DEGENERATION OF THE LERAY SPECTRAL SEQUENCE
FOR CERTAIN GEOMETRIC QUOTIENTS

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Dedicated to V. I. Arnold on the occasion of his 65th birthday

Abstract. We prove that the Leray spectral sequence in rational cohomology for the quotient map $U_{n,d} \to U_{n,d}/G$ where $U_{n,d}$ is the affine variety of equations for smooth hypersurfaces of degree $d$ in $\mathbb{P}^n(\mathbb{C})$ and $G$ is the general linear group, degenerates at $E_2$.

Key words and phrases. Geometric quotient, hypersurfaces, Leray spectral sequence.

1. Introduction

We consider an affine complex algebraic group $G$ which acts on a smooth algebraic variety $X$. Assume that a geometric quotient $f: X \to Y$ for the action of $G$ on $X$ exists (cf. [11, Section 0.1]). We want to give geometric conditions ensuring that the Leray spectral sequence degenerates at $E_2$, knowing the cohomology of the source $X$ is equivalent to knowing that of the target $Y$. As an example of how this could be used, we point out that for any group $G$ acting with finite stabilizers and the Leray spectral sequence for $f$ degenerates at $E_2$, knowing the cohomology of the source $X$ is equivalent to knowing that of the target $Y$. As an example of how this could be used, we point out that for any group $G$ acting with finite stabilizers on a topological space $X$ the equivariant cohomology $H^*_G(X, \mathbb{Q})$ equals $H^*(X/G, \mathbb{Q})$ ([3, §1, Remark 2]) and the former can often be calculated group theoretically. See [3] for examples. So, in these cases one knows $H^*(X, \mathbb{Q})$.

We prove a general result (Theorem 3) giving sufficient geometric conditions for this to happen. These turn out to be satisfied for the group $\text{GL}_{n+1}(\mathbb{C})$ acting on the affine variety $U_{n,d}$ of those homogeneous polynomials of degree $d$ in $(n+1)$ variables which give smooth hypersurfaces in $\mathbb{P}^n$:

**Theorem 1.** Let $d \geq 3$. Then the Leray spectral sequence in rational cohomology for the quotient map $U_{n,d} \to M_{n,d} := U_{n,d}/G$, where $G = \text{GL}_{n+1}(\mathbb{C})$, degenerates at $E_2$.

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Examples. 1. By results of Vassiliev [12] the map
\[ H^* (U_{n,d} ; \mathbb{Q}) \to H^* (GL_{n+1} (\mathbb{C}); \mathbb{Q}) \]
is an isomorphism in the cases \((n, d) = (2, 3), (3, 3)\). Moreover Gorinov [7] has proved the same result for the cases \((n, d) = (4, 3), (2, 5)\). It follows that \(M_{n,d}\) has the rational cohomology of a point in these cases.

2. For the case \((n, d) = (2, 4)\) it follows from [12] and Theorem 2 that the space \(M_{2,4}\) has a cohomology group of dimension 1 in degrees 0 and 6 and has zero rational cohomology in other degrees. This agrees with a result of Looijenga [10] about the Poincaré–Serre polynomial of \(M_{2,4}\):
\[ H^6 (M_{2,4}; \mathbb{Q}) \simeq \mathbb{Q}(-6) \]
and the other cohomology groups are those of a point.

Remark. In [1] there is a description of \(M_{3,3}\) using periods of threefolds. This moduli space turns out to be a certain explicitly described open subset of the quotient of complex hyperbolic 4-space by a certain discrete group. From this description it is quite unexpected that \(M_{3,3}\) has the rational cohomology of a point. It is an interesting question to calculate the cohomology of the various compactifications of \(M_{3,3}\) studied in loc. cit.

2. Generalizing the Leray–Hirsch Theorem

The proof of the Leray–Hirsch theorem as given in [9, p. 229] is valid for a locally trivial fibration \(p : M \to B\). For cohomology with rational coefficients, the same proof applies to a slightly more general situation:

Definition. A continuous map \(p : M \to B\) is a locally trivial fibration, say with fibre \(F\), in the orbifold sense if for every \(b \in B\) there exists a neighbourhood \(V_b\), a topological space \(U_b\), and a topological group \(G_b\) such that

1. \(G_b\) acts on \(U_b\) and on \(F\); the action on \(F\) is by homeomorphisms homotopic to the identity;
2. \(V_b\) is homeomorphic to \(U_b / G_b\);
3. \(p^{-1} V_b\) is homeomorphic to the quotient of \(U_b \times F\) by the product action of \(G_b\).

In this setting, composing the natural quotient map \(F \to F / G_b\) with the homeomorphism \((F / G_b) \xrightarrow{\sim} p^{-1} F\) and the inclusion \(p^{-1} F \hookrightarrow X\), defines the orbifold fibre inclusion \(r_b : F \to X\).

Indeed, in this setting the proof as given in loc. cit. applies starting from the observation that over the rationals we still have graded isomorphisms (replacement of the Künneth formula)
\[ H^*(p^{-1} V_b; \mathbb{Q}) \cong H^*(U_b \times F; \mathbb{Q})^{G_b} \cong H^*(U_b; \mathbb{Q})^{G_b} \otimes H^*(F; \mathbb{Q})^{G_b} \cong H^*(V_b; \mathbb{Q}) \otimes H^*(F; \mathbb{Q}), \]
because \(g \in G_b\) acts trivially on \(H^*(F; \mathbb{Q})\) since it is homotopic to the identity by assumption.

We thus arrive at:
Theorem 2. Let $p: M \to B$ be a fibration which is locally trivial in the orbifold sense. Suppose that for all $q \geq 0$ there exist classes $e_1^{(q)}, \ldots, e_{n(q)}^{(q)} \in H^q(M; \mathbb{Q})$ that restrict to a basis for $H^q(F; \mathbb{Q})$ under the map induced by the orbifold fibre inclusion $r_b: F \to M$. The map $a \otimes r_b^*(e_i) \mapsto p^*a \cup e_i$, $a \in H^r(B; \mathbb{Q})$ extends linearly to a graded linear isomorphism

$$H^r(B; \mathbb{Q}) \otimes H^r(F; \mathbb{Q}) \xrightarrow{\sim} H^r(M; \mathbb{Q}).$$

Example. Let $\phi: X \to Y$ be a geometric quotient for $G$. Suppose that $G$ is connected and that for all $x \in X$, the identity component of the stabiliser $S_x$ of $x$ is contractible (e.g. when $S_x$ is finite). For $y \in Y$ we take for $U_y$ any open slice for the action of $G$ through $x \in \phi^{-1}y$, i.e. a contractible submanifold through $x$ which intersects $Gx$ transversally at $x$. Then, if $gx$ is any other point in same orbit, $gU_y$ is a slice through $gx$ and $gS_xg^{-1} = S_{gx}$ so that for all $g \in G$, the quotient $gU_y/S_{gx}$ gives the same neighbourhood $V_y$ of $y$. We have $(U_y \times G)/S_x = \phi^{-1}(V_y)$. The assumption that $G$ is connected implies that multiplication by $g \in G$ is homotopic to the identity in $G$. So $\phi$ is indeed locally trivial in the orbifold sense (with typical fibre $G$).

We study this example in more detail in the next section.

3. The Case of a Geometric Quotient for a Reductive Group

We assume that $G$ is a reductive complex affine group, that $V$ is a representation space for $G$ and that $X$ is an affine $G$-invariant open subset of $V$ such that the action of $G$ on $X$ is closed. Let $\Sigma = V \setminus X$. For $x \in X$ the orbit map is denoted as follows

$$o_x: G \to X, \quad g \mapsto g(x),$$

and the geometric quotient (which exists in this case, cf. [11, p. 30]) by

$$\phi: X \to Y = X/G.$$

Recall that $H^r(G)$ is an exterior algebra freely generated by classes $\eta_i \in H^{2r_i-1}(G)$. Note also that $V$ being a vector space, we have isomorphisms

$$H^{2r_i-1}(X) \xrightarrow{\sim} H^{2r_i}_G(V).$$

We can now apply the variant of the Leray–Hirsch theorem as stated in the previous section to the geometric quotient $\phi$ and we obtain:

Theorem 3. Suppose that there are schemes $Y_i \subset \Sigma$ of pure codimension $r_i$ in $V$ whose fundamental classes map to a non-zero multiple of $\eta_i$ under the composition

$$H^{2r_i}_Y(V) \to H^{2r_i}_G(V) \xrightarrow{\sim} H^{2r_i-1}(X) \xrightarrow{\sigma_i} H^{2r_i-1}(G).$$

Denote the image of $[Y_i]$ in $H^r(X; \mathbb{Q})$ by $y_i$; then the map $a \otimes \eta_i \mapsto \phi^*a \cup y_i$, $a \in H^r(X/G; \mathbb{Q})$ extends to an isomorphism of graded $\mathbb{Q}$-vector spaces

$$H^r(X/G; \mathbb{Q}) \otimes H^r(G; \mathbb{Q}) \xrightarrow{\sim} H^r(X; \mathbb{Q}).$$
4. Properties of Fundamental Classes

We collect some facts on fundamental classes that we need later on. We refer to [4] for the cohomology version and [5] for the Chow version.

1. For any connected submanifold $Z$ of pure codimension $c$ in a complex algebraic manifold $X$, its fundamental class $[Z] \in H^c_Z(X)(c)$ is the image of $1 \in H^0(Z)$ under the Thom isomorphism $H^*(Z) \xrightarrow{\sim} H^*_Z(X)[2c](c)$. For $Z$ an irreducible subvariety, one still has a fundamental class as above, since restriction to the smooth part of $Z$ induces isomorphisms between the relevant cohomology groups with support in $Z$, respectively the smooth part of $Z$. If $Z = \sum_i n_i Z_i$ is a cycle of codimension $c$ (with $Z_i$ irreducible), with support $|Z|$, there is a cycle class $[Z] \in H^c_Z(X)(c)$. More generally still, one may assume $Z$ to be a complex subscheme of pure codimension $c$ with irreducible components $Z_i$, of multiplicity $n_i$, in $Z$ and define the fundamental class to be the fundamental class of the associated cycle $\sum_i n_i Z_i$. There are natural maps $H^*_Z \to H^*_Z$ and if we identify $[Z_i]$ with their images under these maps we have the equality

$$[Z] = \sum_i n_i[Z_i].$$

2. The fundamental classes behave functorially as follows. Let $f: X \to Y$ be a holomorphic map between complex algebraic manifolds, $Z \subset X$, $W \subset Y$ subschemes such that $Z$ is contained in the scheme-theoretic inverse image $f^{-1}W$. Then $f$ induces $H^*_W(Y) \to H^*_Z(X)$ and if moreover $Z = f^{-1}W$ has the same codimension $c$ as $W$, then $f^*[W] = [Z]$. In particular, if $W$ is irreducible and the cycle associated to $Z = f^{-1}W$ is $\sum n_i Z_i$, we find

$$f^*[W] = [f^{-1}W] = \sum n_i[Z_i] \in H^c_Z(X)(c).$$

3. We can refine the fundamental class of $Z$, a purely $c$-codimensional subscheme of $X$ to a class in the Chow group $A_{n-c}(X)$, $n = \dim(X)$. The Chow group $A_{n-c}(Z)$ is generated by the Chow cycle classes $[Z_i]$ of the irreducible components of $Z$. If the generic point of $Z_i$ has multiplicity $n_i$, then the fundamental class of $Z$ is given by

$$[Z] = \sum n_i[Z_i] \in A_{n-c}(Z).$$

There is a push forward map

$$A_c(Z) \to A_c(X)$$

and a cycle class map

$$A_k(Z) \to H_{2k}^{BM}(Z)(-k)$$

sending the Chow cycle of $Z$ to $[Z]$. Composing this map with Poincaré duality for Borel–Moore homology, which reads

$$H^i_{BM}(Z) \xrightarrow{\sim} H^{2n-i}_Z(n)$$

and taking $\ell = 2k$, we obtain the cycle class map

$$A_k(Z) \to H^{2n-2k}_Z(X)(n-k).$$

Abusing notation, we denote the Chow cycle also by $[Z]$. This is especially useful if $Z$ is the scheme of zeros of a section $s$ of a vector bundle $E$ over $X$. In fact, if
s: \mathcal{E} \to X is the zero section with image, say \{0\}, there is a Gysin isomorphism 
s^*: A_n(E) \to A_(X)[-r] with the property 
\[ A_n(E) \ni \{0\} \mapsto c_r(X) \in A_{n-r}(X). \]
See [5, Example 3.3.2]. This Gysin map is in fact the inverse of the isomorphism 
\[ \pi^*: A_{n-r}(X) \cong A_n(E). \]

5. The Cohomology Ring of the General Linear Group

We turn to \( G = G_n = GL_n(\mathbb{C}) \), \( n \geq 1 \). In this case, by [2], \( H^\ast(G) \) is the exterior algebra with generators \( \eta^{(n)}_\ell \) in all odd degrees \( 2\ell - 1 \), \( \ell = 1, \ldots, n \). In other words \( \tau_1 = 1, \tau_2 = 2, \ldots, \tau_n = n \). Since \( G_n \) is contained in the vector space \( M_n = \text{Mat}_n(\mathbb{C}) \), we have an identification of mixed Hodge structures 
\[ H^\ast(G) \cong H^\ast_{BM}(M_n)[1], \]
where 
\[ D_n = \{ A \in M_n : \det(A) = 0 \} = M_n \setminus G_n, \]
and so \( \eta^{(n)}_\ell \) corresponds to some class in \( H^\ast_{BM}(M_n) \). The goal is to find explicit descriptions of this class as fundamental class of the subvariety \( D_n, \ell \subset D_n \) to be defined below. This will turn out to be essential for the next section. We are going to show this by first defining classes \( \eta^{(n)}_\ell \) that clearly have this property. Then we prove that these classes do generate \( H^\ast(G) \) as an exterior algebra.

We introduce the following notation:

- \( D_{n, \ell} \subset D_n \): the subvariety consisting of those matrices for which the first \( n + 1 - \ell \) columns are linearly dependent. Note that \( D_{n, \ell} \) has codimension \( \ell \) in \( M_n \).
- \( \tilde{D}_n = \{(A, p) \in D_n \times \mathbb{P}^{n-1}(\mathbb{C}) : [p] \subset \text{Ker}(A)\} \) (where \([p]\) stands for the line in \( \mathbb{C}^n \) corresponding to \( p \)) and \( \pi_n: \tilde{D}_n \to D_n \) is the projection to the first factor.
- \( Q_n = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : xy = 1\} \).
- \( \alpha_n: M_{n-1} \to M_n \) is the inclusion which maps a matrix \( A \) to \( \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \).
- \( h: \) the hyperplane class in \( H^2(\mathbb{P}^n(\mathbb{C})) \).

Note that the projection to the second factor turns \( \tilde{D}_n \) into a vector bundle of rank \( n^2 - n \) over \( \mathbb{P}^{n-1}(\mathbb{C}) \), so \( \tilde{D}_n \) is smooth and \( \pi_n \) is a resolution of singularities of \( D_n \).

**Lemma 4.** Let \( X \) be a smooth variety, \( D \subset X \) a subvariety of codimension \( k \) and \( \pi: \tilde{D} \to D \) a resolution of singularities. Then there are natural Gysin maps \( \beta_k: H^{\ell-2k}(\tilde{D})(-k) \to H^{\ell}_D(X) \) which are morphisms of mixed Hodge structures.

**Proof.** Let \( n = \dim(X) \). As \( \tilde{D} \) is smooth, cup product with the fundamental class \( \tilde{D} \) induces an isomorphism 
\[ H^{\ell-2k}(\tilde{D})(-k) \to H^{BM}_{2n-\ell}(\tilde{D})(-n). \]

As Borel–Moore homology is covariant for proper morphisms we have natural maps 
\[ H^{BM}_{2n-\ell}(\tilde{D})(-n) \to H^{BM}_{2n-\ell}(D)(-n). \]
Because $X$ is smooth, Poincaré duality for Borel–Moore homology gives an isomorphism of mixed Hodge structures
\[ H^{2n-\ell}_{BM}(D)(-n) \simeq H^\ell_D(X). \]
by [5, Section 19.1]. The map $\beta_\ell$ is obtained as the composition of these maps. □

Let us apply this to the situation of $\tilde{D}_n \to D_n \hookrightarrow M_n$. We obtain maps
\[ \beta^{(n)}_\ell : H^{2\ell-2}(\mathbb{P}^{n-1}(\mathbb{C}))(1-1) \to H^{2\ell}_{BM}(M_n) \simeq H^{2\ell-1}(G_n) \]
and define for $\ell = 1, \ldots, n$:
\[ \eta^{(n)}_\ell := \beta^{(n)}_\ell \left( \frac{h^{\ell-1}}{2\pi i} \right) \in H^{2\ell-1}(G_n). \]

We observe that the class in $H^{2\ell}_{BM}(M_n)$ corresponding to $\eta^{(n)}_\ell$ is indeed the fundamental class of $D_{n,\ell} \subset D_n$.

**Lemma 5.** The map $\alpha : M_{n-1} \to M_n$ maps $D_{n-1}$ and $G_{n-1}$ to $D_n$ and $G_n$ respectively and $\alpha^*(\eta^{(n)}_\ell) = \eta^{(n-1)}_\ell$ for $\ell = 1, \ldots, n-1$ while $\alpha^*(\eta^{(n)}_n) = 0$.

**Proof.** Observe that $\alpha^{-1}(D_{n,\ell}) = D_{n-1,\ell}$. One checks that this holds not only set theoretically, but even as schemes. Then the lemma follows from property 2 from Section 4. □

Because the classes $\eta^{(n)}_\ell$ are of odd degree, they have square zero and anti-commute, so we have a homomorphism of graded algebras
\[ R_n : \Lambda(z_1, \ldots, z_n) \to H^*(G_n). \]
Here $\Lambda(z_1, \ldots, z_n)$ is the exterior algebra on $n$ generators $z_1, \ldots, z_n$ with $z_i$ of degree $2i - 1$, and $R_n(xz) = \eta^{(n)}_\ell$.

**Theorem 6.** The map $R_n$ is an isomorphism. Moreover, the generators $\eta^{(n)}_\ell \in H^{2\ell-1}(G_n)$ have pure type $(\ell, \ell)$ and map to the fundamental classes $D_{n,\ell}$ under the identification $H^{2\ell-1}(G_n) \simeq H^{2\ell}_{BM}(M_n)$.

**Proof.** By induction on $n$. For $n = 1$ everything is clear. Suppose the map $R_{n-1}$ is an isomorphism. We consider the map
\[ \rho : G_n \to Q_n, \quad \rho(g) = (g(e_1), g^{-1}(e_1)). \]
This is the orbit map of a transitive action of $G_n$ on $Q_n$ and $\alpha(G_{n-1})$ is the isotropy subgroup of $(e_1, e_1) \in Q_n$. Therefore, $\rho$ is also the quotient map for the action of $G_{n-1}$ on $G_n$ by left translation via $\alpha$. As the classes $\eta^{(n-1)}_\ell$ generate the cohomology ring of $G_{n-1}$ and are images of classes on $G_n$, the restriction maps $\alpha^* : H^*(G_n) \to H^*(G_{n-1})$ are surjective. Hence by Theorem 2 we have an isomorphism
\[ H^*(Q_n) \otimes H^*(G_{n-1}) \simeq H^*(G_n). \]
The variety $Q_n$ is homotopy equivalent to a sphere of dimension $2n - 1$ (in fact to its subvariety consisting of pairs $(x, y)$ with $y = \bar{x}$). Moreover, a generator of $H^{2n-1}(Q_n)$ is mapped to a non-zero multiple of $\eta^{(n)}_\ell$ by the map $\rho^*$. This implies the surjectivity and hence bijectivity of $R_n$. □
Remark. For any Lie group \( G \), the map \( g \mapsto g^{-1} \) induces multiplication by \(-1\) on the Lie algebra, hence on \( H^k(G) \) it induces multiplication by \((-1)^k\). The involution \( \sigma : G_n \to G_n \) given by \( \sigma(g) = g^{-1} \) has \( \sigma^*(\eta_n(n)) = (-1)^n \eta_n(n) \). Indeed, if we let \( \sigma : Q_n \to Q_n \) be given by \( \sigma(x, y) = (y, x) \) then \( \rho \) becomes equivariant, and it is an easy exercise to see that \( \sigma^* = (-1)^n \) on \( H^{2n-1}(Q_n) \). We conclude that transposition \( \tau \) on \( G_n \) induces \( \tau^*(\eta_n(n)) = (-1)^{n-1} \eta_n(n) \). As the inclusion \( G_{n-1} \to G_n \) commutes with transposition, we conclude that \( \tau^*(\eta_n(n)) = (-1)^{\ell-1} \eta_\ell(n) \) for all \( \ell \leq n \).

6. Moduli of Smooth Hypersurfaces

We let \( \Pi_{n,d} = \mathbb{C}[x_0, \ldots, x_n]_d \) denote the vector space of homogeneous polynomials of degree \( d \) in \( n+1 \) variables over \( \mathbb{C} \). We let

\[ \Sigma_{n,d} = \{ f \in \Pi_{n,d} : f \text{ has a critical point outside } 0 \}. \]

There exists an irreducible polynomial \( \Delta \) in the coefficients of \( f \in \Pi_{n,d} \) such that \( f \in \Sigma_{n,d} \) if and only if \( \Delta(f) = 0 \). Moreover, \( \Delta \) is homogeneous of degree \((n+1)(d-1)^n\).

We let \( U_{n,d} = \Pi_{n,d} \setminus \Sigma_{n,d} \). The group \( \text{GL}_{n+1}(\mathbb{C}) \) acts on \( U_{n,d} \). For \( d \leq 2 \) or \( d = 3, n = 1 \) it acts transitively, but in the remaining cases it acts with finite isotropy groups, so the action is closed. As \( U_{n,d} \) is affine, by [11, p. 30] we have a geometric quotient \( M_{n,d} \) which is a coarse moduli space for non-singular projective hypersurfaces of degree \( d \) in \( \mathbb{P}^n(\mathbb{C}) \). In our situation we fix a particular \( f = f_{n,d} \in U_{n,d} \), the Fermat hypersurface:

\[ f_{n,d} = x_0^d + \cdots + x_n^d, \]

and the orbit map then extends to a map

\[ \tau_n : M_{n+1} \to \Pi_{n,d}, \quad A \mapsto f_{n,d} \circ A. \]

It induces maps for cohomology with supports:

\[ H^{2\ell}_{\Sigma_{n,d}}(\Pi_{n,d}) \to H^{2\ell}_{D_{n+1}}(M_{n+1}). \]

Let \( e_0, \ldots, e_n \) denote the standard basis vectors of \( \mathbb{C}^{n+1} \). Define for \( \ell = 1, \ldots, n+1 \)

\[ \Sigma_{n,d}^{(\ell)} = \{ f \in \Pi_{n,d} : V(f)^{\text{sing}} \cap \mathbb{P}[e_0, \ldots, e_{n-\ell+1}] \neq \emptyset \}. \]

Then \( \Sigma_{n,d}^{(\ell)} \subset \Sigma_{n,d} \) has codimension \( \ell \) in \( \Pi_{n,d} \). Below we shall prove:

**Lemma 7.** The class \( \tau_n^*[\Sigma_{n,d}^{(\ell)}] \) is a non-zero multiple of \( [D_{n+1,\ell}] \).

Recall from the previous section that \( [D_{n+1,\ell}] \) corresponds to the generator \( \eta_\ell(n) \in H^{2\ell-2}(G) \) and we now apply Theorem 3 to deduce:

**Theorem 8.** Let \( d \geq 3 \). Then the Leray spectral sequence in rational cohomology for the quotient map \( U_{n,d} \to M_{n,d} \) degenerates at \( E_2 \).

Let us proceed to give a proof of Lemma 7. We want to do this by induction on \( n \), so we fix an embedding \( i : \Pi_{n-1,d} \hookrightarrow \Pi_{n,d} \) by posing

\[ i(h) = x_0^d + h(x_1, \ldots, x_n). \]
Note that $\iota(f_{n-1,d}) = f_{n,d}$ and that $\iota(\Pi_{n-1,d}) \cap \Sigma_{n,d} = \iota(\Sigma_{n-1,d})$. The intersection multiplicity however is equal to $d - 1$. Indeed, the multiplicity of a stratum of the discriminant corresponding to hypersurfaces with isolated singularities is equal to the sum of their Milnor numbers, and adding the term $x_0^d$ multiplies the Milnor numbers by $d - 1$. We obtain a commutative diagram

$$
\begin{array}{c}
M_n \xrightarrow{r_{n-1}} \Pi_{n-1,d} \\
\alpha \downarrow \\
M_{n+1} \xrightarrow{r_n} \Pi_{n,d}.
\end{array}
$$

We have a corresponding diagram in cohomology with supports

$$
\begin{array}{c}
H^{2\ell}_{\Sigma_{n,d}}(\Pi_{n,d}) \xrightarrow{\iota^*} H^{2\ell}_{D_{n+1}}(M_{n+1}) \\
\downarrow \\
H^{2\ell}_{\Sigma_{n-1,d}}(\Pi_{n-1,d}) \xrightarrow{\iota^*} H^{2\ell}_{D_n}(M_n).
\end{array}
$$

Observe that $\iota^*(\Sigma_{n,d}^{(\ell)}) = \nu_{n,d}^{(\ell)}|_{\Pi_{n-1,d}}^{(\ell)}$ where $\nu_{n,d}^{(\ell)}$ is the intersection multiplicity of $\Sigma_{n,d}^{(\ell)}$ with $\iota(\Pi_{n-1,d})$ in $\Pi_{n,d}$. In particular, $\nu_{n,d}^{(\ell)}$ is a positive integer.

We can now prove the lemma by induction on $n$ using the above diagram, provided we check the case $\ell = n + 1$ for each $n$.

The variety $S = \Sigma_{n,d}^{(n+1)}$ is the linear space of all polynomials singular at $e_0$. Its pre-image under $r_n$ has two irreducible components: one consists of the matrices whose first column is zero, i.e. with $A(e_0) = 0$; this component is exactly $T_1 = D_{n+1,n+1}$. The other component, $T_2$, which has the same dimension, is the closure of

$$
\{ A \in M_{n+1} : 0 \neq A(e_0) \in V(f_{n,d}), \ \text{Im}(A) = T_A(e_0) V(f_{n,d}) \}.
$$

The component $T_2$ has multiplicity one, whereas $T_1$ has multiplicity $d(d-1)^n$. We have the commutative diagram

$$
\begin{array}{c}
H^{2n+2}_{S}(\Pi_{n,d}) \xrightarrow{r_n^*} H^{2n+2}_{\Sigma_{n,d}}(\Pi_{n,d}) \\
\downarrow \\
H^{2n+2}_{T_1 \cup T_2}(M_{n+1}) \xrightarrow{r_n^*} H^{2n+2}_{D_{n+1}}(M_{n+1})
\end{array}
$$

and therefore

$$
r_n^*([S]) = d(d-1)^n [T_1] + [T_2]
$$

by property 1 in Section 4.

**Claim.** We have

$$
[T_2] = (-1)^n(1 - (1-d)^n)[T_1] \quad \text{in} \quad H^{2n+2}_{D_{n+1}}(M_{n+1}).
$$
Combining the claim with (1) we find:
\[ r_n^*[S] = d(d - 1)^n[T_1] + [T_2] = ((d - 1)^{n+1} + (-1)^n)[T_3] \neq 0, \]
which proves the lemma.

It remains to prove the claim. Let \( T_2' \) denote the image of \( T_2 \) under the transposition map \( \tau \). Then
\[ [T_2] = (-1)^n[T_2'] \quad (2) \]
in \( H^{2n+2}_{D_{n+1}}(M_{n+1}) \) by the remark at the end of Section 5. Let \( \tilde{T}_1 = T_1 \times \{r_0\} \subset \tilde{D}_{n+1} \).

Write \( X = V(f_{n,d}) \subset \mathbb{P}^n \) and let \( \gamma: X \to \mathbb{P}^n \) be the Gauss map, which associates to a point \( p \in X \) the coordinates of its tangent hyperplane, i.e. \( \gamma(p) = \nabla f_{n,d}(p) \).

The space \( \tilde{D}_{n+1} \) is the total space of a vector bundle \( E \) over \( \mathbb{P}^n \) of rank \( r = n(n+1) \). Let
\[ \tilde{T} := \{(A, p) \in M_{n+1} \times X: (df_0)_p \circ s_A = 0 \text{ and } s_A(r_0) = p \}. \]
Then \( \tilde{T} \) is the total space of a vector bundle \( F \) over \( X \) of rank \( r - n + 1 \) which is a subbundle of \( \gamma^*(E) \), because \( (A, p) \in \tilde{T} \) implies that \( A(\gamma(p)) = 0 \). The projection of \( \tilde{T} \) in \( M_{n+1} \) is precisely \( T_2' \).

We will carry out our calculations in Chow groups instead of cohomology groups, using property 3 in Section 4. Consider the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\gamma^*(E)} & E \\
\downarrow & & \downarrow \pi \\
X & = & X \xrightarrow{\gamma} \mathbb{P}^n.
\end{array}
\]

We let \( s \) be the 0-section of \( E \), and \( s' \) that of \( \gamma^*(E) \) and recall from Section 4, property 3, that these induce Gysin maps in Chow groups.

The strategy is to compare the classes \( \tilde{T}_1 \) and \( \tilde{T} \) by pushing them to \( \mathbb{P}^n \). We get two 0-cycles on \( \mathbb{P}^n \) whose degrees we compare. Clearly \( \deg s[\tilde{T}_1] = 1 \) and so it suffices to calculate the degree of \( s^*[\tilde{T}_1] = A_0(\mathbb{P}^n) \).

By [5, Proposition 1.7] we find that
\[ \bar{\gamma}_s \pi^* \alpha = \pi^* \gamma_s \alpha \in A_{i+r}(E) \]
for any \( \alpha \in A_i(X) \). Applying this to \( \alpha = s^*[F] \) we find
\[ \bar{\gamma}_s[F] = \pi^* \gamma_s s^*[F]. \]

Next, applying \( s^* \) to both sides and using that the Gysin map \( s^* \) is in fact the inverse of the isomorphism induced by the bundle projection \( \pi: E \to \mathbb{P}^n \), and similarly for \( s' \), we get
\[ s^* \bar{\gamma}_s[F] = \gamma_s s'^*[F]. \]

We next compute \( s'^*[F] \in A_0(X) \). By [5, Example 3.3.2] applied to the vector bundle \( \gamma^*(E)/F \) we get
\[ s'^*[F] = c_{n-1}(\gamma^*(E)/F). \]
On \( \mathbb{P}^n \) we have the exact sequence

\[
0 \to E \to \mathcal{O}^{(n+1)^2} \to \mathcal{O}(1)^{n+1} \to 0
\]

showing that \( c(E) = (1 + h)^{-n-1} \) where \( h = c_1(\mathcal{O}(1)) \). As \( \gamma^*\mathcal{O}(1) = \mathcal{O}_X(d - 1) \) we get

\[
c(\gamma^*E) = (1 + (d - 1)h_X)^{-n-1}
\]

where \( h_X = c_1(\mathcal{O}_X(1)) \). For the bundle \( F \) we have the exact sequences

\[
0 \to F \to \mathcal{O}_X^{(n+1)^2} \to Q_X \oplus \mathcal{O}_X(d - 1)^n \to 0,
\]

\[
0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X^{n+1} \to Q_X \to 0
\]

so \( Q_X \) is the restriction of the universal quotient bundle to \( X \). Hence we find

\[
c(F) = (1 + (d - 1)h_X)^{-n} c(Q_X)^{-1} = (1 + (d - 1)h_X)^{-n}(1 - h_X)^{-1}
\]

so

\[
c(\gamma^*E/F) = (1 + (d - 1)h_X)^{-1}(1 - h_X)^{-1}.
\]

We find

\[
c_{n-1}(\gamma^*E/F) = \left( \frac{1 - (1 - d)^n}{d} \right) h_X^{n-1}
\]

which has degree equal to \( 1 - (1 - d)^n \). Combining this with (2), the claim then follows.

\[\square\]

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**References**


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