal) μ-action, but X might have a μ_n-action, Y a μ_m-action with $n \neq m$ and then $X \times Y$ admits a μ_r-action with $r = \mathrm{lcm}(m, n)$. The convolution product is defined with the aid of the Fermat curve

$$J_n := \{(u, v) \in (\mathbb{C}^*)^2 \mid u^n + v^n = 1\}.$$

It is invariant under the subgroup $\mu_n^2 \subset (\mathbb{C}^*)^2$. Define

$$J_n(X, Y) := J_n \times_{(\mu_n \times \mu_n)} (X \times Y).$$

(If a group G acts on varieties A and B, then $A \times_G B$ stands for the quotient of $A \times B$ by the equivalence relation $(ga, b) \sim (a, gb)$ with G acting on it by $g[a,b] := [ga,b] = [a,gb]$). The group μ_n acts diagonally on $J_n(X,Y)$, i.e. $\zeta[(u,v),(x,y)] := [(\zeta u, \zeta v),(x,y)]$. Summarizing: from a couple (X,Y) of varieties with μ_n-action I have produced a new variety $J_n(X,Y)$ with μ_n-action. Clearly, $J_n(X,Y) \cong J_n(Y,X)$ and if m is a divisor of n and the action of μ_n on X and Y is through μ_m, then $J_m(X,Y) = J_n(X,Y)$. So this induces a binary operation, the *join*

$$J : \underline{\mathrm{Var}}^{\hat{\mu}} \times \underline{\mathrm{Var}}^{\hat{\mu}} \to \underline{\mathrm{Var}}^{\hat{\mu}}$$

and likewise on the level of K-theory:

$$J : \mathcal{M}^{\hat{\mu}} \times \mathcal{M}^{\hat{\mu}} \to \mathcal{M}^{\hat{\mu}}.$$

To define the convolution, one needs the *augmentation* by passing to the quotient

$$\epsilon : \mathcal{M}^{\hat{\mu}} \to \mathcal{M} \to \mathcal{M}^{\hat{\mu}}, \quad \epsilon[X] = [X/\mu].$$

The last map comes from considering X/μ as a variety with trivial μ-action.

The convolution can now be defined as follows:

$$a * b = -J(a,b) + (\mathbb{L} - 1) \cdot \epsilon(ab), \quad a,b \in \mathcal{M}^{\hat{\mu}}.$$

This definition is set up just so that 1 is a unit for $*$ and ϵ transfers $*$ into the usual product. In the relative situation, say $X \to S$ and $Y \to T$, one has $X \times Y \to S \times T$ and if X and Y have a μ-action, there is an exterior convolution product $[X] * [Y] \in \mathcal{M}^{\hat{\mu}}_{S \times T}$.

Consider now the Thom-Sebastiani property. So let X and Y be smooth connected varieties and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$ non-constant morphisms. Set $X_0 = f^{-1}(0)$ and $Y_0 = g^{-1}(0)$. Note that in general $(f \oplus g)^{-1}(0) \supset X_0 \times Y_0$ but equality does not necessarily hold. This explains that some care has to be taken:

Theorem. *In the preceding situation one has*

$$[\phi_{f \oplus g | X_0 \times Y_0}] = [\phi_f] * [\phi_g] \in \mathcal{M}^{\hat{\mu}}_{X_0 \times Y_0}.$$

LECTURE 10

Motives in the Relative Setting, Topological Aspects

10.1. The Relative Approach

I started the lectures with the motivic topological point of view in the topological category. I can do this in a relative way, i.e. one can consider topological spaces equipped with a continuous map to some fixed base space S. For such a relative topological space $f: X \to S$ one should consider a relative Euler characteristic built from the sheaves $R^k f_* \underline{\mathbb{Q}}_X$. These belong to the category of sheaves of $\mathbb{Q}$-vector spaces on S.

To get a well-defined Euler characteristic, one needs some finite dimensionality condition as in the absolute case. In the algebraic setting this is automatic and I shall not make this condition explicit but loosely refer to it as f *has finite dimensional fibres.* As to functoriality, given a continuous map $f: X \to Y$, to sheaves F, G of $\mathbb{Q}$-vector spaces on X, respectively G, one can associate $f^{-1}G$, the topological pull back, f_*F, the push forward, and, if f has finite dimensional fibres, there is also $f_!F$, the push forward with proper support.

To an abelian category $\mathfrak{A}$ one has associated its derived category $D^{\mathrm{b}}(\mathfrak{A})$ consisting of bounded complexes K in $\mathfrak{A}$. Recall (§ 4.1) that the morphisms between two complexes $K \to L$ are the fractions $[f]/[s]$ where $f: K \to M$ and $s: M \overset{s}{\longleftarrow} K$ are morphisms of complexes, s a quasi-isomorphism. The brackets denote homotopy classes. Here I shall consider

$$\boxed{D^{\mathrm{b}}(S) = D^{\mathrm{b}}(\text{sheaves of } \mathbb{Q}\text{-spaces on } S).}$$

The topological pull back f^{-1} is exact, while the push forward f_* is only left exact. They induce the functors $f^{-1}: D^{\mathrm{b}}(Y) \to D^{\mathrm{b}}(X)$ and $Rf_*: D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$ respectively. The push forward with proper support induces $Rf_!: D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$. Although there is no functor $f^!$ on the level of sheaves, there is a pull back functor with proper support $f^!: D^{\mathrm{b}}(Y) \to D^{\mathrm{b}}(X)$. This is somewhat subtle. See [**P-S**, Chap. 13.1.4].

These four functors are related by *adjoint relations*

$$\left.\begin{array}{r}\mathrm{Hom}(f^{-1}L, K) = \mathrm{Hom}(L, Rf_*K) \\ \mathrm{Hom}(Rf_!K, L) = \mathrm{Hom}(K, f^!L)\end{array}\right\} \quad K \in D^{\mathrm{b}}(S), L \in D^{\mathrm{b}}(T).$$

If you apply the first to $K = f^{-1}L$ and the second one to $K = f^!L$, and with the identity morphism, one gets the *adjunction morphisms*

$$f^{\#}: L \to Rf_*f^{-1}L, \quad f_{\#}: Rf_!f^!L \to L. \tag{28}$$

The preceding constructions should be considered in relation with the Verdier duality operator

$$\boxed{\mathsf{D}_X : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(X)}$$

which is an involution. It intertwines f_* and $f_!$ and also f^{-1} and $f^!$:

$$\boxed{Rf_*\circ\mathsf{D}_X = \mathsf{D}_Y\circ(Rf)_!, \quad f^{-1}\circ\mathsf{D}_Y = \mathsf{D}_X\circ f^!.}$$

The definition of the *Verdier duality operator* is a bit involved. See [**P-S**, Chap. 13.1.3]. The following example explains how it relates to the usual Poincaré duality operator.

Example 10.1.1. The Verdier dual of the constant sheaf, $\mathsf{D}_X\underline{\mathbb{Q}}_X$ by definition computes Borel-Moore homology and ordinary homology:

$$H^{-p}(X, \mathsf{D}_X\underline{\mathbb{Q}}_X) = H_p^{\mathrm{BM}}(X), \quad H_c^{-p}(X, \mathsf{D}_X\underline{\mathbb{Q}}_X) = H_p(X). \tag{29}$$

If X is a smooth complex algebraic variety, $\mathsf{D}_X\underline{\mathbb{Q}}_X$ is the sheaf $\underline{\mathbb{Q}}_X$ placed in degree $-2d$, $d = \dim X$ so that $H_p^{\mathrm{BM}}(X) = H^{-p}(X, \underline{\mathbb{Q}}_X[2d]) = H^{2d-p}(X)$ and $H_p(X) = H_c^{-p}(X, \underline{\mathbb{Q}}_X[2d]) = H_c^{2d-p}(X)$. This shows that Verdier duality generalises Poincaré duality in a natural way.

The four functors just introduced have the following important properties. Details can be gleaned from [**P-S**, Chap. 13.1.4 and Appendix B.2.5–6].

Properties 10.1.2. (1) Hypercohomology comes from the constant map $a_X : X \to \mathrm{pt}$. Indeed for $K \in D^{\mathrm{b}}$ one has

$$\mathbb{H}^k(X, K) = H^k((Ra_X)_*K), \ \mathbb{H}_c^k(X, K) = H^k((Ra_X)_!K)). \tag{30}$$

It follows that for a morphism $f : X \to Y$, the induced map $H^k(Y, K) \to H^k(X, f^{-1}K)$ comes from the adjunction morphism

$$\mathbb{H}^k(Y, L) \xrightarrow{f^\#} \mathbb{H}^k(Y, Rf_*f^{-1}L) = \mathbb{H}^k(X, f^{-1}L).$$

Since for a proper morphism f one has $\mathbb{H}_c^k(Y, Rf_*f^{-1}L) = \mathbb{H}_c^k(X, f^{-1}L)$, this gives in this case an induced map

$$\mathbb{H}_c^k(Y, L) \xrightarrow{f^\#} \mathbb{H}_c^k(X, f^{-1}L).$$

The other adjunction relation gives an induced map

$$\mathbb{H}_c^k(X, f^!L) = \mathbb{H}^k(Y, (Rf)_!f^!L) \xrightarrow{f_\#} \mathbb{H}_c^k(Y, L).$$

(2) Let $i : Z \hookrightarrow X$ be a closed embedding and $U = X - Z$ the complement with embedding $j : U \hookrightarrow X$. Then there is an *adjunction triangle* in $D^{\mathrm{b}}(X)$ which is a distinguished triangle:

$$Ri_*i^!K \to K \to Rj_*j^{-1}K \to [1]. \tag{31}$$

The left exact functor $(Ra_X)_*$ preserves distinguished triangles and so

$$(Ra_Z)_*i^!K \to K \to (Ra_U)_*j^{-1}K \to [1].$$

The cohomology of the first term, $\mathbb{H}^k(Z, i^!K)$ by definition gives $\mathbb{H}_Z^k(X, K)$, the cohomology with support in Z. This yields the long exact sequence

$$\cdots \to \mathbb{H}_Z^k(X, K) \to \mathbb{H}^k(X, K) \xrightarrow{j^*} \mathbb{H}^k(U, K) \to \cdots.$$

For $K = \underline{\mathbb{Q}}_X$ this is the usual long exact sequence for cohomology with supports.

(3) There is a different adjunction triangle which you get applying Verdier duality to (31):

$$(Rj)_! j^{-1}K \to K \to Ri_* i^{-1}K \to [1]. \tag{32}$$

For $K = \underline{\mathbb{Q}}_X$, but now with compactly supported cohomology (i.e. applying the functor $(a_X)_!$) this gives

$$\cdots \to H_c^k(U) \xrightarrow{j^*} H_c^k(X) \xrightarrow{i^*} H_c^k(Z) \to \cdots .$$

This is clearly related to the motivic approach and can be generalised to the relative situation as will be seen in § 10.2.

As a first step, let me introduce *constructible sheaves*: by definition these are sheaves of $\mathbb{Q}$-vector spaces restricting to a local constant system on the open strata with respect to some stratification of S.

Example 10.1.3. In the algebraic category the sheaves $R^k f_* \underline{\mathbb{Q}}_X$ clearly have this property: they are locally constant over the Zariski open subset S^0 over which f is smooth, then restrict f to the inverse image X^1 of the complement $S_{d-1} = S - S^0$. It is of locally differentiable trivial over some Zariski-open subset $S^1 \subset S_{d-1}$ and hence the sheaf $R^k f_* \underline{\mathbb{Q}}_{X(1)}$ will be locally constant over S^1. Continuing in this way one gets the desired stratification $S \supset S^0 \supset S^1 \supset \cdots$.

It is then crucial that local systems as in the preceding system have finite dimensional stalks. However, from the point of view of derived categories it is better to have complexes whose stalks are not necessarily finite dimensional, but whose cohomology is finite dimensional. For instance, in the above example, one considers the entire derived complex $Rf_* \underline{\mathbb{Q}}_X$. In this way one arrives at the following definition

Definition 10.1.4. A complex of sheaves K of $\mathbb{Q}$-vector spaces on S is called a *constructible complex* if each cohomology sheaf $H^k(K)$ is a constructible sheaf of finite dimensional $\mathbb{Q}$-vector spaces and if moreover $H^k(K) = 0$ for $|k| \gg 0$, i.e. K has bounded cohomology. The corresponding derived category is denoted

$$\boxed{D^{\mathrm{b}}_{\mathrm{cs}}(S) = D^{\mathrm{b}}(\text{constructible complexes of sheaves of } \mathbb{Q}\text{-spaces on } S).}$$

If $K \in D^{\mathrm{b}}_{\mathrm{cs}}(S)$, one may put

$$[K] = \sum (-1)^k [H^k(K)] \in \mathrm{K}_0(D^{\mathrm{b}}_{\mathrm{cs}}(S)).$$

Example 10.1.5 (Continuation of Example 10.1.3)**.** The alternating sum $\chi_{\mathrm{top}}(X/S) = \sum (-1)^k [R^k f_* \underline{\mathbb{Q}}_X] = [Rf_* \underline{\mathbb{Q}}_X]$ is a well-defined object in $\mathrm{K}_0(X, D^{\mathrm{b}}_{\mathrm{cs}}(S))$ and it generalises the topological Euler characteristic. However, only the Euler characteristic with compact support behaves motivically, as we have seen, and so I need to refine the preceding theory to incorporate compact supports; one has to replace the functor Rf by the compactly supported derived image functor $Rf_!$ and then

$$\boxed{\chi^c_{\mathrm{top}}(X/S) = [(Rf_! \underline{\mathbb{Q}}_X].} \tag{33}$$

I shall next explain that this has a motivic interpretation.

10.2. Perverse Sheaves

If X is a smooth variety there is a good topological duality theory. For singular varieties one has to replace the constant sheaf $\underline{\mathbb{Q}}_X$ by a complex which takes into account the singularities of X and which behaves well with respect to Verdier duality. So, even if X is smooth, it is better to work with $\underline{\mathbb{Q}}_X[d]$, i.e. one should place the complex in degree $-d$. Indeed, since $\mathsf{D}_X(K[n]) = (\mathsf{D}_X K)[-n]$ the complex $\underline{\mathbb{Q}}_X[d]$ is thus self-dual; it is the first basic example of a perverse sheaf.

To explain what these are in general, recall the constructible complexes introduced in § 10.1. Let X be a possibly singular variety and let $U \subset X$ be a Zariski open subset consisting entirely of smooth points of X. Given any local system V of finite dimensional $\mathbb{Q}$-vector spaces on U such as $\underline{\mathbb{Q}}_U$ there is a canonical way of extending it to a constructible complex $\widetilde{V}$ on X which turns out to be self-dual under Verdier duality provided one places V in degree $-d$, $d = \dim X$. The resulting complex

$$IC_X(V) := \widetilde{V}[d] \in D^{\mathrm{b}}_{\mathrm{cs}}(X) \tag{34}$$

is called the *intersection complex* and one has $\mathsf{D}_X(IC(V)) = IC(V)$. Such a complex is called a *perverse sheaf*. See [**P-S**, Chap. 13.2] for details on this and on what follows. More generally, if V is a local system of finite dimensional $\mathbb{Q}$-vector spaces on a Zariski dense open smooth subset $j : Z^0 \hookrightarrow Z$ of a closed subvariety $i : Z \hookrightarrow X$, its *perverse extension*

$$\boxed{{}^{\pi}V = i_*[\widetilde{V}[d_Z]] = i_*[IC_Z(V)], \quad d_Z = \dim Z}$$

is a perverse sheaf on X. Note that this complex is zero outside of Z: it is entirely supported on Z.

Examples 10.2.1. (1) Let V be a local system on Δ^*. Its perverse extension to Δ is $j_*V[1]$ where $j : \Delta^* \hookrightarrow \Delta$ is the inclusion. Its stalk over 0 is the subspace of invariants under the monodromy acting on the general stalk of V.

(2) This becomes more involved if V is a local system over $(\Delta^*)^d$ with local monodromy operators $T_1, \dots, T_d$ acting on the generic stalk W. The perverse extension over the coordinate hyperplanes is the sheaf $j_*V[d]$ but already the extension to the codimension-2 boundary gives a 2-step complex and so on. For instance, (cf. [**C-K-S87**, § 1]) the fibre at the origin is a certain subcomplex of the full Koszul complex on $N_k = T_k - \mathrm{id}$, $k = 1, \dots, d$. The full complex is as follows:

$$0 \to W \xrightarrow{d_0} U \otimes W \xrightarrow{d_1} \Lambda^2 U \otimes W \to \cdots \to \Lambda^d U \otimes W \to 0,$$

where $U = \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_d$ and

$$d_j(u_{i_1} \wedge \cdots u_{i_j} \otimes w) = \sum_k e_k \wedge u_{i_1} \wedge \cdots u_{i_j} \otimes N_k w.$$

This complex calculates the stalk at the origin of Rj_*V. For instance, if $d = 1$ its H^0 gives the invariants and its H^1 gives the co-invariants.

The stalk at 0 of the perverse extension of V is obtained first by shifting the above complex so that it starts in degree $-d$ and next, one replaces $\Lambda^k U \otimes W$ by

$$\bigoplus_I \mathbb{Q}e_I \otimes N_I W, \quad I = (i_0, \dots, i_k) \text{ (strictly increasing)},$$
$$e_I := e_{i_0} \wedge \cdots \wedge e_{i_k} \quad N_I = N_{i_0} \circ \cdots \circ N_{i_k}.$$

For instance, if $d = 1$ this has the effect of naming the previous H^0 now H^{-1} and replacing the old H^1 by 0.

The perverse sheaves form an abelian subcategory $\mathsf{Perv}(X)$ of $D^{\mathrm{b}}_{\mathrm{cs}}(X)$ and $D^{\mathrm{b}}(\mathsf{Perv}(X)) = D^{\mathrm{b}}_{\mathrm{cs}}(X)$. This is not obvious at all and is explained in [**B-B-D**]. One can also show that in a certain sense the category $\mathsf{Perv}(X)$ is generated by intersection complexes of local systems on the smooth locus of lower dimensional subvarieties:

Proposition 10.2.2. $\mathrm{K}_0(\mathsf{Perv}\, X)$ *is generated by the classes of the perverse extensions of local systems defined on the smooth locus of closed subvarieties of* X*.*

The direct image functors do not preserve perversity: one has to consider the derived functors $Rf_*, Rf_! : D^{\mathrm{b}}_{\mathrm{cs}}(X) \to D^{\mathrm{b}}_{\mathrm{cs}}(Y)$, and so even if $K \in \mathsf{Perv}(X)$, Rf_*K in general belongs to the larger category $D^{\mathrm{b}}_{\mathrm{cs}}(Y)$.

Example 10.2.3. Regardless whether K is perverse or not, in $D^{\mathrm{b}}_{\mathrm{cs}}(X)$ one has $[K] = [i_*i^!K] + [Rj_*j^{-1}K]$.

With triangle (32) one finds $[K] = [i_*i^{-1}K] + [Rj_!j^{-1}K]$. In the special case $K = \underline{\mathbb{Q}}_X$ this reads $[\underline{\mathbb{Q}}_X] = [i_!\underline{\mathbb{Q}}_Z] + [Rj_!\underline{\mathbb{Q}}_U]$ (note that $i_* = i_!$). Since $(Ra_X)_!$ preserves triangles, this gives the relation $[(Ra_X)_!\underline{\mathbb{Q}}_X] = [(Ra_Z)_!\underline{\mathbb{Q}}_Z] + [(Ra_U)_!\underline{\mathbb{Q}}_U]$ which is compatible with the scissor relation.

I shall explain that these considerations can be made relative and thus will lead to a motivic interpretation of the relative Euler characteristic. I can now state and prove the promised motivic interpretation of (33):

Lemma 10.2.4. *The relative Euler class defines a ring homomorphism*

$$\chi^c_{\mathrm{top}} : \mathrm{K}_0(\underline{\mathrm{Var}}_S) \to \mathrm{K}_0(\mathsf{Perv}(S)).$$

Proof: Let K be a constructible complex on X. Apply $Rf_!$ to the distinguished triangle (32). This yields a new distinguished triangle in $D^{\mathrm{b}}_{\mathrm{cs}}(S)$. take now $K = \underline{\mathbb{Q}}_X$. Hence, as in Example 10.2.3 one gets the relation $[Rf_!\underline{\mathbb{Q}}_X] = [(R(f \circ i))_!\underline{\mathbb{Q}}_Z] + [(R(f \circ j))_!\underline{\mathbb{Q}}_U]$ in $\mathrm{K}_0(\mathsf{Perv}(S))$, i.e. $\chi^c_{\mathrm{top}}(X_{/S}) = \chi^c_{\mathrm{top}}(Z_{/S}) + \chi^c_{\mathrm{top}}(U_{/S})$. □

One of the difficulties of working with the categories $\mathsf{Perv}(X)$ and $D^{\mathrm{b}}_{\mathrm{cs}}(X)$ is that the cohomology sheaves of an object in any of these categories rarely are perverse. However, there is a modified cohomology theory which behaves better: for a complex $K \in D^{\mathrm{b}}_{\mathrm{cs}}(X)$ the *perverse cohomology* ${}^{\pi}H^k(K)$ is a perverse sheaf. This is in general not a single sheaf, but a whole complex of sheaves. I shall not give details on how to compute perverse cohomology which is explained for instance in [**deC-M**], but rather give some relevant examples which can be worked out given the explanation in loc. cit.

Examples 10.2.5. (1) If $K \in \mathsf{Perv}(X)$, ${}^{\pi}H^0(K) = K$ and ${}^{\pi}H^k(K) = 0$ for $k \neq 0$.

(2) If K is locally constant on a smooth variety X, one has ${}^\pi H^j(K) = 0$ unless $j = d = \dim X$ and then ${}^\pi H^d(K) = K[d]$. More generally take for K any complex whose cohomology sheaves are locally constant on X. Then ${}^\pi H^j(K) = \big(H^{j-d}(K)\big)[d]$.

(3) This is a relative version of the previous example. Let $f : X \to Y$ be a smooth morphism between complex manifolds and K a local system on X. Then the direct images $R^k f_* K$ are local systems on Y. The perverse cohomology of such a local system is

$$\boxed{{}^\pi R^k f_* K := {}^\pi H^k R f_* K = (R^{k-e} f_* K)[e], \quad e = \dim Y.}$$

(4) Let $f : X \to Y$ be a surjective morphism between smooth varieties. Let Y^0 be the subset of the regular values of f and put $f^0 : X^0 = (f^0)^{-1} Y^0 \to Y^0$. Fix some k and let $V^k = R^k f^0{}_* \underline{\mathbb{Q}}_{X^0}$. This is a local system. Its perverse extension is $\widetilde{V}^k[e]$ (see (34) for the notation). The perverse direct image ${}^\pi R^{k+e} f_* \underline{\mathbb{Q}}_X$ always contains the perverse complex $\widetilde{V}^k[e]$ as a summand but it may contain further summands entirely supported on the discriminant locus $Y - Y^0$. For instance, if $\dim X = 2$, $\dim Y = 1$ and $k = 1$ one has to add the summand $\mathrm{Ker}(R^2 f_* \underline{\mathbb{Q}}_X \to j_* V^2)$ coming from the Leray spectral sequence. This is somewhat involved and I refer to [**deC-M**, Remark 3.4.12] for the calculation.

Recall that Deligne has shown [**Del68**] that the Leray spectral sequence for a projective *smooth* morphism $f : X \to Y$ degenerates at E_2. In fact something stronger holds: one has a non-canonical decomposition

$$R f_* \underline{\mathbb{Q}}_X \simeq \bigoplus R^k f_* \underline{\mathbb{Q}}_X[-k] \tag{35}$$

which generalizes for perverse sheaves such as $K = \underline{\mathbb{Q}}_X[d]$ to the existence of a non-canonical isomorphism

$$R f_* K \simeq \bigoplus {}^\pi R^k f_* K[-k] \tag{36}$$

If f is no longer smooth (35) in general is false, but I shall explain (Theorem 12.3.2) that (36) remains true provided f is still projective. This is a deep theorem, originally proved in [**B-B-D**] using characteristic p-methods and later in [**Sa88**] by analytic means.

To hint at why such a theorem could be true, go back to the last example. The k-th perverse image in general splits into objects constructed from cohomology sheaves in *different* degrees and can become rather complicated. The idea is that these extra pieces need to be present to compensate for the fact that the ordinary Leray spectral sequence in general does not degenerate at E_2.

LECTURE 11

Variations of Hodge Structure

11.1. Basic Definitions

Up to now only one mixed Hodge structure at a time has been considered. Instead, one can also consider mixed Hodge structures on a fixed vector space that depend on parameters. The proper concept is that of a *local system*, i.e. a locally constant sheaf of finite dimensional $\mathbb{Q}$-vector spaces. Such a system can be trivialized over a simply connected base and hence it pulls back to a trivial system over the universal cover. This implies that such a system can be seen as defined by a finite dimensional representation of the fundamental group of the base. The image of the latter in the automorphisms of the vector space is called the *monodromy group* of the local system. Suppose that the local system is such that each stalk carries a Hodge structure. If these fit nicely together, one speaks of a variation of Hodge structure:

Definition 11.1.1. Let S be a complex manifold. A *variation of Hodge structure of weight* k on S is a pair (V, F) consisting of a local system V of finite dimensional $\mathbb{Q}$-vector spaces groups on S and a finite decreasing filtration F on the holomorphic vector bundle $\mathcal{V} = V \otimes_{\mathbb{Q}} \mathcal{O}_S$ by holomorphic subbundles (the Hodge filtration). These data should satisfy the following conditions:

i) for each $s \in S$ the filtration F induces a Hodge structure of weight k on the stalk of V;
ii) the connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega^1_S$ whose sheaf of horizontal sections is $V_{\mathbb{C}}$ satisfies the *Griffiths' transversality condition*

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_S. \tag{37}$$

The notion of a *morphism of variations of Hodge structure* is defined in the obvious way. Moreover, given two variations V, V' of Hodge structure over S of weights k and k', there is an obvious structure of variation of Hodge structure on the underlying local systems of $V \otimes V'$ and $Hom(V, V')$ of weights $k + k'$ and $k - k'$ respectively.

Example 11.1.2. The standard example comes from geometry as follows. Let $f : X \to S$ be a proper and smooth morphism between complex algebraic manifolds. Such a morphism is locally differentiable trivial by the Ehresmann theorem [**Ehr**]. Therefore the cohomology groups $H^k(X_s)$ of the fibres X_s fit together into a local system. By the fundamental results of Griffiths [**Grif68**] this local system underlies a variation of Hodge structure on S such that the Hodge structure at s is just the Hodge structure one has on $H^k(X_s)$.

This leads immediately to the abelian category $\mathfrak{hs}_{/S}$ of variations of Hodge structures over S. If $f : X \to S$ is smooth and projective the relative Euler-class morphism can be refined since by Example 11.1.2 the $R^k f_* \underline{\mathbb{Q}}_X$ carry variations of Hodge structures. Indeed, then the class $[f_! \underline{\mathbb{Q}}_X] = [f_* \underline{\mathbb{Q}}_X] = \sum (-1)^k [R^k f_* \underline{\mathbb{Q}}_X]$ identified as the relative Euler characteristic (see (33)), lives naturally in the K-group of the category of variations of Hodge structures over S.
To go further one has to extend the category to variations of Hodge structures which are supported over a Zariski-open subset consisting of smooth points of a Zariski-closed subvariety $Z \subset S$. This leads to the following category:

Definition 11.1.3. The category $\mathfrak{hs}_{/S}$ of *Hodge structures over* S consists of variations of Hodge structures supported over the smooth points of closed subvarieties of S.

Theorem 11.1.4 (Existence of the relative motivic Hodge characteristic)**.** *There is a morphism of rings*

$$\chi^c_{/S} : \mathrm{K}_0(\underline{\mathrm{Var}}_S) \to \mathrm{K}_0(\mathfrak{hs}_{/S}),$$

which, when composed with the morphism

$$\mathrm{rat}_S : \mathrm{K}_0(\mathfrak{hs}_{/S}) \to \mathrm{K}_0(\mathsf{Perv}(S))$$

induced by the forgetful map, yields χ^c_{top}.

Examples 11.1.5. (1) If $f : X \to S$ is a smooth map between projective manifolds, one has $\chi^c_{/S}(X) = \sum (-1)^k [R^k f_* \underline{\mathbb{Q}}_X] \in \mathrm{K}_0(\mathfrak{hs}_{/S})$ if one equips the direct image sheaves with their natural structure of variation of Hodge structure.

(2) Let $f : X \to S$ be a smooth projective surface fibered over a smooth projective curve. Let S_0 be the set of regular values of f and let $\Delta = S - S_0$. Suppose that $X_p, p \in \Delta$ is irreducible and has only ordinary double points, say N_p of them and that the genus of the normalization of X_p equals g_p. The variation of weight one Hodge structure $R^1 f_* \underline{\mathbb{Q}} | S_0$ is complete determined by the period map $\varphi : S^0 \to \mathfrak{h}_g / \Gamma$ where $g =$ genus of a smooth fibre, $\Gamma =$ the monodromy group of the family. Let me denote this variation by $V^1_{S_0}$. The underlying local system extends to a constructible sheaf whose stalk over any point $p \in \Delta$ is the subspace inside $H^1(X_t)$, t close to p of invariants under the local monodromy-action. This subspace is isomorphic to $H^1(X_p)$. Hodge-theoretically, one has to consider the limit mixed Hodge structure on $H^1(X_t)$ and then $H^1(X_p)$ is a mixed Hodge substructure. So, mixed Hodge structures come up inevitably. By Example 3 in Lecture 3 one has that $\chi_{\mathrm{Hdg}}(X_p) = 1 - N_p - [V_{g_p}]_p + [\mathbb{L}]_p$. Then one finds

$$\chi^c_{/S}(X) = \underline{\mathbb{Q}}_S - V^1_{S_0} + \underline{\mathbb{L}}_S - \sum_{p \in \Delta} \left((N_p)[\mathbb{Q}]_p + [V_{g_p}]_p \right). \tag{38}$$

Here $[M]_p$ denotes the class of the Hodge structure M supported on p and $\underline{\mathbb{L}}_S$ is the constant variation on S with stalk $\mathbb{L}$ at every point $p \in S$. Note that this is indeed an expression involving only variations of pure Hodge structures.

One way to prove the preceding theorem is to note that Theorem 2.1.5 remains true in the relative setting [**Bitt1**, Theorem 5.1]. As a consequence, one only needs to define the relative Hodge characteristic for X, S smooth and f projective. If f is smooth, variations of Hodge structure suffice to define it. To define $\chi^c_{/S}$ one has to deal with the problem posed by the singular fibres and one has to prove that the definition is stable under the blow-up relation. Instead, I shall follow a different route which makes use of mixed Hodge modules. These are the natural objects extending the variations of Hodge structure coming from geometry.

Since it takes several Lectures to complete this route, let me give first an **outline of the proof of Theorem 11.1.4**:
− A variation of Hodge structures contains a rational part, the underlying local system, and a part governed by the flat connection operating on the vector bundle associated to the local system and which shifts the Hodge filtration by one. The action of the flat connection can be seen as providing the bundle with the structure of a so-called D-module. This is explained in § 11.2.
− The D-module structure is linked to the rational structure through the Riemann-Hilbert correspondence. One needs to generalise this to include the so-called perverse sheaves. This is to take care of possibly singular varieties and to describe suitable extensions of variations initially only defined over certain subsets which are locally closed in the Zariski topology. The remainder of Lecture 11 is devoted to this.
− One thus is naturally led to the so-called Hodge modules. These are introduced in Lecture 12. In it a reformulation of Theorem 11.1.4 will be given in terms of these: Theorem 12.2.10.
− But this does not yet suffice. As amply shown in the examples that will be treated below, mixed Hodge structures come up inevitably and one needs to enlarge the setting to mixed Hodge modules. In Lecture 13 these will be explained and the proof of the theorem can be completed.

11.2. D-modules

Let me recall the definition of a *D-module* on a smooth complex variety S. The formal definition is maybe best preceded by an informal local discussion. Let $U \subset \mathbb{C}^d$ be an open set (in the classical topology) and let $(z_1, \ldots, z_n)$ be the standard coordinates on U. Then taking a partial differential with respect to z_k of a holomorphic function on U is an example of a differential operator of order 1, written ∂_k. Using multi-index notation, m-th order operators are of the form

$$\sum_{|I| \le m} P_I \partial^I, \quad \partial^I = \partial_1^{i_1} \cdots \partial_n^{i_n}.$$

Now ∂_k is associated to the constant holomorphic vector field $\dfrac{\partial}{\partial x_k}$ and composing m vector fields in general gives an operator of order $\le m$.

To arrive at a coordinate free definition, observe first that the Leibniz rule can be reinterpreted by saying that for any two holomorphic functions f and g on U, and any vector field X, the map $f \mapsto X(gf) - gX(f)$ is multiplication by $X(g)$. This is a differential operator of order 0. Similarly, starting with two holomorphic vector fields X_1, X_2 the composition of the two $X = X_1X_2$ by the Leibniz rule yields a differential operator $f \mapsto X(gf) - gX(f)$ of order ≤ 1 consisting of the sum of the first order operator $X_1(f)X_2 + X_2(f)X_1$ and multiplication by $X_1X_2(f)$. Continuing in this manner, one finds the following coordinate free definition which proceeds by induction:

- An operator of order 0 is multiplication by a function;
- an operator P is of order $\leq m$ if for all holomorphic functions f, g on U the operator $f \mapsto P(gf) - gP(f)$ is of order $\leq (m-1)$;

The differential operators (of any order) are gotten by taking the union over all orders m. They give a ring D_U and hence a sheaf $\mathcal{D}_U$. This also makes sense for any smooth complex variety S and can be done with the Zariski-topology or with the classical topology. A sheaf of left $\mathcal{D}_S$-modules is simply called a $\mathcal{D}_S$-module, or, if no confusion arises, a D-module.

The definition of coherence for D-modules mimics the definition for ordinary $\mathcal{O}_S$-modules and I will not dwell on this. See for instance [**P-S**, Chap 13.4].

Example 11.2.1. Let $V_{\mathbb{C}}$ be a local system of complex vector spaces on a complex manifold S. Then the corresponding vector bundle $\mathcal{V} = V_{\mathbb{C}} \otimes_{\underline{\mathbb{C}}_S} \mathcal{O}_S$ carries the flat connection $\nabla = 1 \otimes d$, i.e. $\nabla \circ \nabla = 0$. The pair $(\mathcal{V}, \nabla)$ is a (left) $\mathcal{D}_S$-module, i.e. differential operators act on $\mathcal{V}$. Indeed, any local holomorphic vector field ξ acts on germs of sections v of $\mathcal{V}$ as $v \mapsto \nabla_\xi v$ and the Leibniz rule guarantees that this action can be extended as an action of differential operators of any order, just as composition of vector fields acting on holomorphic functions give differential operators of any order. This D-module turns out to be coherent.

The preceding example can be used to rephrase the concept of a variation of Hodge structure with base S in terms of D-modules on S. Suppose that V underlies a variation of Hodge structure. Then the Griffiths transversality condition (37) states that the first order differential operators shift the Hodge filtration by -1. Hence the corresponding increasing filtration $\mathcal{F}_\bullet = \mathcal{F}^{-\bullet}$ is shifted by 1 and such a pair $((\mathcal{V}, \nabla), \mathcal{F}_\bullet))$ is a filtered D-module:

Definition 11.2.2. A *filtered* D-module (M, F), consists of a D-module M equipped with an exhaustive increasing filtration F by coherent $\mathcal{O}_S$-modules such that the operators of order 1 increase the filter degree of F by 1.

***Remark* 11.2.3.** The D-module $(\mathcal{V}, \nabla)$ is quite special since it comes from a local system $V_{\mathbb{C}}$. Let me summarise this as follows: a variation of Hodge structure $(V, F^\bullet)$ on a complex manifold S yields the filtered $\mathcal{D}_S$-module

$$\mathrm{Dmod}(V, F^\bullet) := ((\mathcal{V}, \nabla), F_\bullet). \tag{39}$$

$V_{\mathbb{C}}$ is a resolution of the corresponding *De Rham complex*

$$\mathcal{V} \xrightarrow{\nabla} \mathcal{V} \otimes \Omega^1_S \to \cdots \to \mathcal{V} \otimes \Omega^d_S, \quad d = \dim(S). \tag{40}$$

This leads to the Riemann-Hilbert correspondence, the subject of the next section.

11.3. The Riemann-Hilbert Correspondence

The exposition which follows summarizes [**P-S**, Chap. 13.5–13.6] where further references can be found.

Let me come back to the vector bundle $(\mathcal{V},\nabla)$ coming from a local system $V_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces on a complex manifold S. Let $d=\dim S$. The complex $V_{\mathbb{C}}[d]$ is a perverse sheaf, hence so is the De Rham complex (40) provided one shifts it as follows:

$$\mathsf{Dr}_S(\mathcal{V},\nabla):=[\mathcal{V}\xrightarrow{\nabla}\mathcal{V}\otimes\Omega^1_S\xrightarrow{\nabla}\dots\xrightarrow{\nabla}\mathcal{V}\otimes\Omega^d_S][d]\,.$$

Put

$\mathcal{D}^{\rm b}_{\rm coh}(S)$:	the derived category of bounded complexes of $\mathcal{D}_S$-modules with coherent cohomology.

So the assignment $(\mathcal{V},\nabla)\mapsto\mathsf{Dr}_S(\mathcal{V},\nabla)$ associates to a class of D-modules a class of perverse sheaves in a canonical way. Which class of D-modules is needed to make this a full one to one correspondence? Clearly, in order to make this work, one has to have an analogue of the Verdier duality operator on the level of D-modules. Indeed, there exists such an operator D_S which can be defined on the entire derived category $\mathcal{D}^{\rm b}_{\rm coh}(S)$. This operator has the desired property that the De Rham complex of D_SM is the (derived) dual $M^*=R\,\mathrm{Hom}_{\mathcal{D}_S}(M,\mathcal{O}_S)$:

$$\mathsf{Dr}_S{\circ}\mathsf{D}_SM=M^*. \tag{41}$$

To see what sort of special D-modules one then needs, consider the basic example of the vector bundle $\mathcal{V}$ associated to a local system V. In this case $R\,\mathrm{Hom}_{\mathcal{D}_S}(\mathcal{V},\mathcal{O}_S)=V^*_{\mathbb{C}}$, the dual of the local system $V_{\mathbb{C}}$ so that $D_S\mathcal{V}=\mathcal{V}^\vee$, the dual vector bundle of $\mathcal{V}$. It is a complex concentrated in degree 0 and hence this also holds for its cohomology. Such a $\mathcal{D}_S$-module $M=(\mathcal{V},\nabla)$ is called a *holonomic* $\mathcal{D}_S$-module M:

Definition 11.3.1. A $\mathcal{D}_S$-module M is *holonomic* if the complex D_SM only has cohomology in degree 0.

One now has a crude version of the desired correspondence:

Theorem (Riemann-Hilbert correspondence–I)**.** *Let S be a smooth compact complex algebraic variety. The assignment $M\mapsto\mathsf{Dr}_SM$ establishes a one-to-one correspondence between holonomic $\mathcal{D}_S$-modules and the category $\mathsf{Perv}_{\mathbb{C}}(S)$ of perverse sheaves of $\mathbb{C}$-vector spaces on S.*

If S is a no longer compact one has to modify this construction. First note that S always admits a *good compactification* $\bar S$; i.e. $\bar S$ is smooth and $\bar S-S=D$ is a normal crossing divisor D. Suppose that $V_{\mathbb{C}}$ is a local system of $\mathbb{C}$-vector spaces on S.

Definition 11.3.2. One says that the $\mathcal{D}_S$-module $(\mathcal{V},\nabla)$ admits a *regular* meromorphic extension to $\bar S$ if $\mathcal{V}$ extends to a vector bundle $\widetilde{\mathcal{V}}$ on $\bar S$ and ∇ extends with logarithmic poles along D. This means that ∇ extends to an operator $\widetilde{\mathcal{V}} \to \Omega^1_{\bar{S}}(\log D) \otimes \widetilde{\mathcal{V}}$ which satisfies Leibniz' rule. This property is independent of the chosen compactification. Indeed, one can show that this is the case if for instance all the local monodromy operators for the local system $V_{\mathbb{C}}$ around infinity are quasi-unipotent. This is known to be the case in the geometric situation. Moreover, such a regular extension is holonomic.

This then leads to the following full fledged version of the correspondence:

Theorem (Riemann-Hilbert correspondence–II)**.** *The De Rham-functor gives a one-to-one correspondence*

$$\boxed{\{\text{Regular holonomic } \mathcal{D}_S\text{-modules}\} \xrightarrow{\mathsf{Dr}_S} \mathsf{Perv}_{\mathbb{C}}(S).}$$

The full category $D^{\mathrm{b}}_{\mathrm{cs}}(S) \subset \mathsf{Perv}_{\mathbb{C}}(S)$ on the right corresponds to the subcategory $\mathcal{D}_{\mathrm{rh}}(S)$ of $\mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(S)$ on the left consisting of complexes of $\mathcal{D}_S$-modules whose cohomology groups are regular holonomic:

$$\boxed{\mathcal{D}_{\mathrm{rh}}(S) \xrightarrow{\mathsf{Dr}_S} D^{\mathrm{b}}_{\mathrm{cs}}(S).}$$

This should be complemented by the following assertions which describe the functorial behaviour. If $\varphi : S \to T$ is a morphism of complex algebraic manifolds there are (derived) functors $\varphi_*, \varphi_! : \mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(S) \to \mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(T)$ and $\varphi^*, \varphi^! : \mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(T) \to \mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(S)$ which intertwine the duality operator in that

$$\boxed{\varphi_* \circ \mathsf{D}_T = \mathsf{D}_S \circ \varphi_!, \quad \varphi^* \circ \mathsf{D}_T = \varphi^! \circ \mathsf{D}_S}$$

just as for constructible complexes. Indeed, they are fully compatible with the De Rham functor:

$$\boxed{\varphi^! \circ \mathsf{Dr}_S = \mathsf{Dr}_T \circ \varphi^!, \quad \varphi_* \circ \mathsf{Dr}_T = \mathsf{Dr}_S \circ \varphi_*}$$

and by (41) the duality-operator on the level of regular holonomic complexes corresponds to the Verdier duality operator.

Remark. Usually one employs the notation φ^+ and φ_+ instead of φ^* and φ_* since the latter two have a different meaning for D-modules. However in these Lectures this does not play a role.

Let me summarise what has been done so far. I have introduced the category of varieties over S (Def. 9.1.3) and the relative Euler characteristic (Lemma 10.2.4) which factors over the Hodge theoretic Euler characteristic. This makes use of variations of Hodge structures. Any variation of Hodge structure gives rise to a perverse sheaf and a D-module on the base. These constructions fit into a big commutative diagram

$$\begin{array}{ccccc} \mathrm{K}_0(\underline{\mathrm{Var}}_S) & \xrightarrow{\chi^c_{/S}} & \mathrm{K}_0(\mathfrak{hs}_{/S}) & \xrightarrow{\mathrm{Dmod}_{/S}} & \mathrm{K}_0(\mathcal{D}_{\mathrm{rh}}(S)) \\ & \searrow^{\chi^c_{\mathrm{top}} \otimes 1_{\mathbb{C}}} & \downarrow{\scriptstyle \mathrm{rat}_{/S}} & \swarrow_{\mathsf{Dr}_S} & \\ & & \mathrm{K}_0(\mathsf{Perv}(S)_{\mathbb{C}}). & & \end{array} \tag{42}$$

Example 11.3.3. Let $f : X \to S$ be a smooth projective family. Then $R^k f_* \underline{\mathbb{Q}}_X$ underlies a variation of Hodge structure $V^{(k)}$ of weight k so that

(Example 11.1.5.1)

$$\chi^c_{/S}(X) = \sum(-1)^k[V^{(k)}] \in \mathrm{K}_0(\mathfrak{hs}_{/S}).$$

The vector bundle $\mathrm{Dmod}_{/S}(V^{(k)})$ underlying $V^{(k)}$ is a regular holonomic $\mathcal{D}_S$-module.

To extend example 11.3.3 to a general proper morphism between smooth varieties I need a new concept, that of the *canonical extension* ${}^{\pi}\mathcal{V}$ of a regular holonomic $\mathcal{D}_U$-module $\mathcal{V}$ defined on any Zariski-open dense subset $j : U \hookrightarrow Z$ of a subvariety $i : Z \subset S$ consisting of smooth points. This is done in two steps. First, the $\mathcal{D}_Z$-submodule $\widetilde{\mathcal{V}}$ of $j_*\mathcal{V}$ generated by $\mathcal{V}$ is regular holonomic. Then set

$$\boxed{{}^{\pi}\mathcal{V} := i_*\widetilde{\mathcal{V}} \in \mathcal{D}_{\mathrm{rh}}(S).}$$

Example 11.3.4. Let $f : X \to S$ be a projective morphism between smooth projective manifolds. Fix k and let $\mathcal{V}$ be the D-module underlying the restriction V of $R^k f_* \underline{\mathbb{Q}}_X$ over the open subset consisting of the regular values of f. The De Rham complex of its canonical extension $\widetilde{\mathcal{V}}$ is exactly the perverse extension ${}^{\pi}V$ of V. The pair $(\widetilde{\mathcal{V}}, {}^{\pi}V)$ is an example of a Hodge module of weight k, a concept that will be treated more fully in Section 12.2.

LECTURE 12

Hodge Modules

12.1. Digression: Polarizations

The category of Hodge structures is not semi-simple, but it becomes so if one restricts to those Hodge structures that admit a polarization. To give the definition, suppose V is a Hodge structure of weight k with Hodge decomposition $V_{\mathbb{C}} = \bigoplus H^{p,q}$ and Hodge filtration $F^\bullet$. The *Weil-operator* $C : V_{\mathbb{C}} \to V_{\mathbb{C}}$ is defined to be multiplication by i^{p-q} on $V^{p,q}$.

Definition 12.1.1. A *polarization* of Hodge structure V of weight k is a $\mathbb{Q}$-valued bilinear form

$$Q : V \otimes V \longrightarrow \mathbb{Q}$$

which is $(-1)^k$-symmetric and such that the two Riemann bilinear relations hold:

(1) The orthogonal complement of F^m is F^{k-m+1};
(2) The hermitian form on $V \otimes \mathbb{C}$ given by $(u, v) \mapsto Q(Cu, \bar{v})$ is positive-definite.

A Hodge structure that admits a polarization is said to be *polarizable*.

Clearly, the category of polarizable Hodge structures is semi-simple: if (V, Q) is a polarized Hodge structure and $W \subset V$ a sub Hodge structure, the polarization restricts to a polarization on W and the orthogonal complement $W^\perp$ of W with respect to Q is also a Hodge structure.

The classical example of a polarized Hodge structure is given by the primitive cohomology groups on a complex projective manifold X of dimension d. Recall that primitive cohomology is defined using the L-operator

$$L : H^*(X) \to H^*(X)[2](-2),$$

which comes from cup-product with the hyperplane class.

For details, see [**P-S**, Chap. 1.2 and 2.1]. It is however helpful to give one word of explanation. The hyperplane class should be viewed as an element not of $H^2(X)$ but of $H^2(X; \mathbb{Q}(-1))$, i.e. with coefficients $\frac{1}{2\pi \mathrm{i}}\mathbb{Q}$. This is because the Chern form for the Fubini-Study metric has periods in $\frac{1}{2\pi \mathrm{i}}\mathbb{Z}$ and the class of this form defines the hyperplane class. It has pure type $(1, 1)$ and so defines a generator for $\mathbb{L}$.

$$H^m_{\mathrm{prim}}(X) = \mathrm{Ker}\left(L^{d-m+1} : H^m(X) \to H^{2d-m+2}(X)\right), \quad \text{if } m \le d.$$

If $m > d$ there is no primitive cohomology and one has the *Lefschetz decomposition*

$$H^m(X) = \bigoplus_{r \ge (p+q-d)_+} L^r H^{m-r}_{\mathrm{prim}}(X) = \bigoplus_{r \ge (p+q-d)_+} H^{m-r}_{\mathrm{prim}}(X)(r). \tag{43}$$

The orientation coming from the complex structure defines a preferred generator for the top-cohomology which will be written as

$$\mathrm{Tr}_X : H^{2d}(X) \xrightarrow{\sim} \mathbb{Q}(-d).$$

The polarizing form Q on $H^k(X)$ can then be given by

$$Q(\alpha, \beta) = (-1)^{\frac{1}{2}k(k-1)} \mathrm{Tr}_X \left(\alpha \cup \beta \cup h^{n-k}\right) \quad \alpha, \beta \in H^k(X).$$

The Riemann bilinear relations state exactly that this form restricts to a polarization on $H^k_{\mathrm{prim}}(X)$. To get a polarization on all of $H^k(X)$ one has to change signs on the constituents of the primitive decomposition as follows. Write $a = \sum L^r a_r$, $b = \sum L^r b^r$, $a_r, b_r \in H^{k-2r}_{\mathrm{prim}}(X)$ and then

$$Q(a, b) = (-1)^{\frac{1}{2}k(k-1)} \sum_r (-1)^r \mathrm{Tr}_X \left(L^{d-k+2r}(a_r \cup b_r)\right). \tag{44}$$

***Remark* 12.1.2.** Observe that there is a concise alternative definition of a polarization as a $(-1)^k$ symmetric morphism of Hodge structures $Q : V \otimes V \to \mathbb{Q}(-k)$ which satisfies the positivity condition expressed by the second Riemann bilinear relation. The validity of the first relation is equivalent to Q being a morphism of Hodge structures. Note that the polarization induces an identification $V^* = V(-k)$.

This last version of the definition generalises immediately to a polarized variation:

Definition 12.1.3. A *polarization* of a variation of Hodge structure V of weight k on S is a morphism of variations $Q : V \otimes V \to \underline{\mathbb{Q}(-k)}_S$ which induces on each fibre a polarization of the corresponding Hodge structure

The cohomology groups of the fibres of a proper smooth morphism $f : X \to Y$ of complex algebraic manifolds give the standard example, i.e. the local system $V = R^k f_* \underline{\mathbb{Q}}_X$ carries a natural polarized variation of Hodge structure which comes from the polarization (44) on the fibres $H^k(f^{-1}(s))$ in which L comes from a relative hyperplane section for f.

12.2. Hodge Modules

Polarizable variations of Hodge structure over a smooth compact algebraic variety S form the basic examples of Hodge modules which is the topic of this section.

Here is a summary of what will be done.

- Polarizable variations over smooth algebraic varieties are interpreted as rational D-modules;
- extensions of polarizable variations are rational D-modules; this is done in two steps: first one goes from a Zariski-open subset of smooth points to a compactification and then one considers extensions from locally closed subvarieties;
- Hodge modules correspond precisely to extensions.

So let me start by defining the notion of a rational D-module:

Definition 12.2.1. Let S be a complex manifold A *rational $\mathcal{D}_S$-module* is a triple $(M, \alpha, M_{\mathbb{Q}})$ where M is a regular holonomic $\mathcal{D}_S$-module, $M_{\mathbb{Q}}$ a perverse sheaf on S and $\alpha : M_{\mathbb{C}} = M_{\mathbb{Q}} \otimes \mathbb{C} \to \mathsf{Dr}_S(M)$ a quasi-isomorphism in $\mathsf{Perv}(S)$. Such a rational D-module is said to be *filtered* if M admits a filtration in the sense of Def. 11.2.2. The complex $M_{\mathbb{Q}}$ is called the *rational component* of the rational $\mathcal{D}_S$-module.

By Remark 11.2.3 a variation of Hodge structure $(V, F^\bullet)$ on a smooth compact algebraic variety S gives a filtered rational D-module: just take $((\mathcal{V}, \nabla), F_\bullet = F^{-\bullet})$ as the filtered $\mathcal{D}_S$-module and take $V[d]$ as the rational component. Then (40) shows that the inclusion $\iota = (\mathrm{id} \otimes 1) : V_{\mathbb{C}} \hookrightarrow \mathcal{V} = V_C \otimes \mathcal{O}_S$ gives the desired quasi-isomorphism.

If the base S is no longer complete, one restricts to D-modules $(\mathcal{V}, \nabla)$ on S admitting a regular extension to a good compactification $\bar{S}$, i.e. the D-module should be regular holonomic.

Example 12.2.2. Suppose that the D-module $(\mathcal{V}, \nabla)$ comes from a variation of Hodge structure on S, say (V, F). By Lemma 7.1.1 the local monodromy operators around infinity of $V_{\mathbb{C}}$ are all quasi-unipotent and hence the module $(\mathcal{V}, \nabla)$ is indeed regular holonomic so that (V, F) gives a filtered rational D-module on S.

The preceding example turns out to give a so-called Hodge module provided (V, F) is polarizable. I am not going to give the definition, since it is rather complicated and the only thing which is used in these lectures is the final result Corollary 12.2.9. The full theory can be found in [**Sa88**]; for a first light introduction, see [**P-S**, Chap. 14.2–14.3].

The concept of weight transfers to Hodge modules, provided one adds the dimension of its base. Let me summarise this as follows:

Theorem. *Let S be a smooth complex algebraic variety of dimension d and let $\mathbf{V} = (V, F)$ a polarizable variation of Hodge structure of weight m. Let $\iota = (\mathrm{id} \otimes 1) : V_{\mathbb{C}} \hookrightarrow \mathcal{V} = V_{\mathbb{C}} \otimes \mathcal{O}_S$ be the inclusion. Then*

$$\boxed{\mathbf{V}_S^{\mathrm{Hdg}} = ((\mathcal{V}, \nabla), F_\bullet, \iota, V[d])}$$

is a Hodge module of weight $m + d$.

Example 12.2.3. As before, let S be a smooth algebraic variety. The constant sheaf $\underline{\mathbb{Q}}_S$ underlies a weight 0 variation of Hodge structure on S. The corresponding Hodge module will be denoted $\mathbf{Q}_S^{\mathrm{Hdg}}$, $d = \dim S$. This is consistent with the previous notation for variations of Hodge structures. Note that it has weight d as a Hodge module and that the underlying perverse sheaf is $\underline{\mathbb{Q}}_S[d]$. The constant sheaf $\underline{\mathbb{Q}}_S$ underlies an object in the derived category of Hodge modules, namely the complex $\mathbf{Q}_S^{\mathrm{Hdg}}[-d]$ consisting of a single Hodge module placed in degree d. One also needs a name for it:

$$\underline{\mathbb{Q}}_S^{\mathrm{Hdg}} := \mathbf{Q}_S^{\mathrm{Hdg}}[-d]. \tag{45}$$

I now pass to the construction of extensions of variations (V, F) to any *good* compactification $\bar{S}$ of S. For the D-module extension one takes the extension of $(\mathcal{V}, \nabla)$ to a vector bundle $\widetilde{\mathcal{V}}$ on $\bar{S}$ equipped with the extension of ∇ to a connection $\widetilde{\nabla}$ with regular poles along $D = \bar{S} - S$ (Def. 11.3.2). It is well known that the F filtration also extends in such a way that $\widetilde{\mathcal{V}}$ becomes a filtered D-module on $\bar{S}$. For the rational component one then takes the intersection complex $IC_S(V)$ defined in (34). One can show that its De Rham complex is a resolution of $IC_S(V)$, which provides the comparison map $\iota : IC_S(V) \to \widetilde{\mathcal{V}}[d]$. Hence:

Theorem. *Let U be a smooth complex algebraic variety with good compactification S. Let (V, F) a polarizable variation of Hodge structure of weight m with corresponding rational $\mathcal{D}_U$-module $\mathbf{V}_U^{\mathrm{Hdg}} = ((\mathcal{V}, \nabla), F_\bullet, \iota, V[d])$. Then*

$$\boxed{\mathbf{V}_S^{\mathrm{Hdg}} = ((\widetilde{\mathcal{V}}, \widetilde{\nabla}), F_\bullet, \iota, IC(V))}$$

is a Hodge module of weight $m + d$, the Hodge module extension *of $\mathbf{V}_U^{\mathrm{Hdg}}$.*

For the next step assume that S is just any compactification of U which might singular. We have seen that $IC_S(V)$ still exists and is the rational component of a Hodge module $\mathbf{V}_Z^{\mathrm{Hdg}}$ supported on Z. More generally, if $i : Z \hookrightarrow S$ is the inclusion of a closed subvariety of S then $i_* IC_S(V)$ is still perverse on S. Also Hodge modules behave well under such inclusions:

Lemma 12.2.4. *Let $i : Z \hookrightarrow S$ be an inclusion of a closed subvariety and let $\mathbf{V}$ be a Hodge module on Z. Then $i_*\mathbf{V}$ is a Hodge module on S.*

In fact, although a complete proof will not be given here, this proves the better part of the following result:

Theorem 12.2.5. *Let $i : Z \subset S$ is an irreducible complex subvariety. Let $\mathbf{V} = (V, F)$ be a polarisable variation of Hodge structure of weight m on $U \subset Z$, a Zariski open subset consisting of smooth points. Then*

$$\boxed{\mathbf{V}_S^{\mathrm{Hdg}} = i_* \mathbf{V}_Z^{\mathrm{Hdg}}}$$

is a Hodge module of weight $m + \dim Z$ whose rational component is $i_* IC_Z(V)$.

Set

$$\boxed{\mathsf{HM}_S(k) = \text{Category of Hodge modules of weight } k \text{ supported on } S.}$$

There is a converse to Theorem 12.2.5. To formulate it one needs the concept of strict support:

Definition 12.2.6. A Hodge module has *strict support on Z*, a subvariety of S, if its underlying rational component has support on Z, but no submodule nor quotient module has support on a strictly smaller subvariety.

Theorem 12.2.7. *The category $\mathsf{HM}_S(k)$ is abelian. Any Hodge module $\mathbf{V} \in \mathsf{HM}(k)$ can be written*

$$\mathbf{V} = \bigoplus_Z (i_Z)_* \mathbf{V}_Z^{\mathrm{Hdg}},$$

where $\mathbf{V}_Z^{\mathrm{Hdg}}$ is the Hodge module extension associated to a polarizable variation of weight $k - \dim Z$ supported on a dense open smooth subset of Z and where $i_Z : Z \hookrightarrow S$ is the inclusion of a closed subvariety.

Finally I can state the converse to Theorem 12.2.5:

Theorem 12.2.8. *Any Hodge module of weight m with strict support $Z \subset X$ is of the form $i_* \mathbf{V}_Z^{\mathrm{Hdg}}$ where V is a variation of Hodge structures of weight $m - \dim Z$ with base a Zariski dense open subset of smooth points of Z.*

So, if one wishes, this can be used as an alternative definition for Hodge modules and in this way it will be used in these lectures.

Corollary 12.2.9. *There is a natural functor $\mathfrak{hs}_S \to \mathsf{HM}_S$ which associates to a variation of Hodge structure on a Zariski-dense smooth subset of a subvariety $Z \hookrightarrow S$ the Hodge module extension of its associated Hodge module. This functor is an equivalence of categories.*

This corollary implies that in the statement of Theorem 11.1.4 one should work with Hodge modules, i.e. one should replace $\mathfrak{hs}_S$ by HM_S:

Theorem 12.2.10 (Existence of the relative motivic Hodge characteristic-bis)**.** *There is a morphism of rings $\chi^c_{/S} : \mathrm{K}_0(\underline{\mathrm{Var}}_S) \to \mathrm{K}_0(\mathsf{HM}_{/S})$, fitting into a commutative diagram*

$$\begin{array}{ccc} \mathrm{K}_0(\underline{\mathrm{Var}}_S) & \xrightarrow{\chi^c_{/S}} & \mathrm{K}_0(\mathsf{HM}_{/S}) \\ & \searrow^{\chi^c_{\mathrm{top}} \otimes 1_{\mathbb{C}}} & \downarrow \mathrm{rat}_{/S} \\ & & \mathrm{K}_0(\mathsf{Perv}(S)_{\mathbb{C}}). \end{array}$$

It is this version which will be proved in § 13.4.

12.3. Direct Images

The main theorem as reformulated above as Theorem 12.2.10 partly claims existence of a relative Hodge characteristic. Clearly, the construction should use the direct images of the constant sheaf. Let me first consider what to do in the smooth situation.

Example 12.3.1. Continuing example 11.3.4, consider a proper morphism $f : X \to S$ between smooth algebraic varieties. Let $f^0 : X^0 \to S^0$ be the restriction of f to the set of regular values of f. Then $R^k f^0_* \underline{\mathbb{Q}}_{X^0}$ underlies a polarizable variation V of Hodge structure of weight k and $\mathrm{V}_S^{\mathrm{Hdg}}$ is a Hodge module of weight $k + \dim S$. The underlying perverse sheaf is the perverse extension of $R^k f^0_* \underline{\mathbb{Q}}_{X^0}[d]$, $d = \dim S$. As remarked above (Examples 10.2.5.4) this perverse sheaf in general only gives a direct summand of the perverse direct image ${}^{\pi}R^{k+d} f_* \underline{\mathbb{Q}}_X$. So these two should not be confused.

However, in two cases one gets everything: since $f_* \underline{\mathbb{Q}}_X = \underline{\mathbb{Q}}_S$ and $R^{2n} f_* \underline{\mathbb{Q}}_X = \underline{\mathbb{L}}^n_S$, n the relative dimension of X/S one gets

$${}^{\pi}R^d f_* \underline{\mathbb{Q}}_X = \underline{\mathbb{Q}}_S[d], \quad {}^{\pi}R^{2n+d} f_* \underline{\mathbb{Q}}_X = \underline{\mathbb{L}}^n_S[d].$$

Surprisingly, if f is projective, for all k the complex ${}^{\pi}R^{k+d} f_* \underline{\mathbb{Q}}_X$ is the perverse component of a Hodge module. See the next theorem 12.3.2.

Theorem 12.3.2 (Decomposition theorem, [**Sa88**, Thm. 5.3.1])**.** *Let $f : X \to S$ be a projective morphism between smooth complex algebraic varieties and let $\mathbf{M} = (M, \alpha, M_{\mathbb{Q}})$ be a weight m Hodge module on X. Then*

$H^kf_*\mathbf{M}$ is a Hodge module of weight $m+k$ whose rational component is ${}^\pi H^k f_* M_{\mathbb{Q}}$. There is a non-canonical isomorphism in $D^{\mathrm{b}}(\mathsf{HM})$:

$$Rf_*\mathbf{M} = \bigoplus_k \left(H^k f_*\mathbf{M}\right)[-k].$$

***Remark* 12.3.3.** By Theorem 12.2.7 there is a further decomposition over subvarieties Z of S

$$H^k f_*\mathbf{M} = \bigoplus_Z (i_Z)_* \, (V^k)_Z^{\mathrm{Hdg}}, \tag{46}$$

where $(V^k)_Z^{\mathrm{Hdg}}$ is the Hodge module extension of a variation $V^k_{Z^0}$ of Hodge structure of weight $k+m-\dim Z$ on a Zariski-dense open subset Z^0 of Z consisting of smooth points.

Usually, the conjunction of Theorem 12.3.2 and the above splitting is called the decomposition theorem. It implies a decomposition for the rational component $M_{\mathbb{Q}}$ of $\mathbf{M}$:

$${}^\pi R^k f_* M_{\mathbb{Q}} = \bigoplus_Z (i_Z)_* \, IC_Z(V^k_{Z^0}).$$

Example 12.3.4. Let $f : X \to S$ be a projective map between smooth algebraic varieties. Apply Theorem 12.3.2 to the Hodge module $\mathbf{Q}_X^{\mathrm{Hdg}}$. It follows that

$$Rf_*\mathbf{Q}_X^{\mathrm{Hdg}} = \bigoplus_k \left(H^k f_*\mathbf{Q}_X^{\mathrm{Hdg}}\right)[-k]$$

and hence

$$[Rf_*\mathbf{Q}_X^{\mathrm{Hdg}}] = \sum (-1)^k [H^k f_*\mathbf{Q}_X^{\mathrm{Hdg}}] \in \mathrm{K}_0(\mathsf{HM}_S).$$

The decomposition (46) for $H^k f_*\mathbf{Q}_X^{\mathrm{Hdg}}$ gives contributions from variations supported on various subvarieties of S. By example 12.3.1, one of these is $R^k f^0_* \underline{\mathbb{Q}}_{X^0}$, supported on the set S^0 of regular values of f and further variations all supported on subvarieties of the discriminant locus of f.

Example 12.3.5. It is instructive to go back to Example 11.1.5.2. To the variation of Hodge structure $V^1_{S_0}$ there is associated the Hodge module $\mathbf{V}_S^{\mathrm{Hdg}}$ whose underlying perverse sheaf is $j_*(R^1 f_* \underline{\mathbb{Q}}_X)|S_0)[1]$ with $j : S_0 \hookrightarrow S$ the inclusion. Its stalk at $p \in \Delta$ is the subspace inside $H^1(X_t)$, t close to p of invariants under the local monodromy-action. It is no longer a Hodge module since the restriction functor i_p^* does not preserve Hodge modules. The mixed Hodge structure supported on p that one gets is an example of a mixed Hodge module. These will be treated in Lecture 13. Note that inside $\mathrm{K}_0(\mathsf{HM}(S)$ there still is a decomposition $[\mathbf{V}_S^{\mathrm{Hdg}}] = [\mathbf{V}_{S_0}^{\mathrm{Hdg}}] - \sum_{p\in\Delta}\left(N_p[\mathbb{Q}]_p + V_{g_p}\right)$ into Hodge modules. To compare this with (38), first note that $[Rf_*\mathbf{Q}_X^{\mathrm{Hdg}}] = [Rf_*\underline{\mathbb{Q}}_X^{\mathrm{Hdg}}]$ where the notation (45) is used. Indeed, the two complexes differ by a shift of two. Now, by example 12.3.1, one has $H^1(Rf_*\mathbf{Q}_X^{\mathrm{Hdg}}) = \underline{\mathbb{Q}}_S[1]$, $H^3(Rf_*\mathbf{Q}_X^{\mathrm{Hdg}}) = \mathbb{L}_S[1]$. Moreover, $H^2(Rf_*\mathbf{Q}_X^{\mathrm{Hdg}}) = \mathbf{V}_S^{\mathrm{Hdg}}$ hence

$$[Rf_*\mathbf{Q}_X^{\mathrm{Hdg}}] = \underline{\mathbb{Q}}_S + \mathbb{L}_S + [\mathbf{V}_S^{\mathrm{Hdg}}].$$

Since $[\mathbf{V}_S^{\mathrm{Hdg}}] = [\mathbf{V}_{S_0}^{\mathrm{Hdg}}] - \sum_{p\in\Delta}\left(N_p[\mathbb{Q}]_p + V_{g_p}\right) = -V_{S_0} - \sum_{p\in\Delta}\left(N_p[\mathbb{Q}]_p + V_{g_p}\right)$ this result coincides with the right hand side of (38).

LECTURE 13

Motives in the Relative Setting, Mixed Hodge Modules

13.1. Variations of Mixed Hodge Structure

Suppose that $f : X \to Y$ is a morphism of complex algebraic varieties where X and Y may be singular. The analysis in the previous Lectures leading to the Hodge theoretic character $\chi^c_{/S}$ does not suffice since now the cohomology groups $H^k(f^{-1}s)$ carry mixed Hodge structures instead of pure Hodge structures. One has to enlarge the scope of the study to variations of mixed Hodge structures, a concept which comes up naturally in the geometric setting as I now explain. There exists a Zariski-open dense subset $S \subset Y$ consisting of smooth points of Y such that the restriction to S of $R^k f_* \underline{\mathbb{Q}}_X$ is a local system. The collection of mixed Hodge structures $H^k(f^{-1}s)$, $s \in S$ is the prototype of a variation of mixed Hodge structure:

Definition 13.1.1. Let S be a complex manifold. A *variation of mixed Hodge structure* on S is a triple (V, F, W) consisting of a local system V of finite dimensional rational vector spaces on S, a finite decreasing filtration F of $\mathcal{V} = V \otimes_{\mathbb{Q}} \mathcal{O}_S$ by holomorphic subbundles (the *Hodge filtration*), a finite increasing filtration W of the local system V by local subsystems (the *weight filtration*).
These data should satisfy the following conditions:

i) for each $s \in S$ the Hodge and weight filtrations induce a mixed Hodge structure on the fibre of V at s;
ii) the connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega^1_X$ whose sheaf of horizontal sections is $V_{\mathbb{C}}$ satisfies the *Griffiths' transversality condition*

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_S.$$

The notion of a *morphism of variations of mixed Hodge structure* is defined in the obvious way.
A variation of mixed Hodge structure will be called *graded-polarizable* if the induced variations of pure Hodge structure $\mathrm{Gr}^W_k V$ are all polarizable.

Example 13.1.2. Suppose that one has a morphism $f : X \to S$ as in the beginning of this section but where $R^k f_* \underline{\mathbb{Q}}_X$ is already a local system and where S is smooth. Then it underlies a variation of mixed Hodge structure. If f is quasi-projective, this variation of mixed Hodge structure is graded-polarizable. Let me refer to this situation as a *geometric variation of mixed Hodge structure*. The goal now is to single out the extra properties satisfied by such geometric variations. It turns out to be crucial to study what happens at infinity, i.e. near the divisor which compactifies S to a good compactification.

The first step is to consider a graded polarizable variation of mixed Hodge structure (V, W, F) over the punctured unit disc Δ^*, say with parameter t. As W is a filtration of V by local subsystems, the monodromy operator preserves W; moreover the monodromy theorem (in the weak form as stated as Lemma 7.1.1) guarantees that the monodromy of each $\mathrm{Gr}^W V$ is quasi-unipotent. Hence the monodromy of V is quasi-unipotent. For simplicity one assumes that the monodromy T of V is in fact unipotent. Put $V_0 = (\psi_t V_{\mathbb{Q}})_0$ and $N = \log T : V_0 \to V_0$ and let W denote the induced filtration on V_0. Let M be the weight filtration of the nilpotent endomorphism $\mathrm{Gr}_k(N)$ on $\mathrm{Gr}_k^W V_0$ as defined in Lemma 7.1.2 and let ${}^kM = M[k]$.

Definition 13.1.3. A *weight filtration* of N relative to W is a filtration $\widetilde{W}$ of V such that

i) $N\widetilde{W}_i \subset \widetilde{W}_{i-2}$
ii) $\widetilde{W}$ induces ${}^kM$ on $\mathrm{Gr}_k^W V_0$.

By [**Del80**, (1.6.13)] there is at most one weight filtration of N on V_0 relative to W. If it exists, it is called the *weight filtration of N relative to W* denoted $M = M(N; W)$.

Let $\widetilde{\mathcal{V}}$ denote the canonical extension of $\mathcal{V}$ to a holomorphic vector bundle on Δ such that the connection extends to one with a logarithmic pole at 0 with nilpotent residue. The filtration W extends to $\widetilde{\mathcal{V}}$ and $\mathrm{Gr}_k^W \widetilde{\mathcal{V}}$ is the canonical extension for $\mathrm{Gr}_k^W \mathcal{V}$. Let ${}^kF$ denote the Hodge filtration on $\mathrm{Gr}_k^W \mathcal{V}$. It extends to a filtration ${}^k\widetilde{F}$ of $\mathrm{Gr}_k^W \widetilde{\mathcal{V}}$.

Definition 13.1.4. i) A variation of mixed Hodge structure (V, W, F) over the punctured unit disc Δ^* is called *admissible* if it is graded-polarizable, the monodromy T is unipotent and the weight filtration $M(N, W)$ of $N = \log T$ relative to W exists. Moreover, the filtration F extends to a filtration $\widetilde{F}$ of $\widetilde{\mathcal{V}}$ which induces ${}^k\widetilde{F}$ on $\mathrm{Gr}_k^W \widetilde{\mathcal{V}}$ for each k.

ii) Let Y be a compact complex analytic space and $S \subset Y$ a smooth Zariski-open subset. A graded polarizable variation of mixed Hodge structure (V, W, F) on S is called *admissible* (with respect to the embedding $S \subset Y$) if for every holomorphic map $i : \Delta \to Y$ which maps Δ^* to S and such that i^*V has unipotent monodromy, the variation $i^*(V, W, F)$ on Δ^* is admissible.

The following two results are fundamental:

Theorem 13.1.5 ([**St-Z, ElZ86, Kash86**]). *Geometric variations are admissible.*

Theorem 13.1.6 ([**St-Z, ElZ03**]). *Let V be an admissible variation of mixed Hodge structure on S. Then for each k the vector space $\mathbb{H}^k(S, V)$ carries a canonical mixed Hodge structure.*

13.2. Mixed Hodge Modules

Admissible variations of mixed Hodge structure on S are the basic examples of mixed Hodge modules on S. The precise definition is complicated but it is easy to describe the category $\mathsf{MHW}(S)$ they belong to, at least in the case where S is a smooth algebraic variety.

For the full theory, consult [**Sa90**]; see also [**P-S**, Chap 14] which serves as a guide for what follows.

First introduce the category of *bi-filtered rational D-modules* consisting of a filtered rational D-module $(\mathcal{M}, F, M)$ together with a filtration W on $(\mathcal{M}, M)$, the *weight filtration.* This is a *pair* of filtrations compatible with the comparison isomorphism. Note that $\mathrm{Gr}^W_k(\mathcal{M}, F, M, W)$ is a filtered rational D-module. One further demands that these be polarizable weight k Hodge modules on S. The full subcategory generated by such bi-filtered rational D-modules is the desired category $\mathsf{MHW}(S)$. If S is no longer smooth, one needs to adapt the definition in that one needs to work with D-modules on arbitrary algebraic varieties. This can be done as follows. The concept of a D-module is local and locally the variety S can be embedded in a smooth variety X. By definition, a $\mathcal{D}_S$ module is a $\mathcal{D}_X$-module whose support is contained in S.

Mixed Hodge modules have to satisfy more requirements which I won't make explicit. Suffices to say that an admissible variation is a special case as will be explained next.

Definition 13.2.1. Let $\mathbf{V} = (V, W, F)$ be an admissible variation of mixed Hodge structure on a smooth complex variety S. It gives rise to the holonomic $\mathcal{D}_S$-module $\mathcal{V} = V \otimes_{\mathbb{Q}} \mathcal{O}_S$ which is filtered by $W_k\mathcal{V} = W_k\ V \otimes_{\mathbb{Q}} \mathcal{O}_S$ and $F_p\mathcal{V} = F^{-p}\mathcal{V}$. Together with the comparison isomorphism given by

$$\iota : (V[d], W) \otimes \mathbb{C} \to \mathsf{Dr}_S(\mathcal{V}, W)$$

the triple $(\mathcal{V}, \nabla), F, W, \iota, V[d])$ belongs to $\mathsf{MHW}(S)$. It is a mixed Hodge module $\mathbf{V}^{\mathrm{Hdg}}$. In fact, it is a so called *smooth mixed Hodge module* on S.

To get a feeling for the usefulness of mixed Hodge modules without getting bogged down by their construction, the following axiomatic approach is suitable.

Axioms for mixed Hodge modules

A) For each complex algebraic variety X there exists an abelian subcategory $\mathsf{MHM}(X)$ of $\mathsf{MHW}(X)$, the category of *mixed Hodge modules* on X with the following properties:
   - The functor which associates to a mixed Hodge module its perverse component extends to a faithful functor

$$\mathrm{rat}_X : D^{\mathrm{b}}\mathsf{MHM}(X) \to D^{\mathrm{b}}_{\mathrm{cs}}(X). \tag{47}$$

   One says that

$$\mathbf{M} \in \mathsf{MHM}(X) \textit{ is supported on } Z \iff \mathrm{rat}_X\mathbf{M} \text{ is supported on } Z.$$

   - The functor which associates to a mixed Hodge module its underlying D-module extends to a faithful functor

$$\mathrm{Dmod}_X : D^{\mathrm{b}}\mathsf{MHM}(X) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X). \tag{48}$$

B) The category of mixed Hodge modules supported on a point is the category of graded polarizable rational mixed Hodge structures; the functor “rat” associates to the mixed Hodge structure the underlying rational vector space.

C) The weight filtration W is such that
   - morphisms preserve the weight filtration strictly;
   - the object $\mathrm{Gr}_k^W M$ is semisimple in $\mathsf{MHM}(X)$;
   - if X is a point, the W-filtration is the usual weight filtration for the mixed Hodge structure.
   - A Hodge module $\mathbf{M}$ of weight k is a mixed Hodge module with a one-step weight filtration $W_k\mathbf{M} = \mathbf{M}$ while $W_{k-1}\mathbf{M} = 0$, i.e. a *pure weight* k mixed Hodge module. Conversely, a pure mixed Hodge module of weight k is a Hodge module of weight k. Consequently

$$\mathrm{K}_0(\mathsf{MHM}(X)) = \mathrm{K}_0(\mathsf{HM}(X)) \tag{49}$$

   Since $\mathsf{MHM}(X)$ is an abelian category, the cohomology groups of any complex of mixed Hodge modules on X are again mixed Hodge modules on X. With this in mind, one says that for a *complex* $\mathbf{M} \in D^{\mathrm{b}}\mathsf{MHM}(X)$ the *weight* satisfies

$$\text{weight}[\mathbf{M}] \begin{cases} \le n, \\ \ge n \end{cases} \iff \mathrm{Gr}_i^W H^j(\mathbf{M}) = 0 \begin{cases} \text{for } i > j+n \\ \text{for } i < j+n. \end{cases}$$

D) The duality functor D_X of Verdier lifts to $\mathsf{MHM}(X)$ as an involution, also denoted D_X, in the sense that $\mathsf{D}_X \circ \mathrm{rat}_X = \mathrm{rat}_X \circ \mathsf{D}_X$.

E) For each morphism $f : X \to Y$ between complex algebraic varieties, there are induced functors $f_*, f_! : D^{\mathrm{b}}\mathsf{MHM}(X) \to D^{\mathrm{b}}\mathsf{MHM}(Y)$, $f^*, f^! : D^{\mathrm{b}}\mathsf{MHM}(Y) \to D^{\mathrm{b}}\mathsf{MHM}(X)$ interchanged under duality and which lifts the analogous functors on the level of constructible complexes and D-modules. Moreover, the adjoint relation

$$\mathrm{Hom}(f^*\mathbf{K}, \mathbf{L}) = \mathrm{Hom}(\mathbf{K}, f_*\mathbf{L})$$

   holds in $D^{\mathrm{b}}(\mathsf{MHM}(X))$ and lift the corresponding adjoint relation on the level of constructible complexes and D-modules. The adjunction morphisms (28) extend to

$$f^{\#} : \mathbf{L} \to f_*f^*\mathbf{L}, \quad f_{\#} : f_!f^!\mathbf{L} \to \mathbf{L}. \tag{50}$$

F) The functors $f_!, f^*$ do not increase weights in the sense that if $\mathbf{M}$ has weights $\le n$, the same is true for $f_!\mathbf{M}$ and $f^*\mathbf{M}$.

G) The functors $f^!, f_*$ do not decrease weights in the sense that if $\mathbf{M}$ has weights $\ge n$, the same is true for $f^!\mathbf{M}$ and $f_*\mathbf{M}$.

By way of terminology, $\mathbf{M} \in D^{\mathrm{b}}\mathsf{MHM}(X)$ is *pure of weight* n if it has weight $\ge n$ and weight $\le n$, i.e. for all $j \in \mathbb{Z}$ the cohomology sheaf $H^j(\mathbf{M})$ has pure weight $j+n$: only $\mathrm{Gr}_{j+n}^W H^j(\mathbf{M})$ might be non-zero. A Hodge module $\mathbf{M}$ of weight k is an example, since $H^0(\mathbf{M}) = \mathbf{M}$ has pure weight k and all other cohomology vanishes.

A morphism *preserves weights*, if it neither decreases or increases weights. For proper maps, for the ordinary direct images one has $f_* = f_!$, almost by definition. This holds on the level of constructible complexes, but also on the level of filtered D-modules, hence for Hodge modules. It follows that Axiom F) and G) imply:

H) For proper maps between complex algebraic varieties $f_* = f_!$ and hence preserves weights.

13.3. Some Consequences of the Axioms

The starting point is the observation that the cohomology groups of any complex of mixed Hodge modules $\mathbf{M}$ on X is a mixed Hodge module on X. A consequence of Axiom A) then is:

Lemma 13.3.1. *The cohomology functors* $H^q : D^{\mathrm{b}}\mathsf{MHM}(X) \to \mathsf{MHM}(X)$ *are compatible with the functor* rat_X *in the sense that for any bounded complex* $\mathbf{M}$ *of mixed Hodge modules one has*

$$\mathrm{rat}_X[H^q\mathbf{M}] = {}^{\pi}H^q[\mathrm{rat}_X\mathbf{M}].$$

The right hand side is perverse *cohomology as introduced in § 10.2.*

Axiom E) and B) imply:

Lemma 13.3.2. *Let* $a_X : X \to \mathrm{pt}$ *be the constant map to the point. For any complex* $\mathbf{M}$ *of mixed Hodge modules on* X

$$\mathbb{H}^p(X, \mathbf{M}) := H^p((a_X)_*\mathbf{M}) \tag{51}$$

is a mixed Hodge structure.

Note that Lemma 13.3.1 implies that for a complex of sheaves over a point perverse cohomology is ordinary cohomology. So, by (30) the rational component $H^p((a_X)_*M)$ of the hypercohomology group $\mathbb{H}^p(X, \mathbf{M})$ is just $\mathbb{H}^p(X, M)$ which explains the notation. Hence by axiom B):

Corollary 13.3.3. *Let* $\mathbf{M}$ *be mixed Hodge module whose rational component is the perverse sheaf* $M_{\mathbb{Q}}$*. Then the hypercohomology group* $\mathbb{H}^p(X, M_{\mathbb{Q}})$ *has a natural mixed Hodge structure.*

Let me explain how this leads to mixed Hodge structures on ordinary and compactly supported cohomology. To start with, from axiom A) and B) it follows that there is a unique element $\mathbb{Q}^{\mathrm{Hdg}} \in \mathsf{MHM}(\mathrm{pt}\,)$ whose rational component is the unique Hodge structure on $\mathbb{Q}$ of type $(0,0)$. Define the *complexes* of mixed Hodge modules

$$\underline{\mathbb{Q}}_X^{\mathrm{Hdg}} := a_X^*\mathbb{Q}^{\mathrm{Hdg}}, \quad \mathsf{D}_X\underline{\mathbb{Q}}_X^{\mathrm{Hdg}} := a_X^!\mathbb{Q}^{\mathrm{Hdg}}$$

Note that even if X is smooth projective, these are not in general mixed Hodge modules, e.g. by Example 12.2.3) one knows that $\mathbf{Q}_X^{\mathrm{Hdg}} = \underline{\mathbb{Q}}_X^{\mathrm{Hdg}}[d]$, $d = \dim X$, is a Hodge module and $\underline{\mathbb{Q}}_X^{\mathrm{Hdg}}$ is a complex of Hodge modules concentrated entirely in degree $-d$. In this case this is a pure weight 0 complex, but this is not the case in general since a_X^* and $a_X^!$ do not preserve weights; the axioms imply only that the first has weights ≤ 0 while the second has weights ≥ 0. This can be used to show:

Proposition 13.3.4. (1) $H^k(X)$ *and* $H_k(X)$ *have mixed Hodge structures;*
(2) $H_c^k(X)$ *and* $H_k^{\mathrm{BM}}(X)$ *have mixed Hodge structures of weights* $\leq k$*,respectively* $\geq -k$*.*

(3) *A morphism* $f : X \to Y$ *induces morphisms of Hodge structures* $f^* : H^k(Y) \to H^k(X)$, $f_* : H_k(X) \to H_k(Y)$.
(4) *A proper morphism* $f : X \to Y$ *induces morphisms of Hodge structures* $f^* : H_c^k(Y) \to H_c^k(X)$, $f_* : H_k^{\rm BM}(X) \to H_k^{\rm BM}(Y)$.

Proof: Since a_X lifts the analogous functor on the level of constructible complexes, the rational component of $\underline{\mathbb{Q}}_X^{\rm Hdg}$ is $\mathbb{Q}_X$ and that of $\mathsf{D}_X\underline{\mathbb{Q}}_X^{\rm Hdg}$ its Verdier dual. By Corr. 13.3.3 both $H^p(X)$ and its Verdier dual, which, using (29) is the the Borel-Moore homology group $H^p(X, \mathsf{D}_X\underline{\mathbb{Q}}_X) = H_{-p}^{\rm BM}(X)$, have natural mixed Hodge structures. Similarly, e.g. using (30), ordinary homology $H_p(X)$ as well as its Verdier dual, $H_c^{-p}(X)$, cohomology with compact support, have natural mixed Hodge structures. This proves (1).

As to weights, $(a_X)_! a_X^* \mathbb{Q}^{\rm Hdg}$ has weights ≤ 0 and $(a_X)_* a_X^! \mathbb{Q}^{\rm Hdg}$ has weights ≥ 0. Hence $H_c^k(X) = \mathbb{H}^k((a_X)_! a_X^* \mathbb{Q}^{\rm Hdg})$ has weights $\le k$ and $H_k^{\rm BM}(X) = \mathbb{H}^{-k}((a_X)_* a_X^! \mathbb{Q}^{\rm Hdg})$ has weights $\ge -k$. This proves (2).

As to morphisms, the adjunction morphism (50) applied to $L = \underline{\mathbb{Q}}_Y^{\rm Hdg}$ directly gives morphisms of mixed Hodge structures

$$f^* : H^k(Y, \mathbb{Q}) \to H^k(Y, f_* f^* \underline{\mathbb{Q}}_Y) = H^k(X, \mathbb{Q})$$

and, similarly, for the compactly supported cohomology.

For homology, apply the second relation (50) to $(a_Y)^! \mathbb{Q}^{\rm Hdg}$ to get

$$(a_X)_! \circ (a_X)_! \mathbb{Q}^{\rm Hdg} = (a_Y)_! f_! f^! (a_Y)^! \mathbb{Q}^{\rm Hdg} \to (a_Y)_! (a_Y)^! \mathbb{Q}^{\rm Hdg}$$

and hence

$$f_* : H_{-k}(X) = H^k((a_X)_! \circ (a_X)_! \mathbb{Q}^{\rm Hdg}) \to H_{-k}((a_Y)^! \mathbb{Q}^{\rm Hdg}) = H_{-k}(Y).$$

For Borel-Moore homology, replace $(a_X)_!$ and $(a_Y)_!$ by $(a_X)_*$ and $(a_Y)_*$. □

Another example is cohomology with support. In § 10.1 it was stated that with $i : Z \hookrightarrow X$ a closed embedding, $H_Z^p(X) = \mathbb{H}^p(i_* i^! \underline{\mathbb{Q}}_X)$ which is the rational component of the p-th hypercohomology of the Hodge module $i_* i^! \underline{\mathbb{Q}}_X^{\rm Hdg}$. Since $i^!$ and i_* do not decrease weights, $i_* i^! \underline{\mathbb{Q}}_X^{\rm Hdg}$ has weights ≥ 0 and hence $H_Z^p(X)$ has weights $\ge k$.

***Remark* 13.3.5.** These mixed Hodge structures coincide with the ones constructed by Deligne and which were discussed in Lecture 2. This is not hard to prove if X is a smooth algebraic variety, or if X can be embedded in a smooth algebraic variety. See the remark at end of [**Sa90**, § 4.5]. It is true in general, but highly non-trivial since Saito's approach does not work well in the setting of cubical or simplicial spaces. See [**Sa00**, Cor. 4.3].

13.4. The Motivic Hodge Characteristic

At last, I now can extend the definition from Example 12.3.4 valid for projective maps between smooth varieties to the general setting:

Definition 13.4.1. Let $f : X \to S$ be a morphism between complex algebraic varieties. The axioms imply that $f_! \underline{\mathbb{Q}}_X^{\rm Hdg}$ is a *complex* of mixed Hodge modules. The *motivic Hodge characteristic* is defined by

$$\chi_{/S}^c(X) := [f_! \underline{\mathbb{Q}}_X^{\rm Hdg}] = \sum (-1)^k [H^k f_! \underline{\mathbb{Q}}_X^{\rm Hdg}] \in \mathrm{K}_0(\mathsf{MHM}(S)) = \mathrm{K}_0(\mathsf{HM}_S).$$

The last equality is (49).

This gives the construction of the motivic Hodge characteristic claimed by Theorem 12.2.10. What remains to be shown is compatibility with the scissor relations. To explain how to do this, let me go back to the underlying rational components. I have shown in Example 10.2.3 that for these the scissor relations are respected because of the distinguished triangle (32). The same argument holds for mixed Hodge modules provided I show:

Proposition 13.4.2. *Let* $\mathbf{M}$ *be a mixed Hodge module on* X*. Let* $i : Z \to X$ *be a closed subvariety and let* $j : U \hookrightarrow X$ *be the inclusion of the complement* $U = X - Z$ *into* X*. Then in* $D^{\mathrm{b}}(\mathsf{MHM}(X))$ *one has distinguished triangles:*

$$\left.\begin{array}{l} i_*i^!\mathbf{M} \to \mathbf{M} \to j_*j^*\mathbf{M} \to [1], \\ j_!j^*\mathbf{M} \to \mathbf{M} \to i_*i^*\mathbf{M} \to [1]. \end{array}\right\} \tag{52}$$

Sketch of Proof. The axioms for mixed Hodge modules imply that second triangle is obtained from the first by the duality operator. It suffices therefore to show that the first triangle is distinguished. This can be reduced to a local construction as follows. One can cover X by affine subsets, still denoted by X over which Z is the zero set of some regular functions $f_1, \ldots, f_r$. Put $X_i = X - f_i^{-1}(0)$ and for $I \subset \{1, \ldots, r\}$, set $X_I = \bigcap_{i \in I} X_i$, and let $j_I : X_I \hookrightarrow X$ be the natural inclusion. Note that for $I = \{1, \ldots, r\}$ one gets back $j : U \hookrightarrow X$. There are quasi-isomorphisms in the category $D^{\mathrm{b}}\mathsf{MHM}(X)$

$$i_*i^!\mathbf{M} \xrightarrow{\sim} [\cdots 0 \to M \to B_1 \to B_2 \cdots \to B_r \to 0], \quad B_k = \bigoplus_{|I|=k} (j_I)_* j_I^* \mathbf{M}$$

$$j_*j^*\mathbf{M} \xrightarrow{\sim} [\cdots 0 \to B_1 \to B_2 \to B_3 \cdots \to B_r \to 0], \quad B_k \text{ in degree } k-1.$$

This is a straightforward calculation and using this local description it follows that $j_*j^*\mathbf{M}$ is quasi-isomorphic to the cone on $i_*i^!\mathbf{M} \to \mathbf{M}$, i.e. the first sequence of (52) is distinguished. It remains to show that these locally constructed triangles glue together. See [**Sa90**, 4.4.1] for details. □

LECTURE 14

The Motivic Chern Class Transformation

14.1. Riemann-Roch for Smooth Projective Varieties

If X is any smooth complex algebraic variety and E a vector bundle on X one has *Chern classes*

$$c_k(E) \in H^{2k}(X).$$

The Chern roots $\gamma_k(E)$ are formally introduced by writing

$$1 + tc_1(E) + \cdots t^r c_r(E) = \prod_{k=1}^{r} (1 + t\gamma_k(E)), \quad r = \operatorname{rank}(E).$$

Using these, one defines the Chern and Todd character:

$$\begin{aligned} \operatorname{ch}(E) &= \sum_k \mathrm{e}^{\gamma_k}, \\ \operatorname{td}(E) &= \prod_k Q(\gamma_k), \quad Q(t) := \frac{t\mathrm{e}^t}{\mathrm{e}^t - 1}. \end{aligned}$$

They take values in $H^{\mathrm{even}}(X) = \bigoplus_k H^{2k}(X)$ – denominators force *rational* coefficients. These characters occur in the Hirzebruch-Riemann-Roch theorem [**Hir**, Theorem 21.1.1] valid when X is *compact*

$$\chi(X, E) := \sum (-1)^k \dim H^k(X, E) = \int_X \operatorname{ch}(E) \cdot \operatorname{td}(TX). \tag{53}$$

The Todd class is well defined on the level of $K^0(X)$, the Grothendieck group of vector bundles on X which for X smooth coincides with $K_0(X)$, the Grothendieck group of coherent sheaves on X. The τ-class is defined using the Todd class of X:

$$\tau([E]) = \operatorname{ch}(E) \cdot \operatorname{td}(T_X) \in H^{\mathrm{even}}(X), \quad \tau([\mathcal{O}_X]) = \operatorname{td}(T_X) \tag{54}$$

and Riemann-Roch states that $\chi(X, E) = \tau_d([E])$, $d = \dim X$. The χ-characteristic is the specialization for $y = 0$ of the χ_y characteristic

$$\chi_y(X, E) := \sum_{q \geq 0} \sum_{k \geq 0} (-1)^k \dim H^k(X, E \otimes \Omega^p_X) y^p \in \mathbb{Z}[y]$$

The Chern character is the specialization to $y = 0$ of the generalized Chern character

$$\operatorname{ch}_y(E) := \sum_k \mathrm{e}^{(1+y)\gamma_k}$$

The function $Q(t)$ is a formal power series in t with *rational* coefficients. It is obtaind by setting $y = 0$ in the two following formal power series in two variables

$$Q_y(t) := \frac{t(1+y)}{1-\mathrm{e}^{-t(1+y)}} - ty, \quad \widetilde{Q}_y(t) := \frac{t(1+y\mathrm{e}^{-y})}{1-\mathrm{e}^{-t}}$$

which leads to the following genera in $H^{\mathrm{even}}(X)[y]$:

$$T_y(E) = \prod_k Q_y(\gamma_k) = \sum_{p\geq 0} T^p(E)y^p, \tag{55}$$

$$\widetilde{T}_y(E) = \prod_k \widetilde{Q}_y(\gamma_k) = \sum_{p\geq 0} \widetilde{T}^p(E)y^p. \tag{56}$$

The component $T^p_n(E)$ of $T^p(E)$ in $H^{2n}(X)$ is a polynomial in the Chern classes of E. In the Riemann-Roch theorem only the top-degree expressions occur, those with $n = \dim X$. These one gets by integrating over X. More generally, write $[T_y(E)]_n$ for the power series

$$[T_y(E)]_n := \sum_{p\geq 0} T^p_n(E)y^p \in H^{2n}(X)[y].$$

The formula [**Hir**, 1.8 (15)] can be generalized easily to give the relations

$$\left[\mathrm{ch}(E)\cdot \widetilde{T_y}(T_X)\right]_n = [\mathrm{ch}_y(E)\cdot T_y(T_X)]_n \quad \forall\, n\geq 0$$

The Hirzebruch-Riemann-Roch theorem then implies the *generalized Hirzebruch-Riemann-Roch theorem* (cf. [**Hir**, 12.1 (8)–(10)])

$$\begin{aligned}\chi_y(X,E) &= \sum_p \int_X \left[\mathrm{ch}(E\otimes \Lambda^p\Omega^1_X)\cdot \mathrm{td}(T_X)\right] y^p \\ &= \int_X \left[\mathrm{ch}(E)\cdot \sum_p \mathrm{td}(T_X)\cdot \mathrm{ch}(\Lambda^p\Omega^1_X)y^p\right] \\ &= \int_X \mathrm{ch}(E)\cdot \widetilde{T}_y(T_X) \\ &= \int_X \mathrm{ch}_y(E)\cdot T_y(T_X).\end{aligned}$$

14.2. The Motivic Chern Class Transformation

One of the main results from [**B-S-Y**] is:

Theorem 14.2.1 ([**B-S-Y**, Theorem 3.1])**.** *There exists unique homomorphisms*

$$T_y,\, \widetilde{T}_y : \mathrm{K}_0(\underline{\mathrm{Var}}_S) \to H^{\mathrm{BM}}_{\mathrm{even}}(S)[y]$$

commuting with proper push forwards and satisfying the normalization condition

$$T_y(\mathrm{id}: S\to S) = T_y(T_S)\cap [S], \quad \widetilde{T}_y(\mathrm{id}: S\to S) = \widetilde{T}_y(T_S)\cap [S],$$

whenever S is smooth and equi-dimensional.

The proof of Theorem 14.2.1 uses the existence of the motivic Chern class transformation:

Theorem 14.2.2 ([**B-S-Y**, Thm. 2.1])**.** *There is a unique group homomorphism*

$$\chi_{\mathrm{Ch}} : \mathrm{K}_0(\underline{\mathrm{Var}}_S) \to K_0(S)[y]$$

commuting with proper push forwards and which satisfies the normalization condition

$$\chi_{\mathrm{Ch}}(\mathrm{id} : S \to S) = \sum [\Lambda^q \Omega^1_S] y^q$$

for S smooth and pure dimensional.

Proof: The *definition* of the Chern class character is based on the *filtered De Rham complex* (Ω_X, F) as introduced by Du Bois [**DuB**]. One sets

$$\chi_{\mathrm{Ch}}(\mathrm{id} : S \to S) = \sum (-1)^p [\mathrm{Gr}^p_F\, \Omega_S] y^p$$

where by definition

$$[\mathrm{Gr}^p_F\, \Omega_S] := \sum_j (-1)^j [H^j(\mathrm{Gr}^p_F\, \Omega_S)] \in K_0(S).$$

Note that for S smooth (Ω_S, F) is the usual De Rham complex with the trivial filtration whose graded pieces are locally free sheaves $\Lambda^q \Omega^1_S = \Omega^p_S$ placed in degree p with no higher cohomology sheaves so that $[\mathrm{Gr}^p_F\, \Omega_S] = (-1)^p [\Omega^p_S]$ and $\chi_{\mathrm{Ch}}(\mathrm{id} : S \to S) = \sum [\Omega^p_S] y^p$ which shows the normalization. Then compatibility with proper push forwards $f : X \to S$ forces the definition

$$\chi_{\mathrm{Ch}}[f : X \to S] = \sum (-1)^p f_* [\mathrm{Gr}^p_F\, \Omega_X] y^p, \tag{57}$$

where $f_*[F] = \sum (-1)^j [R^j f_* F]$. To show that this determines the Chern character completely one uses a different representation of $\mathrm{K}_0(\underline{\mathrm{Var}}_S)$ as generated by isomorphism classes of proper $f : X \to S$ modulo the *acyclity relation* defined as follows: $[X'] - [Y'] = [X] - [Y]$ whenever $i : Y \hookrightarrow X$ a closed embedding, $\pi : X' \to X$ a proper morphism, both fitting in a cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow \pi \\ Y & \xrightarrow{\ i\ } & X. \end{array}$$

The proof is similar in spirit as that of Bittner's theorem. This is then used in conjuction with the fact that the filtered Du Bois complex respects the acyclicity relation [**DuB**, Prop. 1.3]. □

Remark. 1) The authors of [**B-S-Y**] use the terminology Chern class transformation mC_*; they explain that the terminology is inspired by the Chern-Schwarz-MacPherson Chern class transformation defined for singular varieties and of which it is a generalization. See the Introduction of loc. cit.
2) The normalization $\chi_{\mathrm{Ch}}(\mathrm{id}_X) = \sum (-1)^q [\Lambda^q \Omega^1_X]$ for non-singular X makes it possible to read off the Chern classes $c_q(X) := c_q(T_X)$ from the classes $[\Lambda^q \Omega^1_X]$ using the easily established formulas

$$c_q(X) = \sum_{k=0}^{q} (-1)^k \operatorname{ch}(\Lambda^k \Omega^1_X) \operatorname{td}(T_X).$$

To be able to prove Theorem 14.2.1, I need a to extend the constructions leading up to Riemann-Roch to include singular varieties. First of all, for possibly singular varieties X one needs to work with $K_0(X)$ and Borel-Moore homology $H^{\mathrm{BM}}_{\mathrm{even}}(X)$. Instead of the top degree one works with degree 0 which is obtained by taking the cap–product with the fundamental class

$[X]$. Next the Todd genus generalizes as follows (see [**B-F-M**]): there is a homomorphism

$$\tau : K_0(X) \to H^{\mathrm{BM}}_{\mathrm{even}}(X)$$

with the property that for smooth X, modulo the Poincar-duality isomorphism $H^{\mathrm{even}}(X) \xrightarrow{\sim} H^{\mathrm{BM}}_{\mathrm{even}}$, this morphism coincides with (54). Moreover, if for arbitrary X one also puts $\mathrm{td}(TX) := \tau([\mathcal{O}_X])$, the Riemann-Roch formula (53) still holds for vector bundles on X, but one needs a variant. First extend the homomorphism τ to

$$\tau_y : K_0(X) \to H^{\mathrm{BM}}_{\mathrm{even}}(X)[y, (1+y)^{-1}],$$

as follows. For $[F] \in K_0(X)$, let $\tau_k[F] \in H^{\mathrm{BM}}_{2k}(X)$ be the degree $2k$ component of $\tau([F])$. Then setting $\tau_y([F]) := \sum_{k \geq 0} \tau_k([F]) \cdot (1+y)^{-k}$, one has

Theorem (The generalized Hirzebruch-Riemann-Roch formula)**.** *For any vector bundle E on a possibly singular complex algebraic variety X one has*

$$[\tau_y([E])]_0 = \left[\mathrm{ch}(E) \cdot \widetilde{T}_y(T_X)\right] \cap [X] = [\mathrm{ch}_y(E) \cdot T_y(T_X)] \cap [X].$$

One further extension is needed: extend τ_y linearly to obtain a homomorphism $\tau_y : K_0(X)[y] \to H^{\mathrm{BM}}_{\mathrm{even}}(X)[y, (1+y)^{-1}]$ and now, finally, I can give the

Proof of Theorem 14.2.1. There is a commutative triangle

$$\begin{array}{ccc} & \mathrm{K}_0(\underline{\mathrm{Var}}_S) & \\ {}^{\chi_{\mathrm{Ch}}}\swarrow & & \searrow^{T_y, \widetilde{T}_y} \\ K_0(S)[y] & \xrightarrow{\quad \tau_y, \tau \quad} & H^{\mathrm{BM}}_{\mathrm{even}}(S)[y, (1+y)^{-1}] \end{array}$$

defining T_y and $\widetilde{T}_y$. Their uniqueness and normalization conditions follow from the uniqueness and normalization for χ_{Ch}. The fact that the maps T_y and $\widetilde{T}_y$ don't need denominators involving $(1+y)$ comes from the fact that these are absent for smooth X by their very definition (see (55) and (56)) and the fact that $\mathrm{K}_0(\underline{\mathrm{Var}}_S)$ is generated by smooth varieties proper over S, by the relative variant of Bittner's theorem. $\square$

Remark. There is a variant of the above in which Borel-Moore homology has been replaced by the total Chow group $\bigoplus_p \mathrm{Chow}(X)$. It uses the Fulton apprach to Chern classes [**Fulton**]. This gives finer invariants.

14.3. Hodge Theoretic Aspects

As an application of the theory of mixed Hodge modules, I shall discuss the Hodge theoretic construction of the Chern class transformation from [**B-S-Y**].

Start by recalling that a mixed Hodge module on S contains as part of its data a filtered (holonomic) $\mathcal{D}_S$-module. Consider the category $F\mathcal{D}_S$ of filtered $\mathcal{D}_S$-modules $(\mathcal{M}, F)$. The morphisms are those $\mathcal{D}_S$-linear maps which respect the filtration. The De Rham complex exists in the context of filtered modules. Loosely speaking, one should consider Ω^k_S as having filtering degree $-k$. More formally, introduce

$$\left.\begin{array}{l} F_\ell(\mathsf{Dr}^\bullet_S\mathcal{M}) \;= \\ \quad = \Big[F_\ell\mathcal{M} \to \Omega^1_S(F_{\ell+1}\mathcal{M}) \to \cdots \to \Omega^{d_X}_S(F_{\ell+d}\mathcal{M})\Big]\,[-d]. \end{array}\right\} \quad (58)$$

This defines a filtration of $\mathsf{Dr}^\bullet_S\mathcal{M}$ by subcomplexes, the *filtered De Rham complex* $\mathsf{Dr}^\bullet_S(\mathcal{M},F)$. It is a filtered complex of $\mathcal{O}_S$-modules whose morphisms are only $\mathbb{C}_S$-linear. However, for a *filtered* $\mathcal{D}_S$-module, the morphisms in any of the associated graded complexes are $\mathcal{O}_S$-linear. This implies the following result.

Lemma 14.3.1. *Let* $\mathrm{K}_0(S)$ *be the Grothendieck group for the category of coherent* $\mathcal{O}_S$*-modules. The De Rham functor induces the* De Rham characteristic

$$\left.\begin{array}{llll} \mathsf{Dr}_S & : & \mathrm{K}_0(F\mathcal{D}_S) & \to & \mathrm{K}_0(S)[y,y^{-1}] \\ & & [(\mathcal{V},F)] & \mapsto & \sum(-1)^j[\mathrm{Gr}^F_j\,\mathsf{Dr}^\bullet_S(\mathcal{V},F)]y^j. \end{array}\right\} \quad (59)$$

This is compatible with proper pushforwards, in the sense that for $\varphi : S \to T$ *a proper map between algebraic manifolds, there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{K}_0(F\mathcal{D}_S) & \to & \mathrm{K}_0(S)[y,y^{-1}] \\ \big\downarrow{\scriptstyle \varphi_*} & & \big\downarrow{\scriptstyle \varphi_*} \\ \mathrm{K}_0(F\mathcal{D}_T) & \to & \mathrm{K}_0(T)[y,y^{-1}]. \end{array}$$

Here I have assumed that S is *smooth.* However, if S is singular, as mentioned before (see the beginning of 13.2) D-modules can be defined. Moreover, this is also true for the De Rham functor and its gradeds give functors

$$\mathrm{Gr}^F_j\,\mathsf{Dr}^\bullet_S : D^{\mathrm{b}}F\mathcal{D}_S \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S).$$

This is very well explained in [**Sa00**, § 1]. It follows that the De Rham characteristic can be defined in the singular case as well.

Proposition 14.3.2. *One has the equality*

$$\chi_{\mathrm{Ch}} = \mathsf{Dr}_S \circ \chi^c_{/S} : \mathrm{K}_0(\underline{\mathrm{Var}}/S) \to \mathrm{K}_0(S)[y].$$

Proof: This follows immediately from the uniqueness statement in Theorem 14.2.2, the fact that by construction the right hand side is compatible with proper push forwards and the fact that for S smooth the values of the right hand side on $[\mathrm{id} : S \to S]$ equals $\sum[\Omega^q_S]y^q$. □

14.4. Stringy Matters

I shall use the results on motivic integration collected in Appendix B. The naive motivic ring $\mathcal{M} := \mathrm{K}_0(\underline{\mathrm{Var}})[\mathbb{L}^{-1}]$ and a certain completion, the so-called dimension completion $\hat{\mathcal{M}}$ (Definition B.3.3) and a certain subring thereof $\hat{\mathcal{M}}_0$ (see (63)) come into play, or rather relative invariants of such. Since y is invertible in $K_0(X)[y,y^{-1}]$ as well as in $H^{\mathrm{even}}_{\mathrm{BM}}(X)[y,y^{-1}]$, sending $\mathbb{L}^{-1}$ to $(-y)$ one can extend the three transformations $\chi_{\mathrm{Ch}}, T_y, \widetilde{T}_y$ to homomorphisms

$$\chi_{\mathrm{Ch}}^{\wedge} : \hat{\mathcal{M}}_S \to K_0(S[y]\,[[y^{-1}]]$$
$$T_y^{\wedge}, \widetilde{T_y}^{\wedge} : \hat{\mathcal{M}}_S \to H_{\mathrm{BM}}^{\mathrm{even}}(s)[y]\,[[y^{-1}]]\,.$$

In this setting, if Z is a subscheme of Y one has introduced the motivic volume

$$\int_{J_\infty(X)} \mathbb{L}^{-\operatorname{ord}_Z} d\mu_Y \in \hat{\mathcal{M}}$$

and in the relative situation one can compose it with any of the three transformations $\chi_{\mathrm{Ch}}^{\wedge}, T_y^{\wedge}, \widetilde{T_y}^{\wedge}$ yielding the *stringy* motivic Chern character, and stringy motivic T-characteristics associated to pairs (Y, Z) defined over S. This applies in particular whenever Y is a resolution of singularities of X, assumed to have only canonical Gorenstein singularities, and Z is the discrepancy divisor (Theorem B.4.4). When the latter has only simple normal crossings, the motivic volume of (Y, Z) does not depend on the resolution; it is the motivic log-volume $\mathrm{vol}(X)$ and the corresponding stringy characters give the invariants $\chi_{\mathrm{Ch}}^{\wedge} \circ \mathrm{vol}(X)$, $T_y^{\wedge} \circ \mathrm{vol}(X)$ and $\widetilde{T_y}^{\wedge} \circ \mathrm{vol}(X)$.

For the relation with the elliptic class introduced by Borisov-Libgober see the discussion in [**B-S-Y**, Example 3.4].

APPENDIX B

Motivic Integration

B.1. Why Motivic Integration?

Motivic integration was introduced by Kontsevich [**Ko**] to prove the following result conjectured by Batyrev: let

$$\begin{array}{ccc} X_1 & & X_2 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & X & \end{array}$$

be two crepant resolutions of the singularities of a projective Calabi-Yau variety X having at most canonical Gorenstein singularities. Recall that a normal projective variety X of dimension n is called *Calabi-Yau* if the canonical divisor K_X is trivial and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$. In the context of mirror symmetry one forgets about these last conditions: it is customary to call X Calabi-Yau as soon as $K_X = 0$ (and the singularities are mild), see [**Baty**]. Crepant (as in *non discrepant*) means that the pullback of the canonical divisor class on X is the canonical divisor class on X_i, i.e. the discrepancy divisor $E_i = K_{X_i} - \pi_i^* K_X$ is numerically equivalent to zero. In this situation Batyrev showed, using p-adic integration, that X_1 and X_2 have the same Betti numbers. Kontsevich used *motivic integration* to show that X_1 and X_2 even have the same Hodge numbers.

This problem was motivated by the *topological mirror symmetry test* of string theory which asserts that if X and X^* are a mirror pair of smooth Calabi-Yau varieties then they have mirrored Hodge numbers

$$h^{i,j}(X) = h^{n-i,j}(X^*).$$

As the mirror of a smooth Calabi-Yau might be singular, one cannot restrict to the smooth case and the equality of Hodge numbers actually fails in this case. Therefore Batyrev suggested, inspired by string theory, that one should look instead at the Hodge numbers of a crepant resolution, if such exists[1]. The independence of these numbers from the chosen crepant resolution is Kontsevich's result. This makes the *stringy Hodge numbers* $h^{i,j}_{\mathrm{st}}(X)$ of X, defined as $h^{i,j}(X')$ for a crepant resolution X' of X, well defined. The mirror symmetry conjecture thus has been modified by asserting that the stringy Hodge numbers of a mirror pair are equal [**Baty-Bor**]. Batyrev's conjecture has been proved as follows:

[1] Calabi-Yau varieties do not always have crepant resolutions.

Theorem B.1.1 (Kontsevich)**.** *Birationally equivalent smooth Calabi-Yau varieties have the same Hodge numbers.*[2]

For a proof of a more general statement see Prop. B.4.7. For the moment, to give an indication where it is all leading to, I give a
Sketch of the proof. The idea is to assign to any variety a *volume* in a suitable ring $\widehat{\mathcal{M}}$ such that the information about the Hodge numbers is retained. The following diagram illustrates the construction of $\widehat{\mathcal{M}}$:

$$\begin{array}{ccccccc}
\underline{\mathrm{Var}} & \longrightarrow & \mathrm{K}_0(\underline{\mathrm{Var}}) & \longrightarrow & \mathcal{M} & \longrightarrow & \widehat{\mathcal{M}} \\
& \searrow^{P_{\mathrm{Hdg}}} & \downarrow & & \downarrow & & \downarrow \\
& & \mathbb{Z}[u,v] & \hookrightarrow & \mathbb{Z}[u,v,(uv)^{-1}] & \hookrightarrow & \mathbb{Z}[u,v,(uv)^{-1}]^{\wedge}
\end{array}$$

The diagonal map is the Hodge number polynomial. The Hodge characteristic factors through the *naive Grothendieck ring* $\mathrm{K}_0(\underline{\mathrm{Var}})$ which is the universal object with the latter property. This explains the left triangle of the diagram.

The bottom row of the diagram is the composition of a localization (inverting uv) and a completion with respect to negative degree. $\widehat{\mathcal{M}}$ is constructed analogously, by first inverting $\mathbb{L}^{-1}$ (a pre-image of uv) and then completing appropriately (negative dimension). Whereas the bottom maps are injective, the map $\mathrm{K}_0(\underline{\mathrm{Var}}) \to \widehat{\mathcal{M}}$ is most likely not injective. The need to work with $\widehat{\mathcal{M}}$ instead of $\mathrm{K}_0(\underline{\mathrm{Var}})$ arises in the setup of integration theory and will become clear later.

Clearly, by construction it is now enough to show that birationally equivalent Calabi-Yau varieties have the same *volume*, i.e. the same class in $\widehat{\mathcal{M}}$. This is achieved via the all important *birational transformation rule* of motivic integration, to be proved below (Theorem B.4.4). Roughly it asserts that for a proper birational map $\pi : Y \to X$ the class $[X] \in \widehat{\mathcal{M}}$ is an *expression* in Y and $K_{Y/X}$ only:

$$[X] = \int_Y \mathbb{L}^{-\operatorname{ord}_{K_{Y/X}}} d\mu_Y$$

To finish the proof let X_1 and X_2 be birationally equivalent Calabi-Yau varieties. Resolve the birational map to a Hironaka hut:

$$\begin{array}{ccc}
& Y & \\
{}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\
X_1 & \dashrightarrow & X_2
\end{array}$$

By the Calabi-Yau assumption one has $K_{X_i} \equiv 0$ and therefore $K_{Y/X_i} \equiv K_Y - \pi_i^* K_{X_i} \equiv K_Y$. Hence the divisors K_{Y/X_1} and K_{Y/X_2} are numerically equivalent. This numerical equivalence implies in fact an equality of divisors $K_{X/X_1} = K_{X/X_2}$ since, again by the Calabi-Yau assumption,

[2] There is also a proof by Ito [**Ito**] of this result using p-adic integration, thus continuing the ideas of Batyrev who proved the result for Betti numbers using this technique. Furthermore the weak factorization theorem allows for a proof avoiding integration of any sort.

$\dim H^0(X, K_Y) = \dim H^0(X_i, \mathcal{O}_{X_i}) = 1$. By the transformation rule $[X_1]$ is an expression depending only on Y and $K_{X/X_1} = K_{X/X_2}$. The same is true for $[X_2]$ and thus we have $[X_1] = [X_2]$ as desired. □

B.2. The Space of Formal Arcs of a Complex Algebraic Manifold

Let X be a complex algebraic manifold of dimension d. Any (germ) of an analytic curve $\gamma : \Delta \to X$ at p, say with $\gamma(0) = p$ defines a *formal arc* at p, i.e. a morphism

$$\gamma : \operatorname{Spec} \mathbb{C} [\![z]\!] \longrightarrow X \quad \text{with} \quad \gamma(0) = p.$$

Dually this corresponds to a morphism of $\mathbb{C}$-algebras $\mathcal{O}_{X,p} \to \mathbb{C} [\![z]\!]$. Composing with the truncation map $\mathbb{C} [\![z]\!] \to C [\![z]\!] /(z^{k+1})$ we get a k-th order arc at p, i.e. a morphism $\operatorname{Spec} \mathbb{C} [\![z]\!] /(z^{k+1}) \to X$ with $\gamma(0) = p$. Varying $p \in X$ gives an affine bundle $J_k(X) \to X$ with fibres $\simeq \mathbf{A}^{dk}$. Subsequent truncations give affine bundles $J_{k+1}(X) \to J_k(X)$ with fibres $\simeq \mathbf{A}^d$ and then the space of all formal arcs in X is

$$J_\infty(X) := \varprojlim_k J_k(X).$$

Mapping arcs at p to the base point p gives the affine bundle

$$\pi : J_\infty(X) \to X.$$

The truncation maps induces surjections

$$\pi_k : J_\infty(X) \to J_k(X).$$

Recall that a a formal arc γ corresponds to a homomorphism $\gamma^* : \mathcal{O}_X \to C [\![z]\!]$. If $Y \subset X$ is a subvariety, or, more generally, a subscheme, the *order of vanishing* $\operatorname{ord}_Y(\gamma)$ of γ of along Y is the supremum of k such that the map

$$\mathcal{O}_X \xrightarrow{\gamma^*} \mathbb{C} [\![z]\!] \to \mathbb{C} [\![z]\!] /(z^k)$$

sends I_Y to zero. For a $(k-1)$-jet $\gamma \in J_{k-1}(X)$ to send I_Y to zero means precisely that $\gamma \in J_{k-1}(Y)$. Thus $\operatorname{ord}_Y(\gamma)$ is the supremum of all k such that the truncation $\pi_{k-1}(\gamma)$ lies in $J_{k-1}(Y)$. One clearly has:

$$\begin{array}{lcl} \operatorname{ord}_Y(\gamma) \neq 0 & \Leftrightarrow & \pi(\gamma) \in Y, \\ \operatorname{ord}_Y(\gamma) \geq k & \Leftrightarrow & \pi_{k-1}(\gamma) \in J_{k-1} Y \text{ and} \\ \operatorname{ord}_Y(\gamma) = \infty & \Leftrightarrow & \gamma \in J_\infty(Y). \end{array} \tag{60}$$

B.3. The Motivic Measure

Spaces of formal arcs don't have a class in $\mathrm{K}_0(\underline{\mathrm{Var}})$, but the spaces of finite order arcs $J_k(X)$ and any subvariety in it has a well defines class in this ring. The goal is to measure certain subsets in $J_\infty(X)$ that are finitely determined, i.e. of the form $C = \pi_k^{-1}(B)$ for a certain non-negative integer k and B any subvariety. The problem is that then also $C = \pi_{k+\ell}^{-1}(B_\ell)$ with B_ℓ the inverse image of B under the projection $J_{k+\ell}(X) \to J_k(X)$. The latter is an affine bundle with fibre $\simeq \mathbf{A}^{d\ell}$ and so the equation $[B] \cdot \mathbb{L}^{-dk} = \cdots = [B] \cdot \mathbb{L}^{d\ell} \cdot \mathbb{L}^{-d(k+\ell)} = [B_\ell] \cdot \mathbb{L}^{-d(k+\ell)}$ shows that the *motivic log-volume*

$$\widetilde{\mu}_X(C) = [B] \cdot \mathbb{L}^{-dk} \in \mathcal{M} = \mathrm{K}_0(\underline{\mathrm{Var}})[\mathbb{L}]^{-1}, \quad C = \pi_k^{-1}(B) \tag{61}$$

is well defined. More generally, this applies to cylinder sets:

Definition B.3.1. A subset $C \subseteq J_\infty(X)$ of the space of formal arcs is called a *cylinder set* if $C = \pi_k^{-1}(B_k)$ for some non-negative integer k and $B_k \subseteq J_k(X)$ a constructible subset (Recall that a subset of a variety is *constructible* if it is a finite, disjoint union of (Zariski) locally closed subvarieties). Its motivic volume is given by (61).

The collection of cylinder sets forms an algebra of sets; that is, $J_\infty(X) = \pi_0^{-1}(X)$ is a cylinder set, as are finite unions and complements (and hence finite intersections) of cylinder sets.

Example B.3.2. For $k > 0$, consider the set $\mathrm{ord}_Y^{-1}(\geq k) = \{\gamma \in J_\infty(X) \mid \mathrm{ord}_Y(\gamma) \geq k\}$ consisting of all formal arcs in X which vanish of order *at least* k along Y. By what has just been observed $\mathrm{ord}_Y^{-1}(\geq k) = \pi_{k-1}^{-1}(J_{k-1}(Y))$ is a cylinder. Therefore, the level set $\mathrm{ord}_Y^{-1}(k)$ is also a cylinder equal to

$$\mathrm{ord}_Y^{-1}(\geq k) - \mathrm{ord}_Y^{-1}(\geq k+1) = \pi_{k-1}^{-1} J_{k-1}(Y) - \pi_k^{-1} J_k(Y).$$

For $k = 0$ one has $\mathrm{ord}_Y^{-1}(\geq 0) = \pi^{-1}(X) = J_\infty(X)$ and $\mathrm{ord}_Y^{-1}(0) = \pi^{-1}(X) - \pi^{-1}(Y)$.

On the other hand, the level set at infinity $\mathrm{ord}_Y^{-1}(\infty) = J_\infty(Y)$ is *not* a cylinder set. One would like to extend the measure to μ_X which is valid on countable unions of disjoint cylinders and which has the defining property

$$\mu_X\left(\bigsqcup_{k \in \mathbb{N}} C_k\right) := \sum_{i \in \mathbb{N}} \mu_X(C_k) = \sum_{i \in \mathbb{N}} \widetilde{\mu}_X(C_k). \tag{62}$$

The problem however is that countable sums are not defined in $\mathcal{M}$. Furthermore it is not clear a priori that the measure as defined by the above formula is independent of the choice of the decomposition into disjoint C_i. To overcome these problems Kontsevich [**Ko**] introduced an auxiliary ring:

Definition B.3.3. Let $\widehat{\mathcal{M}}$ denote the completion of the ring $\mathcal{M}$ with respect to the *dimension-filtration*

$$\cdots \supseteq F_1\mathcal{M} \supseteq F_0\mathcal{M} \supseteq F_{-1}\mathcal{M} \supseteq \cdots$$

where for each $m \in \mathbb{Z}$, $F_m\mathcal{V}$ is the subgroup of $\mathcal{M}$ generated by elements of the form $[V] \cdot \mathbb{L}^{-i}$ for $\dim V - i \leq -m$. In other words, giving $\mathbb{L}$ dimension 1 the dimension function becomes well-defined on $\mathcal{M}$ (on finite formal sums one extends additively) and $\xi \in \mathcal{M}$ has dimension $\leq -m$ if and only if $\xi \in F_m$. Alternatively, with $F^{-m} = F_m$ one has $\dim(\xi) \leq m$ if and only if F^m.

The natural completion map is denoted $\phi : \mathcal{M} \longrightarrow \widehat{\mathcal{M}}$. A series $x_k \in \mathcal{M}$ converges in $\widehat{\mathcal{M}}$ to $\phi(x)$ precisely when $(x - x_k) \in F_{n_k}$ with $\lim_{k\to\infty} n_k = \infty$, which is equivalent to $\lim_{k\to\infty} \dim(x - x_k) = -\infty$.

Examples B.3.4. 1) If $\dim X = d$ as before and $Y \subset X$ smooth of codimension c, then $\dim J_k(Y) = (d-c)(k+1)$ and hence $\dim[J_k(Y)] \cdot \mathbb{L}^{-dk} = d - c(k+1)$, i.e. $[J_k(Y)] \cdot \mathbb{L}^{-dk} \in F_{c(k+1)-d}$ and so the limit when k goes to infinity is $0 \in \widehat{\mathcal{M}}$.
2) (Continuation of Example B.3.2) Under the same assumptions, the level sets $\mathrm{ord}_Y^{-1}(k)$ for $k \geq 1$ have been expressed as cylinders; since Y is smooth,

$J^m(Y)$ is an affine bundle with fibres $\simeq \mathbf{A}^{(d-c)m}$ and $\mu_X(\mathrm{ord}_Y^{-1}(k)) = [Y]\cdot\mathbb{L}^{-ck}\cdot(\mathbb{L}^c-1)$. For $k=0$ one gets $\mu_X(\mathrm{ord}_Y^{-1}(0)) = [X]-[Y]$.

Replace now the measure $\widetilde{\mu}_X$ by the $\widehat{\mathcal{M}}$-valued measure $\phi\circ\widetilde{\mu}_X$. Using (62) this modified measure can indeed be extended to a measure μ_X applicable to countable unions $\left(\bigsqcup_{k\in\mathbb{N}} C_k\right)$ of disjoint cylinders C_k with $\lim\limits_{k\to\infty}\widetilde{\mu}_X(C_k) = 0$ and complements of such sets. The independence of choices is not trivial. See [**D-L99a**, §3.2] or [**Baty**, §6.18]. One has:

Proposition B.3.5. *Let $Y\subseteq X$ be a nowhere dense subscheme of X, then $J_\infty(Y)$ is measurable and has measure $\mu_X(J_\infty(Y))$ equal to zero.*

Sketch of Proof. Since $\mathrm{ord}_Y^{-1}(\infty) = J_\infty(Y)$ is the decreasing intersection of the cylinders $\mathrm{ord}_Y^{-1}(\geq k+1) = \pi_k^{-1}J_k(Y)$ its alleged volume should be obtained as the limit of the volumes of these cylinders. The volume of $\pi_k^{-1}J_k(Y)$ is $[J_k(Y)]\cdot\mathbb{L}^{-dk}$ and the latter tends to zero in $\widehat{\mathcal{M}}$. Therefore, $J_\infty(Y)$, the intersection of these cylinder sets must have volume 0. This argument, valid only for Y smooth can be adapted to singular Y. See [**Blick**, § 4]. □

B.4. The Motivic Integral

As before let X be smooth and Y a subscheme. The *motivic log-volume of the pair* (X,Y) is

$$\int_{J_\infty(X)} \mathbb{L}^{-\mathrm{ord}_Y} d\mu_X = \sum_{k=0}^{\infty}\mu(\mathrm{ord}_Y^{-1}(k))\cdot\mathbb{L}^{-k}.$$

Observe that the level set at infinity is already left out from this summation as it has measure zero. The fact that for cylinder sets $A\subseteq B$ we have $\dim\mu_X(A)\leq\dim\mu_X(B)$ when applied to $\mathrm{ord}_Y^{-1}(k)\subseteq J_\infty(X)$ gives that the dimension of $\mu(\mathrm{ord}_Y^{-1}(k))\cdot\mathbb{L}^{-k}$ is less or equal to $d-k$. The notion of convergence in the ring $\widehat{\mathcal{M}}$ then ensures the convergence of the sum. This is illustrated by the following

Example B.4.1 (Continuation of Examples B.3.4.2)**.** If Y is smooth of codimension c in X, by the calculations of loc. cit. one has

$$\begin{aligned}\textstyle\sum_{k=0}^{\infty}\mu(\mathrm{ord}_Y^{-1}(k))\cdot\mathbb{L}^{-k} &= [X]-[Y]+[Y]\cdot[\mathbb{L}^c-1]\cdot\textstyle\sum_{k\geq 1}\mathbb{L}^{-(c+1)k}\\ &= [X]-[Y]+[Y]\cdot[\mathbb{L}^c-1]\cdot\frac{1}{\mathbb{L}^{c+1}-1}.\end{aligned}$$

Note that for $c=1$ (the case of a smooth divisor) one gets simply $X]-[Y]+[Y]\cdot\dfrac{1}{\mathbb{P}^1}$. The general formula is :

Proposition B.4.2. *Let $Y=\sum_{i=1}^s r_iD_i$ ($r_i>0$) be an effective divisor on X with normal crossing support and such that all D_i are smooth. Then*

$$\int_{J_\infty(X)}\mathbb{L}^{-\mathrm{ord}_Y}d\mu_X = \sum_{J\subseteq\{1,\ldots,k\}}[D_J^\circ](\prod_{j\in J}\frac{\mathbb{L}-1}{\mathbb{L}^{r_j+1}-1}) = \sum_{J\subseteq\{1,\ldots,k\}}\frac{[D_J^\circ]}{\prod_{j\in J}[\mathbb{P}^{r_j}]}.$$

As usual $D_J=\bigcap_{j\in J}D_j$ with $D_\varnothing = X$, and one puts $D_J^\circ = D_J-\bigcup_{j\notin J}D_j$.

For details see either [**Baty**, Theorem 6.28] or [**Craw**, Theorem 1.17].

Corollary B.4.3. *Let Y be an effective divisor on X with normal crossing support. The motivic log-volume of the pair (X, Y) belongs to the subring*

$$\widehat{\mathcal{M}}_0 := \phi(\mathcal{M}) \left[\left\{ \frac{1}{\mathbb{L}^i - 1} \right\}_{i \in \mathbb{N}} \right] \tag{63}$$

of the completed motivic ring $\widehat{\mathcal{M}}$.

The *discrepancy divisor* $K_{X'} - \alpha^* K_X$ of a proper birational morphism $\alpha : X' \to X$ between smooth varieties is the divisor of the Jacobian determinant of α. The next result may therefore be viewed as the "change of variables formula" for the motivic log-volume.

Theorem B.4.4 (Birational Transformation Rule)**.** *Let $\alpha : X' \longrightarrow X$ be a proper birational morphism of between smooth varieties and let $D_\alpha := K_{X'} - \alpha^* K_X$ be the discrepancy divisor. Then the motivic log-volumes for $(X', \alpha^* D + D_\alpha)$ and (X, D) are the same.*

Proof: Composition defines maps $\alpha_t : J_t(X') \to J_t(X)$ for each $t \in \mathbb{Z}_{\geq 0} \cup \infty$. An arc in X which is not contained in the locus of indeterminacy of α^{-1} has a birational transform as an arc in X'. By (60) and Proposition B.3.5, α_∞ is bijective off a subset of measure zero.

The sets $\mathrm{ord}_{D_\alpha}^{-1}(k)$, for $k \in \mathbb{Z}_{\geq 0}$, partition $J_\infty(X') - \mathrm{ord}_{D_\alpha}^{-1}(\infty)$. Thus, for any $s \in \mathbb{Z}_{\geq 0}$ we have, modulo a set of measure zero, a partition

$$\mathrm{ord}_D^{-1}(s) = \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} \alpha_\infty(C_{k,s}) \quad \text{where} \quad C_{k,s} := \mathrm{ord}_{D_\alpha}^{-1}(k) \cap \mathrm{ord}_{\alpha^* D}^{-1}(s). \tag{64}$$

The set $\alpha_\infty(C_{k,s})$ as an image of a constructible set is constructible and in fact a cylinder. Lemma B.4.5 below states that $\mu\big(C_{k,s}\big) = \mu\big(\alpha_\infty(C_{k,s})\big) \cdot \mathbb{L}^k$. Use this identity and the partition (64) to calculate

$$\int_{J_\infty(X)} \mathbb{L}^{-\mathrm{ord}_D} d\mu_X = \sum_{k,s \in \mathbb{Z}_{\geq 0}} \mu\big(\alpha_\infty(C_{k,s})\big) \cdot \mathbb{L}^{-s} = \sum_{k,s \in \mathbb{Z}_{\geq 0}} \mu\big(C_{k,s}\big) \cdot \mathbb{L}^{-(s+k)}.$$

Set $s' := s + k$. Clearly $\bigsqcup_{0 \leq k \leq s'} C_{k,s'-k} = \mathrm{ord}_{\alpha^* D + W}^{-1}(s')$. Substituting this into the above leaves

$$\int_{J_\infty(X)} \mathbb{L}^{-\mathrm{ord}_D} = \sum_{s' \in \mathbb{Z}_{\geq 0}} \mu\big(\,\mathrm{ord}_{\alpha^* D + W}^{-1}(s')\big) \cdot \mathbb{L}^{-s'} = \int_{J_\infty(X')} \mathbb{L}^{-\mathrm{ord}_{\alpha^* D + W}} d\mu_{X'},$$

as required. □

Lemma B.4.5. $\mu\big(C_{k,s}\big) = \mu\big(\alpha_\infty(C_{k,s})\big) \cdot \mathbb{L}^k$.

Sketch of Proof. Both $C_{k,s}$ and $\alpha_\infty(C_{k,s})$ are cylinder sets so there exists $t \in \mathbb{Z}_{\geq 0}$ and constructible sets B'_t and B_t in $J_\infty(X')$ and $J_\infty(X)$ respectively such that the following diagram commutes:

$$\begin{array}{ccc} J_\infty(X') \supset C_{k,s} & \xrightarrow{\alpha_\infty} & \alpha_\infty(C_{k,s}) \subset J_\infty(X) \\ \big\downarrow \pi_t & & \big\downarrow \pi_t \\ J_t(X') \supset B'_t & \xrightarrow{\alpha_t} & B_t \subset J_t(X). \end{array}$$

A local calculation (cf. [**D-L99a**, Lemma 3.4(b)]) then shows that the restriction of α_t to B'_t is a $\mathbb{C}^k$-bundle over B_t. It follows that $[B'_t] = [\mathbb{C}^k] \cdot [B_t]$ and hence

$$\mu\big(C_{k,s}\big) = [B'_t] \cdot \mathbb{L}^{-(n+nt)} = [B_t] \cdot \mathbb{L}^k \cdot \mathbb{L}^{-(n+nt)} = \mu\big(\alpha_\infty(C_{k,s})\big) \cdot \mathbb{L}^k$$

as required. □

Definition B.4.6. Let X denote a complex algebraic variety with at worst Gorenstein canonical singularities. The *motivic log-volume* of X is defined to be the motivic log-volume of the pair (Y, D), where $\varphi : Y \to X$ is any resolution of singularities for which the discrepancy divisor has only simple normal crossings.

Note first that the discrepancy divisor D is effective because X has at worst Gorenstein canonical singularities. The crucial point however is that the motivic log-volume of (Y, D) is independent of the choice of resolution:

Proposition B.4.7. *Let $\varphi_1 \colon Y_1 \longrightarrow X$ and $\varphi_2 \colon Y_2 \longrightarrow X$ be resolutions of X with discrepancy divisors D_1 and D_2 respectively. Then the motivic log-volumes of the pairs (Y_1, D_1) and (Y_2, D_2) are equal.*

Proof: Form a resolution Y_0 which is common to the two resolutions:

$$\begin{array}{ccc} Y_0 & \xrightarrow{\psi_2} & Y_2 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \varphi_2} \\ Y_1 & \xrightarrow{\varphi_1} & X \end{array}$$

and let D_0 denote the discrepancy divisor of $\varphi_0 = \phi_1 \circ \psi_1 = \psi_2 \circ \psi_2$. The discrepancy divisor of ψ_i is $D_0 - \psi_i^* D_i$. Indeed

$$\begin{aligned} K_{Y_0} &= \varphi_0^*(K_X) + D_0 = \psi_i^* \circ \varphi_i^*(K_X) + D_0 = \psi_i^*(K_{Y_i} - D_i) + D_0 \\ &= \psi_i^*(K_{Y_i}) + (D_0 - \psi_i^* D_i). \end{aligned}$$

The maps $\psi_i \colon Y_0 \longrightarrow Y_i$ are proper birational morphisms between smooth projective varieties so Theorem B.4.4 applies:

$$\begin{aligned} \int_{J_\infty(Y_i)} \mathbb{L}^{-\operatorname{ord}_{D_i}} d\mu_{Y_i} &= \int_{J_\infty(Y_0)} \mathbb{L}^{-\operatorname{ord}_{\psi_i^* D_i + (D_0 - \psi_i^* D_i)}} d\mu_{Y_0} \\ &= \int_{J_\infty(Y_0)} \mathbb{L}^{-\operatorname{ord}_{D_0}} d\mu_{Y_0}. \end{aligned}$$

This proves the result. □

B.5. Stringy Hodge Numbers and Stringy Motives

Recall (8) that the Hodge number polynomial P_{Hdg} takes values in the ring $\mathbb{Z}[u, v]$. Clearly, $P_{\mathrm{Hdg}}(\mathbb{L}) = uv$ so that this polynomial extends to

$$P_{\mathrm{Hdg}} : \mathcal{M} = \mathrm{K}_0(\underline{\mathrm{Var}})[\mathbb{L}^{-1}] \to \mathbb{Z}[u, v.(uv)^{-1}].$$

One needs a further extension to the subring $\widehat{\mathcal{M}}_0$ (see (63)). The kernel of the completion map $\phi : \mathcal{M} = \mathrm{K}_0(\underline{\mathrm{Var}})[\mathbb{L}^{-1}] \to \widehat{\mathcal{M}}$ is

$$\bigcap_{m \in \mathbb{Z}} F_m \mathrm{K}_0(\underline{\mathrm{Var}})[\mathbb{L}^{-1}]. \tag{65}$$

For $[V]\cdot\mathbb{L}^{-i} \in F_m\mathrm{K}_0(\underline{\mathrm{Var}})[\mathbb{L}^{-1}]$, the degree of the Hodge number polynomial $P_{\mathrm{Hdg}}\left([V]\cdot\mathbb{L}^{-i}\right)$ is $2\dim V - 2i \leq -2m$. The degree of the Hodge number polynomial of an element Z in the intersection (65) must therefore be $-\infty$; that is, $P_{\mathrm{Hdg}}(Z) = 0$. Thus Hodge number annihilates $\ker\phi$ and hence factors through $\phi(\mathcal{M})$. The result follows upon setting

$$P_{\mathrm{Hdg}}\left(\frac{1}{\mathbb{L}^i - 1}\right) := ((uv)^i - 1)^{-1}, \quad i \in \mathbb{N}.$$

By Corollary B.4.3 the motivic volume of (Y, D) lies in the subring $\widehat{\mathcal{M}}_0$.

Definition B.5.1. Let X be a complex algebraic variety of dimension n with at worst Gorenstein canonical singularities. Let $\varphi\colon Y \to X$ be a resolution of singularities for which the discrepancy divisor $D = \sum_{i=1}^r a_i D_i$ has only simple normal crossings. The *stringy Hodge number function* for X is

$$\begin{aligned} P_{\mathrm{stHdg}}(X) \quad &:= \quad P_{\mathrm{Hdg}}\left(\int_{J_\infty(Y)} \mathbb{L}^{-\operatorname{ord}_D} d\mu_X \cdot \mathbb{L}^{d_X}\right) \\ &= \sum_{J\subseteq\{1,\ldots,r\}} P_{\mathrm{Hdg}}(D_J^\circ)\cdot\left(\prod_{j\in J}\frac{uv-1}{(uv)^{a_j+1}-1}\right), \qquad (66) \end{aligned}$$

where we sum over all subsets $J \subseteq \{1, \ldots, r\}$ including $J = \varnothing$.

***Remark* B.5.2.** That the stringy Hodge function does not change after composing the resolution with a blow up along a smooth centre can be seen by a direct calculation. Hence, if one uses the weak factorization theorem (cf. Appendix 1 to Appendix 2), it follows directly that the stringy Hodge function is independent of choices. This proves Batyrev's conjecture without integration as alluded to in footnote 2.

Theorem B.5.3 ([**Ko**])**.** *Let X be a complex projective variety with at worst Gorenstein canonical singularities. If X admits a crepant resolution $\varphi\colon Y \to X$ then the Hodge numbers of Y are independent of the choice of crepant resolution.*

Proof: The discrepancy divisor $D = \sum_{i=1}^r a_i D_i$ of the crepant resolution $\varphi\colon Y \to X$ is by definition zero, so the motivic log-volume of X is the motivic log-volume of the pair $(Y, 0)$. Since each $a_i = 0$ it's clear that

$$P_{\mathrm{stHdg}}(X) = \textstyle\sum_{J\subseteq\{1,\ldots,r\}} P_{\mathrm{Hdg}}(D_J^\circ) = P_{\mathrm{Hdg}}(Y).$$

The stringy Hodge number function is independent of the choice of the resolution φ. In particular, $P_{\mathrm{Hdg}}(Y) = P_{\mathrm{stHdg}}(X) = P_{\mathrm{Hdg}}(Y_2)$ for $\varphi_2 : Y_2 \to X$ another crepant resolution. It remains to note that $P_{\mathrm{Hdg}}(Y)$ determines the Hodge numbers (here one uses that Y is smooth and projective). □

I finally want to relate this topic with the topic of Grothendieck motives. Recall the homomorphism $\chi^c_{\mathrm{mot}} : \mathcal{M} \to \mathrm{K}_0(\underline{\mathrm{Mot}})$ explained in Theorem A-3.2. At present it is unknown whether or not χ^c_{mot} annihilates the kernel of the natural completion map $\phi : \mathcal{M} \to \widehat{\mathcal{M}}$. Denef and Lœser conjecture that it does (see [**D-L98**, Remark 1.2.3]). If this is true, extend χ_{mot} to a ring homomorphism

$$\chi_{\mathrm{st}} : \widehat{\mathcal{M}}_0 \longrightarrow \mathrm{K}_0(\underline{\mathrm{Mot}}) \cdot \left[\left\{ \frac{1}{\mathbb{L}^i - 1} \right\}_{i \in \mathbb{N}} \right]$$

such that the image of $[D_J^\circ]$ under χ_{st} is equal to $\chi_{\mathrm{mot}}(D_J^\circ)$.

Definition B.5.4. Let X denote a complex algebraic variety with at worst canonical, Gorenstein singularities and let $\varphi : Y \to X$ be any resolution of singularities for which the discrepancy divisor $D = \sum a_i D_i$ has only simple normal crossings. Assume that the above conjecture of Denef and Lœser holds (so that χ_{st} is defined). The *stringy motive* of X is the element in $\mathrm{K}_0(\underline{\mathrm{Mot}}) \cdot \left[\left\{ \frac{1}{\mathbb{L}^i - 1} \right\}_{i \in \mathbb{N}} \right]$ given by the expression

$$\begin{aligned} \chi_{\mathrm{st}}(X) \quad &:= \quad \chi_{\mathrm{st}} \left(\int_{J_\infty(Y)} \mathbb{L}^{-\operatorname{ord}_D} d\mu_X \cdot \mathbb{L}^{d_X} \right) \\ &= \quad \sum_{J \subseteq \{1, \ldots, r\}} \chi^c_{\mathrm{mot}}(D_J^\circ) \cdot \left(\prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j + 1} - 1} \right) \end{aligned}$$

where we sum over all subsets $J \subseteq \{1, \ldots, r\}$ including $J = \varnothing$. As with the definition of the stringy Hodge number function (see Definition B.5.1) one multiplies by $\mathbb{L}^{d_X}$ for convenience.

Remark **B.5.5.** If X has a crepant resolution Y, $D = 0$ and $\chi_{\mathrm{st}}(X) = \sum_{J \subseteq \{1, \ldots, r\}} \chi^c_{\mathrm{mot}}(D_J^\circ) = \chi^c_{\mathrm{mot}}(Y)$. In particular, $\chi^c_{\mathrm{mot}}(Y)$ is independent of the crepant resolution which is a consequence of Konsevich's Theorem.

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