A Campedelli surface with torsiongroup $Z/2$

by F. Oort and C. Peters

Mathematical Institute, Budapestlaan 6, 3508 TA Utrecht, the Netherlands
Mathematical Institute, Wassenaarseweg 80, 2333 AL Leiden, the Netherlands

Communicated by Prof. J.P. Murre at the meeting of November 29, 1980

ABSTRACT

A compact complex surface of general type is constructed, having invariants $p_g = q = 0$, $c_1^2 = 1$, Tor(Pic) $= \mathbb{Z}/2$.

INTRODUCTION

In this note we construct a surface Y of general type having invariants $p_g(Y) = q(Y) = 0, c_1^2(Y) = 1$ and with Tor(Pic $Y) \cong \mathbb{Z}/2$. Here, as usual, $p_g(Y) = h^{0,2}(Y), q(Y) = h^{0,1}(Y)$, and $c_1^2(Y)$ is the Chern number, which can be identified with the self intersection number of the canonical divisor K_Y on Y.

The notion "of general type" means that the Kodaira dimension is two (cf. [7], pp. 125 and 250; cf. [8], p. 500–02). References for facts on surfaces with $p_g(Y) = q(Y) = 0$ can be found in [3], [7] and [8]. In particular, if $c_1^2(Y) = 1$, $|$Tor(Pic $Y)| \leq 5$ and M. Reid [6] has given a complete description of the cases where Y has torsion $\mathbb{Z}/5$, $\mathbb{Z}/4$ and $\mathbb{Z}/3$. The group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ does not occur as torsion group and we know of no example without torsion.*

We notice that in case a plane curve of degree ten could be constructed having certain singularities and decomposing into curves of certain degrees our construction would provide a surface of general type with $p_g(Y) = q(Y) = 0$, $c_1^2(Y) = 1$ and without torsion. Related constructions can be found in [2]. This explains why we call the resulting surfaces "Campedelli surfaces".

* (Added in proof). In the meantime M. Reid informed us of a construction of such a surface by Rebecca Barlow.

Throughout this note we shall use '$\sim$' to mean linear equivalence of divisors and $|\dots|$ will be used to denote a complete linear system of divisors. We work over the complex numbers. All curves considered will consist of components of multiplicity one (i.e. we only consider reduced curves).

1. A PLANE CURVE OF DEGREE 10

We construct a (reducible) plane curve C of degree ten consisting of two cubics C_1, C_2 and two conics Q_1 and Q_2. The curves C_1, Q_1 and Q_2 will be smooth, while C_2 has a unique node at P; the equations for these curves are given at the end of this section. Here an affine coordinate system (X, Y) is chosen and $\alpha \in \mathbb{C}, \alpha \neq 0$. At P the four tangents to the branches of C are distinct. The points $P_2, \dots, P_5$ are triple points resolving into ordinary triple points after one σ-process, while P_1 is a triple point resolving into a node and a smooth branch after two times blowing up.

We take for C_1 the curve with equation

$$Y^2 + X(X-1)(X-\mu) = 0, \text{ shortly } C_1 = 0.$$

In order that C_1 be smooth we must have:

(1.1) $\mu \neq 0, 1.$

The curve C_1 passes through P_1. Intersect C_1 with the line $X - \alpha Y = 0$. We obtain (0,0), P_2 and P_3. Observe that $\alpha^2 C_1 + (X-1)(X-\alpha Y)^2$ is divisible by X, thus of the form XQ_1, for some quadratic form Q_1. The corresponding conic passes through P_1, P_2 and P_3 by construction; in particular it cannot be reducible. Also, by construction $C_1 \cdot Q_1 = 2(P_1 + P_2 + P_3)$.

Replacing Y by $-Y$ we obtain a conic Q_2 through P_1, P_4 and P_5. For the equations of Q_1 and Q_2 we find:

$$Q_1 = \{\alpha^2 Y^2 + (X-1)[(\alpha^2+1)X - \alpha^2\mu - 2\alpha Y] = 0\}$$

and

$$Q_2 = \{\alpha^2 Y^2 + (X-1)[(\alpha^2+1)X - \alpha^2\mu + 2\alpha Y] = 0\}.$$

From these equations it follows that Q_1 and Q_2 intersect only on the X-axis, i.e. in P_1 and $P := (\alpha^2\mu/(1+\alpha^2), 0)$. At P the tangent to Q_1 is neither horizontal, nor vertical, so, by symmetry, it is different from the tangent to Q_2 at P. By Bezout, $i(Q_1, Q_2; P_1) = 3$.

We next try to construct C_2 through $P_2, \dots, P_5$ such that

(i) $i(Q_1, C_2; P_j) = 2, j = 2, 3$ and $i(Q_2, C_2; P_j) = 2$ for $j = 4, 5$,

(ii) C_2 has a node at P.

Observe that, since $(C_2, Q_1) = 6$, these demands automatically imply that the four tangents to the branches in P_6 are distinct. Together with the above remarks this finishes the construction of C.

Notice that setting $Y^2 = X^2/\alpha$ in $C_1 = 0$ gives the vertical lines $X = 0$, $M = 0$ and $M' = 0$. This means that

(1.2) $\alpha^2 MM' = \alpha^2 X^2 + (1 - \alpha^2\mu - \alpha^2)X + \alpha^2\mu$

and we take for C_2 the curve defined by $(\beta-\alpha^4X)C_1+(\alpha^2MM')^2=0$, which in fact is a cubic, since the two termes involving X^4 cancel. Clearly, letting $R=(0:1:0)$, we have $C_1\cdot C_2=2(P_2+P_3+P_4+P_5)+R$. So for all choices (α,β,μ) the cubic C_2 satisfies (i). We show that for some choice of these parameters also (ii) can be satisfied. E.g. take $\alpha=1$ and substitute in (1.2): $(\beta-X)C_1+$ $+(X^2-\mu X+\mu)^2=0$ defines C_2. We let $f(X)$ be the polynomial resulting from the left hand side of this equation after substituting $Y=0$. One checks:
(ii)′ $P=(\mu/2,0)\in C_2\Leftrightarrow f(\mu/2)=0$,
(ii)″ C_2 has a node at P if $f'(\mu/2)=0$, but $f''(\mu/2)\neq 0$.

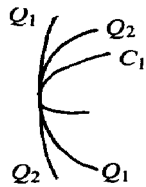
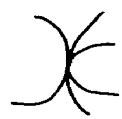
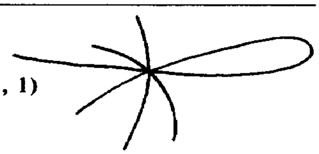
From (ii)′ we find that $\beta=\mu/2+\frac{1}{2}(\mu-4)^2/(\mu-2)$ $(\mu\neq 2)$, while $f'(\mu/2)=0$ if $\beta=\mu-1$, so $\mu=3$ and $\beta=2$. Since $f''(3/2)=-25/4$ we see that (ii)″ is satisfied. Since also (1.1) is satisfied the curve C_1 is smooth for $\mu=3$ and our construction is completed.

Equations for the curves constructed:

$$\begin{aligned} Q_1&: Y^2+(X-1)(2X-3-2Y)=0,\\ Q_2&: Y^2+(X-1)(2X-3+2Y)=0,\\ C_1&: Y^2+X(X-1)(X-3)=0,\\ C_2&: (2-X)C_1+(X^2-3X+3)^2=0 \end{aligned}$$

(for the values $\alpha=1$, $\beta=2$, $\mu=3$).

The singular points of $C=Q_1Q_2C_1C_2$ are:

point	component of Γ through this point	type of singularity	
$P_1=(1,0)$	Q_1, Q_2, C_1	(3; 3; 2,1; 1, 1)*	
$P_2=$, $P_3=$, $P_4=$, $P_5=$ $\left(\frac{3\pm\sqrt{-3}}{2},\frac{3\pm\sqrt{-3}}{2}\right)$	Q_1,C_1,C_2; Q_1,C_1,C_2; Q_2,C_1,C_2; Q_2,C_1,C_2	(3; 3; 1, 1, 1)	
$P=(3/2,0)$	Q_1,Q_2,C_2	(4; 1, 1, 1, 1)	
$R=(0:1:0)$	C_1,C_2	(2; 1, 1)	

* Cf. § 2 for an explanation of this notation.

2. THE SURFACE Y(Γ) AND ITS INVARIANTS

Let there be given a plane curve Γ of even degree. We recall the construction of obtaining a smooth model of the double covering of $\mathbb{P}^2$ branched along Γ (cf. [4]).

A curve C on a surface X is an effective reduced divisor on X. Their singularities we describe by means of repeated blowings up as follows. Let $P \in C$, put $C_0 = C, X_0 = X$ and $P_{0,1} = P$ and inductively define $P_{i,j} \in C_i \subset X_i$ as follows:

$$\pi_{i+1}: X_{i+1} \to X_i,\ i \geq 0$$

is the blowing up of X_i at all points $P_{i,j}$ singular on C_i and C_{i+1} is the strict transform of C_i. Finally we write $P_{i+1,j}$ for the singular points of the reduced curve $\pi_{i+1}^{-1}(C_i)$ lying above P. We write $r_{i,j}$ for the multiplicities of $P_{i,j}$ on C_i and we end the process if all $r_{i,j}$ are 1. (This happens notably after finitely many steps). In this way the singularity $P \in C$ can be characterized by the sequence $(r_{0,1};\ r_{1,1}\ldots;\ r_{2,1}\ldots;\ 1,\ \ldots,\ 1)$. For example

$$(3;\ 3;\ 1,\ 1,\ 1)$$

is the singularity of fig. 1.

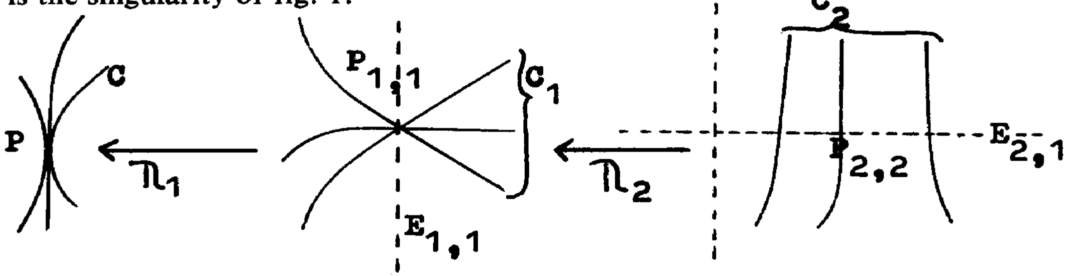


Fig. 1

We write (3; 3; ...) or (4; ...) etc. if the remaining symbols are 2 or 1 (e.g. (3; 3; 2, 1; 1, 1) is a special case). We say that $P \in C$ is a negligible singularity (cf. [5], p. 39, Remark 4) if it is of the form (...) or (3; ...). Cf. fig. 2 for examples.

Negligible singularities

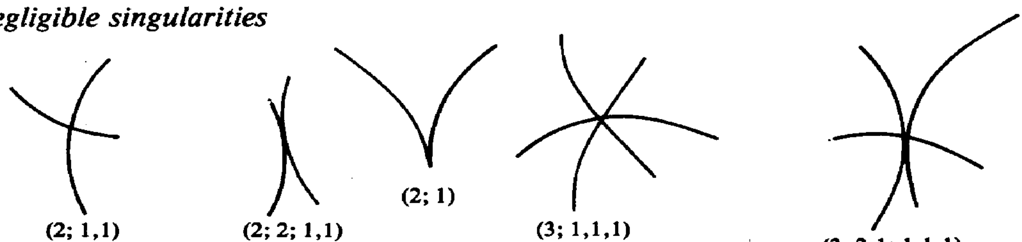


Fig. 2

Let $\Gamma = \Gamma_0$ be a curve on $X_0 = \mathbb{P}^2$. We define $\Gamma_i \subset X_i$ inductively as follows. Let $\{P_{0,j}\}$ be the set of singular points on Γ_0 and let

$$\pi_{i+1}: X_{i+1} \to X_i$$

be the blowing up at the singular points $P_{i,j}$ of X_i and denoting $E_{i,j}=\pi_{i+1}^{-1}$, we define

$$\Gamma_{i+1}=\pi'_{i+1}(\Gamma_i)+\sum s_{i,j}\, E_{i,j}$$

where $s_{i,j}=0$, resp. 1 if mult$(P_{i,j};\ \Gamma_i)$ is even, resp. odd (here π'_{i+1} denotes "strict" transform).

Note that $\Gamma_i \sim 2B_i$ on X_i implies $\Gamma_{i+1}\sim 2B_{i+1}$ for some divisor B_{i+1} on X_{i+1}, so starting with a plane curve Γ of even degree we end up with a smooth curve Γ_n on X_n such that for some divisor B_n we have $\Gamma_n \sim 2B_n$ and we may form the double covering

$$\varrho\colon Z\to X_n$$

branched along Γ_n with *smooth* Z (one takes Z to be the pull back to the total space of the line bundle $[B_n]$ of the section of $[B_n]^{\otimes 2}$ vanishing simply on Γ_n). Blowing down the exceptional curves of Z successively we obtain the minimal surface $Y=Y(\Gamma)$ which we call "Campedelli surface associated to Γ".

PROPOSITION (2.1). *Suppose* $\deg(\Gamma)=10$ *and suppose* Γ *has singularities* P_i *of type* (3; 3; ...) $i=1,\dots,5$, *one singularity* P *of type* (4; ...) *and further at most negligible singularities. Suppose moreover that* $P_1,\dots,P_5,P$ *do not ly on a conic. Then* $Y=Y(\Gamma)$ *is of general type and* $p_g(Y)=q(Y)=0, c_1^2(Y)=1$.

PROOF. We first compute K_Z. We denote by $F_i\subset X_2$ the exceptional divisor lying over $P_i\in\Gamma\subset X_0=\mathbb{P}^2$ obtained by the blowing up $X_2\to X_1$, i.e. if $l_i\in\Gamma_1$ is the singular point of Γ_1 with $\pi_1(l_i)=P_i$, then $F_i=\pi_2^{-1}(l_i)$ $(i=1,\dots,5)$. Furthermore we write $F=\pi_1^{-1}(P)\subset X_1$. We use the same symbols E_i and F for their inverse images on X_n. As in [5], p. 36 we find

$$(2.1)\qquad K_Z\sim\varrho^*(2H-\sum_{i=1}^{5}F_i-F),$$

where H stands for the total transform of a line on $\mathbb{P}^2$. Since $F_i^2=F^2=-1$ we find $K_Z^2=2(4-6)=-4$. The curves $\varrho^{-1}(X_n\to X_1)'(\pi_1^{-1}(P_i))=:\ A_i\subset Z$ are exceptional on Z and blowing them down we obtain a surface Y' with $c_1^2(Y')=+1$. (See [5], pp. 36/37).

We claim:

$$(2.2)\qquad P_2(Z)=P_2(Y')\geq 2 \text{ and } Y' \text{ is minimal.}$$

From this, together with Kodaira's classification theory of surfaces (see e.g. [7], p. 251, Table 20.9) it follows that $Y'=Y(\Gamma)$ is of general type. It then only remains to prove that $p_g(Y)=0$, since then automatically $q(Y)=0$. This together with the inequality $P_2(Y')\geq 2$ follows from a familiar:

LEMMA (2.3). *If* G *is a line bundle on* X_n, *then* $H^0(Z,\varrho^*G)\cong H^0(X_n,G)\oplus\oplus H^0(X_n,G-B_n)$ *is the splitting corresponding to* σ*-invariant and anti-invariant sections* ($\sigma: Z\to Z$ *is the involution defining* ϱ).

Indeed, applying this lemma firstly to $G:=2H-\sum F_i-F$ we find that $h^0(X_n,G)=0=h^0(X_n,G-B_n)$, since $|G|$ corresponds to conics through $P_1,\dots,P_5$ and P. Secondly we apply it to $4H-2\sum F_i-F$ and we find that

$$|2K_Z|=\varrho^*|4H-2\sum F_i-2F|.$$

Let $|D'|$ be the system of quartics $D'\subset\mathbb{P}^2$ having a double point at P and touching P_i at L_i, the tangent cone of $\Gamma_0\subset\mathbb{P}^2$ at $P_i(i=1,\dots,5)$. We verify that

$$(X_n\to X_0)'|D'|+\sum_{i=1}^{5}E_i=|4H-2\sum F_i-2F|,$$

where $E_i=(X_n\to X_1)'(\pi_1^{-1}(P_i))$. So $|2K_Z|=\varrho^*(|D|)+2\sum_{i=1}^{5}A_i$ where

$$D=(X_n\to X_0)'D'$$

and $A_i:=\varrho^{-1}(E_i)$ as before.

The system $\varrho^*|D|$ might contain fixed components, namely possibly in case P_i or P is more complicated than (3; 3; 1, 1, 1) resp. (4; 1, 1, 1, 1). In this case the fixed components do not cut A_i and all have selfintersection ≤ -2, so they do not become exceptional on Y' and Y' must be minimal, because each exceptional curve on Y' has to appear as a fixed component of $|2K_{Y'}|$, the image of $\varrho^*(|D|)$. So $c_1^2(Y')=c_1^2(Y)=1$. Since $P_2(Y')=\dim|D|+1\geq 2$, the assertion (2.2) is proven (in fact we have equality) and the proposition follows.

3. THE TORSION GROUP

We start with a general and more or less well known lemma:

LEMMA (3.1). *Let U be a (possible non-complete) smooth algebraic variety and let σ: $U\to U$ be an involution without fixed points. Denote by V the quotient-variety and let ϱ: $U\to V$ be the canonical projection. There is an exact sequence*

$$0\to\mathbb{Z}/2\to\operatorname{Pic}V\xrightarrow{\varrho^*}(\operatorname{Pic}U)^\sigma\to 0.$$

PROOF. The first direct image sheaf $\varrho_*(\mathcal{O}_U)$ is a locally free rank 2 sheaf with σ-action. The splitting in ± 1 eigensheaves represents this as a direct sum

$$\varrho_*\mathcal{O}_U=\mathcal{O}_V\oplus\mathcal{L},\ \mathcal{L}^{\otimes 2}\cong\mathcal{O}_V,\ \varrho^*\mathcal{L}\cong\mathcal{O}_U.$$

Thus $\mathcal{L}$ generates Ker ϱ^*, and

$$\operatorname{Ker}\varrho^*\cong\mathbb{Z}/2.$$

Secondly we must prove that the image of ϱ^* is $(\operatorname{Pic}U)^\sigma$. Of course, if

$$\mathcal{E}\cong\varrho^*\mathcal{F},\text{ then }\sigma^*\mathcal{E}\cong\mathcal{E},$$

thus Im $\varrho^*\subset(\operatorname{Pic}U)^\sigma$. For the converse it is more convenient to view "Pic" as the group of classes of Weil-divisors with respect to linear equivalence, and to write the group-operation additively. So, suppose that

$$\sigma^*D\sim D,$$

then we have that

$$\sigma^*D - D = \operatorname{div}(f), \text{ for } f \in K_U^*$$

(this is the ring of invertible rational functions on U), and

$$\sigma^*(\sigma^*D - D) = D - \sigma^*D = \operatorname{div}(f^{-1}),$$

hence $\operatorname{div}(\sigma^*f) = \operatorname{div}(f^{-1})$; after multiplying f by a constant we may assume that $\sigma^*f = f^{-1}$. This means that f is a $\mathbb{Z}/2$-cocycle with values in K_U^*, but since

$$H^1(\mathbb{Z}/2, K_U^*) = 0 \text{ (Hilbert's Theorem 90)}$$

we see that f is a 1-coboundary i.e. $f = \sigma^*g \cdot g^{-1}$ for some $g \in K_U^*$. Replacing D by $D + \operatorname{div}(g)$ we see that we actually may assume that $\sigma^*D = D$. Writing $D = D_1 - D_2$ with D_1 and D_2 effective, without common components, it is not hard to see that $\sigma^*D_i = D_i$ $(i = 1,2)$. But an effective divisor, which is σ-invariant of course is the ϱ^*-image of some divisor on V. So $D = \varrho^*C$ for some $C \in \operatorname{Pic} V$. Thus ϱ^* maps Pic V onto $(\operatorname{Pic} U)^\sigma$, and the lemma is proved.

PROPOSITION (3.2) (cf. [1], Section 3, Lemma 2). *Suppose ϱ: $Z \to X$ is a double covering of a surface X with* Tors(Pic X) = 0. *Let I be the set of components of the branch locus Γ, say*

$$I = \{\Gamma_1, \ldots, \Gamma_r\},$$

and define

$$\alpha: \mathbb{Z}^I \to \operatorname{Pic}(X), \alpha_2: \mathbb{Z}^I/2 \cdot \mathbb{Z}^I \to \operatorname{Pic} X/2 \cdot \operatorname{Pic} X$$

by associating to $\sum n_i \Gamma_i$ its class in Pic X. *If $\tau = \alpha(\sum_{i=1}^r \Gamma_i)$, we have*

$$\operatorname{Pic}_2 Z \cong \operatorname{Ker}(\alpha_2)/(\mathbb{Z}/2) \cdot \tau,$$

where the left-hand side is the group of 2*-torsion elements in* Pic Z.

PROOF. We define a map

$$\mu: \operatorname{Ker}(\alpha_2) \to \operatorname{Pic}_2 Z$$

as follows. Let $J \subset I$, and suppose $\sum_{i \in J} \Gamma_i \sim 2A_J$ for some divisor A_J on X. We write $\tilde{\Gamma}_i := \varrho^{-1}(\Gamma_i)$; note that $\sum_{i \in J} \tilde{\Gamma}_i - \varrho^*A_J$ is in $\operatorname{Pic}_2 Z$ since $2\tilde{\Gamma}_i \sim \varrho^*\Gamma_i$; this defines μ.

Let $V := X - \Gamma$ and $U := Z - \varrho^{-1}(\Gamma)$. We obtain the (self explanatory) commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathbb{Z}^I & \longrightarrow & \operatorname{Pic} Z & \xrightarrow{|U} & \operatorname{Pic} U & \longrightarrow & 0 \\
\uparrow {\scriptstyle .2} & & \uparrow {\scriptstyle \varrho^*} & & \uparrow {\scriptstyle (\varrho|U)^*} & & \\
\mathbb{Z}^I & \longrightarrow & \operatorname{Pic} X & \xrightarrow{|V} & \operatorname{Pic} V & \longrightarrow & 0.
\end{array}$$

We claim that Ker $\mu = (\mathbb{Z}/2)\cdot\tau$. Since there exists a double cover branched in $\sum \Gamma_i$, there exists a divisor F on X with

$$\sum_{i\in I} \Gamma_i \sim 2F;$$

then $\sum \tilde{\Gamma}_i \sim \varrho^* F$; hence $\mu(\tau) = 0$. Further, if

$$\sum_{i\in J} \tilde{\Gamma}_i \sim \varrho^* G, \text{ then } (\varrho^* G) \mid U \sim 0,$$

hence $G \mid V \in \mathrm{Ker}((\varrho \mid U)^*)$. By (3.1) this kernel is generated by $F \mid V$, thus

$$\sum_{i\in J} \tilde{\Gamma}_i \sim \varrho^* F \sim \sum_{i\in J} \tilde{\Gamma}_i,$$

hence $J = I$; this proves Ker $\mu = (\mathbb{Z}/2)\cdot\tau$.

Finally we have to prove that μ is surjective. Let $D \in \mathrm{Pic}_2\, Z$. The direct image divisor $\varrho.D$ is torsion in Pic X, hence $\varrho.D \sim 0$, thus

$$\varrho^*(\varrho.D) = D + \sigma^* D$$

is linearly equivalent to zero, hence $D \sim \sigma^* D$. This also holds for the restriction $D \mid U$; we apply (3.1) which shows that there exists a divisor A on X with

$$D \mid U \sim \varrho^* A, \text{ i.e. } D \sim \varrho^* A - \sum_{i\in J} \tilde{\Gamma}_i.$$

Then

$$0 \sim 2D \sim \varrho^*(2A - \sum_{i\in J} \Gamma_i);$$

because $\varrho.\varrho^* C = 2C$ for any divisor C on X, we obtain

$$2(2A - \sum_{i\in J} \Gamma_i) \sim 0,$$

and because Pic X has no torsion, we see that $\sum_{i\in J} \Gamma_i$ is 2-divisible on X. We conclude that μ is onto, and the proposition is proved.

COROLLARY (3.3). *Let $Y = Y(\Gamma)$ be the Campedelli surface constructed in Section 2 with the help of the curve $\Gamma \subset \mathbb{P}^2$ constructed in Section 1. Then*

$$\mathrm{Tor}(\mathrm{Pic}\ Y) \cong \mathbb{Z}/2.$$

PROOF. Notice that torsion in the Picard group remains unchanged under blowing up; hence it suffices to study Tor(Pic Z) where $\varrho\colon Z \to X_n$ is the covering in the Campedelli construction given by Γ. We notice that the elements $\tilde{Q}_1 + \tilde{Q}_2 + \sum_{i=1}^{5} E_i$ and $\tilde{C}_1 + \tilde{C}_2 + E_1$ both give a non-zero contribution to Tor(Pic Z), so (helas) this group is not trivial. By (2.1) we know that Y is of general type, and we know some of its numerical invariants. In the terminology of [4], page 101 therefore it is a "numerical Godeaux surface". By the results of Miyaoka therefore, cf. [4], page 106, Theorem 2′ and Remark, the corollary is proved if we can prove that

$$|3K_Y| \text{ is without base points.}$$

This we prove as follows. We have written $G := 6H - \sum_{i=1}^{5} 3F_i - 3F$, so that $3K_Z \sim \varrho^*G$. Since $B \sim 5H - \sum_{i=1}^{5} (E_i + 3F_i) - 2F - F_7$ (where F_7 comes from (0:1:0)) we have that $G - B \sim H + \sum_{i=1}^{5} E_i + G - F$ and $|3K_Z|$ contains the "anti-invariant divisors" consisting of the branch-locus and the inverse image of a line through P. (Use (2.3).) So the variable part of $|3K_Z|$ can have base points only on the branch locus. Now the "invariant" divisors in $|3K_Z| = |\varrho^*G|$ come from sextics having a triple point at P and double points at $P_1, \ldots, P_5$ with one branch touching the branch locus. This is a bundle of sextics having two base points. Letting $\{Q_3 = 0\}$ be the unique conic through $P, P_2, \ldots, P_5$ one of those sextics is $Q_1Q_2Q_3$. Another one is found by seeking the unique conic Q_4 through P_2, P_3, P_4, P_5 and P_1 which must touch the branch locus at P_1. The sextic $[Q_4C_2 \cdot Y = 0]$ belongs to the system. One verifies that there is a 2-fold base point on X, namely at the intersection of F_1 and the strict transform of Q_3. Since this point does not lie on the branch locus, $\varrho^*|G|$ only has two base points outside the branch locus and $|\varrho^*G|$ has no base points. This ends the proof of Corollary (3.3).

REFERENCES

1. Beauville, A. – Sur le nombre maximum de points doubles d'une surface dans $\mathbb{P}^3$. To appear.
2. Campedelli, L. – Sopra alcuni piani doppi notevoli con curva di diramazioni del decimo ordine. Atti Acad. Naz. Lincei **15**, 536–542 (1932).
3. Dolgachev, I. – Algebraic surfaces with $q = p_g = 0$. Algebraic surfaces, C.I.M.E. 1977. Liguore Ed. Napoli (1979).
4. Miyaoka, Y. – Tricanonical maps of numerical Godeaux surfaces. Invent. Math. **34**, 99–111 (1976).
5. Peters, C. – On two types of surfaces of general type with vanishing geometric genus. Invent. Math. **31**, 43–85 (1976).
6. Reid, M. – Surfaces with $p_g = 0$, $K^2 = 1$. Journ. Fac. Sc., Univ. of Tokyo, Sec IA, Vol. **25**, 75–92 (1978).
7. Ueno, K. – Classification theory of algebraic varieties and compact complex spaces. Lect. Notes Math. 439. Springer Verlag (1975).
8. Ven, A. van de – Some recent results on surfaces of general type. Sém. Bourbaki **29**, no. 500 (1976/77), Springer Lecture Notes in Math. **677**, 155–165 (1978).