On a motivic interpretation of primitive, variable and fixed cohomology

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If M is a smooth projective variety whose motive is Kimura finite-dimensional and for which the standard Lefschetz Conjecture B holds, then the motive of M splits off a primitive motive whose cohomology is the primitive cohomology. Under the same hypotheses on M, let X be a smooth complete intersection of ample divisors within M. Then the motive of X is the sum of a variable and a fixed motive inducing the corresponding splitting in cohomology. I also give variants with group actions.

1 Introduction

This note aims to address the motivic nature of some classical cohomological results of Lefschetz. The first is the Lefschetz decomposition of the cohomology of a smooth projective manifold. The second is a consequence of Lefschetz' hyperplane theorem, namely the splitting of the cohomology of a complete intersection into a summand which comes from the surrounding variety, the "fixed part", and a supplementary summand, the "variable" part. Explicitly, fix an $(d+r)$-dimensional projective manifold M and an ample line bundle $\mathcal{L}$ on M; let $X = H_1 \cap \cdots \cap H_r$ be a smooth complete intersection of r divisors $H_j \in |\mathcal{L}|$, $j = 1, \ldots, r$ and let $i : X \hookrightarrow M$ be the inclusion. With

$$\begin{aligned} H^d(X)_{\rm fix} &:= \operatorname{Im}(i^* : H^d(M) \to H^d(X)) \\ H^d(X)_{\rm var} &:= \operatorname{Ker}(i_* : H^d(X) \to H^{d+2r}(M)) \end{aligned} \tag{1}$$

there is an orthogonal direct sum decomposition

$$H^d(X) = H^d(X)_{\rm fix} \oplus H^d(X)_{\rm var}. \tag{2}$$

In general it seems hard to show the motivic nature of these results and some conditions will be needed, Clearly, a first ingredient one needs is the existence of a correspondence inducing the inverse of the Lefschetz operator on $H^*(M)$. This is Lefschetz' conjecture $B(M)$. The second comes from a concept introduced by Kimura [3] and O'Sullivan, the concept of finite-dimensionality for motives. These authors conjecture that all motives are finite-dimensional. The main result of this note is that the primitive decomposition for the cohomology of M as well as the splitting (2) is motivic provided these two conjectures hold for M. [1] In fact, only a consequence of finite dimensionality is used, namely a certain nilpotency result which is stated as (3).

It is known that both Kimura's conjecture and conjecture B are verified for example for M a projective space, or an abelian variety. For these examples the motive of M is well understood and the primitive decomposition is probably well known. See e.g. Diaz' explicit results [2] for abelian varieties. The motivic nature of the splitting (2) for complete intersections $X \subset M$ shows that the relevant motivic information is hidden in the variable motive.

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[1] For the comfort of the reader some facts about Chow motives are placed together in Section 2.

This can be taken advantage of in situations where the motive of M is too large. Let me illustrate this starting from the Bloch conjecture [1] for surfaces S. Recall that the latter states that if $p_g(S) = 0$, then $\mathsf{CH}_0(S)$ is "small" in the sense that the degree 0 part of the Chow group is just the Albanese variety of S. In the present setting, assuming that one has a *complex* complete intersection surface $S \subset M$, such that $h^{2,0}(M) \neq 0$, then, by Lefschetz' theorem on hyperplane sections $h^{2,0}(S) \neq 0$, and then, by a result of Mumford [5], the Chow group of zero cycles on S is huge. However, it may happen that the variable submotive of S, or a submotive T thereof does satisfy the condition $h^{2,0}(T) = 0$ of Bloch's conjecture. This observation can indeed be put to use as is shown in the examples of [4]; the present note sets up the proper theoretical framework.

Convention. Varieties are taken over a fixed algebraically closed field (of any characteristic).

Notation.

- H^* denotes a fixed Weil cohomology theory; CH_* denotes Chow groups with $\mathbf{Q}$-coeffcients.
- A degree k (Chow) correspondences from X to Y from a smooth projective variety X to a smooth projective variety Y is a cycle class

$$\mathrm{Corr}^k(X,Y) := \mathsf{CH}^{\dim X + k}(X \times Y).$$

- For a smooth projective manifold X, its Chow motive is denoted $\mathsf{h}(X)$.

2 Motives

A correspondence of degree k induces a morphism on Chow groups of the same degree and on cohomology groups (of double the degree). Correspondences can be composed and these give the morphisms in the category of Chow motives. Let me elaborate briefly on this but refer to [6] for more details.

Precisely, *an effective Chow motive* consists of a pair (X, p) with X a smooth projective variety and p a degree zero correspondence which is a projector, i.e., $p^2 = p$. Morphism between motives are induced by degree zero correspondences compatible with projectors. This procedure defines the category of effective Chow motives. Every smooth projective variety X defines a motive

$$\mathsf{h}(X) = (X, \Delta), \quad \Delta \in \mathsf{CH}^{\dim X}(X \times X) \text{ the class of the diagonal}$$

and a morphism $f : X \to Y$ between smooth projective varieties defines a morphism $h(Y) \to \mathsf{h}(X)$ given by the transpose of the graph of X.

The direct sum $(X, p) \oplus (X', p')$ of two effective motives plays a role further on. It consists of the disjoint union of X and X' equipped with the union of the projectors p and p'.

One can also use correspondences of arbitrary degrees provided one uses triples (X, p, k) where p is again a projector, but a morphism $f : (X, p, k) \to (Y, q, \ell)$ is a correspondence of degree $\ell - k$ compatible with projectors. Such triples define the *category of Chow motives*.

It should be recalled that motives, like varieties have their Chow groups and cohomology groups:

$$\mathsf{CH}^m(X,p,k) := \mathrm{Im}\left(\mathsf{CH}^{m+k}(X) \xrightarrow{p_*} \mathsf{CH}^{m+k}(X)\right),$$
$$H^m(X,p,k) := \mathrm{Im}\left(H^{m+2k}(X) \xrightarrow{p_*} H^{m+2k}(X)\right).$$

Kimura [3] has introduced the concept finite-dimensionality for motives and he has shown that it implies the following nilpotency result.

$$N \in \mathrm{Corr}^0(M,M) \text{ such that } N = 0 \text{ on } H^*(M) \implies N \text{ is nilpotent.} \quad (3)$$

3 The primitive motive

3.1 Primitive cohomology

Let M be a smooth projective variety. For ease of presentation, introduce

$$H_M := H^*(M)(\dim M),$$

the cohomology of M centered at 0. The Hard Lefschetz theorem then is the statement that there are isomorphisms

$$L^j : H_M^{-j} \xrightarrow{\sim} H_M^j, \quad j = 0, \dots, \dim M. \tag{4}$$

Over the complex numbers this result is a consequence of the Lefschetz decomposition. The proof requires subtle properties of the formal adjoint of the Lefschetz operator L. If one instead uses a Weil cohomology, one assumes the Hard Lefschetz theorem and tries to derive the analogue of the Lefschetz decomposition. Hard Lefschetz (4) indeed makes it possible to construct a replacement $\lambda : H_M \to H_M$ for the formal adjoint of L, an operator of degree -2 which makes the following diagrams commutative. It is uniquely defined in this way. [2]

$$\begin{array}{ccc} H_M^{-j} & \xrightarrow[\sim]{L^j} & H_M^j \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle L} \\ H_M^{-j-2} & \xrightarrow[\sim]{L^{j+2}} & H_M^{j+2} \end{array} \qquad \text{for } -j = -d+2, \dots, 0$$

$$H_M^{-1} \underset{\lambda}{\overset{L}{\rightleftarrows}} H_M^1 \quad (\sim)$$

$$\begin{array}{ccc} H_M^{-j+2} & \xrightarrow[\sim]{L^{j-2}} & H_M^{j-2} \\ {\scriptstyle L}\uparrow & & \uparrow{\scriptstyle\lambda} \\ H_M^{-j} & \xrightarrow[\sim]{L^j} & H_M^j, \end{array} \qquad \text{for } j = 2, \dots, d.$$

We set $\lambda = 0$ on H_M^{-d} and H_M^{-d+1}. By definition, setting $H_M^{\leq 0} = \oplus_{j=0}^d H_M^{-j}$, we have

$$H_M^{\mathrm{pr}} = \operatorname{Ker} \lambda \subset H_M^{\leq 0}.$$

From the preceding definition it follows that for all integers $r \geq 1$ one has

$$\begin{aligned} \lambda^r \circ L^r &= \text{id on the image of } \quad \lambda^r \\ L^r \circ \lambda^r &= \text{id on the image of } \quad L^r. \end{aligned} \tag{5}$$

This implies that $L^r \circ \lambda^r$ and $\lambda^r \circ L^r$ are projectors. Consider the special case $r = 1$ and write

$$u = (u - L\lambda(u)) + L\lambda(u).$$

This gives the Lefschetz decomposition

$$H_M = H_M^{\mathrm{pr}} \oplus LH_M. \tag{6}$$

Indeed, $L \circ \lambda(u) = 0$ if u is primitive, while if $u = Lu'$, using (5) one sees that $L \circ \lambda(u) = L \circ \lambda \circ L(u') = L(u') = u$ and so $\operatorname{Im}(L) \subset \operatorname{Im}(L \circ \lambda)$ and hence $\operatorname{Im}(L) = \operatorname{Im}(L \circ \lambda)$. Consequently,

$$\pi^{\mathrm{pr}} := \mathrm{id} - L \circ \lambda \tag{7}$$

is a projector onto the primitive cohomology.

[2] Over $\mathbf{C}$ it differs slightly from the formal adjoint of L, usually called the Λ-operator. In fact, from the proof of [7, Cor. 1.25] one sees that $\Lambda \circ L|_{L^r H_M^{j-2r,\mathrm{pr}}} = r(j-r) \cdot \mathrm{id}$ while $\lambda \circ L = \mathrm{id}$ on $\operatorname{Im} L$.

3.2 Construction of the "primitive" Chow projector

We next explain under what conditions these projectors can be lifted to correspondences. First note that $L \in \mathrm{Corr}^1(M,M)$. *Lefschetz' conjecture* $B(M)$ states that there is a correspondence, say $\Lambda \in \mathrm{Corr}^{-1}(M,M)$ inducing λ. More will be needed, namely the existence of a lift of $L^r{\circ}\lambda^r$ to a (Chow) projector. Because operators in general don't commute, this motivates the following variant of the Lefschetz conjecture $B(M)$.

Conjecture 3.1. Property $B(M)^*$ holds if for all $r \geq 1$ there are correspondences Λ_r and $\widetilde{\Lambda}_r$ in $\mathrm{Corr}^{-r}(M,M)$ such that

- $L^r{\circ}\Lambda_r \in \mathrm{Corr}^0(M,M)$ is a projector inducing $L^r{\circ}\lambda^r$ in cohomology.
- $\widetilde{\Lambda_r}{\circ}L^r \in \mathrm{Corr}^0(M,M)$ is a projector inducing $\lambda^r{\circ}L^r$ in cohomology.

It is not clear whether $B(M)^* \implies B(M)$ even under the assumption that $\mathsf{h}(M)$ is finite dimensional, but the converse holds:

Lemma 3.2. *If* $\mathsf{h}(M)$ *is finite dimensional, then* $B(M)$ *implies* $B(M)^*$.

Proof: I shall follow the proof of [6, Lemma 5.6.10] in detail. First I shall construct Λ_r. Using the correspondence Λ, let $e = L^r{\circ}\Lambda^r \in \mathrm{Corr}^0(M,M)$. Since this is a cohomological projector, (3) implies that $e^2 - e$ is nilpotent, say $(e^2-e)^N = 0$. Introduce

$$\begin{aligned} E := (1-(1-e)^N)^N &= (P(e)\cdot e)^N\,, \quad (P \text{ some polynomial}) \\ &= e^N \cdot P(e)^N \\ &= L^r{\circ}\Lambda^r \cdot e^{N-1}\cdot P(e)^N. \end{aligned}$$

In cohomology this induces the same operator as e. One has

$$E = (1-(1-e)^N)^N = 1 + \sum_{j=1}^{N} (-1)^j \binom{N}{j} (1-e)^{jN}$$

and so, since $e^N \cdot P(e)^N = e^N \cdot P(e)^N$, for some polynomial Q one has

$$\begin{aligned} E{\circ}E &= E{\circ}(1 + \sum_{j=1}^{N} (-1)^j \binom{N}{j} (1-e)^{jN}) \\ &= E + P(e)^N{\circ}e^N{\circ}(1-e)^N{\circ}Q(e) \\ &= E \quad (\text{since } e^N{\circ}(1-e)^N = 0). \end{aligned}$$

This is thus a projector inducing the same operator as e in cohomology. Now set $\Lambda_r := \Lambda^r \cdot e^{N-1}\cdot P(e)^N$. By construction $E = L^r{\circ}\Lambda_r$ induces the same operator as e in cohomology, i.e. the operator $L^r{\circ}\lambda^r$.

To show the second claim, exchange the order of L^r and Λ^r. □

For $r = 1$, this yields:

Theorem 3.3. *Suppose that Lefschetz' conjecture* $B(M)$ *holds and that the Chow motive* $\mathsf{h}(M)$ *is Kimura finite dimensional. There is a correspondence* $\Lambda_1 \in \mathrm{Corr}^{-1}(M,M)$ *such that one has a direct sum decomposition of motives*

$$\mathsf{h}(M) = (M, L{\circ}\Lambda_1) \oplus (M, \Pi^{\mathrm{pr}}), \quad \Pi^{\mathrm{pr}} := \Delta_M - L{\circ}\Lambda_1.$$

Referring to (7)*, the projector* Π^{pr} *induces the projector* π^{pr} *onto* H^{pr}_M *in cohomology and* $L{\circ}\Lambda_1$ *induces the projector onto* LH_M.

4 The variable and fixed motive

4.1 Construction of the projectors

Let $i : X \hookrightarrow M$ be a d-dimensional smooth complete intersection of r hypersurfaces. Note that the graph $\Gamma_i \in X \times M$ of i induces the Lefschetz correspondence (I am ignoring multiplicative constants here)

$$L^r = i_* \circ i^* \in \mathrm{Corr}^r(M, M).$$

Set

$$p_r := L^r \circ \Lambda_r \in \mathrm{Corr}^0(M, M),$$

which by construction (cf. Lemma 3.2) is a projector. One has

Lemma 4.1. *Assume $B(M)$ and that $h(M)$ is finite dimensional. Then the correspondences*

$$\pi^{\mathrm{fix}} := i^* \circ \Lambda_r \circ p_r \circ i_* \in \mathrm{Corr}^0(X, X)$$

and

$$\pi^{\mathrm{var}} := \Delta_X - \pi^{\mathrm{fix}}$$

are commuting projectors.

Proof: It suffices to show that π^{fix} is a projector. Then

$$\begin{aligned}(\pi^{\mathrm{fix}})^2 &= i^* \circ \Lambda_r \circ p_r \circ L^r \circ \Lambda_r \circ p_r \circ i_* \\ &= i^* \circ \Lambda_r \circ p_r^3 i_* \\ &= i^* \circ \Lambda_r \circ p_r i_* \\ &= \pi^{\mathrm{fix}}. \quad \square\end{aligned}$$

4.2 Cohomological action

The inclusion $i : X \hookrightarrow M$ induces maps $i^* : H^*(M) \to H^*(X)$ of degree 0 and $i_* : H^*(X) \to H^*(M)$ of degree $2r$ with $i_* \circ i^* = L^r$ and $i^* \circ i_* = (L|X)^r$.

Lemma 4.2. *For the action on $H^d(X)$ one has $p_r \circ i_* = i_*$ and π^{fix} induces the projector $i^* \circ \lambda^r \circ i_*$.*

Proof: By definition of the fixed and variable cohomology (1), one has

$$i_* H^d(X) = i_* H^d_{\mathrm{fix}}(X) = i_* \circ i^* H^d(M) = L^r H^d(M).$$

By equality (5), $L^r \circ \lambda^r = \mathrm{id}$ on the image of L^r. One then has $p_r \circ i_* = L^r \circ \lambda^r \circ i_* = i_*$ and $\pi^{\mathrm{fix}} = i^* \circ \lambda^r \circ i_*$. $\square$

Corollary 4.3. *The cohomological projectors π^{fix} and π^{var} induce projection onto the fixed and variable cohomology.*

Proof: Let $x \in H^d(X)$. Then $\pi^{\mathrm{fix}}(x) = i^*(\lambda^r \circ i_* x) \in H^d_{\mathrm{fix}}(X)$. Since $i_*(x - i^* \lambda^r i_* x) = i_* x - L^r \lambda^r i_* x = i_* x - i_* x = 0$, one has $x - \pi^{\mathrm{fix}} x \in H^d_{\mathrm{var}}(X)$. To complete the proof I need to show that fixed and variable cohomology intersection only in 0. So assume that $\pi^{\mathrm{fix}}(u) = \pi^{\mathrm{var}}(v) = v - \pi^{\mathrm{fix}}(v)$. Then

$$\pi^{\mathrm{fix}}(u + v) = v \implies \pi^{\mathrm{fix}}(u + v) = \pi^{\mathrm{fix}}(v),$$

since π^{fix} is a projector. Hence $\pi^{\mathrm{fix}}(u) = 0$ as was required to show.[3] $\square$

[3] Observe that the result also follows from the direct sum decomposition (2).

4.3 The motives

Now define the *fixed* and *variable submotive* of X by means of

$$\mathsf{h}(X)^{\rm fix} = (X, \pi^{\rm fix}), \quad \mathsf{h}(X)^{\rm var} = (X, \pi^{\rm var}).$$

Then, Lemma 4.1 and Corollary 4.3 can be summarized as follows.

Theorem 4.4. *Let M be a smooth projective manifold for which $B(M)$ holds and suppose that $\mathsf{h}(M)$ is finite dimensional. Let $X \subset M$ be a smooth d-dimensional complete intersection. Then $\pi^{\rm fix}$ is a projector inducing in cohomology projection onto the fixed part of the cohomology and $\pi^{\rm var}$ is a projector commuting with $\pi^{\rm fix}$ and inducing projection on the variable cohomology. There is a direct sum splitting of motives*

$$\mathsf{h}(X) = \mathsf{h}(X)^{\rm fix} \oplus \mathsf{h}(X)^{\rm var}.$$

Remark 4.5. Let X be a surface. Then by [6, §6.3] there is a self dual Chow-Lefschetz decomposition of the diagonal

$$\Delta = \pi_0 + \pi_1 + \underbrace{\pi_2^{\rm alg} + \pi_2^{\rm tr}}_{\pi_2} + \pi_3 + \pi_4.$$

This decomposition is compatible with the splitting into variable and fixed motives. This is because one has a splitting

$$\pi_2^{\rm alg} = \pi_2^{\rm alg,fix} + \pi_2^{\rm alg,var}, \quad \pi_2^{\rm alg,fix} := \pi_2^{\rm alg}\circ\pi_2^{\rm fix},\ \pi_2^{\rm alg,var} = \pi_2^{\rm alg}\circ\pi_2^{\rm var}. \tag{8}$$

Indeed, the construction of the projector $\pi_2^{\rm alg}$ as given in loc. cit. proceeds by first taking an orthogonal basis for the algebraic classes of X, say $d_1, \ldots, d_\rho$ with $\pi_1(d_j) = 0$ for $j = 1, \ldots, \rho$, and then one sets

$$\pi_2^{\rm alg} = \sum_{i=1}^{\rho} \frac{1}{d_i \cdot d_i}\, d_i \times d_i \in {\rm Corr}^0(X, X).$$

Since the motive $(X, \pi_2^{\rm alg})$ is a Lefschetz motive one may identify it with the corresponding cohomological motive. In cohomology one has

$$\pi_2^{\rm alg}\circ\pi_2^{\rm fix} = \pi_2^{\rm fix}\circ\pi_2^{\rm alg} \implies \pi_2^{\rm alg,fix} \text{ is a projector.}$$

Secondly, the splitting in variable and fixed parts is an orthogonal splitting which implies the splitting (8). One then puts $\pi_2^{\rm tr} = \pi_2 - \pi_2^{\rm alg}$ and hence, defining $\pi_2^{\rm tr,var} := \pi^{\rm var} - \pi_2^{\rm alg,var}$ and $\pi_2^{\rm tr,fix} := \pi_2^{\rm tr} - \pi_2^{\rm tr,var}$, one gets a refinement of the above Chow-Lefschetz decomposition

$$\Delta = \pi_0 + \pi_1 + \underbrace{\pi_2^{\rm alg,fix} + \pi_2^{\rm tr,fix}}_{\pi_2^{\rm fix}} + \underbrace{\pi_2^{\rm alg,var} + \pi_2^{\rm tr,var}}_{\pi_2^{\rm var}} + \pi_3 + \pi_4.$$

Theorem 4.4 asserting the splitting into variable and fixed motives has the following consequence which states that the characterization for fixed and variable cohomology has a motivic analog:

Lemma 4.6. *Same assumptions as before.*
1. *For $k \le d$ we have*

$$\mathsf{CH}_k(\mathsf{h}(X)^{\rm var}) = {\rm Ker}(p_r\circ i_* : \mathsf{CH}_k(X) \to \mathsf{CH}_k(M)).$$

2. *We have an injective morphism*

$$\mathsf{CH}_k(\mathsf{h}(X)^{\rm fix}) \hookrightarrow i^*(\mathsf{CH}_{k+r}(M)) \tag{9}$$

Proof: 1. By definition the left hand side consists of cycles of the form $y = z - i^*\Lambda_r p_r i_* z$ for some $z \in \mathsf{CH}_k(X)$. Clearly, if $p_r\circ i_* u = 0$, u is of this form and conversely, if y is of this form, we have $i_* y = i_* z - i_* i^* \Lambda_r p_r\circ i_* z = i_* z - L^r\circ\Lambda_r p_r\circ i_* z = i_* z - p_r\circ i_* z$ since p_r is a projector and applying p_r this vanishes.
2. By definition the projector $\pi^{\rm fix}$ puts the "fixed" cycles all in the image of i^*. □

Remark. One expects equality in (9) as is the case for cohomology. I have not been able to show this.

4.4 A variant with group actions

In the preceding set-up, suppose that a finite group G acts on M and that X is invariant under the action of G. In particular, g commutes with i and with L_X and L_M. Let Γ_g be the graph of the action of g on X. For $\chi = \sum_g \chi(g) \cdot g \in \mathbf{Q}[G]$ we set

$$\pi_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g) \Gamma_g.$$

This is a projector and defines the motive (X, π_χ).

Replacing Λ_r by $\widehat{\Lambda_r} := \frac{1}{|G|} \sum_{g \in G} \Gamma_g \circ \Lambda_r \circ \Gamma_{g^{-1}}$ the operator $L^r \circ \widehat{\Lambda_r}$ remains a projector since L commutes with the G-action and it still lifts λ^r. With this new lift, $\pi_\chi \circ \pi^{\mathrm{fix}} = \pi^{\mathrm{fix}} \circ \pi_\chi$ and hence, π^{fix} also commutes with π^{var} and both $\pi^{\mathrm{fix}} \circ \pi_\chi$ and $\pi^{\mathrm{fix}} \circ \pi_\chi$ are projectors. For any $\mathbf{Q}$-vector space on which G acts, setting

$$V^\chi := \{x \in V \mid g(x) = \chi(g)x \text{ for al } g \in G\},$$

one has

$$H^k(X, \pi_\chi) = H^k(X)^\chi.$$

Since $X \subset M$ is left invariant by the G-action, the variable and fixed motives are G-stable and one sets

$$\mathsf{h}(X, \pi_\chi)^{\mathrm{fix}} := (X, \pi^{\mathrm{fix}} \circ \pi_\chi), \quad \mathsf{h}(X, \pi_\chi)^{\mathrm{var}} := (X, \pi^{\mathrm{var}} \circ \pi_\chi).$$

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