Complete Intersections of Quadrics and Complete Intersections on Segre Varieties with Common Specializations

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Abstract. We investigate whether surfaces that are complete intersections of quadrics and complete intersection surfaces in the Segre embedded product $\mathbf{P}^1 \times \mathbf{P}^k \hookrightarrow \mathbf{P}^{2k+1}$ can belong to the same Hilbert scheme. For $k = 2$ there is a classical example; it comes from K3 surfaces in projective 5-space that degenerate into a hypersurface on the Segre threefold. We show that for $k \geq 3$ there is only one more example. It turns out that its (connected) Hilbert scheme has at least two irreducible components. We investigate the corresponding local moduli problem.

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1 Introduction

This note is motivated by exercise 11, Ch. VIII in [2], where two types of surfaces of degree 8 in $\mathbf{P}^5$ are compared: those that are complete intersections of three quadrics and those that arise as smooth hypersurfaces of bidegree $(2, 3)$ in the Segre embedded $\mathbf{P}^1 \times \mathbf{P}^2$. The exercise asks to show that the latter arise as limits of some well-chosen complete intersection of quadrics. The limit surfaces in the example form a divisor in the boundary of the 19-dimensional family consisting of complete intersections of three quadric hypersurfaces in $\mathbf{P}^5$ and the problem is to make this explicit. We have included a construction in Section 3 where it appears as Theorem 3.1.

We have simplified our original construction following hints from the referee, to whom we express our gratitude. [1]

This phenomenon is restricted to surfaces: if we want to construct higher dimensional examples in a similar fashion, we are doomed to fail since by the Lefschetz hyperplane theorem complete intersections of dimension ≥ 3 have the same second Betti number as the surrounding variety in which they are embedded and so cannot live in a non-trivial product of projective spaces. The simplest generalization for surfaces amounts to a comparison of complete intersection quadrics in $\mathbf{P}^{2k+1}$ and complete intersection surfaces lying on $\mathbf{P}^1 \times \mathbf{P}^k$ for $k \geq 3$ using the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^k \hookrightarrow \mathbf{P}^{2k+1}$. One could of course also compare complete intersection quadrics with complete intersection surfaces lying on products of projective spaces, but computations become quickly very involved. That examples are hard to come by is already apparent within our modest search area. Indeed, the first main new result of this paper is as follows.

Theorem (=Theorem 4.2, Proposition 4.3). *Assume that T is smooth surface with ample canonical system, embedded as complete intersection in $\mathbf{P}^1 \times \mathbf{P}^k$, whose image via the Segre embedding is in the same Hilbert scheme of a complete intersection S of quadrics in $\mathbf{P}^{2k+1}$.*
Then $k = 3$, $c_1^2(S) = c_1^2(T) = 2^7$, $c_2(S) = c_2(T) = 2^8$, and T is a complete intersection of type $(0,4),(4,2)$.

Our next result concerns the moduli of such surfaces. For surfaces of general type one usually fixes several numerical invariants and searches for a moduli space for all surfaces with the given invariants. In our case the surfaces are simply connected and we fix the Chern numbers. Our second main result is as follows.

Theorem (=Theorem 5.11). *Let $\mathcal{M}$ be the moduli space of simply connected minimal surfaces of general type with $c_1^2 = 2^7$ and $c_2 = 2^8$.*
(i) *The Kuranishi space for smooth complete intersections S of five quadrics in $\mathbf{P}^7$ is smooth of dimension 92. The corresponding component $\mathcal{M}_S$ of $\mathcal{M}$ is smooth of the same dimension.*
(ii) *The (restricted) Kuranishi space $\mathcal{M}'_T$ for smooth complete intersections T of type $(0,4)$, $(4,2)$ in $P = \mathbf{P}^1 \times \mathbf{P}^3$ is smooth of dimension 95. The Kuranishi space $\mathcal{M}_T$ itself is smooth.*
(iii) *The closure of the component $\mathcal{M}_T$ in $\mathcal{M}$ does not meet the component $\mathcal{M}_S$: no smooth type T surface degenerates into an intersection of 5 quadrics in $\mathbf{P}^7$.* [2]

Let us compare this result to previous results on moduli of complete intersection surfaces of general type. Of course one always has the component describing smooth complete intersections of the same type and there is a nice

[1] We remark that Beauville's exercise is closely related to Saint-Donat's work on projective models of K3-surfaces [23]; see also [19, Chap. 7].

[2] But smooth complete intersections of quadrics might degenerate into a type T surface.

general theory for this component. See e.g. Benoist's work [3]. However, to our knowledge, no systematic search for further components has been done. There is a famous example due to E. Horikawa [11] concerning quintics in $\mathbf{P}^3$, surfaces with invariants $c_1^2 = 5$, $c_2 = 55$. Here the relevant moduli space has two irreducible components of dimension 40 meeting in a divisor. The components correspond to quintics and double covers of quadric surfaces respectively, and certain "limit" surfaces parametrized by the common divisor. There is no reason to believe that similar behaviour might not occur for other complete intersections. Our example illustrates this for the invariants $c_1^2 = 2^7$ and $c_2 = 2^8$. This example is easier to understand than Horikawa's example, but we want to point out a somewhat unexpected phenomenon: besides the obvious deformations of a type T surface, namely the complete intersections in P of the same type which account for 65 moduli, there are 30 more parameters corresponding to surfaces in P that are not complete intersections. So this is another instance of the phenomenon R. Vakil described in [26].

We also want mention that our results are somewhat orthogonal to the existing classification results for surfaces of general type as described e.g. in [6]. With the exception of the other (older) results [12, 13, 14, 15] of Horikawa, all classification results either concern surfaces with (very) small invariants, or Beauville-type surfaces constructed from group quotients of products of curves.

We finish this introduction with some remarks and questions.
1) As noted above, there are members of the second family that are not complete intersections. Can we write explicit equations for those surfaces?
2) Can we decide whether the second family is complete? In other words, can complete intersections in $\mathbf{P}^1 \times \mathbf{P}^3$ be deformed to surfaces that are not contained in $\mathbf{P}^1 \times \mathbf{P}^3$?
3) What about the topological and differentiable classification of our surfaces? Here a further invariant of the surface S plays a decisive role, the parity of $c_1(S) \in H^2(S, \mathbf{Z})$, that is, the parity of the integer k appearing in the expression $c_1 = k \cdot d$, d primitive. Indeed, since the surfaces are simply connected, the invariants c_1^2, c_2 together with the "parity" of c_1 completely determine the topological type of the surface. See e.g. [1, Chap. VIII, Lemma 3.1, Chap. IX.1]. If k is even, such a surface admits a spin structure. Our construction necessarily gives only surfaces with a spin structure and whenever two of those have the same Chern classes, they must be oriented homeomorphic. The differentiable classification is far more difficult since the only known computable differentiable invariant is the divisibility of the canonical class (under some extra hypotheses that are usually satisfied). This is often used to show that two surfaces are *not* diffeomorphic. See for example [7, 21]. For our examples the divisibility is the same and so this invariant cannot be used. Consequently, the two might or might not be diffeomorphic.
4) Can we decide if the two families have a common (smooth) member? If this is the case, the two types of surfaces are diffeomorphic as well.

2 Numerical invariants

As is well known (see e.g. [1, Ch. IV, §2]) for a complex projective surface S, the *basic triple invariant* $\{b_1(S), c_1^2(S), c_2(S)\}$ completely determines the complex invariants $K_S^2 = c_1^2(S), p_g(S)$ and $q(S)$. In particular, if S is simply connected (and hence $b_1(S) = 0$), the Chern classes suffice for that purpose. Suppose that S comes with a preferred embedding $S \hookrightarrow \mathbf{P}^{n+2}$ as a codimension n submanifold, and $H = \mathcal{O}_{\mathbf{P}^{n+2}}(1)$ is the hyperplane bundle. Then the embedding yields two more invariants, the *basic embedding invariants*:

$$\deg(S) = [S] \cdot H^2 \text{ and } K_S \cdot H|_S \text{ both in } H^{2n+4}(\mathbf{P}^{n+2}) = \mathbf{Z}. \tag{1}$$

Lemma 2.1. *The invariants* (1) *together with the basic triple invariant determine the Hilbert scheme of* $S \hookrightarrow \mathbf{P}^{n+2}$.

Proof. By Hartshorne [10], p. 366, Exercise 1.2, the Hilbert polynomial for a surface S is

$$P_S(z) = \frac{1}{2}az^2 + bz + c, \quad a = \deg S,\ b = \frac{1}{2}\deg S + 1 - \pi,\ c = \chi(\mathcal{O}_S) - 1,$$
$$\pi = \text{genus of the curve } (S \cap H) = \frac{1}{2}(K_S \cdot H + \deg S + 2).$$

∎

Remark 2.2. By [9] the Hilbert scheme H_S of $S \subset \mathbf{P}^{n+2}$ is connected. This implies that if $S' \in H_S$, the surface S can be deformed into S'. In fact Hartshorne in loc. cit. proves that this deformation can be done via a linear deformation. Suppose that the resulting family is through *smooth* surfaces, then by [8] they would be diffeomorphic. In general, all one can say is that S and S' deform to the same surface which may or may not be singular.

We next calculate the basic triples and embedding invariants for smooth complete intersections in a projective space or in a product of two projective spaces. First of all, Lefschetz' theorems imply that these are all simply connected and so $b_1 = 0$.

Example 2.3. 1. *Surface complete intersections of quadrics.* Let $j : S \hookrightarrow P = \mathbf{P}^{2k+1}$ be a smooth complete intersection of $2k-1$ quadrics. We have

$$\deg S = 2^{2k-1}.$$

With H the class of a hyperplane in $H^*(P, \mathbf{Z})$, and $h = j^*H$, the Whitney product relation gives:

$$\begin{aligned} c_1(S) &= -2(k-2)\,h && (2)\\ c_1^2(S) &= 4(k-2)^2 \cdot 2^{2k-1} && (3)\\ c_2(S) &= (2k^2 - 5k + 5)\,h^2 = (2k^2 - 5k + 5) \cdot 2^{2k-1} && (4)\\ -c_1(S) \cdot h &= 2(k-2) \cdot 2^{2k-1}. && (5) \end{aligned}$$

2. *Surface complete intersections in* $P = \mathbf{P}^1 \times \mathbf{P}^k$. The Picard group is given by $= \mathbf{Z}H_1 + \mathbf{Z}H_2$ where H_j is the pull back of the generator of the j-th factor, $j = 1, 2$. For any complete intersection surface $j : T \hookrightarrow P$ we write $h_k = j^*H_k$, $k = 1, 2$. The cohomology class of a complete intersection T of $k-1$ hypersurfaces of bidegrees $(a_1, b_1), \dots, (a_{k-1}, b_{k-1})$ is given by:

$$[T] = (\underbrace{a_1H_1 + b_1H_2}_{F_1}) \cdots (\underbrace{a_{k-1}H_1 + b_{k-1}H_2}_{F_{k-1}}) \in H^*(P, \mathbf{Z}).$$

The intersection table in $H^*(T, \mathbf{Z})$ becomes:

	h_1	h_2
h_1	0	$\mathbf{b}$
h_2	$\mathbf{b}$	$\mathbf{c}$

$$\mathbf{b} := b_1 \cdots b_{k-1},\ \mathbf{c} := \sum_{j=1}^{k-1} a_j \cdot (b_1 \cdots \widehat{b_j} \cdots b_{k-1}) \tag{6}$$

Now $h_1 + h_2$ comes from an ample class, so $0 < h_1 \cdot (h_1 + h_2) = \mathbf{b}$ implies that all $b_i \geq 1$. Since we wish to compare with a surface of general type (a complete intersection of quadrics), the case $a_i = 0$ $(i = 1, \dots, k-1)$ is not of interest to us. Indeed, in our computations later on we will assume and use that al least one of the a_i is positive. In summary:

$$\mathbf{b} \neq 0, \quad \mathbf{c} \neq 0. \tag{7}$$

Let $s : P = \mathbf{P}^1 \times \mathbf{P}^k \hookrightarrow \mathbf{P}^{2k+1}$ be the Segre embedding and let $h = s^*H = h_1 + h_2$, where, as before, H is the hyperplane class of P. Setting

$$\left.\begin{array}{|lcl||lcl|}
\hline
\alpha & = & \sum_{j=1}^{k-1} a_j & \beta & = & \sum_{j=1}^{k-1} b_j \\
\gamma & = & \sum_{i \neq j} a_i b_j & \delta & = & \sum_{i<j} b_i b_j \\
x & = & (\alpha-1)(2\beta-k-1)+ & y & = & \beta^2+ \\
 & & \quad (k+1)-\gamma & & & \quad +(k+1)(\frac{1}{2}k-\beta)-\delta \\
u & = & 2(\alpha-2)(\beta-(k+1)) & v & = & (\beta-(k+1))^2 \\
\hline
\end{array}\right\} \tag{8}$$

one finds first of all

$$\begin{aligned} j^*c(P) &= (1+2h_1)(1+(k+1)h_2 + \frac{1}{2}k(k+1)h_2^2) \\ &= 1 + 2h_1 + (k+1)h_2 + \\ &\qquad + 2(k+1)\,h_1h_2 + \frac{1}{2}k(k+1)\,h_2^2, \end{aligned}$$

$$j^*(1+F_1)\cdots j^*(1+F_{k-1}) = 1 + \alpha h_1 + \beta h_2 + \gamma h_1h_2 + \delta h_2^2,$$

and thus, by the Whitney formula:

$$\begin{aligned} c_1(T) &= (-\alpha+2)h_1 + (-\beta+(k+1))h_2 && (9)\\ c_1^2(T) &= u\mathbf{b} + v\mathbf{c} && (10)\\ c_2(T) &= x\mathbf{b} + y\mathbf{c} && (11)\\ \deg T = h^2 &= 2\mathbf{b} + \mathbf{c} && (12)\\ -c_1(T)\cdot h &= (\alpha+\beta-(k+3))\mathbf{b} + (\beta-(k+1))\mathbf{c}. && (13) \end{aligned}$$

3 Beauville's exercise (the case $k = 2$)

We consider the case $k = 2$. So S is a smooth complete intersection of three quadrics in $\mathbf{P}^5$ which is a $K3$ surface. On the other hand, T is a smooth hypersurface of bidegree $(2, 3)$ in $\mathbf{P}^1 \times \mathbf{P}^2$. By (9) it follows that T is also a K3-surface. Consider its Segre image $s(T)$ in $\mathbf{P}^5$. From from (12) and Table 6 we see that $\deg s(T) = 8 = \deg S$. The equations describing the image of $\mathbf{P}^1 \times \mathbf{P}^2$ in $\mathbf{P}^5$ will appear below after analyzing bihomogeneous polynomials of bidegree $(2, 3)$. The surfaces S and T have the same Hilbert polynomial and so by Lemma 2.1 they belong to the same connected Hilbert scheme. The component to which T belongs has dimension $3 \cdot 10 - 1 = 29$ with the bi-projective group of dimension $3 + 8 = 11$ acting, while standard calculations for complete intersections show that the component to which S belongs has bigger dimension $18 \cdot 3 = 54$. The projective group of dimension $6^2 - 1 = 35$ then acts on the Hilbert scheme with 19-dimensional quotient. This calculation shows that in moduli, the surfaces T give a divisor on the 19-dimensional moduli space of those projective K3 surfaces that have a genus 5 hyperplane section. So there is only one component of the Hilbert scheme and the following theorem is a consequence. We want however to give a constructive proof.

Theorem 3.1. *There exists a one parameter family $\{S_t\}$ whose fibers S_t for small $t \neq 0$ are smooth complete intersections of three quadrics and whose special fiber S_0 is the given Segre embedded surface $s(T)$.*

Proof. Let $R = \mathbf{C}[u, v] \otimes \mathbf{C}[x_1, x_2, x_3]$, the homogeneous coordinate ring of $\mathbf{P}^1 \times \mathbf{P}^2$. Consider a bihomogeneous polynomial of bidegree $(2, 3)$ defining the surface T:

$$F = u^2 C_{11} + uvC_{12} + v^2 C_{22},$$

where the $C_{ij} \in \mathbf{C}[x_1, x_2, x_3]$ are homogeneous cubics. These can be written

$$C_{ij} = \sum_\alpha Q_{ij}^\alpha x_\alpha,$$

for some homogeneous quadratic polynomials[3] Q_{ij}^α. The latter determine a bilinear form q_{ij}^α for which $Q_{ij}^\alpha = q_{ij}^\alpha(\mathbf{x}, \mathbf{x})$, $\mathbf{x} = (x_1, x_2, x_3)$. Then we have

$$\begin{aligned} u^2 C_{11} &= \sum_\alpha x_\alpha q_{11}^\alpha(u\mathbf{x}, u\mathbf{x}) \\ v^2 C_{22} &= \sum_\alpha x_\alpha q_{22}^\alpha(v\mathbf{x}, v\mathbf{x}) \\ uvC_{12} &= \sum_\alpha x_\alpha q_{12}^\alpha(u\mathbf{x}, v\mathbf{x}). \end{aligned}$$

It follows that F can be written in the form $F = \sum_\alpha x_\alpha Q^\alpha$ where $Q^\alpha = q_{11}^\alpha(u\mathbf{x}, u\mathbf{x}) + q_{22}^\alpha(v\mathbf{x}, v\mathbf{x}) + q_{12}^\alpha(u\mathbf{x}, v\mathbf{x})$.

[3] Note that the Q_{ij}^α are not uniquely determined by the C_{ij}.

Next, observe that the Segre embedding is induced by the homomorphism h from the homogeneous coordinate ring $\mathbf{C}[X_1, X_2, X_3, X_1', X_2', X_3']$ of $\mathbf{P}^5$ to R given by

$$(X_1, X_2, X_3, X_1', X_2', X_3') \mapsto (x_1u, x_2u, x_3u, x_1v, x_2v, x_3v).$$

With A_1, A_2, A_3 the subdeterminants of $\begin{pmatrix} X_1 & X_2 & X_3 \\ X_1' & X_2' & X_3' \end{pmatrix}$ obtained by omitting the first, second and third column, respectively, the ideal of $s(\mathbf{P}^1 \times \mathbf{P}^2)$ is generated by A_1, A_2, A_3. Since $h(\sum X_\alpha Q^\alpha) = uF$ and $h(\sum X_\alpha' Q^\alpha) = vF$, the ideal of $s(T)$ is generated by the polynomials $\sum X_\alpha Q^\alpha$, $\sum X_\alpha' Q^\alpha$ and the ideal (A_1, A_2, A_3) of $s(\mathbf{P}^1 \times \mathbf{P}^2) \subset \mathbf{P}^5$.

As a second step, we recall how Pfaffians can be used to describe ideals. Let $M = (M_{ij})$ be a skew symmetric $m \times m$ matrix with entries in a field k, $V = k^m$ with standard basis $\{e_1, \dots, e_m\}$, and set

$$\omega(M) = \sum_{i<j} M_{ij} e_i \wedge e_j \in \Lambda^2 V.$$

If m is even, there is an associated Pfaffian $\operatorname{Pf} M$ given by

$$\underbrace{\omega(M) \wedge \cdots \wedge \omega(M)}_{m/2 \text{ copies}} = \operatorname{Pf} M \cdot e_1 \wedge e_2 \wedge \cdots \wedge e_m \in \Lambda^m V.$$

If m is odd, the Pfaffian is zero by definition. However, for any even subset of $\{1, \dots, m\}$, the corresponding basis elements span an even dimensional subspace of V and the above procedure gives an associated Pfaffian. We apply this construction to the 5×5 skew symmetric matrix

$$M = \begin{pmatrix} 0 & t & X_1 & X_2 & X_3 \\ -t & 0 & X_1' & X_2' & X_3' \\ -X_1 & -X_1' & 0 & Q^3 & -Q^2 \\ -X_2 & -X_2' & -Q^3 & 0 & Q^1 \\ -X_3 & -X_3' & Q^2 & -Q^1 & 0 \end{pmatrix}.$$

By construction, A_1, A_2, A_3 occur in the expression for the 4-th order Pfaffians. Indeed, setting

$$Q_t^\alpha := tQ^\alpha + (-1)^\alpha A_\alpha,$$

these are given by

$$\operatorname{Pf}_{1234} M = Q_t^3 \qquad \operatorname{Pf}_{1235} M = Q_t^2 \qquad \operatorname{Pf}_{1245} M = Q_t^1$$

$$\operatorname{Pf}_{1345} M = \sum_{\alpha=1}^{3} X_\alpha Q^\alpha \qquad \operatorname{Pf}_{2345} M = \sum_{\alpha=1}^{3} X_\alpha' Q^\alpha.$$

For $t = 0$ this gives exactly the ideal of $s(T)$. To treat the case $t \neq 0$, we observe that since the quadrics A_1, A_2, A_3 obey the two relations $\sum (-1)^\alpha X_\alpha A_\alpha = 0$

and $\sum(-1)^\alpha X'_\alpha A_\alpha = 0$, we have two further relations $\sum X_\alpha Q_t^\alpha = t\sum X_\alpha Q^\alpha$ and $\sum X'_\alpha Q_t^\alpha = t\sum X'_\alpha Q^\alpha$. So for $t \neq 0$, the ideal generated by the Pfaffians is the ideal generated by the three quadrics Q_t^α, $\alpha = 1, 2, 3$.
Now invoke the Buchbaum–Eisenbud theorem [5, Sect. 3] stating that such codimension 3 ideals have an explicit 3-step resolution determined by the 4-th order Pfaffians. In our case this resolution is given by

$$0 \to \Lambda^5\mathscr{F} \xrightarrow{{}^{\mathsf{T}}f} \Lambda^4\mathscr{F} \xrightarrow{g} \mathscr{F} \xrightarrow{f} \mathscr{O}_{\mathbf{P}^5}, \quad \mathscr{F} = \oplus^3\mathscr{O}_{\mathbf{P}^5}(-2) \oplus \oplus^2\mathscr{O}_{\mathbf{P}^5}(-3),$$

where $f = (\mathrm{Pf}_{1234}, \mathrm{Pf}_{1235}, \mathrm{Pf}_{1245}, \mathrm{Pf}_{1345}, \mathrm{Pf}_{2345})$ and g is given by a 5×5-matrix built from the Pfaffians. In particular, the generators as well as the syzygies depend polynomially on t and so the family of subvarieties in $\mathbf{P}^5$ it defines, is flat. In particular, for small t, the complete intersection given by (Q_t^1, Q_t^2, Q_t^3) is smooth, independent of which choice we take for the Q_{ij}^α. □

4 Comparison of basic invariants for $k \geq 3$

The integers introduced in the lists (6), (8) come up in the relevant formulae below. Moreover, we recall that $K_T = ah_1 + bh_2$ where

$$\begin{aligned} a &= \alpha - 2 = \sum a_j \,-\, 2, \\ b &= \beta - (k+1) = \sum b_j \,-\, (k+1). \end{aligned}$$

Comparing the two examples 2.3 we find:

Lemma 4.1. *1. The topological invariants $c_1^2(T), c_2(T)$ equal those of a smooth complete intersection S of $(2k-1)$ quadrics in $\mathbf{P}^{2k+1}$ precisely if*

$$2ab\ \mathbf{b} + b^2\ \mathbf{c} = 2^{2k-1}(2(k-2))^2, \tag{14}$$
$$x\mathbf{b} + y\,\mathbf{c} = 2^{2k-1}(2k^2 - 5k + 5). \tag{15}$$

Suppose that K_T is ample. Then $a \geq 0$ and $b \geq 1$. If such a T exists with even a and b it is oriented homeomorphic to a smooth complete intersection of $(2k-1)$ quadrics in $\mathbf{P}^{2k+1}$.
2. The surfaces S and T belong to the same Hilbert scheme if, moreover,

$$2\mathbf{b} + \mathbf{c} = 2^{2k-1}, \tag{16}$$
$$(a+b)\mathbf{b} + b\,\mathbf{c} = 2^{2k-1}(2(k-2)). \tag{17}$$

We first consider the case $k = 3$ and then we have:

$$\mathbf{b} = b_1b_2 = \delta, \quad \gamma = a_1b_2 + a_2b_1 = \mathbf{c}. \tag{18}$$

Theorem 4.2. *A smooth complete intersection T of two hypersurfaces of type $(4,2)$ and $(0,4)$ in $\mathbf{P}^1 \times \mathbf{P}^3$ is oriented homeomorphic to a smooth complete intersection S of 5 quadrics in $\mathbf{P}^7$. This is the only possibility among complete intersections of $\mathbf{P}^1 \times \mathbf{P}^3$. Both surfaces are simply connected, spin and have invariants $c_1 = 2^7$, $c_2 = 2^8$.*
The two types of surfaces belong to the same Hilbert scheme of $\mathbf{P}^7$ when we consider T as embedded in $\mathbf{P}^7$ through the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^3 \hookrightarrow \mathbf{P}^7$. In particular they deform to the same, possibly singular, surface. [4]

Proof. Because of (18), our system of equations reduces to

$$\begin{gathered} 2ab\delta + b^2\gamma = 2^7, \\ (2ab + 4a + 2b + 8 - \gamma)\delta + (b^2 + 4b + 6 - \delta)\gamma = 2^8. \end{gathered}$$

By (7), $\gamma = \mathbf{c} \neq 0$. Rewriting the first equation as $b(2a\delta + b\gamma) = 2^7$, we see that b is a power of 2 and that $b^2 \leq b(2a\delta + b\gamma) = 2^7$ so we conclude that $b = 2^\ell$ with $\ell = 0, 1, 2, 3$. Hence

$$2a\delta + b\gamma = 2^{7-\ell}.$$

Subtracting this twice from the second equation, after some rewriting, yields,

$$(\gamma - (2^\ell + 4))(\delta - (2^\ell + 3)) = (2^\ell + 4)(2^\ell + 3) + 2^{7-\ell} - 2^6.$$

The right-hand side equals $84, 30, 24, 84$, respectively, for $\ell = 0, 1, 2, 3$, respectively.

- Case $\ell = 0$, i.e. $b = 1$. Then $(\gamma - 5)(\delta - 4) = 84 = 7 \cdot 4 \cdot 3$. Now $0 \leq \delta = b_1 b_2 = b_1(b + 4 - b_1) = b_1(5 - b_1) \leq 6$, and $\gamma \geq 0$, so both factors $\gamma - 5$ and $\delta - 4$ must be positive. But then δ must be 6 and γ must be 47. But the equation $2ab\delta + b^2\gamma = 2^7$ reduces to $12a + 47 = 128$ which has no integer solutions.

- Case $\ell = 1$, i.e. $b = 2$. Then $(\gamma - 6)(\delta - 5) = 30 = 2 \cdot 3 \cdot 5$. The solution $\gamma = 0$ and $\delta = 0$ is ruled out, since we saw that $\gamma \neq 0$.

  Otherwise $1 \leq \delta = b_1(6 - b_1) \leq 9$ so that $-4 \leq \delta - 5 \leq 4$. For divisibility reasons, the only possibility for δ is 8 and thus $\gamma = 16$. Then the equation $2ab\delta + b^2\gamma = 2^7$ reduces to $a = 2$. We get

  $$\begin{aligned} \delta &= b_1 b_2 &&= 8, \\ \gamma &= a_1 b_2 + a_2 b_1 &&= 16, \\ a + 2 &= a_1 + a_2 &&= 4. \end{aligned}$$

  The first equation has solutions $(b_1, b_2) = (1, 8), (2, 4)$. The first is incompatible with the other two equations. The second leads to the only solution $(a_1, a_2) = (4, 0)$, $(b_1, b_2) = (2, 4)$ compatible with the three equations.

[4] If it can be chosen to be smooth, the two types of surfaces are diffeomorphic.

- Case $\ell = 2$, i.e. $b = 4$. Then $(\gamma - 8)(\delta - 7) = 24 = 2^3 \cdot 3$. The equation $2ab\delta + b^2\gamma = 2^7$ reduces to $a\delta + 2\gamma = 16$. Now $\delta = b_1(8 - b_1)$ can only assume the values $0, 1 \cdot 7, 2 \cdot 6, 3 \cdot 5$ and $4 \cdot 4$. From divisibility the only possibility left for δ is 15. But then $a\delta + 2\gamma = 16$ implies $a = 0$ and $\gamma = 8$. But $\gamma \neq 8$ because the factor $\gamma - 8$ must be nonzero.
- Case $\ell = 3$. Here we have $8(2a\delta + 8\gamma) = 2^7$ so that $a\delta + 4\gamma = 8$. Since $a \geq 0, \gamma \geq 1$, the only possibilities for γ are 1 and 2, but that conflicts with $(\gamma - 12)(\delta - 11) = 84$.

Concluding, we have shown that the only solution to the first two equations is as stated. However, for this solution, $a = b = 2$ the remaining equations are identical to the first equation and so T and S belong to the same Hilbert scheme. □

We complete the above result by showing that the phenomenon of Theorem 4.2 does not occur for $k \geq 4$:

Proposition 4.3. *If $k \geq 4$ there cannot exist two surfaces S and T of the above type which belong to the same Hilbert scheme.*

Proof. The idea here is to consider the three equations (14), (16), (17) as a system of equations for $\mathbf{b}$, $\mathbf{c}$ with coefficients involving a and b. By (7), if an integer solution exists the rank of the coefficient matrix has to be at most 1. This means that $ab = b^2, a + b = 2b$ and so $a = b$. But then the equations imply that $a = b = 2(k - 2)$. To exclude this solution, argue as follows:

$$\sum_{j=1}^{k-1} a_j = a + 2 = 2(k-1)$$
$$\sum_{j=1}^{k-1} b_j = b + (k+1) = 3(k-1).$$

The maximal value of $\mathbf{b}$ can be computed with the methods of Lagrange multipliers: the maximum for $\mathbf{b}$ occurs for $b_j = 3$ and equals 3^{k-1}. Note that there is an extremal value for $\mathbf{c}$ when $a_j = 2$, $b_j = 3$ but this is not a maximum. But we may use that $\sum_{j \neq i} b_j \leq 3(k-1) - 1$ since $b_i \geq 1$. We then use the Langrange multiplier method for the product of $(k-2)$ different b_j. This gives:

$$\prod_{j \neq i} b_j \leq \left(3 + \frac{2}{k-2}\right)^{k-2}$$

and hence

$$\mathbf{c} \leq \left(3 + \frac{2}{k-2}\right)^{k-2} \cdot \left(\sum_{j=1}^{k-1} a_j\right) = \left(3 + \frac{2}{k-2}\right)^{k-2} \cdot (2(k-1)).$$

But this would imply

$$4^{k-1} = \frac{1}{2}(2\mathbf{b}+\mathbf{c}) \leq 3^{k-2}\left(3+\left(1+\frac{2}{3(k-2)}\right)^{k-2}\cdot(k-1)\right)$$

which is false as soon as $k \geq 6$. To exclude $k = 4, 5$ we have to use that the b_j are positive integers summing up to $3(k-1)$. For $k = 5$, writing down all possibilities for the quadruple (b_1, b_2, b_3, b_4), we see that the product of three among them can be 48 for $(1, 3, 4, 4)$, 45 for $(1, 3, 3, 5)$ and at most 40 for all other quadruples. The first quadruple gives, using that $a_1 + a_2 + a_3 + a_4 = 8$,

$$4^4 = 256 = \frac{1}{2}\mathbf{c}+\mathbf{b} = 24a_1 + 8a_2 + 6(a_3 + a_4) + 48 = 18a_1 + 2a_2 + 6\cdot 8 + 48$$

and so $80 = 9a_1 + a_2$ which has no solutions since $a_1 + a_2 \leq 8$. For $(1, 3, 3, 5)$ we find

$$\begin{aligned} 256 &= \frac{1}{2}(45a_1 + 15a_2 + 15a_3 + 9a_4) + 45 \\ &= \frac{1}{2}(36a_1 + 6a_2 + 6a_3 + 9\cdot 8) + 45 \\ &= 18a_1 + 3(a_2 + a_3) + 81, \end{aligned}$$

which gives a contradiction modulo 3.
In the other cases, we have

$$256 = \frac{1}{2}\mathbf{c}+\mathbf{b} \leq \frac{1}{2}40\cdot 8 + 3^4 = 241,$$

and hence no solution either.
For $k = 4$ there is a solution to $\mathbf{b}+\frac{1}{2}\mathbf{c} = 4^3$, namely $(a_1, a_2, a_3) = (0, 0, 6)$, and $(b_1, b_2, b_3) = (4, 4, 1)$. This can be seen to be the only one: we only have to test whether for each of the values of the triples $(b_1, b_2, b_3) = (1, 1, 7)$, $(1, 2, 6)$, $(1, 3, 5)$, $(1, 4, 4)$, $(2, 2, 5)$, $(2, 3, 4)$, $(3, 3, 3)$, i.e., the positive integral solutions of $b_1 + b_2 + b_3 = 9$, one can find a triple (a_1, a_2, a_3) with $a_1 + a_2 + a_3 = 6$ such that

$$\frac{1}{2}(a_1b_2b_3 + a_2b_1b_3 + a_3b_1b_2) + b_1b_2b_3 = 64.$$

This gives for $(1, 4, 4)$ one solution only, which is the one we had (up to renumbering). For the other triples the argument resembles the one for $k = 5$. To test for instance $(b_1, b_2, b_3) = (1, 2, 6)$, one gets

$$6a_1 + 3a_2 + a_3 + 12 = 5a_1 + 2a_2 + 18 \leq 5\cdot(a_1 + a_2) + 18 \leq 48 < 64.$$

It remains to exclude the solution we found, $(a_1, a_2, a_3) = (0, 0, 6)$, $(b_1, b_2, b_3) = (4, 4, 1)$. For this we observe that it does not satisfy the remaining equation (15) since $x = 22, y = 10$ while $\mathbf{b} = 16, \mathbf{c} = 96$ and thus (15) would give

$$22\cdot 16 + 10\cdot 96 = 17\cdot 128,$$

which is false. $\square$

Remark 4.4. For $k \geq 4$, there could still be solutions to the "topological" equations (14),(15). We have not tested this since these equations become unwieldy. Some experimentation suggest that existence of solutions is very unlikely.
Referring to Remark 2.2, if these do exist, they would give other examples of complete intersection surfaces in $\mathbf{P}^1 \times \mathbf{P}^k$ oriented homeomorphic to complete intersections of quadrics.

Remark 4.5. If we replace $\mathbf{P}^1 \times \mathbf{P}^k$ by $\mathbf{P}^2 \times \mathbf{P}^k$, and try to compare complete intersection surfaces in the latter space with complete intersections of quadrics in $\mathbf{P}^{3k+2}$, we are led to introduce a new variable, since the intersection table corresponding to Table 6 no longer contains a 0. In this case we need to bring more equations into play than the ones corresponding to (16) and (17). The new set of equations doesn't look promising to handle.
In the case $k = 2$, however, a simple argument can be given to exclude solutions based on the fact that the analogs of equations (10), (12), (13) lead to the system of equations

$$\begin{pmatrix} 1 & 2 & 1 \\ a & a+b & b \\ a^2 & 2ab & b^2 \end{pmatrix} \begin{pmatrix} b_1 b_2 \\ a_1 b_2 + a_2 b_1 \\ a_1 a_2 \end{pmatrix} = \begin{pmatrix} 2^6 \\ 3 \cdot 2^6 \\ 3^2 \cdot 2^6 \end{pmatrix},$$

where $a = a_1 + a_2 - 3, b = b_1 + b_2 - 3$.

5 Moduli

5.1 Generalities

We refer to [16, 6.2], [18], [20], [25] for the basics of deformation theory for compact complex manifolds. For the convenience of the reader, we recall a few salient facts we make freely use of below. For a compact complex manifold X, the deformations are governed by the vector spaces $H^i(X, \Theta_X)$, $i = 0, 1, 2$, where Θ_X denotes the holomorphic tangent sheaf of X. To any deformation of X over an analytic space (germ) $(S, 0)$, one associates the *Kodaira–Spencer map* $\kappa : T_0(S) \longrightarrow H^1(\Theta_X)$. In a deformation $\{X_t\}$, $t \in S$, the dimensions $\dim H^1(\Theta_{X_t})$ may jump. It this is not the case, the deformation is said to be *regular*.
The *Kuranishi deformation* of X is a semi-universal deformation. Its Kodaira–Spencer map then is an isomorphism. In fact, its base space, the *Kuranishi space* of X, as a germ can be realized as an analytic subspace of an open neighborhood of 0 in $H^1(\Theta_X)$ with Zariski tangent space $H^1(\Theta_X)$. Smoothness of the Kuranishi space is equivalent to the latter having dimension $H^1(\Theta_X)$.
In our situation we consider deformations of X as a submanifold of a fixed manifold Y. For such an "embedded" family parametrized by S one has a *characteristic map*

$$\sigma : T_0(S) \to H^0(N_{X/Y}) \tag{19}$$

fitting into the following commutative diagram

$$\begin{tikzcd} T_0S \arrow[d, "\sigma"'] \arrow[dr, "\kappa"] & \\ H^0(N_{X/Y}) \arrow[r, "\delta"'] & H^1(\Theta_X), \end{tikzcd}$$

where δ is the connecting homomorphism in the long exact sequence associated to the normal bundle sequence of X in Y. The subspace of *embedded infinitesimal deformations* of X is defined as

$$H^1(\Theta_X)_Y := \mathrm{Im}(\delta) \subset H^1(\Theta_X). \tag{20}$$

Example 5.1 (Hilbert schemes). For this example we refer to [24, Thm. 4.3.5]. Let Y be a smooth projective variety. The Hilbert space parametrizing subschemes of Y with the same Hilbert polynomial $P = P(X)$ as the subvariety $X \subset Y$ exists as a scheme $H^{(P)}$. Moreover, one has a tautological family over it, i.e. the fiber of this family over $[X'] \in H^{(P)}$ is the variety X'. One has

$$T_{[X]}(H^{(P)}) = H^0(X, N_{X/Y}) \text{ and}$$
$$h^0(X, N_{X/Y}) - h^1(X, N_{X/Y}) \leq \dim H_{X/Y} \leq h^0(X, N_{X/Y}).$$

Moreover, if $H^{(P)}_{[X]}$ is the germ of $H^{(P)}$ at $[X]$, we have

$$h^1(X, N_{X/Y}) = 0 \implies H^{(P)} \text{ is smooth at } [X] \text{ and } \dim H^{(P)}_{[X]} = h^0(X, N_{X/Y}).$$

This implies that locally at $[X]$ the tautological family is regular. Conversely, if we have a regular deformation of $X \subset Y$ with smooth base S and surjective characteristic map $\sigma : T_{[X]}S \to H^0(N_{X/Y})$, then the tautological family is regular over $H^{(P)}$, a scheme smooth at $[X]$ and of dimension $h^0(X, N_{X/Y})$. Compare this with Proposition 5.2.

The following result is in essence due to Kodaira and Spencer [18, Chapter V,VI]:

Proposition 5.2. *Let X be a compact complex submanifold of the compact complex manifold Y. Assume that $H^0(\Theta_X) = 0$, that the restriction map $H^0(Y, \Theta_Y) \to H^0(X, \Theta_Y|X)$ is an isomorphism, and that there exists a regular deformation of X within Y with smooth base and with surjective characteristic map*
Then there is a subdeformation with smooth base for which the Kodaira-Spencer map is an isomorphism onto $H^1(\Theta_X)_Y$. This deformation is locally universal for deformations of X within Y. Its dimension equals $\dim H^0(N_{X/Y}) - \dim H^0(\Theta_Y)$.

The condition $H^0(\Theta_X) = 0$ is know to imply that the Kuranishi family is locally universal. Next, consider the long exact sequence associated to the normal bundle sequence, under the hypothesis that $H^0(\Theta_X) = 0$:

$$0 \to H^0(\Theta_Y|X) \to H^0(N_{X/Y}) \xrightarrow{\delta} H^1(\Theta_X) \to H^1(\Theta_Y|X) \to H^1(N_{X/Y}) \cdots .$$

Since the image of δ has codimension $\leq h^1(\Theta_Y|X)$ within $H^1(\Theta_X)$ we conclude:

Corollary 5.3. *Assume that the conditions of Proposition 5.2 hold. If, moreover, $H^0(\Theta_X) = H^1(\Theta_Y|X) = 0$, then the above locally universal embedded deformation is the Kuranishi deformation. Moreover, its base is smooth and of dimension*

$$h^1(\Theta_X) = h^0(N_{X/Y}) - h^0(\Theta_Y|X).$$

5.2 Auxiliary vanishing results

Our situation concerns complete intersection surfaces S in products P of projective spaces[5] , say of codimension c. Then $N_{S/P}$ is the restriction to S of a vector bundle N on P. We make use of the Koszul resolution for $\mathcal{O}_S$ given by

$$0 \to N_c^* \to N_{c-1}^* \to \cdots \to N_1^* \to \mathcal{O}_P \to \mathcal{O}_S \to 0, \quad N_j^* = \Lambda^j N^*. \tag{21}$$

The Koszul sequence gives a resolution of the ideal sheaf $\mathcal{J}_S$.

Lemma 5.4. *Let $\mathcal{F}$ be a locally free sheaf on P. Set $N_0^* = \mathcal{O}_P$.*

1. *If $h^j(\mathcal{F} \otimes N_{j+1}^*) = 0$ for $j = 0, \dots, c-1$, then $h^0(\mathcal{F} \otimes \mathcal{J}_S) = 0$ and if $h^j(\mathcal{F} \otimes N_j^*) = 0$ for $j = 1, \dots, c$ then $h^1(\mathcal{F} \otimes \mathcal{J}_S) = 0$.*
2. *If $h^{j+1}(\mathcal{F} \otimes N_j^*) = 0$ for $j = 0, \cdots, c$, then $h^1(\mathcal{F}|S) = 0$.*
3. *If $h^j(\mathcal{F} \otimes N_j^*) = 0$ for $j = 1, \cdots, c$, then $h^0(\mathcal{F}|S) = \sum_{j=0}^{c} (-1)^j h^0(\mathcal{F} \otimes N_j^*)$.*

Proof. Tensor the long exact sequence (21) with $\mathcal{F}$ and break up the resulting sequence in short sequences, the first of which reads

$$0 \to \mathcal{F} \otimes \mathcal{J}_S \to \mathcal{F} \to \mathcal{F}|S \to 0,$$

the last is of the form

$$0 \to \mathcal{F} \otimes N_c^* \to \mathcal{F} \otimes N_{c-1}^* \to K_{c-1} \to 0,$$

and the intermediate steps $j = 1, \dots, c-2$ are of the form

$$0 \to K_{j+1} \to \mathcal{F} \otimes N_j^* \to K_j \to 0.$$

Now use descending induction. [6] $\square$

[5] For a single projective space see also [22, 24].
[6] This can also be seen using spectral sequences

The vanishing results we need are as follows:

Proposition 5.5. (a) (Cf. [10, III, Theorem 5.1]) $h^i(\mathbf{P}^k, \mathcal{O}(\lambda)) = 0$ *for all* λ *if* $i \neq 0, k$; $h^0(\mathbf{P}^k, \mathcal{O}(\lambda)) = 0$ *if* $\lambda < 0$ *and* $h^k(\mathbf{P}^k, \mathcal{O}(\lambda)) = 0$ *if* $\lambda > -k-1$.
(b) (Cf. [4]) *We have*

- $h^0(\mathbf{P}^k, \Theta_{\mathbf{P}^k}(\lambda)) = 0$ *if* $\lambda \leq -2$.
- $h^q(\mathbf{P}^k, \Theta_{\mathbf{P}^k}(\lambda)) = 0$ *for all* λ *and for* $1 \leq q \leq k-2$.
- $h^{k-1}(\mathbf{P}^k, \Theta_{\mathbf{P}^k}(\lambda)) = 0$ *if* $\lambda \neq -k-1$.
- $h^k(\mathbf{P}^k, \Theta_{\mathbf{P}^k}(\lambda)) = 0$ *for* $\lambda \geq -k-2$.

5.3 Deformations of complete intersections of quadrics

We use:

Corollary 5.6. *Let* $S \subset \mathbf{P}^{2k+1}$ *be a complete intersection of* $(2k-1)$ *quadrics with* $k \geq 3$. *We have*

1. *The restriction map* $H^0(\Theta_{\mathbf{P}^{2k+1}}) \to H^0(\Theta_{\mathbf{P}^{2k+1}}|S)$ *is an isomorphism.*
2. $h^0(N_{S/\mathbf{P}^{2k+1}}) = (2k-1) \cdot \left(\binom{2k+3}{2} - (2k-1)\right)$ *and* $h^0(\Theta_{\mathbf{P}^{2k+1}}|S) = (2k+2)^2 - 1$.
3. $h^1(N_{S/\mathbf{P}^{2k+1}}) = h^1(\Theta_{\mathbf{P}^{2k+1}}|S) = 0$.
4. *The Kuranishi space is smooth of dimension* $h^1(\Theta_S) = 4k^3 - 3k - 7$. *In particular, for* $k = 3$ *we find* $h^1(\Theta_S) = 92$.

Proof. In (21) take $N = \oplus^{2k-1}\mathcal{O}_{\mathbf{P}^{2k+1}}(2)$.
1. In Lemma 5.4.1 take $\mathcal{F} = \Theta_{\mathbf{P}^7}$. The required vanishing conditions follow from Proposition 5.5.
2. and 3. The assertion for $h^i(N_{S/\mathbf{P}^{2k+1}})$, $i = 0, 1$, follows from Lemma 5.4 with $\mathcal{F} = N = \oplus^{2k-1}\mathcal{O}_{\mathbf{P}^{2k+1}}(2)$ since $N_j^* \otimes N$ is a direct sum of line bundles of strictly negative degrees for $j \geq 2$, while $N^* \otimes N$ is a trivial bundle of rank $(2k-1)^2$. To calculate $h^i(\Theta_{\mathbf{P}^{2k+1}}|S)$ for $i = 0, 1$, restrict the Euler sequence

$$0 \to \mathcal{O}_{\mathbf{P}^{2k+1}} \to \oplus^{2k+2}\mathcal{O}_{\mathbf{P}^{2k+1}}(1) \to \Theta_{\mathbf{P}^{2k+1}} \to 0$$

to S and use the corresponding long exact sequence in cohomology. Since $h^1(\mathcal{O}_S) = 0$ and $h^0(\mathcal{O}_S) = 1$, it thus suffices to show that $h^0(\mathcal{O}_S(1)) = h^0(\mathcal{O}_{\mathbf{P}^{2k+1}}(1)) = 2k+2$ and $h^1(\mathcal{O}_S(1)) = 0$. This follows as before from the Koszul resolution since $N_j^* \otimes \mathcal{O}(1)$ is a sum of line bundles of strictly negative degrees for all $j \geq 1$.
4. First observe that the family of smooth complete intersection surfaces of fixed type is regular. Since S is a surface of general type, as noticed before, one has $h^0(\Theta_S) = 0$. Now apply Proposition 5.2 and our previous calculation. $\square$

Remark 5.7. 1. By [3] the relevant component of the moduli space is an affine variety of the given dimension.
2. The exception $k = 2$ in Corollary 5.6 covers the complete intersections S of 3 quadrics in $\mathbf{P}^5$. One has $h^1(\Theta_{\mathbf{P}^5}|S) = 1$.

5.4 Deformations of the complete intersection surface $T \subset P$

Recall that $P = \mathbf{P}^1 \times \mathbf{P}^3$ and let $T \subset P$ be our complete intersection surface. With $p : \mathbf{P}^1 \times \mathbf{P}^3 \to \mathbf{P}^1$, $q : \mathbf{P}^1 \times \mathbf{P}^3 \to \mathbf{P}^3$ the two projections, for any coherent sheaf $\mathcal{F}$ on P, write

$$\mathcal{F}(a,b) = \mathcal{F} \otimes (p^*\mathcal{O}(a) \otimes q^*\mathcal{O}(b)).$$

Note that
$$\Theta_P(a,b) = \mathcal{O}_P(a+2,b) \oplus q^*\Theta_{\mathbf{P}^3}(a,b).$$

We shall be needing the following χ-characteristics.

$$\chi(a,b) = \chi(a)\chi(b) = (a+1)\cdot \binom{b+3}{3},$$
$$\chi(\Theta_P(a,b)) = (2ab+8a+3b+9)\cdot \frac{(b+3)(b+2)}{3}.$$

In our case we have $N = \mathcal{O}_P(4,2) \oplus \mathcal{O}_P(0,4)$ and the Koszul resolution for $\mathcal{J}_T$ gives two exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{F}\otimes \mathcal{J}_T & \to & \mathcal{F} & \to & \mathcal{F}|T & \to & 0 \\ 0 & \to & \mathcal{F}\otimes \Lambda^2 N^* & \to & \mathcal{F}\otimes N^* & \to & \mathcal{F}\otimes \mathcal{J}_T & \to & 0, \end{array} \tag{22}$$

which we use for $\mathcal{F} = N$ and $\mathcal{F} = \Theta_P$. This uses in turn the following numbers:

$$\chi(N) = 85, \quad \chi(N\otimes N^*) = -28, \quad \chi(N^*) = -1$$
$$\chi(\Theta_P) = 18, \quad \chi(\Theta_P \otimes N^*) = -2, \quad \chi(\Theta_P\otimes \Lambda^2 N^*) = 28.$$

Furthermore, one needs the following vanishing results. These can be deduced from Proposition 5.5 together with the Künneth formula for a sheaf $\mathcal{F} = p^*\mathcal{F}_1 \otimes q^*\mathcal{F}_2$ on P, which in our case reads

$$h^{j+1}(\mathcal{F}(a,b)) = h^0(\mathcal{F}_1(a))\cdot h^{j+1}(\mathcal{F}_2(b)) + h^1(\mathcal{F}_1(a))\cdot h^j(\mathcal{F}_2(b)).$$

One gets:

$h^j(\mathcal{O}_P(a,b))$ vanishes	provided
$j=0$	$a<0$ or $b<0$
$j=1$	$a>-2$ or $b<0$
$j=2$	always
$j=3$	$a<0$ or $b>-4$
$j=4$	$a>-4$ or $b>-4$

$h^j(\Theta_{\mathbf{P}^1\times\mathbf{P}^3}(a,b))$ vanishes	provided
$j=0$	if $a<-2$ or $b<-1$
$j=1$	if $a>0$ or $b<-1$
$j=2$	if $a<0$ or $b\neq -4$
$j=3$	$a<-2$ and $b<-4$ or $b>-4$
$j=4$	$a>-2$ or $b>-4$

Using this, the χ-characteristic, as well as the sequences (22), we find:

h^j	$N\otimes$ $\Lambda^2 N^*$	$N\otimes$ N^*	$N\otimes$ $\mathscr{J}_T$	N	$N\vert T$	$\Theta_P\otimes$ $\Lambda^2 N^*$	$\Theta_P\otimes$ N^*	$\Theta_P\otimes$ $\mathscr{J}_T$	Θ_P	$\Theta_P\vert T$
0	0	2	2	85	113	0	0	0	18	18
1	0	30	30	0	1	0	0	0	0	1
2	0	0	1	0	0	0	1	1	0	31
3	1	0	0	0	0	0	3	31	0	0
4	0	0	0	0	0	28	0	0	0	0

We use these calculations to determine the space of embedded infinitesimal deformations of T within P (see (20)).

Corollary 5.8. *We have* $h^1(\Theta_T)_P = 95$.

Proof. This is a consequence of the long exact sequence for the tangent bundle sequence $0 \to \Theta_T \to \Theta_P|T \to N_{T/P} \to 0$ since

$$\dim H^1(\Theta_T)_P = \dim H^0(N_{T/P}) - H^0(\Theta_P|T) = 113 - 18 = 95.$$

$\square$

Next, let us determine a deformation for the surfaces T which is locally universal for deformations within P. We have $83 = (5 \cdot 10 - 1) + (35 - 1)$ parameters for the complete intersections. However, from the above table we see that

$$h^0(T, N|T) = h^0(N) - h^0(N \otimes \mathscr{J}_T) + h^1(N \otimes \mathscr{J}_T) = 83 + 30.$$

So the 83 parameters we found account only for a part of the moduli, the so-called "natural" moduli which come from varying the global equations. The second term in the above expression shows that there are 30 supplementary deformation parameters. We show that these come from deformations of the rank two vector bundle N on P. To do this we invoke some fundamental deformation results which have been collected in Appendix A. See especially Remark A.6.

Proposition 5.9. *The (restricted) Kuranishi space* $\mathscr{M}'_T$ *for deformations of complete intersections* T *of type* $(0,4)$, $(4,2)$ *within* $\mathbf{P}^1 \times \mathbf{P}^3$ *is a smooth variety of dimension* 95.

Proof. As announced, we need to consider the deformations of the bundle N on P. Since

$$\begin{aligned} h^1(\mathrm{End}(N)) &= h^1(N \otimes N^*) = 30, \\ h^2(\mathrm{End}(N)) &= h^2(N \otimes N^*) = 0, \\ h^1(N) &= 0, \end{aligned}$$

the conditions of Corollary A.5 are satisfied; indeed, the remaining conditions on sections of $L_1 \otimes L_2^*$ and its dual are trivially satisfied. Hence there is a regular family of embedded deformations whose characteristic map gives a surjection onto $H^0(T, N_{T/P})$, i.e., one has a complete family of embedded deformations of T. From Theorem 5.2 it follows that the Kuranishi space for the embedded deformations of T within P is smooth of dimension $113 - 18 = 95$. □

Remark 5.10. (1) What about non-embedded deformations? From the long exact sequence for the normal bundle sequence of T in P we find that

$$0 \longrightarrow H^1(\Theta_T)_P \longrightarrow H^1(\Theta_T) \longrightarrow H^1(\Theta_P|T) \xrightarrow{\alpha} H^1(N|T)$$

with $h^1(\Theta_P|T) = h^1(N|T) = 1$. We conclude

$$h^1(\Theta_T) = \dim(\text{embedded defs}) + \dim\ker\alpha = \begin{cases} 95 & \text{if } \alpha \text{ is an isomorphism} \\ 96 & \text{if } \alpha = 0. \end{cases}$$

We were not able to decide which alternative holds. This problem is related to deformability of pairs (T, L) where L is the restriction to T of a line bundle on P: the first alternative holds if and only if all such pairs (T, L) deform. This is true precisely if the cup product pairing

$$\begin{array}{ccc} H^1(\Theta_T) & \xrightarrow{\mu} & \mathrm{Hom}(\mathrm{Pic}(T), H^2(\mathcal{O}_T)) \\ \theta & \mapsto & \mu_\theta, \quad \mu_\theta(L) = \theta \cup c_1(L) \end{array}$$

is identically zero, where we view $c_1(L)$ as a class in $H^1(\Omega^1_T)$. See e.g. [25, Sect. 3.3.3]. Note that the canonical bundle K_T always deforms with T and hence so does any power of K_T.

Supposing that the second alternative holds, i.e., $\alpha = 0$, there would be a 1-dimensional subspace $\mathbf{C}\theta \subset H^1(\Theta_T)$ with $\mathbf{C}\theta \oplus H^1(\Theta_T)_P = H^1(\Theta_P)$ and such that μ_θ vanishes on multiples of the canonical bundle but not on other line bundles. So this subspace would correspond to infinitesimal surface deformations generically having Picard number 1 and not 2, like for K3-surfaces. Since a threefold F of bidegree $(0, 4)$ in $P = \mathbf{P}^1 \times \mathbf{P}^3$ is a product $\mathbf{P}^1 \times S$, with S a K3-surface, it is tempting to make use of the non-deformability of line bundles on S. This points towards the second alternative, suggesting that there are deformations which do *not* stay confined to P.

Regardless of the alternative, one can show, using a spectral sequence argument and the formalism [20] of differential graded Lie-algebra structures on tangent cohomology that the possible extra infinitesimal deformation is not obstructed and thus gives a genuine deformation parametrized by a smooth curve. Moreover, it is then realizable in the canonical embedding $\kappa : T \to \mathbf{P}^{30}$ for which $\kappa^*\mathcal{O}(1) = (2, 2)$ since a standard computation gives that $H^1(\Theta_{\mathbf{P}^{30}}|T) = 0$.

5.5 Comparison of the local moduli calculations

Theorem 5.11. *Let $\mathcal{M}$ be the moduli space of simply connected minimal smooth surfaces of general type with $c_1^2 = 2^7$ and $c_2 = 2^8$.*
(i) *The Kuranishi space for smooth intersections S of five quadrics in $\mathbf{P}^7$ is smooth of dimension $h^1(\Theta_S) = 92$. The corresponding component $\mathcal{M}_S$ of $\mathcal{M}$ is smooth of the same dimension.*
(ii) *The (restricted) Kuranishi space $\mathcal{M}'_T$ for smooth complete intersections T of type $(0,4)$, $(4,2)$ within $P = \mathbf{P}^1 \times \mathbf{P}^3$ is smooth of dimension $h^1(\Theta_T)_P = 95$. The Kuranishi space $\mathcal{M}_T$ itself is smooth.*
(iii) *The closure of the component $\mathcal{M}_T$ in $\mathcal{M}$ does not meet the component $\mathcal{M}_S$: no smooth type T surface degenerates into an intersection of 5 quadrics in $\mathbf{P}^7$.*[7]

Proof. We only have to show the last part. To see this, we may use upper semicontinuity of $h^1(\Theta)$: in a neighborhood of a given point t in the Kuranishi space $h^1(\Theta_t)$ can only decrease. □

We come back to the Hilbert scheme H^P for surfaces $X \subset \mathbf{P}^7$ whose Hilbert polynomial P is determined by $c_1^2 = 2^7, c_2 = 2^8$, $c_1 \cdot h = -2^6, \deg X = 2^5$.

Corollary 5.12. *The Hilbert scheme H^P has at least two components, one of dimension 155 and one of dimension at least 158.*

Proof. In Example 5.1 we recalled some facts about the dimension of the local Hilbert scheme $H^P_{[X]}$ of $X \hookrightarrow \mathbf{P}^n$. Applying this to H^P_S where S is a complete intersection of 5 quadrics in $\mathbf{P}^7$, we find $\dim H^P_{[S]} = h^0(N_{S/\mathbf{P}^7}) = 155$.
For type T surfaces we have constructed a family of deformations of T within P whose characteristic map is an isomorphism and hence the dimension of the component of the local Hilbert scheme at $[T]$ equals $h^0(T, N_{T|P}) = 113$. Segre-embeded in $\mathbf{P}^7$ this yields get a family with smooth base and dimension $113 + (63 - 18) = 158$ (take into account the group of automorphisms). Its characteristic map is an injection onto a subspace of $H^0(T, N_{T|\mathbf{P}^7})$ of dimension 158. So $H^P_{[T]}$ has dimension at least 158. □

Remark 5.13. To determine $\dim H^P_{[T]}$ one can consider the exact sequence coming from the normal bundle sequence in $\mathbf{P}^7$:

$$0 \to H^0(\Theta_{\mathbf{P}^7}|T) \to H^0(N_{T/\mathbf{P}^7}) \to H^1(\Theta_T) \to H^1(\Theta_{\mathbf{P}^7}|T) \xrightarrow{\beta} H^1(N_{T/\mathbf{P}^7}).$$

Using the restriction to T of the Euler sequence, one can show that $h^0(\Theta_{\mathbf{P}^7}|T) = 63$ and $h^1(\Theta_{\mathbf{P}^7}|T) = 1$. We have seen that $h^1(\Theta_T) = 95$ or $= 96$. So, either β is injective, or $\beta = 0$ and in that case $h^1(\Theta_T) = 96$ and $h^0(T, N_{T|\mathbf{P}^7}) = 158$, the local Hilbert scheme H^P at $[T]$ is smooth and of dimension 158. Otherwise, if β is injective, $h^0(T, N_{T|\mathbf{P}^7}) = 158$ or $= 159$. Either way, $H^P_{[T]}$ is smooth of dimension 158 or 159.

[7] But smooth complete intersections of quadrics might degenerate into a type T surface.

A On deformations of vector bundles

Recall that a *deformation* of a vector bundle E on a projective manifold M parametrized by (V, o), is a vector bundle $\mathscr{E}$ on $M \times V$ such that $\mathscr{E}|M \times \{o\} = E$. Then $E_v := \mathscr{E}|M \times \{v\}$ is the deformation of E defined by $v \in V$. First order deformations are those for which we take $(V, o) = (\mathbf{o}, o)$, the thick point, i.e., the one-pointed space with structure sheaf $\mathbf{C}[\epsilon]/\epsilon^2$.

Lemma A.1. *There is a one-to-one correspondence between (isomorphism classes of) first order deformations of a vector bundle E on M and elements of $H^1(M, \mathrm{ad}(E))$, where the trivial deformation corresponds to the origin.*

Sketch of Proof : A first order deformation of E is a vector bundle over $M \times \mathbf{o}$ with the property that it restricts to E over $M \times o$. It is given by a 1-cocycle, say

$$\tilde{\varphi}_{\alpha\beta} = \varphi_{\alpha\beta} + \epsilon \tilde{E}_{\alpha\beta} \in \mathrm{GL}_r(\mathcal{O}(U_{\alpha\beta})) + \epsilon\, \mathrm{End}(\mathcal{O}(U_{\alpha\beta})^{\oplus r}).$$

The cocycle relation yields

$$\tilde{E}_{\alpha\gamma} = \varphi_{\alpha\beta} \tilde{E}_{\beta\gamma} + \tilde{E}_{\alpha\beta} \varphi_{\beta\gamma}.$$

Setting

$$E_{\alpha\beta} = \varphi_{\beta\alpha} \tilde{E}_{\alpha\beta}$$

and making use of the cocycle relations for the $\varphi_{\alpha\beta}$, this yields

$$E_{\alpha\gamma} = E_{\beta\gamma} + \mathrm{ad}(\varphi_{\gamma\beta}) E_{\alpha\beta}.$$

The $\{E_{\alpha\beta}\}$ give a 1-cocycle $\{e_{\alpha\beta}\}$ with values in $\mathrm{ad}(E)$. It is then a standard verification that cocycles which differ by a coboundary yield isomorphic deformations. Reversing the above argument shows that any cohomology class determines a unique first order deformation of E up to isomorphism. □

It follows that to a deformation of E over V one can associate a Kodaira-Spencer map

$$\kappa_E : T_o V \to H^1(M, \mathrm{ad}(E)). \tag{23}$$

Hence giving an element $\sigma \in H^1(M, \mathrm{ad}(E))$ is equivalent to giving a deformation E_σ over $\mathbf{o}$.

Next, we consider deformations of sections following Sernesi's treatment [25, Prop.3.3.4] for the case of line bundles. First observe that as vector bundles the bundles $\mathrm{ad}(E)$ and $\mathrm{End}\, E$ are the same and so $H^q(M, \mathrm{ad}(E)) = H^q(M, E^* \otimes E)$.

Lemma A.2. *Let $\sigma \in H^1(M, \mathrm{ad}(E)) = H^1(M, E^* \otimes E)$ and let E_σ be the corresponding first order deformation of E. A section s of E extends to a section[8] of E_σ if and only if $\sigma \cup s = 0$ where the cup product is the natural product*

$$H^1(M, E^* \otimes E) \otimes H^0(M, E) \to H^1(M, E).$$

[8] Of course if an extension exists, it is in general not unique: think of a trivial deformation of E.

Proof. An extension of s exists precisely if a section $\tilde{s} = s + \epsilon t$ of E_σ exists. In the above trivializations one represents s and t by vector valued functions $s_\alpha \in \mathcal{O}(U_\alpha)^{\oplus r}$ and $t_\alpha \in \mathcal{O}(U_\alpha)^{\oplus r}$ respectively. The condition is that the $s_\alpha + \epsilon t_\alpha$ glue together to give a section of E_σ, which means

$$\begin{aligned} s_\alpha + \epsilon t_\alpha &= (\varphi_{\alpha\beta} + \epsilon \tilde{E}_{\alpha\beta})(s_\beta + \epsilon t_\beta) \\ &= \varphi_{\alpha\beta} s_\beta + \epsilon(\tilde{E}_{\alpha\beta} s_\beta + \varphi_{\alpha\beta} t_\beta) \implies t_\alpha = \tilde{E}_{\alpha\beta} s_\beta + \varphi_{\alpha\beta} t_\beta. \end{aligned}$$

The 1-cocycle described by $u_{\alpha\beta} = \tilde{E}_{\alpha\beta} s_\beta$ actually is the coboundary given by $\{t_\alpha - \varphi_{\alpha\beta} t_\beta\}$. On the other hand, $\{u_{\alpha\beta}\}$ represents the cup product of the class σ with s. □

To extend the above results to deformations over arbitrary parameter spaces, note that the obstructions to extending first order deformations of vector bundles E lie in $H^2(M, \mathrm{ad}(E))$ and obstructions to extending sections are measured by the cup product

$$H^1(M, E^* \otimes E) \otimes H^1(M, E) \to H^2(M, E). \tag{24}$$

Indeed, the following well-known basic result holds (compare e.g. [17, Ch. VII, Theorem 3.23], [25, Prop. 3.3.6]).

Proposition A.3. i) *Suppose that $H^2(M, \mathrm{ad}(E)) = 0$. Then a deformation $\mathcal{E}$ of E exists for which the Kodaira-Spencer map (23) is an isomorphism.*
ii). *Let $\mathcal{E}_\sigma$ be a 1-parameter deformation induced by the family $\mathcal{E}$ from i) with Kodaira-Spencer class σ. Let $s \in H^0(M, E)$ be a section with $\sigma \cup s = 0$ for all $\sigma \in H^1(M, \mathrm{ad}(E))$. If the cup product (24) vanishes, s extends to $\mathcal{E}_\sigma$.*

We now apply this to the particular case where E is a rank 2 split vector bundle $E = L_1 \oplus L_2$ having a section $s = (s_1, s_2)$ whose scheme of zeros is a smooth codimension 2 subvariety Z_s of M, the complete intersection of the zero sets of the sections $s_1 \in H^0(L_1)$ and $s_2 \in H^0(L_2)$. In this case there is a Koszul resolution for the ideal sheaf $\mathcal{J}_{Z_s}$ of Z_s which, after applying $\mathrm{Hom}_{\mathcal{O}_M}(-, \mathcal{O}_{Z_s})$ gives

$$0 \to \mathrm{Hom}(\mathcal{J}_Z/\mathcal{J}_Z^2, \mathcal{O}_{Z_s}) \to E \otimes \mathcal{O}_{Z_s} \xrightarrow{\begin{pmatrix} -s_2 \\ s_1 \end{pmatrix}} \mathrm{Hom}(\Lambda^2 E^*, \mathcal{O}_{Z_s})$$

where the second map is the zero map since $\mathcal{J}_{Z_s} = (s_1, s_2)$. Hence a canonical isomorphism

$$N_{Z_s/M} \simeq E|_{Z_s}.$$

Let δ be the connecting homomorphism in the long exact sequence for

$$0 \to E \otimes \mathcal{J}_{Z_s} \to E \to E|_{Z_s} \to 0$$

and let

$$H^1(E \otimes E^*) \xrightarrow{\cdot s} H^1(E \otimes \mathcal{J}_{Z_s})$$

be the map induced by the Koszul resolution for $\mathcal{J}_{Z_s}$. The preceding two maps fit into the commutative diagram

$$\begin{tikzcd} H^0(E|_{Z_s}) \arrow[r, "\delta"'] & H^1(E \otimes \mathcal{J}_{Z_s}) \arrow[r, equal] & H^1(L_1 \otimes \mathcal{J}_{Z_s}) \oplus H^1(L_2 \otimes \mathcal{J}_{Z_s}) \arrow[r] & H^1(E) \\ & H^1(E \otimes E^*) \arrow[u, "\cdot s"'] \arrow[r, equal] & H^1(L_1 \otimes L_2^*) \oplus H^1(L_1^* \otimes L_2). \arrow[u] & \end{tikzcd} \tag{25}$$

Proposition A.4. *As above, let $Z_s \subset M$, $s = (s_1, s_2)$, where the $s_j \in H^0(M, L_j)$, $j = 1, 2$ vanish along hypersurfaces that intersect transversely in a smooth manifold Z_s. Suppose that there is a section*

$$t \in H^0(Z_s, E|_{Z_s})$$

and some element

$$\sigma \in H^1(M, E \otimes E^*)$$

for which $\sigma \cdot s = -\delta(t)$. Let E_σ be the infinitesimal deformation of E defined by σ. Then there exists a section s_t of E_σ extending s such that its scheme of zeros is precisely $Z_{s,t}$, the infinitesimal deformation of Z_s defined by t.

Proof. First we recall the procedure from [25, Prop. 3.2.1] to describe an element $t \in H^0(E|_{Z_s})$ as an infinitesimal deformation $Z_{s,t}$ of Z_s in M. One has

$$t|_{U_\alpha} \iff (t_\alpha^1, t_\alpha^2) \in \mathcal{O}_{Z_s}(U_\alpha \cap Z_s)^{\oplus 2} \tag{26}$$

transforming in the right way on overlaps. Now lift this last vector valued function to $(\tilde{t}_\alpha^1, \tilde{t}_\alpha^2) \in \mathcal{O}(U_\alpha)^{\oplus 2}$. Following the proof of [25, Prop. 3.2.1] one sees that the ideals

$$(s_\alpha^1 + \epsilon \tilde{t}_\alpha^1, s_\alpha^2 + \epsilon \tilde{t}_\alpha^2) \subset \mathcal{O}(U_\alpha \times \mathbf{o}) \tag{27}$$

glue together and define $Z_{s,t}$. Note also that any two lifts of t over U_α and U_β differ by an element in $(\mathcal{J}_{Z_s} \cap U_{\alpha\beta})^{\oplus 2}$. This yields the 1-cocycle

$$-\delta(t)_{\alpha\beta} = (-\tilde{t}_\beta^1 + \lambda_{\beta\alpha}^1 \tilde{t}_\alpha^1, -\tilde{t}_\beta^2 + \lambda_{\beta\alpha}^2 \tilde{t}_\alpha^2) \tag{28}$$

whose class in $H^1(E \otimes \mathcal{J}_{Z_s})$ by definition represents $-\delta(t)$.
The assumption that $E = L_1 \oplus L_2$ makes it possible to describe the deformations of E by means of extension classes of line bundles. For simplicity, suppose that $H^1(M, \mathcal{O}_M) = 0$, then

$$H^1(\mathrm{ad}(E)) = H^1(L_1 \otimes L_2^*) \oplus H^1(L_2 \otimes L_1^*) = \mathrm{Ext}^1(L_2, L_1) \oplus \mathrm{Ext}^1(L_1, L_2).$$

The direct sum splitting of the spaces on the right of diagram (25) implies that we may assume that the extension class belongs to $\mathrm{Ext}^1(L_2, L_1)$ and so $-\delta(t) = \sigma \cdot s = (*, 0)$.

To continue, let $\{U_\alpha\}$ be a Zariski-open cover and choose trivializations of the line bundles L_j, $j = 1, 2$, so that their transition functions give rise to the 1-cocycle $\lambda^j_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$. Then $\sigma \in \mathrm{Ext}^1(L_2, L_1)$ can be represented by a 1-cocycle $\{e_{\alpha\beta}\}$ with associated first order deformation the vector bundle E_σ given by a 1-cocycle

$$\begin{pmatrix} \lambda^1_{\alpha\beta} & 0 \\ 0 & \lambda^2_{\alpha\beta} \end{pmatrix} + \epsilon \begin{pmatrix} 0 & \lambda^1_{\alpha\beta} E_{\alpha\beta} \\ 0 & 0 \end{pmatrix}. \tag{29}$$

We claim that the searched for extension s_t of the section s to the vector bundle E_σ can be given on $U_\alpha \times \mathbf{o}$ explicitly by

$$(s^1_\alpha + \epsilon \tilde{t}^1_\alpha, s^2_\alpha), \tag{30}$$

where the $\tilde{t}_\alpha$ are the lifts of t_α we used before to describe $Z_{s,t}$. Indeed by (28) and since $(\sigma \cdot s)_{\alpha\beta} = E_{\alpha\beta} s^1_\beta = -\delta(t)_{\alpha\beta}$, we have

$$-\delta(t)_{\alpha\beta} = (-\tilde{t}^1_\beta + \lambda^1_{\beta\alpha} \tilde{t}^1_\alpha, 0) = (E_{\alpha\beta} s^1_\beta, 0).$$

This implies

$$\begin{pmatrix} \lambda^1_{\alpha\beta} & \epsilon\lambda^1_{\alpha\beta} E_{\alpha\beta} \\ 0 & \lambda^2_{\alpha\beta} \end{pmatrix} \begin{pmatrix} s^1_\beta + \epsilon \tilde{t}^1_\beta \\ s^2_\beta \end{pmatrix} = \begin{pmatrix} s^1_\alpha + \epsilon \tilde{t}^1_\alpha \\ s^2_\alpha \end{pmatrix},$$

and so the local sections (30) glue together to a global section s_t. Finally, to finish the proof, note that in this situation the characteristic element (26) is given by $\{(t_\alpha, 0)\}$ and yields a deformation $Z_{s,t}$ with ideal of the form $(s^1_\alpha + \epsilon \tilde{t}^1_\alpha, s^2_\alpha)$ which is precisely the ideal of the variety given by the vanishing of the section s_t. ∎

The preceding construction can be made over a "multidimensional thick point", a point with structure sheaf $\mathbf{C}[t_1, \ldots, t_n]/(t_i t_j)_{1 \le i \le j \le n}$ to give a deformation $E_{\sigma_1, \ldots, \sigma_n}$, $n = h^1(\mathrm{ad}(E))$ where $\{\sigma_j\}$ is a basis incorporating independent directions of deformations. Under the condition that $h^2(\mathrm{ad}(E)) = 0$ this bundle extends to give a vector bundle $\mathcal{E}$ over an open neighborhood of the origin in $H^1(M, \mathrm{ad}(E))$ and, under the condition that $h^1(E) = 0$, any section t of $E|_{Z_s}$ extends to a section s_t of $\mathcal{E}$. Under conditions given below, the dimensions $h^0(E|_{Z_s})$ are constant when s varies in a suitable Zariski-open neighborhood of 0 in $H^0(M, E)$ and then the zero-schemes of s_t for varying s and t yield a regular deformation of $Z = Z_{s_0}$ where s_0 is some fixed section of E transversal to the zero-section. We get:

Corollary A.5. *Let M be a complex projective manifold, and let L_1, L_2 be line bundles with sections $s_j \in H^0(M, L_j)$, $j = 1, 2$ that vanish along two hypersurfaces that intersect transversely in a smooth manifold Z. Set $E = L_1 \oplus L_2$. Suppose*

$$\begin{array}{lclcl} H^1(M, E) & = & H^2(M, E^* \otimes E) & = & 0, \\ H^0(M, \Lambda^2 E^* \otimes E) & = & H^1(M, \Lambda^2 E^* \otimes E) & = & 0, \\ H^0(L_1^* \otimes L_2) & = & H^0(L_1 \otimes L_2^*) & = & 0. \end{array}$$

Then there exists a regular *deformation of Z parametrized by an open neighborhood of the origin in $H^0(M,E) \oplus H^0(M,E^* \otimes E)$ such that its characteristic map surjects onto $H^0(Z,E|Z)$.*

Proof. Proposition A.3 exhibits conditions on E guaranteeing that all obstructions to the relevant embedded deformations vanish. These are fulfilled in our setting. Since $h^0(L_1^* \otimes L_2) = h^0(L_1 \otimes L_2^*) = 0$ we see that $h^0(Z,E|Z) = h^0(M,E) - 2$ and diagram (25) then shows that the other vanishing assumptions imply that $h^0(Z_s, N_{Z_s/M}) = h^0(M,E) - 2 + h^1(M,E^* \otimes E)$ and so this dimension does not depend on s. Hence the embedded deformation whose construction has been outlined above, gives a regular family. □

Remark A.6. For those infinitesimal deformations for which the characteristic element (see (19)) is in the image of the restriction map

$$r_{Z_s} : H^0(M,E) \to H^0(Z_s, E|_{Z_s}),$$

one gets the *natural deformations*, those coming from varying the global sections of E. Explicitly, the varieties $Z_{(s_1+t_1, s_2+t_2)}$ for (t_1,t_2) varying over a suitably small ball centered at the origin of $H^0(M,E)$, define a family whose characteristic map surjects onto the image of r_{Z_s}. Under the hypothesis $H^1(M,E) = 0$, the commutative diagram (25) shows that a complement to this subspace under δ maps bijectively onto its image and, by construction, the infinitesimal deformations of Z_s described by Proposition A.4 cover also this complement.

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