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★**Lectures on the theory of pure motives.**
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Since Grothendieck came up with the notion of motives in the 1960's, few books have been written on the subject and fewer books have constituted, as does the one under review, a clear introduction to the topic. Here is a good initiation to the theory of motives, seen from a geometric point of view. The reader is advised to refer to Y. André's book [*Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panor. Synthèses, 17, Soc. Math. France, Paris, 2004; MR2115000 (2005k:14041)] for the arithmetic side of the theory. Accessible to people with a fair knowledge of algebraic cycles and intersection theory as in W. Fulton's book [*Intersection theory*, second edition, Ergeb. Math. Grenzgeb. (3), 2, Springer, Berlin, 1998; MR1644323 (99d:14003)], it essentially deals with motives à la Grothendieck and is a good preparation for the study of mixed motives; it actually ends with V. Voevodsky's constructions of triangulated categories of motives [in *Cycles, transfers, and motivic homology theories*, 188–238, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000; MR1764202].

The numerous conjectures inherent in the world of motives are carefully treated. The standard conjectures are well set and explained, with an emphasis on the Künneth type of conjectures, to which the first author has contributed a lot. The relations between the different conjectures neatly presented in a table on page 103 are well outlined. Nevertheless the book is far from being an exposition of conjectures. Starting with the theorem that the category of motives modulo numerical equivalence is an abelian semi-simple category [U. Jannsen, Invent. Math. **107** (1992), no. 3, 447–452; MR1150598 (93g:14009)], Jannsen's contributions, published or not, are present throughout the book. Not all statements are proven, but the ingredients and references are clearly given.

In Chapter 1, the authors recall the different notions of equivalence relations on algebraic cycles: rational, algebraic, homological, numerical and even the less known smash nilpotence equivalence. In the appendices can be found a survey on known properties of Chow rings, Picard and Albanese varieties, and a proof of a theorem of Voevodsky [Internat. Math. Res. Notices **1995**, no. 4, 187–198; MR1326064 (96c:14003)] and C. Voisin [in *Moduli of vector bundles (Sanda, 1994; Kyoto, 1994)*, 265–285, Lecture Notes in Pure and Appl. Math., 179, Dekker, New York, 1996; MR1397993 (97e:14013)] on the comparison between algebraic and smash nilpotence equivalences. Chapter 2 is devoted to the construction of Grothendieck motives based on smooth projective varieties, correspondences modulo equivalence and pseudo-abelianization. Chow motives are defined using rational equivalence. As examples are treated the Lefschetz and Tate motives, and the motives of smooth projective curves. The chapter ends with Manin's Identity Principle. Chapter 3 provides a clear exposition of the three types of the standard conjectures: Künneth, Lefschetz and Hodge; and of Jannsen's theorem previously quoted.

Much less classical is the notion of finite-dimensionality of motives developed in Chapters 4 and

5. As defined by S. Kimura [Math. Ann. **331** (2005), no. 1, 173–201; MR2107443 (2005j:14007)], a motive is said to be evenly (resp. oddly) finite-dimensional if some exterior (resp. symmetric) power of it vanishes. The dimension is the maximal power which is nonzero. A finite-dimensional motive is the sum of two motives, one evenly and the other oddly dimensional. Kimura and O'Sullivan conjectured that any Chow motive is finite-dimensional.

The following three chapters deal with the Künneth decomposition of the diagonal and related topics such as the Bloch-Beilinson filtration. Chapter 6 consists mainly of results by the first author; some give an unconditional definition of the Picard and Albanese motives. In Chapter 7, the authors discuss further conjectures such as the vanishing of projectors operating on Chow groups. Relations between different conjectures are explained and consequences are given on surfaces and their products. In Chapter 8, the previous work is extended to the relative case on a not necessarily smooth variety over the field of complex numbers. This requires a survey on perverse sheaves and the decomposition theorem. In appendix D, the case of surfaces fibered over a curve is treated. Chapter 9 goes beyond projective varieties and pure motives. Two extensions are presented; the virtual motives suggested by Serre involve K-theoretic constructions such as the motivic Euler characteristic of F. Bittner [Compos. Math. **140** (2004), no. 4, 1011–1032; MR2059227 (2005d:14031)] and the weight complex of H. A. Gillet and C. Soulé [J. Reine Angew. Math. **478** (1996), 127–176; MR1409056 (98d:14012)]. Finally comes a survey on the construction of the categories of motivic complexes by Voevodsky, and their relations to the category of Chow motives.

Reviewed by *Florence Lecomte*