Review of "Mixed Hodge Structures" by C. Peters and J. Steenbrink

Hodge structures (HS) have long been a fundamental tool of transcendental algebraic geometry, with origins in the 19th Century study of integrals of rational 1-forms on Riemann surfaces, and subsequent work of Poincaré and Lefschetz on normal functions. The theory began in earnest with Hodge's decomposition (c. 1947) for Kähler manifolds, which records the complex analytic structure on their cohomology groups against the integral structure coming from topology. It was further enriched by Griffiths's invention of Abel-Jacobi maps and *variations* of HS for (families of) smooth projective varieties. In modern treatments, for a HS of weight d one speaks of a free abelian group $V_{\mathbb{Z}}$, an (increasingly) filtered complex vector space $(V_{\mathbb{C}}, F^{\bullet})$ (with the filtration satisfying $V_{\mathbb{C}} = F^r V_{\mathbb{C}} \oplus \overline{F^{d-r+1} V_{\mathbb{C}}}$), and a comparison isomorphism $V_{\mathbb{Z}} \otimes \mathbb{C} \xrightarrow{\cong} V_{\mathbb{C}}$. In the variational setting, $V_{\mathbb{Z}}$ is replaced by a local system and $V_{\mathbb{C}}$ by a holomorphic vector bundle with integrable connection satisfying Griffiths's transversality condition.

The "mixed" aspect of the theory enters with the introduction of a rationally defined *weight filtration*, used first by Deligne to generalize Hodge theory to the cohomology of incomplete and singular algebraic varieties. Weight filtrations were later employed to put mixed Hodge structures (MHS) on the cohomology of degenerating families of smooth algebraic varieties (the "limit" mixed Hodge structure of Griffiths [conj.], Schmid, Steenbrink and Clemens) and on homotopy groups (Morgan and later Hain, using Chen's iterated integrals and results of Sullivan); fundamental work on extensions of MHS was done by Carlson. The variational picture was enriched by the definition and study (Steenbrink/Zucker, Kashiwara) of admissible variations of MHS. The ultimate generalization (which includes admissible variations) is M. Saito's theory of *mixed Hodge modules* on a complex algebraic variety X, in which (leaving out the weight filtration for comparison to the definitions mentioned above) one has a perverse "sheaf" $\mathcal{V}^{\bullet}_{\mathbb{Q}}$, a rational filtered regular holonomic $\mathcal{D}$-module $(\mathcal{V}_{\mathcal{O}}, \mathcal{F}^{\bullet})$, and a comparison quasi-isomorphism $\mathcal{V}^{\bullet}_{\mathbb{Q}} \otimes \mathbb{C} \xrightarrow{\simeq} DR^{\bullet}_{X}(\mathcal{V}_{\mathcal{O}})$.

These developments have been motivated by applications of MHS to algebraic cycles and their generalizations, singularity theory, period maps and period domains, and other areas (e.g. the interaction between algebraic geometry and string theory in mirror symmetry). Introductory accounts of (mixed) Hodge theory and (some of) these "applied" aspects include the books of Voisin [*Hodge theory and complex algebraic geometry. I, II*, trans. L. Schneps, Cambridge Univ. Press, Cambridge, 2002/3; MR1967689, MR1997577], Lewis [*A survey of the Hodge conjecture (2nd Ed.)*, AMS, Providence, RI, 1999; MR1683216], Kulikov-Kurchanov ["Complex algebraic varieties: periods of integrals and Hodge structures", in *Algebraic Geometry, III*, 1-217, 263-270, Springer, Berlin, 1998; MR1602375], and Carlson/Müller-Stach/Peters [*Period mappings and period domains*, Cambridge Univ. Press, Cambridge, 2003; MR2012297]. The book under review, which (apart from some nice sections on singularity theory and vanishing theorems) focuses mainly on the "pure" story just summarized, is aimed at graduate students and researchers with such an introduction already under their belts.

I would like to single out a few particular features of the present text which, in my view, make it a "must-buy" for researchers. First, there is the use of *mixed Hodge complexes of sheaves* (the authors' technical simplification of an idea of Deligne) to unify the constructions of MHS on the cohomology of arbitrary varieties and on the nearby- and vanishing-cycle spaces. In particular, to deal with singular (and relative) varieties, they give a very thorough treatment (Ch. 5) of Navarro Aznar's elegant procedure using cubical hyperresolutions. Chapter 11 on limit MHS, which combines the second author's original approach with the Fontaine-Illusie-Kato notion of logarithmic structures (to furnish the underlying rational vector space), is an absolute tour-de-force. It contains complete, detailed proofs, and is probably the jewel of the book. Finally, Chapters 13-14 give a remarkably accessible crash-course (many proofs either skipped or sketched) on perverse sheaves, holonomic $\mathcal{D}$-modules, and mixed Hodge modules.

We now turn to a summary of the book's contents, which are divided into four parts, starting with "basics" on pure Hodge theory and the definitions in the mixed case. At the outset, Chern's theorem (operators on forms commuting with holonomy commute with the Laplacian) leads to a nice treatment of the classical Hodge theorems for Kähler manifolds. The hard Lefschetz theorem (and resulting decomposition) is set up additionally by a thorough review of finite-dimensional Lie-algebra representations of $\mathfrak{sl}_2(\mathbb{R})$. Several important applications — including the Hodge-Riemann bilinear relations, Barth's theorem, Deligne's criterion for degeneration of the Leray spectral sequence, and the global invariant cycle theorem — round out the first chapter. Next, the formal side of the theory is presented: Hodge structures, morphisms, polarizations, semisimplicity and the like. The authors treat the representation-theoretic aspect of HS's, and here the exposition contains errors; in particular, the section on Mumford-Tate groups should be read alongside §I.3 of [Deligne, Milne, Ogus and Shih, *Hodge cycles, motives, and Shimura varieties*, LNM 900, Springer, Berlin, 1982; MR0654325]. They then return to the geometric setting, covering for example the Fröhlicher spectral sequence, Hodge complexes of sheaves, and refined Thom classes of subvarieties; the most interesting results are that the Hodge decomposition, Dolbeault isomorphisms, etc. hold for (not necessarily projective) compact algebraic manifolds, and (in a weaker sense) for almost-Kähler V-manifolds. Chapter 3 finally introduces weights: mixed Hodge structures, Tate twists, Deligne's splitting (the $I^{p,q}$'s, which it is nice to see at the beginning), strictness of morphisms, and the abelian categorical structure on MHS. Mixed Hodge complexes of sheaves (MHCS) are defined, and the authors prove that their hypercohomology groups receive a natural mixed Hodge structure (using Deligne's lemma of the 3 filtrations). Finally, they explain extensions in the categories MHS and D^bMHS, as well as Beilinson's absolute Hodge cohomology.

The next major part is concerned with attaching MHS to algebro-geometric data. The first step, done in Chapter 4, is to augment Deligne's logarithmic de Rham complex to a MHCS, and so put a MHS on the cohomology of a smooth quasi-projective variety (thought of as a normal crossing divisor complement; of course, independence of the compactification is checked). The applications include the theorem of the fixed part and a description of the Leray spectral sequence for Lefschetz pencils. Also, logarithmic structures are discussed and used to give an algebraic description of the rational part of the MHCS. (Note that these have recently become very important due to their prominent use in the fundamental work [K. Kato and S. Usui, *Classifying spaces of degenerating polarized Hodge structures*, Ann. Math. Stud. 169, Princeton Univ. Press, 2009 (to appear)] on partial compactifications of period domains.) Chapter 5 further expands MHS to the cohomology groups of possibly singular varieties, starting with the definition of (semi-)simplicial and cubical spaces and their geometric realizations, along with sheaf cohomology. Cubical hyperresolutions (which satisfy cohomological descent) are constructed for arbitrary complex algebraic varieties; along with the log de Rham MHCS construction of Ch. 4, this is enough to put a well-defined, functorial MHS on their cohomology. Compatibility with discriminant squares and Künneth is verified; the construction is also extended to relative varieties. The next Chapter further pursues compatibility of MHS with standard algebro-topological constructions: various cup and cap products, Poincaré duality maps, and so on. Two of the more interesting parts are: (a) a digest of [D. Arapura, "The Leray spectral sequence is motivic", Invent. Math 160 (2005), no. 3, 567-589; MR2178703], showing how to put MHS on the Leray filtrands of $H^k(X)$ for a morphism of quasi-projective varieties $X \to Y$; and (b) a MHS on the link of a subvariety of X (assumed to contain its singular set), constructed by carrying out an embedded resolution and defining a semi-simplicial MHCS on the resulting NCD. Chapter 7 begins with a brief tour of the most basic material on Hodge-theoretic invariants of algebraic cycles (Deligne cycle-class map, which includes Abel-Jacobi) and a more interesting take on the generalized Hodge conjecture — including a homological version for singular varieties, and the equivalence of GHC and HC considered over all smooth projective varieties. Finally, a very general vanishing theorem for certain hypercohomology groups involving the filtered de Rham complex of du Bois is given, with applications to the local cohomology groups of (du Bois) singularities.

Chapters 8 and 9 give two different approaches to constructing MHS on the homotopy groups of a smooth projective variety X, due respectively to Hain and Morgan. On the higher homotopy groups (assuming $\pi_1 = \{0\}$) these MHS agree, and are compatible with the Whitehead product and Hurewicz homomorphism. For π_1, Hain's approach gives a pro-MHS on the completion of $\mathbb{Q}\pi_1(X, x)$ (the base point matters) with respect to the augmentation ideal. The idea behind his approach is roughly this: by a theorem of Borel and Serre, homotopy groups are dual to (the indecomposables in) cohomology groups of path space, which are computed by iterated integrals — or equivalently (in light of Chen's work) the bar construction on the de Rham algebra of X (with a natural product induced by the shuffle product). The second method uses a rational Postnikov tower of X to build a (quasi-isomorphic) minimal model of the Sullivan-de Rham (differential graded) algebra of the rational polynomial forms; Morgan proved that this minimal model admits a MHS, and one knows that the homotopy groups are dual to its indecomposables. Many of the harder proofs in this part are sketched.

Part IV begins with variations of Hodge structures (VHS) and associated concepts — local systems, holomorphic vector bundles, the Gauss-Manin connection, transversality, and semisimplicity. Brief treatments are given of the locus of Hodge classes (including recent work of Voisin), the relation between Mumford-Tate groups and monodromy, and criteria for stable irreducibility of the complex monodromy representation. (The authors' treatment of this material has also appeared in their article ["Monodromy of variations of Hodge structure", on first author's website].) The end of Chapter 10 and beginning of the next give a first approximation to the Riemann-Hilbert correspondence, explaining in detail (for a NCD complement $X \setminus D$) the equivalence between local systems, holomorphic vector bundles with integrable connection, and (after Malgrange) regular meromorphic extensions of the latter to X. The authors then restrict to the setting of a semistably degenerating family X of varieties over a disk (Schmid's more abstract result is only stated), giving a thoroughly motivated explanation of the nearby and vanishing cycle functors (ψ_f, ϕ_f). The second author's de Rham theoretic treatment of the canonically extended Hodge sheaves is then developed into a MHCS for $\psi_f\underline{\mathbb{Q}_X}$ (using log structures), computing the limit MHS and action of the monodromy logarithm. This leads to proofs of the monodromy and local invariant cycle theorems and the Wang and Clemens-Schmid exact sequences. A formula for the nearby-cycle Euler-Hodge polynomials produces some interesting computations of the $I^{p,q}$-ranks of LMHS's. The short following chapter gives applications of Ch. 11 to spectra of singularities and to Milnor fibers; and of semisimplicity (from Ch. 10) of polarized VHS to present Grothendieck's inductive approach to the GHC for (mainly) projective hypersurfaces of low degree.

At this point (still in the fourth part) the text switches back into a more expository (but no less technical) mode. The goal of Ch. 13 is to present the Riemann-Hilbert correspondence, under which the de-Rham-complex functor maps regular holonomic $\mathcal{D}$-modules to $\mathbb{C}$-perverse complexes, and its compatibility with direct and inverse images and Verdier duality. A review of the latter for sheaf complexes feeds into a presentation of perverse complexes, intermediate extensions of local systems, and the decomposition theorem of Beilinson-Bernstein-Deligne(-Gabber). The rest of the chapter is devoted to $\mathcal{D}$-modules; this becomes a bit of a technical rush through various definitions (solution complexes, characteristic varieties, regularity) and operations, but as a collecting together of what is needed to explain the structure of Saito's theory it is quite a service to researchers. That remark applies even more to the final chapter, which begins with an *axiomatic* presentation of mixed Hodge modules. From here they can use MHM to rather painlessly produce (a) the MHS on the *perverse Leray* filtrands of a morphism, (b) the pure HS on on the (middle perversity) intersection cohomology groups of compact varieties, and more generally (c) reproduce essentially all the MHS previously encountered in the book. Only then do they sketch the construction: the categories of polarizable Hodge modules and mixed Hodge modules are built out of (perverse/canonical) extensions of VHS and admissible VMHS (and limits of the latter are briefly treated); the Riemann-Hilbert correspondence is central in defining operations. There are a number of further interesting applications which we won't enumerate.

The book begins with a brief historical survey; each chapter is headed by a good summary of its contents and concluded by historical remarks (with references). There are three *excellent* appendices on homological algebra and category theory, sheaf cohomology, and the topology of stratified spaces and degenerations — these make the book essentially self-contained. As explained above, it is consistent with the main theme of the book not to consider classifying spaces for (M)HS, polylogarithm VMHS, normal functions, Nori connectivity, arithmetic aspects (p-adic Hodge theory, higher Abel-Jacobi maps, CM Hodge structures), etc.; the one disappointment is that they don't really treat degenerations of MHS or multivariable degenerations of variations of HS (beyond giving references). Finally, the prospective reader should be aware that there are quite a few typos in some sections, though we don't feel this diminishes this work as a reference. It is a through-readable and very up-to-date account of mixed Hodge theory, written by masters of the subject, and will undoubtedly serve as a basic reference for years to come.