Introduction

Brief History of the Subject

One can roughly divide the history of mixed Hodge theory in four periods; the period up to 1967, the period 1967–1977, the period 1977–1987, the period after 1987.

The **first period** could be named *classical*. The "prehistory" consists of work by Abel, Jacobi, Gauss, Legendre and Weierstrass on the periods of integrals of rational one-forms. It culminates in Poincaré's and Lefschetz's work, reported on in Lefschetz's classic monograph [Lef]. The second landmark in the classical era proper is Hodge's decomposition theorem for the cohomology of a compact Kähler manifold [Ho47]. To explain the statement, we begin by noting that a complex manifold always admits a hermitian metric. As in differential geometry one wants to normalise it by choosing holomorphic coordinates in which the metric osculates to second order to the constant hermitian metric. This turns out not be always possible and one reserves for such a special metric the name *Kähler metric*. The existence of such a metric implies that the decomposition of complex-valued differential forms into type persists on the level of cohomology classes. We recall here that a complex form α has type (p, q), if in any local system of holomorphic coordinates $(z_1, \ldots, z_n)$, the form α is a linear combinations of forms of the form (differentiable function)$\cdot(dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge \overline{dz}_{j_1} \wedge \cdots \wedge \overline{dz}_{j_q})$. Indeed, Hodge's theorem (See Theorem 1.8) states that this induces a decomposition

$$H^m(X; \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(X), \qquad \text{(HD)}$$

where the term on the right denotes cohomology classes representable by closed forms of type (p, q). The space $H^{p,q}(X)$ is the complex conjugate of $H^{q,p}$, where the complex conjugation is taken with respect to the real structure given by $H^m(X; \mathbb{C}) = H^m(X; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. A decomposition (HD) with this reality constraint by definition is the prototype of *a weight m Hodge structure*.

The Hodge decomposition fails in general, as demonstration by the Hopf manifolds, complex m-dimensional manifolds homeomorphic to $S^1 \times S^{2m-1}$. Indeed H^1 being one-dimensional for these manifolds, one can never have a splitting $H^1 = H^{1,0} \oplus H^{0,1}$ with the second subspace the complex conjugate of the first. It follows that complex manifolds do not always admit Kähler metrics. A complex manifold which does admit such a metric is called a *Kähler manifold*. Important examples are the complex projective manifolds: the Fubini-Study metric (Examples 1.5.2) on projective space is Kähler and restricts to a Kähler metric on every submanifold.

It is not hard to see that the fundamental class of a complex submanifold of a Kähler manifold is of pure type (c, c), where c is the codimension (Prop. 1.14). This applies in particular to submanifolds of complex projective manifolds. By the GAGA-principle these are precisely the algebraic submanifolds. Also singular codimension c subvarieties can be shown to have a fundamental class of type (c, c), and by linearity, so do cycles: finite formal linear combinations of subvarieties with integral or rational coefficients. Hodge's famous conjecture states that, conversely, any rational class of type (c, c) is the fundamental class of a rational cycle of codimension c. This conjecture, stated in [Ho50], is one of the millennium one-million dollar conjectures of the Clay-foundation and is still largely open.

The **second period** starts in the late 1960's with the work of Griffiths [Grif68, Grif69] which can be considered as neo-classical in that this work goes back to Poincaré and Lefschetz. In the monograph [Lef], only weight one Hodge structures depending on parameters are studied. In Griffiths's terminology these are weight one *variations of Hodge structure*. Indeed, in the cited work of Griffiths this notion is developed for any weight and it is shown that there are remarkable differences with the classical weight one case. For instance, although the ordinary Jacobian is a polarized abelian variety, their higher weights equivalents, the intermediate Jacobians, need not be polarized. Abel-Jacobi maps generalize in this set-up (see § 7.1.2) and Griffiths uses these in [Grif69] to explain that higher codimension cycles behave fundamentally different than divisors.

All these developments concern smooth projective varieties and cycles on them. For a not necessarily smooth and/or compact complex algebraic variety the cohomology groups cannot be expected to have a Hodge decomposition. For instance H^1 can have odd rank. Deligne realized that one could generalize the notion of a Hodge structure to that of a mixed Hodge structure. There should be an increasing filtration, the *weight filtration*, so that m-th graded quotient has a pure Hodge structure of weight m. This fundamental insight has been worked out in [Del71, Del74].

Instead of looking at the cohomology of a fixed variety, one can look at a family of varieties. If the family is smooth and projective all fibres are complex projective and the cohomology groups of a fixed rank m assemble to give the prototype of a *variation of weight m Hodge structure*. An important observation at this point is that giving a Hodge decomposition (HD) is equivalent to giving a *Hodge filtration*

$$F^pH^m(X;\mathbb{C}) := \bigoplus_{r\geq p} H^{r,s}(X), \quad F^p \oplus \overline{F}^{m-p+1} = H^m(X;\mathbb{C}), \tag{HF}$$

where the last equality is the defining property of a Hodge filtration. The point here is that the Hodge filtration varies holomorphically with X while the subbundles $H^{p,q}(X)$ in general don't.

If the family acquires singularities, one may try to see how the Hodge structure near a singular fibre degenerates. So one is led to a *one-parameter degeneration* $X \to \Delta$ over the disk Δ, where the family is smooth over the punctured disk $\Delta^* = \Delta - \{0\}$. So for $t \in \Delta^*$ cohomology group $H^m(X_t;\mathbb{C})$ has a classical weight m Hodge structure. In order to capture the degeneration Hodge theoretically this classical structure has to be replaced by a *mixed* Hodge structure, the so-called *limit mixed Hodge structure*. Griffiths conjectured in [Grif70] that the monodromy action defines a weight filtration which together with a certain limiting Hodge filtration should give the correct mixed Hodge structure. Moreover, this mixed Hodge structure should reveal restrictions on the monodromy action, and notably should imply a local invariant cycle theorem: all cohomology classes in a fibre which are invariant under monodromy are restriction from classes on the total space. In the algebraic setting this was indeed proved by Steenbrink in [Ste76]. Clemens [Clem77] treated the Kähler setting, while Schmid [Sch73] considered abstract variations of Hodge structure over the punctured disk. We should also mention Varchenko's approach [Var80] using asymptotic expansions of period integrals, and which goes back to Malgrange [Malg74].

The **third period**, is a period of on the one hand consolidation, and on the other hand widening the scope of application of Hodge theory. We mention for instance the extension of Schmid's work to the several variables [C-K-S86] which led to an important application to the Hodge conjecture [C-D-K]. In another direction, instead of varying Hodge structures one could try to enlarge the definition of a variation of Hodge structure by postulating a second filtration, the weight filtration which together with the Hodge filtration (HF) on every stalk induces a mixed Hodge structure. Indeed, this leads to what is called a *variation of mixed Hodge structure*. On the geometric side, the fibre cohomology of families of possible singular algebraic varieties should give such a variation, which for obvious reasons is called "geometric". These last variations enjoy strong extra properties, subsumed in the adjective *admissible*. Their study has been started by Steenbrink and Zucker [St-Z, Zuc85], and pursued by Kashiwara [Kash86].

On the abstract side we have Carlson's theory [Car79, Car85b, Car87] of the *extension classes* in mixed Hodge theory, and the related work by Beilinson on *absolute Hodge cohomology* [Beil86]. Important are also the *Deligne-Beilinson cohomology groups*; these can be considered as extensions in the category of pure Hodge complexes and play a central role in unifying the classical class map and the Abel-Jacobi map. For a nice overview see [Es-V88].

Continuing our discussion of the foundational aspects, we mention the alternative approach [G-N-P-P] to mixed Hodge theory on the cohomology of a singular algebraic variety. It is based on cubical varieties instead of simplicial varieties used in [Del74]. See also [Car85a].

In this period a start has been made to put mixed Hodge structures on other geometric objects, in the first place on *homotopy groups* for which Morgan found the first foundational results [Mor]. He not only put a mixed Hodge structure on the higher homotopy groups of complex algebraic manifolds, but showed that the minimal model of the Sullivan algebra for each stage of the rational Postnikov tower has a mixed Hodge structure. The fundamental group being non-abelian a priori presents a difficulty and has to be replaced by a suitably abelianized object, the De Rham fundamental group. Morgan relates it to the 1-minimal model of the Sullivan algebra which also is shown to have a mixed Hodge structure. In [Del-G-M-S] one finds a striking application to the formality of the cohomology algebra of Kähler manifolds. For a further geometric application see [C-C-M]. Navarro Aznar extended Morgan's result to possibly singular complex algebraic varieties [Nav87]. Alternatively, there is Hain's approach [Hain87, Hain87b] based on Chen's iterated integrals. At this point we should mention that the Hurewicz maps, which are natural maps from homotopy to homology, turn out to be morphisms of mixed Hodge structure.

A second important development concerns *intersection homology and cohomology* which is a Poincaré-duality homology theory for singular varieties. The result is that for any compact algebraic variety X the intersection cohomology group $IH^k(X;\mathbb{Q})$ carries a weight k pure Hodge structure compatible with the pure Hodge structure on $H^k(\tilde{X};\mathbb{Q})$ for any desingularization $\pi : \tilde{X} \to X$ in the sense that π^* makes $IH^k(X;\mathbb{Q})$ a direct factor of $IH^k(\tilde{X};\mathbb{Q}) = H^k(\tilde{X};\mathbb{Q})$.

There are two approaches. The first, which still belongs to this period uses L_2-cohomology and degenerating Hodge structures is employed in [C-K-S87] and [Kash-Ka87b]. The drawback of this method is that the Hodge filtration is not explicitly realized on the level of sheaves as in the classical and Deligne's approach. The second method remedies this, but belongs to the next period, since it uses D-modules.

We now come to this last period, the **post D-modules period**. Let us explain how D-modules enter the subject. A variation of Hodge structure with base a smooth complex manifold X in particular consists of an underlying local system $\mathbb{V}$ over X. The associated vector bundle $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_X$ thus has a canonical flat connection. So one has directional derivatives and hence an action of the sheaf $\mathcal{D}_X$ of germs of holomorphic differential operators on X. In other words, $\mathcal{V}$ is a a $\mathcal{D}_X$-module.

At this point we have a pair $(\mathcal{V}, \mathbb{V})$ consisting of a $\mathcal{D}_X$-module and a local system which correspond to each other. A Hodge module as defined by Saito incorporates a third ingredient, a so called "good" filtration on the $\mathcal{D}_X$-module. In our case this is the Hodge filtration $\mathcal{F}^\bullet$ which for historical reasons is written as as increasing filtration, i.e. one puts $\mathcal{F}_k = \mathcal{F}^{-k}$. The axiom of Griffiths tranversality just means that this filtration is good in the technical sense. The resulting triple $(\mathcal{V}, \mathcal{F}_\bullet, \mathbb{V})$ indeed gives an example of a Hodge module of weight n. It is called a smooth Hodge module. [1]

Saito has developed the basic theory of Hodge modules in [Sa87, Sa88, Sa90]. The actual definition of a Hodge module is complicated, since it is by induction on the dimension of the support. To have a good functorial theory of Hodge modules, one should restrict to polarized variations of Hodge structure and their generalizations the polarized Hodge modules. If we are "going mixed", any polarized admissible variation of mixed Hodge structure over a smooth algebraic base is the prototype of a mixed Hodge module. But, again, the definition of a mixed Hodge module is complex and hard to grasp. Among the successes of this theory we mention the existence of a natural pure Hodge structure on intersection cohomology groups, the unification of the proofs of vanishing theorems, and a nice coherent theory of fundamental classes.

A second important development that took place in this period is the emergence of non-abelian Hodge theory. Classical Hodge theory treats harmonic theory for maps to the abelian group $\mathbb{C}^*$ which governs line bundles: in contrast, non-abelian Hodge theory deals with harmonic maps to non-abelian groups like $\mathrm{GL}(n)$, $n \geq 2$. This point of view leads to so-called *Higgs bundles* which are weaker versions of variations of Hodge structure that come up when one deforms variations of Hodge structure. It has been developed mainly by Simpson, [Si92, Si94, Si95], with contributions of Corlette [Cor]. This work leads to striking limitations on the kind of fundamental group a compact Kähler manifold can have. A similar approach for the mixed situation is still largely missing.

There are many other important developments of which we only mention two. The first concerns the relation of Hodge theory to the logarithmic structures invented by Fontaine, Kato and Illusie, which was studied in [Ste95]. A second topic is mixed Hodge structures on Lawson homology, a subject whose study started in [F-M], but which has not yet been properly pursued afterwards.

Contents of the Book

The book is divided in four parts which we now discuss briefly. The first part, entitled *basic Hodge theory* comprises the first three chapters.

In Chapter 1 in order to motivate the concept of a Hodge structure we give the statement of the Hodge decomposition theorem. Likewise, polarizations are motivated by the Lefschetz decomposition theorem. It has a surprising

[1] If you want such a triple to behave well under various duality operators it turns out to be better to replace $\mathbb{V}$ by a complex placed in degree $-n = -\dim X$ so that it becomes a perverse sheaf. See Chapter 13 for details.

topological consequence: the Leray spectral sequence for smooth projective families degenerates at the E_2-term. In particular, a theorem alluded to in the Historical Part holds in this particular situation: the invariant cycle theorem (cycles invariant under monodromy are restrictions of global cycles).

Chapter 2 explains the basics about pure Hodge theory. In particular the crucial notions of a Hodge complex of weight m and a Hodge complex of sheaves of weight m are introduced. The latter makes Hodge theory local in the sense that if a cohomology group can be written as the hypercohomology groups of a Hodge complex of sheaves, such a group inherit a Hodge structure. This is what happens in the classical situation, but it requires some work to explain it. In the course of this Chapter we are led to make an explicit choice for a Hodge complex of sheaves on a given compact Kähler manifold, the *Hodge-De Rham complex of sheaves* $\mathbb{Z}_X^{\mathrm{Hdg}}$. Incorporated in this structure are the *Godement resolutions* which we favour since they behave well with respect to filtrations and with respect to direct images. The definition and fundamental properties are explained in Appendix B.

These abstract considerations enable us to show that the cohomology groups of X can have pure Hodge structure even if X itself is *not* a compact Kähler manifold, but only bimeromorphic to such a manifold. In another direction, we show that the cohomology of a possibly singular V-manifold posses a pure Hodge structure.

The foundations for mixed Hodge theory are laid down in Chapter 3. The notions of Hodge complexes and Hodge complexes of sheaves are widened to mixed Hodge complexes and mixed Hodge complexes of sheaves. The idea is as in the pure case: the construction of a mixed Hodge structure on cohomological objects can be reduced to a local study. Crucial here is the technique of spectral sequences which works well because the axioms imply that the Hodge filtration induces only one filtration on the successive steps in the spectral sequence (Deligne's comparison of three filtrations). Next, the important construction of the cone in the category of mixed Hodge complexes of sheaves is explained. Since relative cohomology can be viewed as a cone this paves the way for mixed Hodge structures on relative cohomology, on cohomology with compact support, and on local cohomology. The chapter concludes with Carlson's theory of extensions of mixed Hodge structures and Beilinson's theory of absolute Hodge cohomology.

The second part of the book deals with *mixed Hodge structures on cohomology groups* and starts with Chapter 4 on smooth algebraic varieties. The classical treatment of the weight filtration due to Deligne is complemented by a more modern approach using logarithmic structures. This is needed in Chapter 11 which deals with variations of Hodge structure.

Chapter 5 treats the cohomology of singular varieties. Instead of Deligne's simplicial approach we explain the cubical treatment proposed by Guillén, Navarro Aznar, Pascual-Gainza and F. Puerta.

The results from Chapter 5 are further extended in Chapter 6 where Arapura's work on the Leray spectral sequence is explained, followed by a treatment of cup and cap products and duality. This chapter ends with an application to the cohomology of two geometric objects, halfway between an algebraic and a purely topological structure: deleted neighbourhoods and links of closed subvarieties of a complex algebraic variety.

In Chapter 7 we give applications of the theory which we developed so far. First we explain the Hodge conjecture as generalized by Grothendieck, secondly we briefly discuss Deligne cohomology and the relation to algebraic cycles. Finally we introduce Du Bois's filtered de Rham complex and give applications to singularities.

The third part is entitled *mixed Hodge structures on homotopy groups*. We first give the basics from homotopy theory enabling to make the transition from homotopy groups to Hopf algebras. Next, we explain Chen's homotopy de Rham theorem and Hain's bar construction on Hopf algebras. These two ingredients are necessary to understand Hain's approach to mixed Hodge theory on homotopy which we give in Chapter 8. The older approach, due to Sullivan and Morgan is explained in Chapter 9.

The fourth and last part is about *local systems in relation to Hodge theory* and starts with the foundational Chapter 10. In Chapter 11 Steenbrink's approach to the limit mixed Hodge structure is explained from a more modern point of view which incorporates Deligne's vanishing and nearby cycle sheaves. The starting point is that the cohomology of any smooth fibre in a one-parameter degeneration can be reconstructed as the cohomology of a particular sheaf on the singular fibre, the nearby cycle sheaf. So a mixed Hodge structure can be put on cohomology by extending the nearby cycle sheaf to a mixed Hodge complex of sheaves on the singular fibre. This is exactly what we do in Chapter 11. Important applications are given next: the monodromy theorem, the local invariant cycle theorem and the Clemens-Schmid exact sequence.

Follows Chapter 12 with applications to singularities (the cohomology of the Milnor fibre and the spectrum), and to cycles (Grothendieck's induction principle).

The fourth part is leading up to Saito's theory which, as we explained in the historical part, incorporates D-modules into Hodge theory through the Riemann-Hilbert correspondence. This is explained in Chapter 13, where the reader can find some foundational material on D-modules and perverse sheaves. In the final Chapter 14 Saito's theory is sketched. In this chapter we axiomatize his theory and directly deduce the important applications we mentioned in the Historical Part. We proceed giving ample detail on how to construct Hodge modules as well as mixed Hodge modules, and briefly sketch how the axioms can be verified. Clearly, many technical details had to be omitted, but we hope to have clarified the overall structure. Many mathematicians consider Saito's formidable work to be rather impenetrable. The final chapter is meant as an introductory guide and hopefully motivates an interested researcher to penetrate deeper into the subject by reading the original articles.

The book ends with three appendices: Appendix A with basics about derived categories, spectral sequences and filtrations, Appendix B where several fundamental results about the algebraic topology of varieties is assembled, and Appendix C about stratifications and singularities.

Finally a word about what is *not* in this book. Due to incompetence on behalf of the authors, we have not treated mixed Hodge theory from the point of view of L_2-theory. Hence we don't say much on Zucker's fundamental work about L_2-cohomology. Neither do we elaborate on Schmid's work on one-parameter degenerations of abstract variations of Hodge structures, apart from the statement in Chapter 10 of some of his main results. In the same vein, the work of Cattani-Kaplan-Schmid on several variables degenerations is mostly absent. We only give the statement of the application of this theory to Hodge loci (Theorem 10.15), the result about the Hodge conjecture alluded to in the Historical Part.

The reader neither finds many applications to singularities. In our opinion Kulikov's monograph [Ku] fills in this gap rather adequately. For more recent applications we should mention Hertling's work, and the work of Douai-Sabbah on Frobenius manifolds and tt^*-structures [Hert03, D-S03, D-S04].

Mixed Hodge theory on Lawson homology is not treated because this falls too far beyond the scope of this book. For the same reason non-abelian Hodge theory is absent, as are characteristic p methods, especially motivic integration, although the motivic nearby and motivic vanishing cycles are introduced (Remark 11.27).