Rigidity, Past and Present

Chris PETERS

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Introduction

This note is an elaborated version of the talk with the same title I gave in the January 2006 workshop at the HRI. The talk was aimed at (complex) algebraic geometers with little background in Hodge theory. In the talk, I have tried to give as much details from Hodge theory so as to allow the audience to follow the main arguments. Much of these details are presented here as well. I try to avoid to become too technical by substituting references to the existing literature.

The point of departure is a beautiful result of Arakelov [Arak] which is explained below in § 1.2. A brief sketch of the original proof is given in § 1.3. Since its appearance, this theorem motivated many people to look for analogs in the number theoretic setting. Analogs which eventually led Faltings to his proof [Fa83b] of the Mordell conjecture conjecture. Indeed, a version of Arakelov's theorem for Abelian varieties appears by the same author [Fa83a] in the same journal: it is the article which is just preceding the famous article in which the Mordell conjecture has been proven.

In [Pe90] I proved a generalization of Faltings' result for variations of Hodge structures. At the request of the organizers of the conference I explained this proof and added a few recent developments. Among the latter I mention the classification of non-rigid variations of K3-surfaces [S-Zu] due to M.-H. Saito and S. Zucker, as well as for Abelian varieties [Sa], due to M.-H. Saito. The reader will find a few new proofs of some recent results as well. For example, one of the main results of [L-T-Y-Z]: any family of Calabi-Yau's with maximally unipotent local monodromy at some point of the boundary must be rigid (Cor 3.5). The same approach gives an easy proof of a result from [VZ]: a variation with maximal Higgs field is rigid (Prop. 3.7).

Contrary to when I wrote [Pe90], nowadays some good introductory works to Griffiths' theory have appeared such as [V] and [C-P-M]. The latter treats the Lie-theoretic background which I need. For that reason I often refer to it for details omitted in the presentation below.

At the conference H. Shiga pointed out to me that he together with Y. Imayoshi in [I-S] proved Arakelov's rigidity result by analytic means.

1 Variations on Arakelov's Theorem

1.1 *Families*

The statement of Arakelov's theorem uses the concept of a family of complex algebraic varieties:

Definition 1.1. 1) A *family of algebraic varieties* is a proper and flat morphism $f : X \to Y$ between complex manifolds whose fibres are (irreducible) algebraic varieties. If $\dim Y = n$ one speaks of an *n-dimensional family*. Note that neither X nor Y are assumed to be algebraic. Indeed, Y could be a disc or a polydisc.

2) The locus over which f is *not* of maximal rank is called *discriminant locus* and is denoted $\Delta(f)$; on the complement the fibres are smooth. If the discriminant locus is empty one says that f is a *smooth family*.

3) A morphism from the family $f' : X' \to Y'$ to the family $f : X \to Y$ is given by a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{\tilde{f}} & X \\ \downarrow f' & & f \downarrow \\ Y' & \xrightarrow{f} & X \end{array}$$

If $f = \mathrm{id}$ and $\tilde{f}$ is an *isomorphism*, one says that f and f' are isomorphic; if $f = \mathrm{id}$ but $\tilde{f}$ is only a bimeromorphic map, we say that f and f' are *bimeromorphic*.

Obvious examples are given by products: $X_0 \times Y \to Y$, where X_0 is some smooth algebraic variety. Any family isomorphic to such a family is called *trivial*.

Whenever we are given a holomorphic map between manifolds $g : Y' \to Y$, there is a standard procedure to obtain a new family over Y': one takes the fibre product $X' := X \times_Y Y' = \{(x, y') \mid f(x) = g(y')\}$ which comes with a natural morphism $X' \to Y'$. This family is called the *pull back of f by g*. A family is called *isotrivial* if it becomes bimeromorphically trivial after pulling back through some *finite* covering map $Y' \to Y$.

1.2 *Arakelov's theorem*

Consider a class $\mathcal{C}$ of smooth projective varieties and fix a smooth projective variety $\bar{S}$, the *base manifold* and a closed subvariety $\Sigma \subset \bar{S}$, the *degeneracy locus*. The *Shafarevich problem* for the triple $(\mathcal{C}, \bar{S}, \Sigma)$ consists in determining the isomorphism classes of families over a given base $\bar{S}$ of which the fibres over $\bar{S} - \Sigma$ belong to $\mathcal{C}$. In other words, one asks how many families

f there are over $\bar{S}$ with smooth members in $\mathcal{C}$ and with $\Delta(f) \subset \Sigma$, possibly with strict inclusion.

Arakelov's theorem answers this question for 1-dimensional families of curves of a fixed genus $g \geq 2$.

Theorem ([Arak]). *Let $\bar{S}$ be a complete curve and a finite set Σ of points on $\bar{S}$. There are at most finitely many non-isotrivial families of curves of given genus $g \geq 2$ over $\bar{S}$ that are smooth over $\bar{S} - \Sigma$.*

In order to sketch Arakelov's proof let us return to the Shafarevich problem for a class $\mathcal{C}$ of varieties which has a good moduli space M. For simplicity of exposition this shall mean that M is some quasi-projective variety, possibly singular, such that the points of M are in one to one correspondence with isomorphism classes of varieties in $\mathcal{C}$. In addition, given a family $f : X \to S$, there is an induced algebraic *moduli map* $\mu_f : S \to M$ which sends $s \in S$ to the isomorphism class of the fibre $X_s := f^{-1}s$ in M. Usually, some more functorial properties are needed which are formalised in the concept of a *coarse moduli space*. For details, see [Nst]. In many cases these exist.

Examples 1.2. 1. Fix an integer $g \geq 0$. Smooth projective curves of genus g (=compact Riemann surfaces of genus g) have a quasi-projective moduli space M_g, as is well known.

2. Fix an integer $g \geq 1$. Abelian varieties (complex tori embeddable in projective space) with a fixed (principal) polarization have a quasi-projective moduli space A_g as is equally well known.

3. A K3-surface is a simply connected compact complex surface with trivial canonical bundle. More generally, a Calabi-Yau manifold is a simply connected compact complex manifold with trivial canonical bundle. Projective Calabi-Yau's with fixed cohomology ring, fixed Chern classes and a fixed polarization admit a coarse moduli space. This is less well known. See for example [L-T-Y-Z, Thm. 8].

It would be even more ideal if there exists a universal family $\mathcal{C}_M \to M$ of varieties in $\mathcal{C}$ so that f is isomorphic to its pull back under the moduli map. In this situation one speaks of a *fine moduli space*. This rarely occurs; for instance, μ_f then would be the constant map if and only if the family f is trivial, but it is easy to construct 1-dimensional non-trivial families of curves all of which are isomorphic (see e.g. [B-H-P-V, Ch. V.§5]). This is related to the existence of automorphisms of the fibres which in fact make the family a fibre-bundle with non-trivial monodromy. The fact that a fine moduli space of curves does not exist also is related to the fact that isotrivial families enter the statement of Arakelov's theorem: it is another way of saying that the moduli map is constant. In terms of moduli maps, the theorem states that there are only finitely many non-constant maps from a quasi-projective curve to M_g.

Generalizing to any algebraic variety M, one is led to consider

$$\mathrm{Mor}(S, M) = \{\mu : S \to M \mid \mu \text{ algebraic.}\}$$

This set has a natural structure of a topological space with possibly infinitely many components, and one expects that each of these has an algebraic structure. In the curve case it is not hard to see that $\mathrm{Mor}(S, M_g)$ indeed has such a structure.

The first step in the proof of Arakelov's theorem is to show that there are in fact only *finitely* many components. This is a *boundedness* statement. The second step consists of establishing *rigidity*, i.e. to show that the individual components corresponding to non-isotrivial families are points. In this survey I'll concentrate on this second aspect.

1.3 *Deforming Maps and Rigidity*

Details of what follows can be found in [Pe90, § 2]. Let me start with the basic

Definition 1.3. Let $\mu : S \to M$ be a morphism between complex varieties. A *deformation* of μ (keeping base and target fixed) parametrized by a germ of a complex space (T, t) is a holomorphic map $m : S \times T \to M$ such that $m|S \times \{t\} = \mu$.

Taking for T the point with ring of functions the dual numbers $\mathbb{C}[\epsilon]/\epsilon^2$ we get the *infinitesimal deformation* which are classified by the vector space $H^0(S, \mu^* T_M)$. Intuitively, infinitesimally small deformations of μ are given

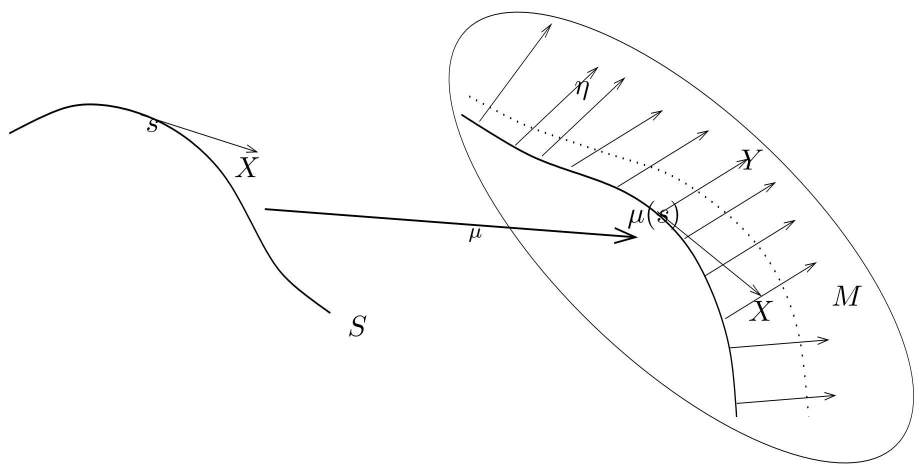

Figure 1: Infinitesimal deformations

by a vector field in M only defined at points of the image $\mu(M)$ pointing in a potential deformation direction. The set of deformations of μ can be shown to be a complex analytic space whose Zariski tangent space at μ is the space of infinitesimal deformations:

$$T_\mu \mathrm{Def}(\mu) = H^0(S, \mu^* T_M). \tag{1}$$

However, in the situations we are interested in, S is quasi-projective (we speak of an *algebraic family*) and in this case not all deformations are of interest: only the analytic subspace

$$\mathrm{Def}(\mu)^{\mathrm{alg}} = \mathrm{Def}(\mu) \cap \mathrm{Mor}(S, M) \subset \mathrm{Def}(\mu) \tag{2}$$

of *algebraic* deformations count. To detail this further, in this situation S can be compactified by some complex projective manifold $\bar{S}$ such that $\Sigma = \bar{S} - S$ is a divisor with normal crossings and the algebraic infinitesimal deformations of f correspond to global sections of some suitable extension of $\mu^* T_M$ with certain controlled growth conditions "near infinity", i.e. near the points of Σ. Usually the bundle $\mu^* T_M$ will extend to a bundle on all of $\bar{M}$ and the sections of $\mu^* T_M$ having the desired growth conditions correspond precisely to sections of this extension.

Let me come back to the case of the situation in Arakelov's theorem where the moduli map $\mu = \mu_f : S \to M_g$ comes from the restriction to S of a family $\bar{f} : \bar{X} \to \bar{S}$ of curves of genus g over the compact curve $\bar{S}$. There is a canonical identification $\left[\mu^* T_{M_g}\right]_t = H^1(X_t, T_{X_t})$ where, I recall $X_t = f^{-1}t$. This globalizes to $\mu^* T_{M_g} = R^1 f_* T_{X/S}$ and one aims to calculate its global algebraic sections over S. In this case $T_{X/S}$ is a line bundle on X which extends naturally to a line bundle

$$L := T_{\bar{X}/\bar{S}}$$

and the looked for space of infinitesimal deformations is $H^0(\bar{S}, R^1 \bar{f}_* L)$. The Leray spectral sequence for $\bar{f}$ also involves $H^1(\bar{S}, \bar{f}_* L)$, but since $g \geq 2$ the fiber of $f_* L$ at $t \in S$ which is $H^0(X_t, T_{X_t})$ vanishes. The upshot is that

$$T_\mu \mathrm{Def}(\mu)^{\mathrm{alg}} = H^0(\bar{S}, R^1 \bar{f}_* L) = H^1(\bar{X}, L).$$

Now L^{-1} is the relative canonical bundle on $\bar{X}$ (with respect to $\bar{f}$) and to show rigidity, by Kodaira vanishing, it suffices to show that L^{-1} is ample. Arakelov does this by a clever geometric argument involving the locus of Weierstraß points on the fibers.

1.4 *Towards Hodge Theory*

Starting with the family of curves $f : X \to S$, one may look at the associated family of Jacobians $J_t = \mathrm{Jac}(X_t)$. Recall (example 1.2.2) that g-dimensional

Abelian varieties have a moduli space A_g. This leads to a commutative diagram in which the moduli map μ_f and the *period map* p_f figure:

$$\begin{array}{ccc} S & \xrightarrow{\mu_f} & M_g \\ & {\scriptstyle p_f}\searrow \quad \swarrow{\scriptstyle \iota} & \\ & A_g. & \end{array}$$

Torelli's theorem implies that ι is an immersion and so rigidity for the moduli map is implied by rigidity for the period map.

To explain the transition to Hodge theory, recall that there is a Hodge decomposition for 1-cohomology $H^1(X_t) \otimes_{\mathbb{Z}} \mathbb{C} = H^{1,0}(X_t) \oplus H^{0,1}(X_t)$ and the position of the g-dimensional subspace $H^{1,0}(X_t)$ in $H^1(X_t) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}^{2g}$ is described by a point $\widetilde{p_f} t)$ in the Siegel upper half space $\mathfrak{h}_g$. The assignment $t \mapsto \widetilde{p_f}(t)$ is ambiguous because of the monodromy action of the fundamental group $\pi_1(S)$ through the properly discontinuous action of $\mathrm{Sp}(2g)$ on $\mathfrak{h}_g$ and the quotient $A_g = \mathfrak{h}_g/\mathrm{Sp}(2g)$ is the moduli space considered in 1.2.2 and $p_f(t)$ is the equivalence class of $\widetilde{p_f}(t)$.

There is a description of the tangent space to $\mathrm{Def}(p_f)$ entirely in terms of Hodge theory as follows. The 1-dimensional cohomology of a genus g curve is modelled on a symplectic lattice $H := (\mathbb{Z}^{2g}, Q)$ carrying a weight one Hodge structure. More generally, recall:

Definition 1.4. Let H be a free $\mathbb{Z}$-module of finite rank. A *weight k Hodge structure* on H is a direct sum decomposition on its complexification

$$H_{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

such that $H^{q,p} = \overline{H^{p,q}}$ where complex conjugation is with respect to the natural underlying real structure.

In the curve case there is an extra ingredient coming from the cup-product pairing $Q = \langle -,- \rangle$. It is an example of a polarization:

Definition 1.5. A non-degenerate integral bilinear form Q on H, symmetric for k even and skew-symmetric for k odd *polarizes* the Hodge structure on H if the two Riemann bilinear relations hold:

$$Q(H^{p,q}, H^{r,s}) = 0 \qquad \text{if } (r,s) \neq (q,p) \tag{3}$$

$$Q(Cu, \bar{u}) > 0 \qquad \text{if } u \neq 0 \text{ and where } C|H^{p,q} = \mathrm{i}^{p-q}. \tag{4}$$

The pair (H, Q) is called a weight k *polarized Hodge structure.* The example of H^1 for a curve is by no means special:

Example 1.6. Let X be a projective manifold. Cup product with the hyperplane class in $H^2(X)$ defines the Lefschetz operator L on integral cohomology and it is well known that there is a splitting (over $\mathbb{Q}$) of the integral cohomology $H^k(X)$ into basic building blocks of the form $L^m H^{k-2m}_{\text{prim}}(X)$, where the subscript denotes *primitive cohomology.* Primitive cohomology is defined over $\mathbb{Z}$ and comes with a polarization Q. The Hodge structure on primitive cohomology is polarized by Q.

Given a polarized Hodge structure (H, Q), its endomorphism group $\text{End}(H, Q)$ consists of integral endomorphisms which are skew-symmetric with respect to Q. Its complexification has a canonical decomposition:

$$\text{End}(H_{\mathbb{C}}, Q_{\mathbb{C}}) = \bigoplus_j \text{End}^{-j,j} H_{\mathbb{C}}$$

where $\text{End}^{-j,j} H_{\mathbb{C}}$ is the complex vector space of Q-skew endomorphisms X of H_C for which $XH^{p,q} = H^{p+j,q-j}$, $j \in \mathbb{Z}$. It is not hard to see that this is a weight 0 Hodge structure.

After this digression, coming back to the curve case, we note that the vector space $\text{End}(H^1_{\mathbb{C}}(X_t), Q)$ decomposes in three pieces: a real piece $\text{End}^{0,0} H^1_{\mathbb{C}}(X_t)$, which corresponds essentially to the endomorphisms of the Jacobian of X_t, $\text{End}^{-1,1} H^1_{\mathbb{C}}(X_t)$ and $\text{End}^{1,-1} H^1_{\mathbb{C}}(X_t)$ which is the complex conjugate of $\text{End}^{1,-1} H^1_{\mathbb{C}}(X_t)$.

The next step is to let t vary over the base S. The $\mathbb{Z}$-modules $H^1_{\mathbb{C}}(X_t)$ give a local system $\underline{H}_S$ on S and the subspaces $H^{1,0}(X_t)$ form a holomorphic subbundle of the associated vector bundle $\mathcal{H} = \underline{H}_S \otimes_{\mathbb{Z}} \mathcal{O}_S$. The holomorphicity is equivalent to the period map $S \to A_g$ being holomorphic.The bundle $\mathcal{E}nd(\mathcal{H}, Q)$ is the bundle of Q-skew endomorphisms of $\mathcal{H}$ and it splits further into types; it is not hard to show that

$$p_f^* T_{A_g} = \mathcal{E}nd^{-1,1}(\mathcal{H}, Q).$$

As I noted before, one really only is interested in those deformations that give families of curves over $\bar{S}$ and so one needs some growth conditions as I now explain briefly.

For simplicity let me suppose that $\dim S = 1$ so that $S = \bar{S} - \{p_1, \ldots, p_N\}$. Each of the punctures p_j is called a "point at infinity". So locally at such a point the smooth part of the family lives over a punctured disc Δ^*. Now, quite generally, if we have a smooth family of varieties $\{X_t\}$ over the punctured disc, going once around the puncture in the positive direction induces on $H^k(X_t)$, $t \in \Delta^*$ the *local monodromy operator* T. The crucial result is the

Theorem 1.7 (Monodromy Theorem)**.** *The operator T acts quasi-unipotently on $H^k(X_t)$: for some integers ℓ, M, $(T^\ell - I)^M = 0$. Moreover $M \leq k + 1$.*

Choose ℓ such that T^ℓ is unipotent. The smallest number M for which $(T^\ell - I)^M = 0$ is called the *order of unipotency* of T, which by the previous theorem at most equals $k+1$. The quasi-unipotency of T makes it possible to extend the vector bundle $\mathcal{H}^k$ whose fiber at $t \in \Delta^*$ is $H^k(X_t)$ to a fiber bundle on all of Δ, the so-called *quasi-canonical extension* $\mathcal{H}^k_{\mathrm{can}}$ of $\mathcal{H}$. For details I refer to [Del70, p. 94].
In the case at hand there is a vector bundle $\mathcal{H}_{\mathrm{can}}$ on $\bar{S}$ and one can show that the Hodge spaces $H^{1,0}(X_t)$, which make up a holomorphic subbundle, also extend to a subbundle $\mathcal{H}^{1,0}_{\mathrm{can}}$ of $\mathcal{H}_{\mathrm{can}}$. Likewise $\mathcal{E}nd(\mathcal{H}, Q)$ and $\mathcal{E}nd(\mathcal{H}, Q)^{-1,1}$ extend to $\mathcal{E}nd(\mathcal{H}, Q)_{\mathrm{can}}$ and $\mathcal{E}nd(\mathcal{H}, Q)^{-1,1}_{\mathrm{can}}$ respectively. The crucial observation now is:

Lemma 1.8. *Any holomorphic section of the* vector bundle $\mathcal{E}nd(\mathcal{H}, Q)_{\mathrm{can}}$ *over $\bar{S}$ restricts over S to a global section of the* local system $\underline{\mathrm{End}}(\underline{H}_S, Q)$ *of S. Such a global section is precisely a Q-skew endomorphism of any of its fibers $H^1(X_t)$ commuting with the monodromy.*

This is by no means trivial and depends on the existence of a special metric on the bundle $\mathcal{E}nd(\mathcal{H}, Q)$ as will be explained below (Proposition 2.9) which it inherits from the period domain. The metric in this case is any invariant metric on the Siegel upper half space (see Cor. 2.10)

From the preceding discussion a criterion for rigidity emerges:

Criterion 1.9. *One has*

$$\begin{aligned} T_{p_f}\mathrm{Def}^{\mathrm{alg}}(p_f) &= T_{p_f}\mathrm{Mor}(S, A_g) = H^0(\bar{S}, \mathcal{E}nd^{-1,1}(\mathcal{H}, Q)_{\mathrm{can}}). \\ &= Q\text{-skew endomorphisms of } H^1_{\mathbb{C}}(X_t) \text{ of type} \\ &\quad\; (-1,1) \text{ commuting with the monodromy group.} \end{aligned} \tag{5}$$

In particular p_f is rigid if the bundle $\mathcal{E}nd(\mathcal{H}, Q)_{\mathrm{can}}$ only admits global endomorphisms over $\bar{S}$ of type $(0,0)$. It follows that p_f is rigid if and only if the global (skew) endomorphisms of the bundle of 1-cohomology of X_t consists of the endomorphisms of any smooth member of the family of Jacobians J_t which commute with monodromy.

1.5 *Rigidity Results for Abelian Varieties*

The criterion 1.9 is essentially due to Faltings [Fa83a]. Indeed, there is nothing peculiar to Jacobians here and the criterion applies to any family of Abelian varieties. In [Fa83a] an example is given of a family of smooth 8-dimensional Abelian varieties which is not isotrivial but still non-rigid. In fact, the example has no non-isotrivial factors.

M.-H. Saito in [Sa] has taken up a systematic study of rigidity for Abelian varieties *parametrized by a curve* and he shows that families of g-dimensional Abelian varieties without non-isotrivial factors are rigid if $g < 7$ so that Faltings' example is in the lowest possible dimension. In [Sa] Faltings' example is shown to belong to a wide class of Kuga fiber spaces. In the following description non-compact Hermitian symmetric spaces are needed. One type already came up: type III spaces are just the bounded incarnations of the Siegel upper half spaces $\mathfrak{h}_g$ and have dimension $\frac{1}{2}g(g+1)$. Type II symmetric spaces depend on a parameter $h \geq 2$ and will be denoted II_h. They have dimension $\frac{1}{2}h(h-1)$ (see [Sa, §6]). Type I symmetric spaces depend on two parameters p, q and will be denoted I_{pq}; they have dimension pq.

Examples 1.10. 1. Fix natural numbers t, t', n and m, and consider the arithmetic quotients

$$\mathfrak{h}_n^{t-t'}/\Gamma, \quad II_m^{t'}/\Gamma.$$

Over the product there is a Kuga fiber space which is a particular family of Abelian varieties of relative dimension $g = 2tmn$. Fixing a point h in the first factor gives a Kuga fiber space over $III_m^{t'}/\Gamma$ which has no isotrivial factor if $m \geq 2$ and $t' \geq 1$. Moreover, all such spaces obtained by varying h are isomorphic. So, if $n \geq 1$ and $t - t' \geq 1$ they are not rigid. Similarly, fixing a point in the second factor gives a Kuga fiber space over $\mathfrak{h}_n^{t-t'}/\Gamma$ which has no isotrivial factor if $n \geq 1$ and $t - t' \geq 1$, and it is non-rigid if $m \geq 2$. In conclusion, this gives examples for all even $g \geq 8$.

2. A similar construction is possible for type I spaces. Again integers t, t', n, m are given with $t' \leq t$. In addition one has pairs of integers (p_j, q_j) $j = 1, \ldots, t$ for which $p_j + q_j = n$ for $j \leq t'$ and $p_j + q_j = m$ for $j > t'$. As base one takes the product of the two spaces

$$(I_{p_1 q_1} \times \cdots \times I_{p'_t q'_t})/\Gamma, \quad (I_{p'_{t'+1} q'_{t'+1}} \times \cdots \times I_{p_t q_t})/\Gamma.$$

There is a Kuga fiber space over the product of relative dimension $g = tmn$ which over each the two factors give families without isotrivial factors as soon as the base dimension is > 0 and deformable as soon as the other factor has positive dimension. This gives examples of all dimensions $g \geq 8$ which factor as $g = tnm$, $t, n, m \geq 2$.

As to rigidity results, Saito shows that his general classification implies:

Theorem. • *A 1-parameter family of Abelian varieties of prime dimension whose generic fiber is simple and has no isotrivial factors must be rigid.*

• *Assume S is a non-compact curve, the local monodromy around at least one point at infinity has infinite order and the generic fiber is a simple Abelian variety. Then the family is rigid.*

Referring to the Monodromy theorem 1.7, in the weight one case the order of unipotency is at most 2 and so either T is finite or has maximal order of unipotency 2, which puts the previous result into perspective.

Remark 1.11. The results in [Sa] have been explained geometrically by Ben Moonen [Mo] in terms of Shimura theory.

1.6 *Rigidity Results for K3-surfaces*

It is easy to construct examples starting from products of modular families of elliptic curves, say $\{E_\tau \times E_\sigma \mid (\sigma, \tau) \in C \times D\}$, C and D modular curves. Fixing $\tau = \tau_0$ gives a family of abelian surfaces over C with deformation parameter $\tau \in D$. This does not contradict the previous non-rigidity results for Abelian varieties, since the family has a trivial factor. However, the associated family of Kummer surfaces is a genuine 1-parameter family of K3's with a deformation parameter. The results of [S-Zu] generalize this construction. The upshot is that the base of any non-rigid families of K3's must be of the form $\mathfrak{h}/\Gamma$, where Γ is a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ acting freely on $\mathfrak{h}$ and there is one deformation parameter as in the preceding example. This is deduced from the very special structure of the $\mathbb{Q}$-algebra

$$\mathsf{End} = \left\{ \begin{array}{l} \text{endomorphisms of } H^2(X_t; \mathbb{Q}) \\ \text{commuting with the monodromy group.} \end{array} \right\} \qquad (6)$$

Note that one takes *all* endomorphisms instead of just the Q-skew endomorphisms.

To explain the main result concerning this algebra, I briefly have to say something about the structure of the cohomology groups $H^2(X_t, \mathbb{Q})$ of the fibres. The cup product form on this $\mathbb{Q}$-vector space induces an orthogonal direct sum decomposition

$$H^2(X_t, \mathbb{Q}) = S_t \oplus T_t, \quad S_t = \mathsf{NS}(X_t) \otimes \mathbb{Q},\, T_t = S_t^\perp,$$

where $\mathsf{NS}(X_t)$ is the Néron-Severi group of the surface X_t. It can be shown that there is a local system $\underline{S}$ over S, the *algebraic part* of the local system

$$H_S := \{H^2(X_t, \mathbb{Q})\}_{t \in S}$$

such that $\underline{S}_t = \mathsf{NS}(X_t)$ for t in the complement of a countable set of points in S. The rank of the system $\underline{S}$ is therefore called the *generic Picard number* ρ of the family of K3's. The orthogonal complement of $\underline{S}$ is called the *transcendental part* $\underline{T}$ of the local system H_S.

Theorem 1.12. *The algebra* (6) *is a quaternion-algebra over a totally real number field* Z. *If* $[Z : \mathbb{Q}] = g$, *the local system* $\underline{T}$ *over* S *is a rank-1* End*-module and hence in particular has dimension* $4g$.

Since $\dim H^2(X_t, \mathbb{Q}) = 22$, one has in particular

Corollary 1.13. *If* $22 - \rho$ *is not divisible by* 4, *the family is rigid.*

The monodromy group Γ of the family acts always as a finite group on the algebraic part (the algebraic part has signature $(1, \rho - 1)$, but a polarization is always fixed). However, it might act non-finitely on the transcendental part $\underline{T}$. The induced group is a subgroup of the algebra End and one has [S-Zu, The. 5.6.2]:

Theorem 1.14. *Suppose that the family of K3's is non-isotrivial and non-rigid. Then the monodromy group* Γ *either acts as finite group on* $\underline{T}$ *or as an arithmetic subgroup of* $\mathrm{SL}(2, \mathbb{R})$ *commensurable with* $\mathrm{SL}(2, \mathbb{Z})$*. In particular, if there is an infinite order local monodromy operator, one has* $\mathsf{End} = M_2(\mathbb{Q})$ *and* $\rho = 18$*.*

Recalling the Monodromy Theorem 1.7 the maximal order of unipotency on 2-cohomology is 3. So a local monodromy operator T can be finite, or has order of unipotency 2 or 3. In the situation of Theorem 1.14 the order of unipotency must be at most 2 and so one has:

Corollary 1.15. *If there is a local monodromy operator at infinity* T *having maximal order of unipotency* 3*, the family must be rigid.*

2 A Hodge Theoretic Approach

2.1 *Period Maps*

Fix a free $\mathbb{Z}$-module H of finite rank, an integer k, the *weight*, and positive integers $h^{p,q}$, $p + q = k$, the *Hodge numbers* adding up to $\mathrm{rank}(H)$ and assembled in a *Hodge vector* $\vec{h} = (\ldots, h^{p,k-p}, h^{p+1,k-1-p}, \ldots)$.

The set of all possible Hodge structures on H whose Hodge numbers make up the fixed Hodge vector $\vec{h}$ forms a nice algebraic variety, a partial flag variety. In order to make this identification it is essential to pass from the Hodge *decomposition* $H = \bigoplus H^{r,s}$ to the corresponding Hodge *filtration* $\cdots F^p \supset F^{p-1} \supset \cdots$, where

$$F^p = \bigoplus_{r \geq p} H^{r,k-r}. \tag{7}$$

The condition $H^{p,q} = \overline{H^{q,p}}$ translates into $H \otimes \mathbb{C} = F^p \oplus \overline{F^{k-p+1}}$. On can also go in the reverse direction since $H^{p,q} = F^p \cap \overline{H^q}$. This shows that instead of Hodge decompositions, one may equivalently speak of Hodge filtrations.

The Hodge structures coming from geometry are polarized in the sense of Definition 1.5. So assume that a non-degenerate $\mathbb{Z}$-valued form Q on H is given which is symmetric for k even, and skew-symmetric otherwise and which satisfies the two Riemann bilinear relations. These imply a condition on the signature of Q: it should be equal to $\sum (-1)^p h^{p,k-p}$.

The group G of isometries (with respect to Q) of the real vector space $H_{\mathbb{R}} := H \otimes_{\mathbb{Z}} \mathbb{R}$ acts on the *period domain* $D = D(H, Q, \vec{h})$ consisting of possible Hodge structures of Hodge vector $\vec{h}$ polarized by Q. The pair $(\vec{h}, Q)$ will be called the *polarization type*. The form Q induces a hermitian metric on $H_{\mathbb{C}} =: H \otimes_{\mathbb{Z}} \mathbb{C}$, the *Hodge metric* defined by

$$h(u,v) := Q(Cu, \bar{v}), \quad u, v \in H_{\mathbb{C}}. \tag{8}$$

It turns out that G acts transitively on D. At a reference point $F_0 \in D$ corresponding to a Hodge decomposition $\bigoplus H^{p,q}$ the isotropy group is

$$V = \{g \in G \mid gH^{p,q} \subset H^{p,q}\}.$$

Hence the period domain is the homogeneous space $D = G/V$ and so it is in particular a manifold, but one cannot expect this to be an algebraic variety because of non-algebraic nature of the second Riemann relation (4). However, it embeds naturally in a projective manifold, the *compact dual* $\check{D}$ of D which by definition is the set $\check{D} = \check{D}(H, Q, \vec{h})$ of Hodge decompositions on H with given Hodge vector $\vec{h}$ which only satisfy the first Riemann equation (3). The "complex" group

$$G_{\mathbb{C}} = \mathrm{Aut}(H \otimes_{\mathbb{Z}} \mathbb{C}, Q \otimes 1)$$

turns out to act transitively on $\check{D}$ with isotropy group (at a reference point $F_0 \in \check{D}$ corresponding to a Hodge decomposition $\bigoplus H^{p,q}$) the subgroup

$$B = \{g \in G_{\mathbb{C}} \mid gH^{p,q} \subset H^{p,q}\}.$$

So, again, the compact dual

$$\check{D} = G_{\mathbb{C}}/B,$$

a homogeneous variety, and in this case $\check{D}$ is even a smooth projective variety in which D sits as an open (but not Zariski open) subset.

Note that $\check{D}$ is the base of a principal bundle $B \to G_{\mathbb{C}} \to \check{D}$ and the holomorphic tangent bundle $T(\check{D})$ is the vector bundle which is associated to the through the adjoint representation of B on $\mathrm{Lie}\, G_{\mathbb{C}}/\mathrm{Lie}\, V$. The tangents coming from geometry turn out to be very special. To describe these, introduce

$$\begin{aligned} \mathfrak{g} &:= \mathrm{Lie}\, G_{\mathbb{C}} \\ \mathfrak{g}^{j,-j} &:= \{X \in \mathfrak{g} \mid XH^{p,k-p} \subset H^{p+j,k-p-j} \quad \text{for all } p \in \mathbb{Z}.\} \end{aligned}$$

Clearly, $\mathfrak{g} = \mathrm{Lie}\, G \otimes \mathbb{C} = \bigoplus \mathfrak{g}^{j,-j}$ is a Hodge structure on $\mathrm{Lie}\, G$ which depends on the point $F_0 \in \check{D}$. The subbundle $T^{\mathrm{hor}} \subset T(\check{D})$ is given at F_0 by

$$T^{\mathrm{hor}}_{F_0} := \mathrm{Lie}\, B + \mathfrak{g}^{1,1}/\mathrm{Lie}\, B.$$

This $G_{\mathbb{C}}$-equivariant subbundle of $T(\check{D})$ is called the *horizontal bundle*. We have a natural adjoint action of $\operatorname{Lie} B$ on $\mathfrak{g}^{-1,1}$ as well and hence an associated vector bundle on $\check{D}$ with fiber at F_0 given by $\mathfrak{g}^{-1,1}$ and we fix an identification of this bundle with T^{hor}:

$$T^{\text{hor}}_{F_0} \simeq \mathfrak{g}^{-1,1}. \tag{9}$$

Let S be a complex manifold and suppose that a representation of $\pi_1(S,*)$ in $G_{\mathbb{Z}}$, the group of isometries of (H,Q) is given. So over some unramified cover $\tilde{S} \to S$ the representation becomes trivial. The minimal such cover is called the *monodromy cover*, S^{mon}. The image G^{mon} of $\pi_1(S,*)$ in $G_{\mathbb{Z}}$ is called the *monodromy group*.

By definition, a *variation of Hodge structures* on S of polarization type $(\vec{h},Q)$ is a holomorphic map $p : S^{\text{mon}} \to D$ which is *horizontal* in the sense that $p_*[T_s S^{\text{mon}}] \subset T^{\text{hor}}_F D$, $F = p(s)$. Such a map is called *period map*. Equivalently, it is a holomorphic map $p : S \to D/G^{\text{mon}}$ lifting to a horizontal map $\tilde{S} \to D$ on a suitable unramified cover $\tilde{S} \to S$. There is also a direct definition:

Definition 2.1. A *variation of Hodge structure* on S of weight k is a local system $\underline{H}_S$ of free $\mathbb{Z}$-modules of finite rank on S such that each fiber over $t \in S$ of the complexification admits a Hodge structure of weight k and such that
— the associated Hodge flag $F_t^\bullet$ depends holomorphically on t (this is the holomorphicity of the period map)
— the flat connection ∇ satisfies *Griffiths' horizontality condition*:

$$\nabla_\xi F_t^q \subset F_t^{q-1}, \quad \xi \text{ a germ of a holomorphic tangent field at } t\ .$$

(this last condition is the horizontality of the period map).

The Hodge structure is *polarized* by a flat bilinear integral form Q if Q induces a polarization on the Hodge structures on each fibres of $\underline{H}_S$.

This reformulation leads to the basic examples provided by the primitive cohomology group of fixed rank of smooth projective varieties varying in a family:

Example 2.2. Let $p : X \to S$ be a smooth family of projective varieties. For simplicity, assume that every fiber X_t can be embedded in the same projective space so that it makes sense to speak of "the" cohomology class $h \in H^2(X_t)$ of a hyperplane section and hence there for each positive integer k there is an associated local system of primitive cohomology groups $H^k_{\text{prim}}(X_t)$. For later reference, let me observe that since these form a local system, there is a canonically defined flat connection ∇, the *Gauss-Manin*-connection.

By Example 1.6 each of the modules $H^k_{\text{prim}}(X_t)$ carries a polarized Hodge structure. Pulling back to the monodromy cover, this local system trivializes and a deep result of Griffiths states that Hodge filtration (7) varies holomorphically, i.e. the associated period map $p : S^{\text{mon}} \to D$ is holomorphic and, horizontal. See [C-P-M] and the references therein.

To a polarized variation of Hodge structures on a local system H_S at any point $t \in S$ one can associate its *infinitesimal variation*, consisting of the defining polarized Hodge structure on (H, Q) together with the tangents at t to the period map, i.e.

$$\sigma : T = T_t S \to \text{End}^{1,1}(H, Q) \otimes \mathbb{C}. \tag{10}$$

Pick $\eta \in T$ and write

$$\sigma_{p,q}(\eta) = \sigma | H^{p,q} \to H^{p-1,q+1}.$$

One can show (cf. [C-P-M, Theorem 5.3.4]) that in the geometric situation (example 2.2) this map just comes from cup-product with the image of η under the Kodaira-Spencer map $\rho : T \to H^1(X_t, X_t)$:

$$\sigma_{p,q} : H^q(X_t, \Omega^p_{X_t}) \xrightarrow{\cup \rho(\eta)} H^{q+1}(X_t, \Omega^{p-1}_{X_t}) \tag{11}$$

Infinitesimal variations have a rich multi-linear algebra structure. The maps $\sigma_{p,q}(\eta)$ can be composed into

$$\tau(\eta_1, \ldots, \eta_k) = \sigma_{0,k}(\eta_1) \circ \cdots \circ \sigma_{k,0}(\eta_k) : H^{k,0} \to H^{0,k}$$

By [C-P-M, § 5.5] this endomorphism is Q-symmetric and depends symmetrically on the η_j, thereby producing a homomorphism

$$\tau_t : \text{Sym}^k T \to \text{Sym}^2(H^{k,0})^\vee \subset \text{End}^{-k,k}(H \otimes \mathbb{C}) \tag{12}$$

Remark 2.3. In the geometric situation this map, the *Griffiths-Yukawa coupling* can be defined in concrete terms using the Gauss-Manin connection (cf. Examples 2.2) as follows. Recall that the Gauss-Manin connection is the flat connection ∇ on the local system of the primitive cohomology groups $H^k_{\text{prim}}(X_t; \mathbb{C})$. Assume that S is a polydisc with local coordinates $(t_1, \ldots, t_n)$. Then $\nabla_{\partial/\partial t_j}$ acts as differentiation on cohomology classes. It will be abbreviated as $\frac{\partial}{\partial t_j}$. With this convention, at the origin, for any class $\omega \in H^{k,0}_{\text{prim}}(X_t; \mathbb{C})$ there is the following formula for the Yukawa-coupling in terms of the polarisation Q:

$$\tau\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_k}\right)\omega = Q(\omega, \frac{\partial}{\partial t_1} \circ \ldots \circ \frac{\partial}{\partial t_k}\omega) \tag{13}$$

2.2 *Higgs Bundles*

An important property of variation of Hodge structures is their complete reducibility [Del71, 4.2.6]:

Theorem 2.4. *A polarized variation of Hodge structures over a quasi-projective manifold is direct sum of irreducible ones.*

It turns out that a further reduction is often needed. For this, one needs *complex systems of Hodge bundles.* To explain this notion, recall that a Hodge structure on a real vector space leads to a grading on the associated complex vector space and hence to a grading $H_S \otimes \mathcal{O}_S = \bigoplus_p \mathcal{H}_S^{p,k-p}$ on the bundle associated to a variation of Hodge structure. This grading is real in the sense that $\mathcal{H}_S^{p,k-p} = \overline{\mathcal{H}_S^{k-p,p}}$. One can identify $\mathcal{H}_S^{p,k-p}$ with the *holomorphic* bundle $\mathcal{F}^p/\mathcal{F}^{p+1}$. The horizontality just means that ∇ induces $\mathcal{O}_S$-linear bundle maps $\sigma_p : \mathcal{H}_S^{p,p-k} \to \mathcal{H}_S^{p-1,p+1-k} \otimes \Omega_S^1$. The pair

$$(\mathcal{H}_S, \sigma), \quad \text{with } \mathcal{H}_S := \bigoplus_p \mathcal{H}_S^{p,k-p}, \quad \text{and } \sigma := \bigoplus_p \sigma_p$$

is a *complex system of Hodge bundles.* It is an example of a Higgs bundle:

Definition. A *Higgs bundle* over S is a couple $(\mathcal{H}, \sigma)$ consisting of a holomorphic vector bundle $\mathcal{H}$ over S and

$$\sigma \in \mathrm{End}^{-1,1}(\mathcal{H}) \otimes \Omega_S^1 \iff \sigma : TS \to \mathrm{End}^{-1,1}(\mathcal{H}), \quad \text{the } (\textit{Higgs field})$$

having the he property that $\sigma \wedge \sigma = 0$ which is just the flatness of the connexion ∇.

Remarks 2.5.
1. Note that since there is no a priori real structure on a Higgs bundle so that the additional reality constraint on a the complexified variation of Hodge structure is an extra ingredient.
2. In general an irreducible variation gives a complex system of Hodge bundles which splits further into Higgs bundles which themselves are complex systems of Hodge bundles.

A flat bilinear *hermitian* form $Q = \sum Q_p$ on a complex system of Hodge bundles $\mathcal{H} = \bigoplus \mathcal{H}^p$ is called a *polarization* if the splitting is Q-orthogonal and if $(-1)^p Q_p$ is positive definite on $\mathcal{H}^p$. If $\mathcal{H}$ comes from a polarized variation of Hodge structures, the polarization indeed gives a polarization for $\mathcal{H}$.

The complete reducibility result stated for polarized variations of Hodge structures remains true for polarized complex systems of Hodge bundles [Del87, § 1.11]:

Theorem 2.6. *A polarized system of complex systems of Hodge bundles over a quasi-projective manifold is an orthogonal direct sum of irreducible polarized complex systems of Hodge bundles.*

This implies that the complexification of an irreducible variation of Hodge structures may further split into irreducible Higgs bundles.

Example 2.7. Suppose that H_S is a local system of $\mathbb{Q}$-vector spaces underlying a variation of Hodge structures of weight k. Let H be the typical fiber of H_S. Let $A \in \mathsf{End}(H, Q) \otimes \mathbb{C}$ be a non-zero flat endomorphism of H. Then $\operatorname{Ker} A$ defines a non-trivial subsystem of H_S. As remarked above, complex systems of Hodge bundles are completely reducible and so $H_S \otimes \mathbb{C} = \operatorname{Ker}(A) \oplus M_S$ where M_S is a complex variation of Hodge structures.

Suppose that A is pure of type $(-1, 1)$. If $H_S \otimes \mathbb{C}$ has a non-trivial part of type $(0, k)$, the preceding splitting is a non-trivial splitting since $AH_S^{0,k} \subset H_S^{-1,k+1} = 0$. This splitting in general is not a splitting of variations of Hodge structures since $H_S^{k,0}$ does not have to belong to $\operatorname{Ker}(A)$.

2.3 *Curvature*

The goal is to describe the curvature of a suitable connection on the horizontal tangent bundle $T^{\text{hor}}D$. It is a subbundle of the full tangent bundle and thus one can calculate the curvature of the hermitian metric defined by the classical *Killing form* on $\mathfrak{g}$

$$B(X, Y) = \operatorname{Tr} X \circ Y, \qquad X, Y \in \mathfrak{g}.$$

This a symmetric form, invariant under the adjoint action of $G_{\mathbb{C}}$ on $\mathfrak{g}$ and it is real on $\operatorname{Lie}(G)$. The involution θ on $\mathfrak{g}$ which is just multiplication with $(-1)^j$ on $\mathfrak{g}^{-j,j}$ can be shown to be real on $\operatorname{Lie} G$ and the expression

$$h(X, Y) := -B(\theta X, \bar{Y}), \qquad X, Y \in \mathfrak{g} \tag{14}$$

being invariant under the adjoint action of V defines a sesquilinear form on $T_{F_0}D = \operatorname{Lie} G / \operatorname{Lie} V$. It turns out (see [C-P-M, Prop. 12.2.5]) that this form is in fact a hermitian metric on $T_{F_0}D$, which by construction extends to a G-invariant hermitian metric on TD.

Recall that the *Chern connection* D_h is the unique metric connection on TD whose $(0,1)$-part is just the operator $\bar{\partial}$. The curvature of this connection is an $\operatorname{End}(TD)$-valued 2-form F_h and when evaluated on germs of holomorphic vector fields X and Y at $F_0 \in D$ yields $F_h(X, Y)$, an endomorphism of $T_{F_0}D$.

To describe it on horizontal directions, one resorts to the identification (9). To start, note that for $X \in \mathfrak{g}$, by definition $Q(Xu, v) + Q(u, Xv) = 0$ for all $u, v \in H_{\mathbb{C}}$, i.e. $\mathfrak{g}$ consists of the Q-skew endomorphisms. This does not mean that these are skew with respect to the Hodge metric h (8), since here the Weil-operator C intervenes which depends entirely on the Hodge structure. Indeed, for $X \in \mathfrak{g}$ we may define $X^* \in \mathfrak{g}$ as the h-adjoint of X, i.e. $h(Xu, v) = h(u, X^*v)$ for all $u, v \in H_C$. If $\bar{X}$ is conjugation with respect to the real structure coming from $\mathfrak{g} = \mathrm{Lie}\, G \otimes \mathbb{C}$, i.e. $\bar{X}(u) = \overline{X(\bar{u})}$, $u \in H_{\mathbb{C}}$ we have $X^* = \bar{X}$ if $X \in \mathfrak{g}^{-j,j}$ with j odd, while $X^* = -\bar{X}$ if j is even (see [C-P-M, Cor. 12.2.3], the transpose-sign should be deleted, it has no meaning). This clarifies the relation between the Hodge metric and the global complex structure on $\mathfrak{g}$.

Coming back to the curvature calculation, assume now that X and Y are germs of horizontal vector fields near F_0 so that $Y^* = \bar{Y} \in \mathfrak{g}^{1,-1}$ and $[X, Y^*] \in \mathfrak{g}^{0,0} = \mathrm{Lie}\, V$. The adjoint action of this element on $\mathfrak{g}^{-1,1}$ gives an endomorphism on $T^{\mathrm{hor}}_{F_0} D = \mathfrak{g}^{-1,1}$ and by [C-P-M, § 13.3] we have

$$F_h(X, Y) = -\frac{1}{2}\,\mathrm{ad}[X, Y^*]. \tag{15}$$

On horizontal directions X, Y the metric (14) becomes $h(X, Y) = B(X, Y^*) = \mathrm{Tr}\, X \circ Y^*$. Using (15) and the Jacobi-identity, as in [Pe90, Corr. 1.8] one finds

Lemma 2.8. *Let $X, Y \in \mathfrak{g}^{-1,1}$ then*

$$h(F_h(X, X^*)Y, Y) = 2\|[X, Y]\|_h - \|[X^*, Y]\|_h$$

so in particular, if $[X, Y] = 0$, one has $h(F_h(X, X^*)Y, Y) \le 0$.

2.4 *Deforming Period Maps*

Let $p : S^{\mathrm{mon}} \to D$ be a period map. To simplify notation, assume that $S^{\mathrm{mon}} = S$ so that the source of the period map remains algebraic. In particular, S has a suitable smooth compactification $\bar{S}$. As $p^*T^{\mathrm{hor}}D$ is contained as the $(-1, 1)$-part of an endomorphism bundle of Hodge structures, one can take its canonical extension

$$E := [f^*T^{\mathrm{hor}}D]_{\mathrm{can}},$$

a holomorphic object E on $\bar{S}$ (see the end of § 1.3). Actually E generally only is a coherent sheaf, but I neglect details like this and I assume for simplicity that E is a vector bundle. The metric (14) then is defined on $E|S$.

Notation. A metric h on a holomorphic vector bundle E can be used to contract a pair of E-valued differential forms Ω, Θ to an ordinary differential form denoted $\mathrm{Tr}(\Omega \wedge \Theta)$: on decomposables $\Omega = e \otimes \alpha$, $\Theta = f \otimes \beta$ we put $\mathrm{Tr}(\Omega \wedge \Theta)(e \otimes \alpha, f \otimes \beta) := h(e, f)\alpha \wedge \bar{\beta}$; in general extend this in a sesquilinear fashion.

As an example, if $\xi \in H^0(\bar{S}, E)$ is a global section, the curvature form with respect to h is an E-valued $(1,1)$-form $F_h\xi$ on S so that $\mathrm{Tr}(F_h\xi \wedge \xi)$ is an ordinary $(1,1)$-form on S. It can be evaluated on the pair (X, Y), where $X \in \mathfrak{g}^{1,1}$ comes from a tangent vector $X' \in T_sS$ and the tangent Y at the point $F_0 = p(s)$ comes from the deformation. One has:

$$\mathrm{Tr}(F_h\xi \wedge \xi)(X, Y) = h(F_h(X, X^*)Y, Y). \tag{16}$$

Of course period maps can a priori be deformed into maps $S \to D$ that are no longer horizontal, but these are not coming from geometry. For this reason only those deformations $P : S \times T \to D$ are considered which are horizontal themselves. This implies by [C-T, Prop. 5.2.] that any two vectors in $P_*(T_{(s,t)})$ when considered in the Lie-algebra $\mathfrak{g}^{1,1}$ (see (9)) must commute. This applies to X and Y so that from Lemma 2.8 and (16) one concludes that $\mathrm{Tr}(F_hs \wedge s)$ is negative semi-definite. We now invoke the following principle (see [C-P-M, 11.1.9]:

Proposition 2.9 (Principle of Plurisubharmonicity)**.** *Let (E, h) be a Hermitian holomorphic vector bundle over a compact complex manifold and let s be a* holomorphic *section of E. Suppose that the $(1,1)$-form $\mathrm{Tr}(F_hs \wedge s)$ is negative semi-definite. Then s is* flat *with respect to the Chern connection D_h, i.e. $D_hs = 0$.*

So from Lemma 2.8 one concludes that the section ξ is flat with respect to D_h and this proves the analog of (5) in the Hodge theoretic setting of a period map $\mu = \tilde{p} : \tilde{S} \to D$:

Corollary 2.10. *The tangent space at μ to the space of period map deformations is*

$$T_\mu \mathrm{Def}(\mu) = \left\{ \begin{array}{c} Q\text{-skew endomorphisms of } H \text{ of type } (-1,1) \\ \text{commuting with the monodromy group.} \end{array} \right\}$$

The left hand side is also equal to the algebra of global flat endomorphisms of the underlying local system (H_S, Q) which are of type $(-1,1)$. To the variation one has associated a Higgs bundle (see § 2.2). Its Higgs field at every point $t \in S$ can be shown to be just the infinitesimal variation of Hodge structure (10) at that point. So one can reformulate Cor. 2.10 in terms of Higgs bundles as follows:

Theorem 2.11. *Tangents to algebraic deformations of a given variation of Hodge structure on S correspond to flat global type $(-1,1)$ endomorphisms of the underlying local system (H_S, Q) which are at the same time Higgs fields for the associated Higgs bundle.*

3 Back To Geometry

Let me now come back to the setting of the Shafarevich problem. The class $\mathcal{C}$ of projective manifolds is now assumed to have a nice quasi-projective coarse moduli space M. The base manifold $\bar{S}$ is fixed as well as the degeneracy locus Σ which will be assumed to be a strictly normal crossing divisor. Take a family $\bar{X} \to \bar{S}$ with fibers X_t, $t \in S = \bar{S} - \Sigma$ in $\mathcal{C}$ so that, by assumption, there is a moduli map $\mu_f : S \to M$. On the other hand, one can apply Example 2.2 to $f : X \to S$, the restriction of $\bar{f}$ to $X = f^{-1}S$ so that there is a period map $p_f : S \to D/\Gamma$, where D is the period domain associated to the primitive cohomology groups of the fibers of some fixed rank, and where Γ is the monodromy group. These two maps fit into a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\mu_f} & M \\ & {\scriptstyle p_f}\searrow \quad \swarrow{\scriptstyle \iota} & \\ & D/\Gamma. & \end{array}$$

Clearly, if one wishes to draw geometric conclusions from properties of the variation of Hodge structure, one likes to know that the variation determines the family, at least locally. If this is the case, one says that the *(local) Torelli theorem holds* for $\mathcal{C}$. It means that the map ι is an immersion.

Examples 3.1. The Torelli theorem holds for curves. The local Torelli theorem holds for hypersurfaces in projective $(n+1)$-space of degree $\geq n+1$. It also holds or any projective manifold with trivial canonical bundle such as an Abelian variety, a K3-surface, or any other Calabi-Yau manifold. In fact, if n is the dimension, it suffices to look at the variation of $H^{n,0}$ inside primitive cohomology. See [C-P-M] for an explanation and references to the original works.

Under the hypothesis that ι is an immersion, the above diagram gives an inclusion $\mu_f^* T_M \subset p_f^* T_D^{\text{hor}}(D/\Gamma))$. Recall that the right hand side is just the $(-1,1)$-part of the endomorphism bundle of the variation of Hodge structure and has a quasi-canonical extension (see directly below the statement of Thm. 1.7). By definition, the quasi-canonical extension of $\mu_f^* T_M$ is just the closure of this bundle inside the quasi-canonical extension.

As shown before, the tangent space to deformations of p_f as a variation of Hodge structure, is just

$$\mathsf{End}^Q = \left\{ \begin{array}{l} \text{endomorphisms of } (H^k_{\text{prim}}(X_t;\mathbb{Q}), Q) \\ \text{commuting with the monodromy group,} \end{array} \right\}$$

i.e.,

$$H^0(\bar{S}, \left[\mu_f^* T_M\right]_{\text{can}}) = \left[\mathsf{End}^Q \otimes \mathbb{C}\right]^{-1,1}.$$

To bring the geometric deformations into the picture, note that being a coarse moduli space implies that at any smooth point $m \in M$ corresponding to a variety X_m, there is a canonical identification

$$T_m M \simeq H^1(X_m, TX_m)$$

and hence $\mu_f^* T_M$ can be identified as the bundle whose fiber at t equals $H^1(X_s, TX_t)$. Any geometric deformation η of f thus defines at each point $t \in S$ an element $\theta_t \in H^1(X_t, TX_t)$, the associated *infinitesimal Kodaira-Spencer class* of f at t. By (11) cup product with it gives an infinitesimal variation of Hodge structure at the point t:

$$\sigma(\eta_t) \in \mathrm{End}^{-1,1} H^k_{\mathrm{prim}}(X_t; \mathbb{C}).$$

Such an infinitesimal variation can be called the infinitesimal variation in the *geometric direction defined by* η. Tangents to geometric deformations correspond global flat endomorphisms inducing such infinitesimal variations at each point $t \in S$ and vice versa. Summarizing, one has:

Proposition 3.2. *If the (local) Torelli property holds for $\mathcal{C}$, tangents η to geometric deformations correspond to those flat endomorphisms $\left[\mathsf{End}^Q \otimes \mathbb{C}\right]^{-1,1}$ which at each point defines an infinitesimal variation of the Hodge structure in the geometric direction defined by η, and vice versa.*

Let me now consider the kernel of such flat endomorphisms $\sigma(\eta)$. In Example 2.7 we have seen that the complexified variation splits as $\mathrm{Ker}(\sigma) \oplus M_S$ and $\mathrm{Ker}(\sigma)$ contains the $(0,k)$-part of the variation. If $K = M_S^{k,0}$ and if $A_1, \ldots A_k \in \mathrm{End}\, H \otimes \mathbb{C}$ all have pure type $(-1,1)$, their composition $A_1 \circ \ldots \circ A_k$ maps K to the $(0,k)$-part in M_S which is zero by construction. Applying this to the maps which compose into the Yukawa coupling (13), we conclude that any non-trivial deformation forces the Yukawa-coupling to vanish on K. If for instance $h^{k,0} = 1$ the entire $(k,0)$-part of the variation is contained in K and so:

Corollary 3.3. *In the above situation, if $h^{k,0} = 1$, any non-trivial geometric deformation forces the Griffiths-Yukawa coupling to vanish.*

If I combine this result with

Proposition 3.4. *Let there be a given a weight k polarized variation of Hodge structures over a non-compact algebraic curve S with $h^{k,0} = h^{0,k} = 1$, and suppose hat there is a point at infinity where the local monodromy operator has maximal order of unipotency $(k+1)$ (see Theorem 1.7). Then the Griffiths-Yukawa coupling is non-zero.*

I deduce the following criterion:

Corollary 3.5. *Let $f : X \to S$ be a non-isotrivial family of Calabi-Yau's over a non-compact curve S and suppose hat there is a point at infinity where the local monodromy operator has maximal order of unipotency $(k+1)$ (see Theorem 1.7). Then f is rigid.*

Sketch of the proof of Prop. 3.4. Suppose (Δ, t) is a local coordinate disc centered at a puncture where the local monodromy T has order of unipotency $(k+1)$. For simplicity assume that the monodromy is unipotent of order $(k+1)$ and introduce

$$N = \log(T - I) = \sum \frac{(-1)^{\ell+1}}{\ell} T^{\ell}$$

an operator with $N^k \neq 0$, $N^{k+1} = 0$. Schmid in [Sch] has introduced a mixed Hodge structure on the typical fiber H of the local system H_S underlying the variation of Hodge structure whose weight filtration $W_0 \subset \cdots \subset W_{2n}$ is determined by N and whose Hodge structure is given by the fiber at 0 of the quasi-canonical extensions of the Hodge filtration bundles. In the case at hand $[\mathcal{H}^{k,0}]_{\rm can}$ is a line bundle trivialized by a section whose value at 0 can thus be seen as an element $\omega \in H \otimes \mathbb{C}$ The map N is a type $(-1,-1)$-morphism of $H \otimes \mathbb{C}$. In particular $N^k : W_{2k}/W_{2k-1} \xrightarrow{\sim} W_0$ maps the pure (k,k)-type Hodge structure of the left hand side to the $(0,0)$-type Hodge structure on the right hand side. Since $N^k \neq 0$ both sides are 1-dimensional and necessarily $\omega \in W_{2k}$ represents a generator. Now one needs to use that N is essentially the residue at 0 of the logarithmic connection on the canonical extension $\mathcal{H}_{\rm can}$ and after a suitable renormalization one then has $N\omega = \sigma(t[\partial/\partial t])\omega$ and in particular, the Griffiths-Yukawa coupling $\sigma^k(t[\partial/\partial t])$ does not vanish. □

Remark 3.6. In [L-T-Y-Z, Theorem 38] the proof appears to have a gap. The preceding correction has been communicated to me by Kang Zuo. I also want to remark that this gives another proof of Cor. 1.15 in the special case of K3-surfaces.

I want to finish by showing how the above ideas give a simpler proof of [VZ, Lemma 4.3]. For this S is a curve. At any point $t \in S$, the Higgs field gives an endomorphism σ of the variation which is well defined up to a multiplicative constant. A variation of Hodge structure of weight k is said to be *of Hodge-Lefschetz type a* if

- for some integer a with $0 \leq a \leq 2k$ one has $h^{k,0} = \cdots = h^{k-a+1,a+1} = 0$
- the maps $\sigma : H^{k-j,j} \to H^{k-j-1,j-k+1}$ $j = a, \ldots, k-a$ are isomorphisms at a generic point $t \in S$.

This implies that the length of the Hodge filtration is exactly $k - 2a + 1$ and all non-zero Hodge numbers are equal. Any variation of Hodge structures of weight k which is the direct sum

$$H_S \otimes \mathbb{C} = F_0 \oplus \cdots \oplus F_k, \quad F_a \text{ Hodge-Lefschetz of type } a$$

is called (strictly) *maximal*. Now if A is a flat type $(-1,1)$ endomorphism of such a system, $\ker A$ contains the $(0,k-a)$-part of F_a, hence, since $\ker A$ is a Higgs subfield, it must contain the entire Hodge-Lefschetz subsystem F_a, hence $H_S \otimes \mathbb{C} = \ker A$, i.e. $A = 0$. This argument shows the announced result loc. cit.:

Proposition 3.7. *A variation over a curve with strictly maximal Higgs field is rigid.*

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