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Holomorphic Automorphisms of Compact Kähler Surfaces and Their Induced Actions in Cohomology

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For any compact complex manifold X we may ask whether the group $\mathrm{Aut}(X)$ of holomorphic automorphisms of X acts faithfully on the cohomology ring $H^*(X;A)$ with values in some ring A. If the identity component of $\mathrm{Aut}(X)$ contains elements g different from 1 then g acts trivially in cohomology. So the answer is "no" if the Lie-algebra of Aut(X) doesn't reduce to $\{0\}$- or equivalently if X admits a non-zero holomorphic vectorfield. This happens if e.g. X is biholomorphically isomorphic to $Y \times \mathbb{P}^n$.

Now, let me look at the case $\dim_{\mathbb{C}} X = 1$, i.e. X is a compact Riemann surface. Because of the reason given before, if the genus of X is 0 or 1 the answer is negative. However, a well-known theorem – going back to Hurwitz – states that in all other cases, i.e. if the genus is at least 2, the group $\mathrm{Aut}(X)$ *does* operate faithfully on $H^1(X, \mathbb{Z})$. It is instructive to look at the proof of this, since it contains some of the ingredients of the main theorem stated below.

So, suppose X is a compact Riemann surface of genus $\geqq 2$, and assume $1 \neq g \in \mathrm{Aut}\, X$ acts trivially on $H^1(X, \mathbb{Z})$. Now the canonical system on X is free of base points, so for any $p \in X$ there exists a holomorphic 1-form ω which does not vanish at p. Since the vector space of holomorphic 1-forms on X is a direct factor of $H^1(X, \mathbb{C})$ we must have that $g^*\omega = \omega$. In particular, if $p \in X$ were a fixed point of g, the induced map on the cotangent space at p would be the identity. But then $g = 1$, contrary to our assumptions. So g acts fixed point free, and the Lefschetz fixed point formula implies that $\mathrm{Trace}\, g^* | H^1(X, \mathbb{Z}) = 2$. However $g^* = \mathrm{id}$, so $\mathrm{Trace}\, g^* | H^1(X, \mathbb{Z}) = \mathrm{rank}\, H^1(X, \mathbb{Z}) > 3$, since the genus of X is at least 2. This contradiction completes the proof.

Now we go over to the case of compact complex 2-dimensional manifolds, to be called *surfaces*. For the sake of completeness let me recall what is known in this situation.

For $K3$-surfaces X the group $\mathrm{Aut}(X)$ operates faithfully on $H^2(X, \mathbb{Z})$ (cf. Burns-Rapoport, [2], Prop. 1.1) and a similar statement is true for Enriques surfaces (cf. Ueno, [7]). Notice that, whereas in the first case $H^2(X, \mathbb{Z})$ has no torsion, in the second case it *does* have torsion. In fact there exists an Enriques

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surface X for which $\mathrm{Aut}(X)$ does not operate faithfully on $H^2(X,\mathbb{Q})$. (Cf. the example below.) Finally the only other case where $\mathrm{Aut}(X)$ was known to operate faithfully on $H^2(X,\mathbb{Q})$ was if the canonical bundle K_X is very ample. Indeed, let $\mathbb{P}^N=\mathbb{P}(H^0(X,K_X)^\vee)$ and $X\to\mathbb{P}^N$ the resulting embedding. Since $H^0(X,K_X)=H^{2,0}$ is a direct factor of $H^2(X,\mathbb{C})$-by Hodge theory (cf. Weil, [8]), any g which induces the identity on $H^2(X,\mathbb{Q})$, acts trivially on $H^{2,0}$, hence on $\mathbb{P}^N$, so g is the identity.

Example (due to D. Lieberman). Let E be the elliptic curve of modulus $i=\sqrt{-1}$ and τ the unique nonzero point of order 2 on E with $i\tau=\tau$.

Let $X_1=E\times E$ and let X_2 be the K-3 surface obtained by resolving the Kummer surface $(X_1/\pm)$. The automorphism $\lambda\colon (a,b)\to(a+\tau,\ -b+\tau)$ of X_1 induces a fix point free involution on X_2 and the quotient by this action is X_3, an Enriques surface. The automorphism $g=(i,i)$ of X_1 induces automorphisms of X_2 and X_3 and we claim that g induces the identity on $H^2(X_3,\mathbb{Q})$. This is easily seen by identifying $H^2(X_3,\mathbb{Q})$ with the subspace of $H^2(X_2,\mathbb{Q})$ invariant under λ. A basis for this subspace is provided by algebraic cycles of the form $(E/\pm)\times 0$, $0\times(E/\pm)$ and $C_j+C_{i(\tau,\tau)}$ where C_j is the exceptional curve on X_2 associated with the point of order 2, j on X_1. These cycles are g-invariant.

Let me now state the main result:

Theorem. *Let X be a Kähler surface with $H^0(X,T_X)=0$ and such that $|K_X|$ is without base points and fixed components. Suppose $g\in\mathrm{Aut}(X)$ acts trivially on $H^2(X,\mathbb{Q})$. Then $g=1$ unless X is a surface of general type and either*

(i) *$c_1^2(X)=2c_2(X)$ and $\#g$ is a power of 2, or*

(ii) *$c_1^2(X)=3c_2(X)$, $\#g$ is a power of 3 and moreover g acts trivially on all $H^*(X,\mathbb{Q})$.*

Here T_X is the holomorphic tangent bundle and K_X as before $\det(T_X^\vee)$, the canonical bundle. The numbers $c_1^2(X)$, resp. $c_2(X)$ are as usual the Chern numbers of X.

First a *remark* concerning the exceptions mentioned in the theorem. The first exception really occurs: take the direct product of two hyperelliptic curves and let g act as the hyperelliptic involution on each factor. Then $g^*=\mathrm{id}$ on H^2. However $g^*=-\mathrm{id}$ on H^1 and I have not been able to find a surface X with $c_1^2(X)=2c_2(X)$ carrying an involution which acts trivially on all of $H^*(X,\mathbb{Q})$. Also I do not know whether the second exception really occurs.

Before I give the proof of the theorem let me first give an *application:* In general, if X is a polarized algebraic variety (that is, in addition to being a smooth Kähler manifold) Popp has shown ([5], Lecture 10) that there exists a fine moduli space (in the category of algebraic spaces) for the set of isomorphy classes of polarized algebraic varieties over $\mathbb{C}$ having the same Hilbert polynomial as X together with a so-called "level n-structure"-*provided* $\mathrm{Aut}(X)$ *operates faithfully on the free part* of $H^*(X,\mathbb{Z})$. In particular this applies to the algebraic surfaces satisfying the conditions of our theorem.

The following notation is employed throughout. If $g\in\mathrm{Aut}(X)$ acts on a vector space V we let V^{inv} be the invariant subspace. We set:

$e(X)=$ the Euler-Poincare characteristic of X.
$b_j(X)=\dim_{\mathbb{Q}} H^j(X,\mathbb{Q})$,
$b_j^{\mathrm{inv}}(X)=\dim_{\mathbb{Q}} H^j(X,\mathbb{Q})^{\mathrm{inv}}$,
$q(X)=\dim_{\mathbb{C}} H^{1,0}=\dim_{\mathbb{C}} H^{0,1}$, where $H^{p,q}$ are the Hodge-components – cf. Weil, [8],
$\delta(X)=q(X)-\dim_{\mathbb{C}}\{H^{1,0}\}^{\mathrm{inv}}$,
$\chi(X)=1-q(X)+\dim_{\mathbb{C}} H^{2,0}$.

In the sequel l will be a fixed prime number and ρ will be a fixed primitive l-th root of unity.

Lemma 1. *g has finite order.*

Proof. Since X is Kähler, a result of Lieberman ([4], Prop. 2.2) applies which states that the subgroup G of $\mathrm{Aut}(X)$ fixing a Kähler class has only finitely many components. Since $H^0(X,T_X)=0$, this implies that G and hence $g\in G$ has finite order.

Lemma 2. *Let $g(\neq 1)$ have prime order l. The fixed point set of g consists of finitely many points. If p is a fixed point, local coordinates (ξ_1,ξ_2) centered at p can be found such that the action of g is given by $(\xi_1,\xi_2)\to(\rho^k\xi_1,\rho^{-k}\xi_2)$ with $k\neq 0 \bmod l$. In particular p is an isolated simple transversal fixed point.*

Proof. Let p be a fixed point of g. Since $|K_X|$ does not have fixed points or fixed components there exists a holomorphic 2-form ω on X which does not vanish at p. Now $H^{2,0}$ is a direct factor of $H^2(X,\mathbb{C})$, by Hodge theory (cf. Weil, [8], Ch. V) and can be identified with the vector space of holomorphic 2-forms on X (loc. cit. p. 70 Coll. 3). So $g^*\omega=\omega$ and in particular the jacobian of g at p equals 1. Moreover, one can linearize the action of g around p (cf. [9], p. 97) and by a further linear change of coordinates one can diagonalize this action to obtain the coordinates (ξ_1,ξ_2). Together with the previous remark this implies that p is a simple isolated transversal fixed point.

Lemma 3. *Under the assumptions of Lemma 2, the number n of fixed points of g equals $c_2(X)+4\left(\frac{l}{l-1}\right)\delta(X)$.*

Proof. We apply the Lefschetz fixed point formula:

$$\sum_{k=0}^{4}(-1)^k\,\mathrm{Trace}(g^*|H^k(X,\mathbb{Q}))=n. \tag{1}$$

We first compute the action on $H^1(X,\mathbb{Q})$. Observe that $H^1(X,\mathbb{Q})=H^1(X,\mathbb{Q})^{\mathrm{inv}}\oplus V$, where V is a direct sum of dimension $(l-1)$-dimensional representations of trace -1. So we find that

$$\begin{aligned}\mathrm{Tr}(g^*|H^1)&=b_1^{\mathrm{inv}}-(1/l-1)(b_1-b_1^{\mathrm{inv}})\\&=b_1-(l/l-1)(b_1-b_1^{\mathrm{inv}})=b_1-(2l/l-1)\delta,\end{aligned}$$

where the last equality follows since $H^1 \otimes \mathbb{C}$ the direct sum of the G-stable subspace $H^{1,0}$ and its complex conjugate $H^{0,1}$. Since $g^*|H^2(X,\mathbb{Q})=1$, we find for the left hand side of (1):

$$2-2b_1+b_2+4\frac{l}{l-1}\delta=e+4\frac{l}{l-1}\delta.$$

Here we used, that H^1 and H^3 are dual G-vector spaces. Since $e(X)=c_2(X)$, the lemma follows.

Lemma 4. *Still under the assumptions that* $g \neq 1$, $\# g = l$ *we have*

$$c_1^2(X)-l c_2(X)=4\frac{l}{l-1}(l-2)\cdot\delta(X).$$

Proof. We apply the holomorphic Lefschetz fixed point formula (Atiyah-Bott, [1]) for $k \neq 0 \bmod l$:

$$1-\operatorname{Tr}(g^k|H^{0,1})+\operatorname{Tr}(g^k|H^{0,2})=\sum_{p|g(p)=p} 1/\{\det(1-d_p(g^k)\}^{1} \tag{2}$$

where $d_p(g^k)\colon T_p(X)\to T_p(X)$ is the action induced by g^k on the tangent space at a fixed point p.

Now add these equalities for $k=1,\ldots,l-1$ and finally add $1-\dim H^{0,1}+\dim H^{0,2}=\chi(X)$ to both sides. Observe that $\dim V^{\mathrm{inv}}=(1/l)\sum_{k=0}^{l-1}\operatorname{Tr}(g^k|V)$ for any g-module V. So the left hand side of (2) sums up to

$$l(1-\dim(H^{0,1})^{\mathrm{inv}}+\dim(H^{0,2})^{\mathrm{inv}})=l\{\chi(X)+\delta(X)\}. \tag{3}$$

For the right hand side we need the following equality

$$\sum_{k=1}^{l-1}(1-\rho^k)^{-1}(1-\rho^{-k})^{-1}=[(l^2-1)/12]. \tag{4}$$

This, one can prove as follows. Consider

$$\begin{aligned} f(z)=\sum_{k=1}^{l-1}(z-\rho^k)^{-1}&=\frac{d}{dz}\log(z^{l-1}+z^{l-2}+\cdots+1)\\ &=\left\{\sum_{j=1}^{l-1}(jz^{j-1})\right\}\{z^{l-1}+z^{l-2}+\cdots+1\}^{-1}.\end{aligned}$$

Now

$$-\sum_{k=1}^{l-1}(\rho^k-1)^{-1}=f(1)=\tfrac{1}{2}(l-1)$$

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[1] Observe that the fixed point sets of g and g^k ($k \neq 0 \bmod l$) are equal, since l is prime

and

$$-\sum_{k=1}^{l-1} (\rho^k-1)^{-2}=f'(1)=\tfrac{1}{12}(l-1)(l-5).$$

Adding both equalities one gets the identity (4).

Using (4) and the value of n found in Lemma 3 we find that the right hand side sums up to:

$$\chi(X)+\frac{l^2-1}{12}\left[c_2+4\frac{l}{l-1}\delta\right]. \tag{5}$$

Comparing the right hand side of (3) with (5) and using the Riemann-Roch formula for surfaces:

$$\chi(X)=\tfrac{1}{12}[c_1^2+c_2]$$

(after some elementary manipulations) we find the equality stated in the Lemma.

Proof of the Main Theorem. Fix an automorphism g of X which acts trivially on $H^2(X,\mathbb{Q})$. Replacing g by a suitable power, we may assume that $|g|=l$, a prime number, and we reduce the statement of the theorem to:

If $g\neq 1$, *then* X *is of general type and either* $l=2$ *and* $c_1^2=2c_2$ *or* $l=3$ *and* $c_1^2=3c_2$, $\delta=0$.

Secondly, the assumptions on $|K_X|$ imply that X is minimal, in fact, any exceptional curve is contained in the fixed part of the canonical system.

Thirdly, we observe that $|K_X|$ defines a holomorphic map $f\colon X\to Y$, where Y is a point, a curve or a surface. If Y is a point, i.e. K_X is trivial, we argue as follows: X is either a K-3 surface or a torus (cf. Kodaira, On the Structure of Compact Complex Analytic Surfaces I, Am. Journal of Math. 86 (1964), p. 1423). Since a torus has vectorfields, the last case is ruled out. For a K-3 surface $c_2(X)=24$ (cf. [2]), whereas Lemma 4 shows that $c_2(X)\leqq 0$. So this case is ruled out as well. The remaining two cases are treated separately as follows:

Case 1. Y is a curve.

We shall see that X is in fact a minimal elliptic surface[2]. Since K_X is the inverse image of a line bundle on Y we have that $0=(K_X,K_X)=c_1^2(X)$ and moreover $(K_X,F')=0$, where F' is a general fibre of f. Now apply Stein factorization to f to obtain a connected holomorphic map $p\colon X\to C$, whose general fibre F still satisfies $(K_X,F)=0$. The adjunction formula gives that F is a smooth elliptic curve, so X is indeed (minimal) elliptic and p is an elliptic fibration.

Let me compute the Euler number $e(X)$ in terms of this fibration. If $F_t=p^{-1}(t)$ is any fibre over $t\in C$ the result is: $e(X)=\sum_{t\in S} e(F_t)$, where S is the projection onto C of the points where p is not of maximal rank. So $c_2(X)=e(X)\geqq 0$ with equality if and only if p has only multiple non-singular fibres over S. On the other hand the equality of Lemma 4 gives $c_2(X)\leqq 0$, so indeed we have equality.

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[2] This also follows by the classification theory of surfaces

Claim. X carries a non-zero vector field.

This we see as follows. First suppose $p: X \to C$ has a section – so in particular has no multiple fibres. Then X is a smooth elliptic curve over C and admits a translation invariant non-trivial vector field parallel to the fibres of p. The general case can be reduced to this situation as follows. First, if p has no multiple fibre, but not necessarily a section we reduce to the case where p has a section by a "cutting and repasting"-procedure which preserves the local fibre structure, as described in Kodaira [3], §9. Secondly, if p has multiple fibres C admits a branched covering C' such that the resulting fibration $p': X' \to C'$ is free from multiple fibres (Loc. cit. Thm 6.3). Since X' has been shown to admit a non-trivial vector field parallel to the fibres of p' the image under the covering map $X' \to X$ will be a non-trivial vector field on X. This completes the proof of the Claim.

But this would imply that $H^0(X, T_X) \neq 0$, contrary to the assumptions. This settles Case 1.

Case 2. Y is a surface.

By definition, then X is a (minimal) surface of general type. We have thus the fundamental bound

$$c_1^2(X) \leqq 3c_2(X)$$

due to Miyaoka, [6].

Together with Lemma 4 this implies that $l=2$ and $c_1^2 = 2c_2$ *or* $l=3$ and $c_1^2 = 3c_2$, $\delta = 0$.

This completes the proof in this case.

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