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The Local Torelli Theorem

I. Complete Intersections

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§ 0. Introduction

In [4] and [5], Griffiths constructs a generalised period map for algebraic manifolds and asks whether this map is locally injective. This is a problem related to the usual Torelli theorem for curves and therefore called Torelliproblem. In the papers mentioned Griffiths gives a cohomological criterion for solving this problem, provided the moduli are defined for the manifolds in question.

By means of this criterion we prove the local Torelli theorem for complete intersections in projective n-space relative to the holomorphic k-forms, where k is the dimension of the manifold. Because there are no such forms if the canonical bundle is negative, we only have to deal with the case of trivial and ample canonical bundle. In case of surfaces with ample canonical bundle, this provides new examples for which the conjecture stated in [6], Problem 6.1 is true.

As to the organisation of the paper, we collect the necessary background material on deformation theory and the period map in the first two sections. In the next one the moduli are computed for the complete intersections mentioned. Then, in Section 4 and 5, the local Torelli problem is reduced to a question about polynomial ideals. Here we essentially use the criterion given by Griffiths (cf. Section 2). The question on polynomials is solved in Section 6.

We employ the following (partly standard) notations and conventions: If W is a complex manifold, V a submanifold of W and F any holomorphic vector bundle on W, we set:

$F|V$: the restriction of F to V.

Moreover:

0_W: the trivial bundle on W.

T_W: the holomorphic tangent bundle on W.

Ω^d_W: the bundle of holomorphic d-forms on W.

K_W: the canonical bundle on W, i.e. Ω^n_W, where $n = \dim W$.

$N_{V/W}$: the normal bundle of V in W, i.e. the quotient bundle arising in the exact sequence

$$0 \to T_V \underset{i}{\to} T_W | V \to N_{V/W} \to 0. \tag{0.1}$$

Dualising this we find ($p = \operatorname{codim}_W V$):

$$K_V \cong K_W \otimes \wedge^p N_{V/W} \quad \text{(adjunction formula)}. \tag{0.2}$$

We shall often identify a holomorphic vector bundle with its sheaf of holomorphic sections.

In case $W = \mathbb{P}_n$ we drop the subscripts W in the notations 0_W, T_W, etc. If $V \subset \mathbb{P}_n$ is a complete intersection of s hypersurfaces V_k of degree n_k ($k = 1, \ldots, s$) we put: $V = V(n_1, \ldots, n_s) = V(1, \ldots, s)$.

H denotes the hyperplane bundle on $\mathbb{P}_n$, and $F(k) = F \otimes H^k$. Note that the normal bundle of $V(1, \ldots, s)$ in $\mathbb{P}_n$ is isomorphic to $\bigoplus_{k=1}^s 0_V(n_k)$, so by (0.2):

$$K_V \cong 0_V(n_1 + \ldots + n_s - n - 1). \tag{0.3}$$

Therefore we put $\lambda = \sum_{k=1}^s n_k - (n+1)$.

The submodule of the polynomial ring $\mathbb{C}[\xi_0, \ldots, \xi_n]$ consisting of homogeneous polynomials of degree k is denoted as σ_k.

Remark that we may identify $H^0(\mathbb{P}_n, 0(k))$ and σ_k after the choice of a fixed system of homogeneous coordinates on $\mathbb{P}_n$:

$$i_k : \sigma_k \xrightarrow{\sim} H^0(\mathbb{P}_n, 0(k)).$$

We abbreviate:

$$\partial_k F = \partial / \partial \xi_k F \quad \text{for any} \quad F \in \mathbb{C}[\xi_0, \ldots, \xi_n].$$

CAM: connected compact complex manifold.
CAKM: Kähler *CAM*.
PAM: projective manifold.

§ 1. Deformation Theory

We collect some results on deformation theory from [7], [11], and [16]. *A family of CAM's* is a triple $(\mathscr{V}, \pi, B)$ of analytic spaces $\mathscr{V}, B$ and a proper, simple, connected morphism $\pi : \mathscr{V} \to B$. Put $V_b = \pi^{-1}(b)$. *A family of deformations of a CAM* V_0 is a family $(\mathscr{V}, \pi, B)$ with a distinguished point $b \in B$, together with an isomorphism $i : V_0 \to V_b$. Notation $(\mathscr{V}, \pi, B, i, b)$ or $\mathscr{V} \searrow B$ if no confusion arises. A morphism of families of deformations is required to be compatible with the given isomorphisms.

If $\mathscr{V} \searrow B$ is as above and $f : (B', b') \to (B, b)$ is a morphism of pointed spaces $\mathscr{V}_f = \mathscr{V} x_B B'$ becomes, in a natural way, a family of deformations of V_0 over B'; this is *the family induced by f*.

We are only interested in the behavior near the distinguished point of B, which from now on shall be denoted as 0. So we consider two families over B as isomorphic as soon as their restrictions over some neighborhood of $0 \in B$ are isomorphic.

A family $\mathscr{V}\searrow B$ is called *complete* if for any family $\mathscr{W}\searrow B'$ there is a morphism f of some open neighborhood U of $0 \in B'$ into B, such that $\mathscr{W}|U$ and $\mathscr{V}_f$ are isomorphic over U. If the germ of f is unique, i.e. if $\mathscr{V}_g \simeq \mathscr{V}_f$ implies $f = g$, then we call $\mathscr{V}\searrow B$ a *modular family* and $(B, 0)$ a *space of moduli* for V_0. Such modular families do not always exist, but Kuranishi proved that complete families always exist [13]. He constructs a particular family $\mathscr{X}\searrow T$, which is complete. This we call Kuranishi's family. Recall [7]:

Lemma 1.1. *The Zariski tangent space at* $0 \in T$ *is of dimension* $\dim H^1(T_V)$; *furthermore* $\operatorname{codim} T \leqq \dim H^2(T_V)$, *where* T *is considered as an analytic subspace of a polycylinder in* $H^1(T_V)$.

Corollary 1.2. *If* $H^2(T_V) = 0$ *then* T *is non-singular at* 0 *and* $\dim T = \dim H^1(T_V)$.

Corollary 1.3. *If* $\mathscr{V}\searrow B$ *is a modular family, it is isomorphic to* $\mathscr{X}\searrow T$.

Proof. Cf. [16], lemma on page 404.

In [16] some necessary and sufficient conditions are given for V_0 to have a space of moduli. In particular [16], corollary to Theorem 4.2:

Lemma 1.4. *If* T *is reduced and* $\dim H^1(T_{V_t})$ *is independent of* $t \in T$, *there exists a modular family for* V_0.

Remark 1.5. If $H^2(T_V) = 0$ then V need not have a space of moduli. However, in the cases at hand we shall also have $H^0(T_V) = 0$ and it is well known that we have a space of moduli, even a smooth one (cf. [16]).

Next we recall the definition of the Kodaira-Spencer map [11], [7]: Let $(\mathscr{V}, \pi, B)$ be a complex family. Let T_π be the subbundle of $T_{\mathscr{V}}$ consisting of vectorfields tangent along the fibres of π. (Remark that these notions still make sense if B has singularities). There is a bundle sequence on $\mathscr{V}$:

$$0 \to T_\pi \to T_{\mathscr{V}} \to \pi^* T_B \to 0\,.$$

Its restriction to V_t gives:

$$0 - T_{V_t} \to T_{\mathscr{V}}|V_t \to \pi^* T_B| V_t \to 0\,.$$

And the corresponding cohomology sequence provides us with maps:

$$\delta_{*t}\colon H^0(\pi^* T_B | V_t) \to H^1(T_V)\,.$$

Let $T_t(B)$ be the Zariski tangent space at t. Since V_t is compact and connected the map $\pi^*\colon T_t(B) \to H^0(\pi^* T_B | V_t)$ is an isomorphism and we identify these two vectorspaces by means of π^*. So we obtain the map:

$$\varrho_t^{\mathscr{V}}\colon T_t(B) \to H^1(T_{V_t})\,.$$

This is the *Kodaira-Spencer map*. It is easy to see that, if $(F, f)\colon \mathscr{V}\searrow B \to \mathscr{W}\searrow B'$ is a morphism of families of deformations of V_0, there is a commutative diagram:

$$\begin{tikzcd} T_0(B) \arrow[rr, "df"'] \arrow[dr, "\varrho_0^{\mathscr{V}}"'] & & T_0(B') \arrow[dl, "\varrho_0^{\mathscr{W}}"] \\ & H^1(T_{V_0}) & \end{tikzcd}$$

Furthermore it is well known that $\varrho_0^{\mathscr{X}}$ is an isomorphism for the Kuranishi family $\mathscr{X}\searrow T$ (cf. [13], [16]).

Lemma 1.6. *Let B be non-singular. If ϱ_b is an isomorphism for all $b \in B$, then $\mathscr{V}\searrow B$ is a modular family.*

Proof. There is a morphism $(F, f): \mathscr{V}\searrow B \to \mathscr{X}\searrow T$. The above diagram shows that df is injective, hence is bijective because $\dim B = \dim H^1(T_{V_0}) = \dim T$ (cf. 1.1). So (F, f) is a local isomorphism (cf. [7]) and T is non-singular. Now apply 1.4.

Sometimes we shall need the *principle of upper-semicontinuity:*

Theorem 1.7. (cf. [3], [14a]). *Let $\mathscr{E}$ be a coherent analytic sheaf on $\mathscr{V}$. Let $\mathscr{V}\searrow B$ be a family of CAM's. Put $E_b = \mathscr{E}/\mathscr{E} \cdot m_b$ where m_b is the maximal ideal of $b \in B$. There is a neighborhood of $0 \in B$ such that* $\dim H^p(V_b, E_b) \leqq \dim H^p(V_0, E_0)$ *in this neighborhood.*

From now on all our varieties are assumed to be non-singular. If $\mathscr{X}\searrow T$ is a non-singular pair, forming a modular family of V_0 we call $\dim T$ *the number of moduli of* V_0, notation $\mu(V_0)$.

We call $\mathscr{V}\searrow B$ a family of deformations of V in W if

(i) It is a family of deformations of $V = V_0$.

(ii) There is a holomorphic map $\phi: \mathscr{V} \to W$ such that $\phi | V_t$ is an embedding of V_t in W. There is an exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{V_t} & \longrightarrow & T_{\mathscr{V}}|V_t & \longrightarrow & (\pi^* T_B)|V_t & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle d\phi} & & \Big\downarrow & & \\
0 & \longrightarrow & T_{V_t} & \longrightarrow & \phi^* T_W | V_t & \longrightarrow & \phi^* N_{V_t/W} & \longrightarrow & 0.
\end{array}$$

And in cohomology we obtain the commutative diagram:

$$\begin{tikzcd}
T_t(B) \arrow[rr, "\varrho_t"] \arrow[dr, "\sigma_t"'] & & H^1(T_{V_t}) \\
& H^0(N_{V_t/W}) \arrow[ur, "\delta_{*t}"'] &
\end{tikzcd}$$

We quote from [10]:

Theorem 1.8. *If $H^1(N_{V/W}) = 0$, there exists a family of deformations of V in W, say $\mathscr{V}\searrow B$, such that σ_t is an isomorphism for all $t \in B$.*

Corollary 1.9. *If $H^1(N_{V/W}) = H^1(T_W|V) = H^0(T_V) = 0$, then there exists a modular family for V and* $\mu(V) = \dim H^1(T_V)$.

Proof. We take the family from 1.8. Consider the exact sequences

$$0 \to T_{V_t} \to T_W | V_t \to N_{V_t/W} \to 0$$

this gives:

$$0 \to H^0(T_{V_t}) \to H^0(T_W|V_t) \to H^0(N_{V_t/W}) \xrightarrow[\delta_{*t}]{} H^1(T_{V_t}) \to H^1(T_W|V_t) \to \dots$$

Now, for t near $0 \in B$ we have by 1.7 that $H^0(T_{V_t}) = H^1(T_W|V_t) = 0$. Because σ_t is an isomorphism, $\dim H^0(N_{V_t/W})$ is equal to $\dim B$.

So we have: $\dim H^1(T_{V_t}) = \dim B - \dim H^0(T_W|V_t)$, hence by 1.7 $\dim H^1(T_{V_t})$ is constant for $t \in B$. This implies [3], [14a], that $\mathscr{H} = \bigcup_{t \in B} H^1(T_{V_t})$ is a holomorphic vectorbundle. Now $\varrho_0 = \delta_{*0} \cdot \sigma_0$, so is onto and we may choose a submanifold A of B through 0, such that the Kodaira-Spencer map of $\mathscr{V} \searrow A$ is an isomorphism at 0. Now the ϱ_t fit together to give a bundle map $\varrho: T_A \to \mathscr{H}$, which is an isomorphism at 0, hence in a neighborhood of $0 \in A$. So we may apply 1.6 to obtain the result.

For future reference we state a slightly modified form of 1.8 in a special case and its analogous implication:

Theorem 1.8(a) (cf. [12]). *If* $\operatorname{codim}_W V = 1$ *and* $H^1([V]) = 0$, *the conclusion of* 1.8 *holds. Here* $[V]$ *denotes the linebundle on* W *defined by the divisor* V *of* W.

Corollary 1.9(a). *If* $H^1([V]) = H^1(T_W|V) = H^0(T_V) = 0$, *then there exists a modular family for* V *and* $\mu(V) = \dim H^1(T_V)$.

§ 2. The Period Map for Algebraic Manifolds

We recall Griffith's results on the period mapping, see [4] and [5]; A small deformation of a *CAKM* is always a *CAKM* (cf. [11], Theorem 3.1). So if V is a *PAM* we may assume that for any family of deformations of $V = V_0$, B is so small that

(i) All V_t are *CAKM* ($t \in B$).

(ii) $\varphi: \mathscr{V} \xrightarrow{\sim} V \times B$ differentiably.

So we have natural diffeomorphisms $\varphi_t: V_t \to V_0$, the inverse of which we denote by ψ_t. If $\dim V = m$, set $\mathfrak{X} = H^m(V, \mathbb{C})$. Remark $\psi_t^*: H^m(V_t, \mathbb{C}) \xrightarrow{\sim} \mathfrak{X}$. The Hodge decomposition [9]: $H^m(V_t, \mathbb{C}) \simeq \bigoplus_{p+q=m} H^{p,q}(V_t)$ gives a subspace $S_t = \psi_t^* H^{m,0}(V_t)$ of $\mathfrak{X}$. Now $\dim H^m(V_t) = \sum_{p+q=m} \dim H^{p,q}(V_t)$ is constant, whereas each $\dim H^{p,q}(V_t)$ is upper-semicontinuous (1.7), hence is locally constant. So we may assume:

(iii) $\dim H^{m,0}(V_t)$ is locally constant, say k.

Under the assumptions (i) up to (iii) we obtain a map from B into the Grassmann-manifold of k-planes in $\mathfrak{X}$:

$$\Omega: B \to Gr(\mathfrak{X}, k); \quad t \mapsto S_t .$$

this is the period map for $\mathscr{V} \searrow B$. In case there exists a modular family we ask whether Ω is locally injective for this family. This is the Torelli problem. We recall ([5], Theorem 1.1) that Ω is a holomorphic map and we may study the infinitesimal map:

$$\Omega_*: T_0(B) \to T_{\Omega(0)}(Gr(\mathfrak{X}, k)) .$$

Now by [4], Lemma 4.2, $T_{\Omega(0)}(Gr(\mathfrak{X}, k)) \simeq \operatorname{Hom}(H^{m,0}, H^{m-1,1} \oplus \ldots \oplus H^{0,m})$ in a canonical way and we thus obtain:

$$\Omega_*: T_0(B) \to \operatorname{Hom}(H^{m,0}, H^{m-1,1} \oplus \ldots \oplus H^{0,m}) .$$

The pairing $T_V \otimes K_V \to \Omega_V^{m-1}$ defines a cup product (cf. appendix):

$$\cup: H^1(T_V) \otimes H^0(K_V) \to H^1(\Omega_V^{m-1}) .$$

So we get a map:

$$H^1(T_V)\to \mathrm{Hom}\,(H^0(K_V), H^1(\Omega_V^{m-1}))=\mathrm{Hom}\,(H^{m,0}, H^{m-1,1})$$
$$\theta\mapsto\{\hat{\theta}:\kappa\to\theta\cup\kappa\}\,.$$

A central role is played by Proposition 1.20 of [5]:

For all $\lambda\in T_0(B)$, $\Omega_*(\lambda)=\widehat{\varrho_0(\lambda)}$, where ϱ_0 is the Kodaira-Spencer map. In case the moduli are defined and equal to $\dim H^1(T_V)$, the map ϱ_0 is an isomorphism (cf. remark before 1.6). So Ω_* is injective if and only if the cup-product $\cup$ has the following property:

$$\theta\cup\kappa=0 \quad \text{for all} \quad \kappa, \quad \text{implies} \quad \theta=0\,.$$

We call this property, *non-degenerate in the first factor.*

Dualising we find that the map $\theta\mapsto\{\kappa\mapsto\theta\cup\kappa\}$ is into if and only if the dual map:

$$\{H^1(\Omega_V^{m-1})\}^*\otimes H^0(K_V)\mapsto\{H^1(T_V)\}^*$$
$$\varphi\otimes\kappa\mapsto\{\theta\mapsto\varphi(\theta\cup\kappa)\} \quad \text{is onto}\,.$$

In the appendix it is proven that this map corresponds to cup-product if we take the Serre-dual spaces, i.e. the above map is the cup-product

$$\cup_1: H^{m-1}(\Omega_V^1)\otimes H^0(K_V)\to H^{m-1}(\Omega_V^1\otimes K_V)\,.$$

So the above discussion shows that $\cup$ is non-degenerate in the first factor if and only if $\cup_1$ is onto. Resuming:

Proposition 2.1. *Assume V has a modular family $(\mathscr{X}, T)$, where T is a manifold of dimension* $\dim H^1(T_V)$. *Then the period map for this family is locally injective if one of the following equivalent conditions holds:*

(i) $\cup: H^1(T_V)\otimes H^0(K_V)\to H^1(\Omega_V^{m-1})$

is non-degenerate in the first factor.

(ii) $\cup_1: H^{m-1}(\Omega_V^1)\otimes H^0(K_V)\to H^{m-1}(\Omega_V^1\otimes K_V)$

is onto.

Example 2.2. Let V be a *PAM* with $K_V=0_V$ such that $\mu(V)=\dim H^1(T_V)$. Then condition (iii) is trivially fulfilled. In Section 3 we shall prove that complete intersections in $\mathbb{P}_n$ of hypersurfaces of degrees $n_1,\ldots,n_s$ such that $\sum_{k=1}^{s} n_k-(n+1)=0$ belong to this type.

We shall need a special case of 2.1 (i), namely assume V is a submanifold of W. Assume $H^0(K_V)\neq 0$ and study the cohomology diagram of the bundle diagram:

$$\begin{tikzcd}
0 \arrow[r] & T_V \arrow[r, "i"'] \arrow[d, ".s"] & T_W|V \arrow[r, "j"'] \arrow[d, ".s"] & N_{V/W} \arrow[r] \arrow[d, ".s"] & 0 \\
0 \arrow[r] & T_V\otimes K_V \arrow[r, "i\otimes 1"'] & T_W|V\otimes K_V \arrow[r, "j\otimes 1"'] & N_{V/W}\otimes K_V \arrow[r] & 0
\end{tikzcd}$$

where $.s$ denotes the product with $s \in H^0(K_V)$. One finds:

$$\begin{tikzcd}
0 \arrow[r] & H^0(T_V) \arrow[r] \arrow[d, "\cup s"] & H^0(T_W|V) \arrow[r, "j_*"] \arrow[d, "\cup s"] & H^0(N_{V/W}) \arrow[r, "\delta_*"] \arrow[d, "\cup s"] & H^1(T_V) \arrow[r] \arrow[d, "\cup s"] & H^1(T_W|V) \arrow[d, "\cup s"] \\
0 \arrow[r] & H^0(T_V \otimes K_V) \arrow[r] & H^0(T_W|V \otimes K_V) \arrow[r, "(j\otimes 1)_*"] & H^0(N_{V/W} \otimes K_V) \arrow[r, "\delta_*"] & H^1(T_V \otimes K_V) \arrow[r] & \dots
\end{tikzcd}$$

Theorem 2.3. *Assume* $\mu(V) = \dim H^1(T_V)$. *Suppose moreover:*

$$H^0(T_V) = H^0(T_V \otimes K_V) = H^1(T_W|V) = 0$$

then Torelli holds locally if and only if the next condition is fulfilled:

Suppose for $v \in H^0(N_{V/W})$ *we have* $v \cup s \in \operatorname{Im}(j \otimes 1)_*$ *for all* $s \in H^0(K_V)$, then $s \in \operatorname{Im}(j_*)$.

Proof. From the above diagram we see that for any $\theta \in H^1(T_V)$ there is a $v \in H^0(N_{V/W})$ such that $\delta_* v = \theta$. Because of commutativity and exactness:

$$\theta \cup s = 0 \leftrightarrow v \cup s \in \operatorname{Im}(j \otimes 1)_*$$

and

$$\theta = 0 \leftrightarrow v \in \operatorname{Im}(j_*) .$$

Together with 2.1(i) this proves the theorem.

§ 3. Vanishing Theorems for Complete Intersections

We let $V = V(1, \dots, s) = \bigcap_{k=1}^{s} V_k$ where $V_k \subset \mathbb{P}_n$ is a non-singular hypersurface defined by $\varphi_k = 0$ ($\varphi_k \in \sigma_{n_k}$). Put $\mathfrak{p} = \text{ideal}\ (\phi_1, \dots, \phi_s)$. Recall *Bott's theorem* [1]:

Theorem 3.1. $H^p(\mathbb{P}_n, \Omega^q(k)) = 0$, *except for:*
(i) $p=0 \quad k>q$,
(ii) $p=n \quad k<q-n$,
(iii) $p=q \quad k=0$.

The Serre-dual of $H^p(\mathbb{P}_n, T(k))$ is

$$H^{n-p}(\mathbb{P}_n, \Omega^1 \otimes K(-k)) = H^{n-p}(\mathbb{P}_n, \Omega^1(-k-n-1)) ,$$

so we obtain from 3.1

Corollary 3.2. $H^p(\mathbb{P}_n, T(k)) = 0$ *except for:*
(i) $p=n, \quad k<-n-2$,
(ii) $p=0, \quad k>-2$,
(iii) $p=n-1, \quad k=-n-1$.

By means of induction on the codimension of V and the exactness of the sequences:

$$0 \to F(k - n_{\sigma+1}) \xrightarrow[\cdot \phi_{\sigma+1}]{} F(k) \xrightarrow[r]{} F(k)/V(1, \dots, \sigma+1) \to 0$$

valid for any vector bundle F on $V(1, \dots, \sigma)$ we may verify the first two assertions of the next lemma; the last one can be found in [8], Theorem 22.12.

Lemma 3.3. (i) $H^p(V, H^k|V)=0$ *for* $1 \leqq p \leqq \dim V-1$

(ii) $H^p(V, T(k)|V)=0$ *for* $1 \leqq p \leqq \dim V-2$ *and for* $p=\dim V-1$ *in case* $k \neq \lambda$.

(iii) $H^{p,q}(V)=0$ *if* $p \neq q$ *and* $p+q \neq n$.

By means of 3.3(i) we obtain inductively:

Corollary 3.4. *There exists an isomorphism* j_k *such that the next diagram commutes:*

$$\begin{array}{ccc} \sigma_k & \xrightarrow[i_k]{\sim} & H^0(\mathbb{P}_n, H^k) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r_*} \\ \sigma_k/\mathfrak{p} \cap \sigma_k & \xrightarrow[j_k]{\sim} & H^0(V, H^k|V). \end{array}$$

Here r_* *is the restriction map,* i_k *is the map defined in section* 0, *and* q *is the natural quotient-map.*

Lemma 3.3 can also be used to solve the moduli-problem for $V(1, \ldots, s)$ with $\sum n_k-(n+1) \geqq 0$:

Theorem 3.5. *If* V *is a complete intersection of hypersurfaces of degree* n_k $(k=1, \ldots, s)$ *such that* $\lambda=\sum_{k=1}^{s} n_k-(n+1) \geqq 0$, *then* $\mu(V)$ *is defined and equal to* $\dim H^1(T_V)$.

Proof. (i) If V is a curve this is classical (cf. [7])

(ii) If $\dim V \geqq 2$, $\lambda=0$ we have that $H^2(T_V)$ is dual to $H^{m-2}(\Omega_V^1)$ $(m=\dim V)$ [recall that $K_V \cong 0_V(\lambda)$, cf. (0.3]. So $H^2(T_V)=0$, except if $m=3$, by 3.3 (iii). So if $m \neq 3$ we may use 1.5. In case $m=3$ we have that $H^0(T_V)$ is dual to $H^{3,1}(V)=0$, by 3.3 (iii). Moreover $H^1(T|V)=0$ by 3.3 (ii) and $H^1(N_{V/\mathbb{P}_n})=0$, because $N_{V/\mathbb{P}_n} \simeq \bigoplus_{k=1}^{s} \mathcal{O}(n_k)$ and $H^1(V, 0_V(n_k))=0$. Hence we may apply 1.9.

(iii) In case $\lambda>0$ we have $H^1(T|V)=H^1(N_{V/\mathbb{P}_n})=0$ as in (ii). Moreover $H^0(T_V)$ is dual to $H^m(\Omega_V^1 \otimes K_V)$, now K_V is ample, so by [14], Theorem 7.9 we have that $H^{1,m}(K_V)=0$, so $H^0(T_V)=0$, apply then 1.9.

Corollary 3.6. *In case* V *is as in* 3.5 *with* $\lambda=0$, *then the local Torelli theorem holds for* V.

Proof. Use 3.5 and 2.2.

§ 4. Two Fundamental Diagrams

On $\mathbb{P}_n$ we have the exact sequence:

(I) $0 \to 0_{\mathbb{P}_n} \xrightarrow{v} \bigoplus^{n+1} \mathcal{O}(1) \xrightarrow{\pi} T \to 0$

with $v(f)=(f\xi_0, \ldots, f\xi_n)$ and $\pi(L_0, \ldots, L_n)=\sum_{k=0}^{n} L_k \partial/\partial\xi_k$. Indeed exact ness turns out to be equivalent with Euler's relation: if $F \in \sigma_\alpha$, then $\sum_{k=0}^{s} \xi_k \partial_k F=\alpha F$.

Restricting (I) to V we get sequence $(\mathrm{I})_V$ and tensoring with K_V we obtain $(\mathrm{II})_V$. Recall that $K_V \simeq 0_V(\lambda)$. Then by 3.3 and 3.4 we find for the cohomology sequences of (I) and $(\mathrm{I})_V$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r, "\alpha"'] \arrow[d, equal] & \bigoplus^{n+1} \sigma_1 \arrow[r] \arrow[d, "\wr\, r_*"] & H^0(T) \arrow[r] \arrow[d, "\wr\, r_*"] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r, "\nu_*"'] & \bigoplus^{n+1} H^0(\mathcal{O}_V(1)) \arrow[r, "\pi_*"'] & H^0(T|V) \arrow[r] & 0
\end{tikzcd}$$

where $\alpha(c)=(c\xi_0, \dots, c\xi_n)$.

Similarly for $(\mathrm{II})_V$ we have:

$$\begin{tikzcd}
0 \arrow[r] & \sigma_\lambda/\mathfrak{p}\cap\sigma_\lambda \arrow[r] \arrow[d, "\wr\, j_\lambda"] & \bigoplus^{n+1} (\sigma_{\lambda+1}/\mathfrak{p}\cap\sigma_{\lambda+1}) \arrow[d, "\wr\, r_{2*}"] & & \\
0 \arrow[r] & H^0(\mathcal{O}_V(\lambda)) \arrow[r, "\nu_{2*}"'] & \bigoplus^{n+1} H^0(\mathcal{O}_V(\lambda+1)) \arrow[r, "\pi_{2*}"'] & H^0(T_V\otimes K_V) \arrow[r] & 0
\end{tikzcd}$$

where $\alpha_2(f)=(f\xi_0, \dots, f\xi_n)$.

We now use the fact that $j_\lambda : \sigma_\lambda/\mathfrak{p}\cap\sigma_\lambda \xrightarrow{\sim} H^0(K_V)$ (cf. 3.4) to see that from these two diagrams we may derive the next proposition:

Propositions 4.1. *In the preceding diagrams, put* $\beta=\pi_*\circ r_*$, *resp.* $\beta_2=\pi_{2*}\circ r_{2*}$. *Let* $\mathfrak{m}_1$ *be the subspace of* $\bigoplus^{n+1}\sigma_1$ *generated by* $(\xi_0, \dots, \xi_n)$, *resp.* $\mathfrak{m}_2$ *the subspace of* $\bigoplus^{n+1}\sigma_{\lambda+1}$ *spanned by* $\sigma_\lambda(\xi_0, \dots, \xi_n)$ *and* $\bigoplus^{n+1}(\mathfrak{p}\cap\sigma_\lambda)$. *If* $s\in H^0(K_V)$ *and* $S\in\sigma_\lambda$ *such that* $j_\lambda(S)=s$, *there exists a commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}_1 \arrow[r] \arrow[d, "\cdot S"] & \bigoplus^{n+1}\sigma_1 \arrow[r, "\beta"'] \arrow[d, "\cdot S"] & H^0(T|V) \arrow[r] \arrow[d, "\cup s"] & 0 \\
0 \arrow[r] & \mathfrak{m}_2 \arrow[r] & \bigoplus^{n+1}\sigma_{\lambda+1} \arrow[r, "\cdot\beta_2"'] & H^0(T|V\otimes K_V) \arrow[r] & 0\,.
\end{tikzcd}$$

Similarly we have:

Proposition 4.2. *With s and S as in* 4.1 *we have a commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{k=1}^{s}(\mathfrak{p}\cap\sigma_{n_k}) \arrow[r] \arrow[d, "\cdot S"] & \bigoplus_{k=1}^{s}\sigma_{n_k} \arrow[r] \arrow[d, "\cdot S"] & H^0(N_V) \arrow[r] \arrow[d, "\cup s"] & 0 \\
0 \arrow[r] & \bigoplus_{k=1}^{s}(\mathfrak{p}\cap\sigma_{n_k+\lambda}) \arrow[r] & \bigoplus_{k=1}^{s}\sigma_{n_k+\lambda} \arrow[r] & H^0(N_V\otimes K_V) \arrow[r] & 0\,.
\end{tikzcd}$$

§ 5. Reduction of Torelli to a Polynomial Problem

From now on we assume that dim $V\geqq 2$ *and* $\lambda>0$. We study the diagram:

$$\begin{tikzcd}
0 \arrow[r] & T_V \arrow[r, "i"'] \arrow[d, "\cdot s"'] & T|V \arrow[r, "j"'] \arrow[d, "\cdot s"'] & N_{V/\mathbb{P}_n} \arrow[r] \arrow[d, "\cdot s"'] & 0 \\
0 \arrow[r] & T_V\otimes K_V \arrow[r, "i\otimes 1"'] & T|V\otimes K_V \arrow[r, "j\otimes 1"'] & N_{V/\mathbb{P}_n}\otimes K_V \arrow[r] & 0
\end{tikzcd} \qquad (s\in H^0(K_V))\,.$$

We want to apply 2.3. In the proof of 3.5 we derived that $H^0(T_V)=H^1(T|V)=0$; we need only to see that $H^0(T_V\otimes K_V)=0$. But $T_V\otimes K_V\simeq\Omega_V^{m-1}$, so $H^0(\Omega_V^{m-1})=0$ by 3.3. So applying 3.5 we find:

Lemma 5.1. *Torelli holds for V if and only if the next condition is fulfilled: Suppose $v \in H^0(N_{V/\mathbb{P}_n})$ is such that $v \cup \kappa \in \operatorname{Im}(j \otimes 1)_*$ for all $\kappa \in H^0(K_V)$, then $v \in \operatorname{Im} j_*$.*

We want to specify $\operatorname{Im} j_*$ and $\operatorname{Im}(j \otimes 1)_*$. We use 4.1 and 4.2:

Proposition 5.2. *There are commutative diagrams:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}_1 & \longrightarrow & \bigoplus^{n+1} \sigma_1 & \xrightarrow{\beta} & H^0(T|V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle J_1|\mathfrak{m}_1} & & \downarrow{\scriptstyle J_1} & & \downarrow{\scriptstyle j} & & \\ 0 & \longrightarrow & \bigoplus_{k=1}^{s} (\mathfrak{p} \cap \sigma_{n_k}) & \longrightarrow & \bigoplus_{k=1}^{s} \sigma_{n_k} & \xrightarrow{q_1} & H^0(N_{V/\mathbb{P}_n}) & \longrightarrow & 0, \end{array} \qquad \text{(I)}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}_2 & \longrightarrow & \bigoplus^{n+1} \sigma_{\lambda+1} & \xrightarrow{\beta_2} & H^0(T|V \otimes K_V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle J_2|m_2} & & \downarrow{\scriptstyle J_2} & & \downarrow{\scriptstyle (j\otimes 1)_*} & & \\ 0 & \longrightarrow & \bigoplus_{k=1}^{s} (\mathfrak{p} \cap \sigma_{n_k+\lambda}) & \longrightarrow & \bigoplus_{k=1}^{s} \sigma_{n_k+\lambda} & \xrightarrow{q_2} & H^0(N_{V/\mathbb{P}_n} \otimes K_V) & \longrightarrow & 0. \end{array} \qquad \text{(II)}$$

Here $J_i(F_0, \ldots, F_n) = (\sum_k F_k \partial_k \phi_1, \ldots, \sum_k F_k \partial_k \phi_s)$ $(i=1,2)$ where for $i=1$, $F_k \in \sigma_1$ and for $i=2$, $F_k \in \sigma_{\lambda+1}$ for $k=0, \ldots, n$.

Moreover each $s \in H^0(K_V)$ and each S such that $j_\lambda(S) = s$ (notation see 3.4) give rise to a morphism of diagram (I) *to* (II) *in the obvious sence.*

Proof. The usual sequences:

$$0 \to T_{V_k} \to T|V_k \xrightarrow{j_k} 0_{V_k}(n_k) \to 0$$

where

$$j_k\left(\sum_{t=0}^{n} \tau_t \partial/\partial t\right) = \sum_{t=0}^{n} \tau_t \partial_t \phi_k$$

combine to give a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_V & \longrightarrow & T|V & \xrightarrow{\oplus j_k} & N_{V/\mathbb{P}_n} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot s} & & \downarrow{\scriptstyle \cdot s} & & \downarrow{\scriptstyle \cdot s} & & \\ 0 & \longrightarrow & T_V \otimes K_V & \longrightarrow & T|V \otimes K_V & \xrightarrow{j \otimes 1} & N_{V/\mathbb{P}_n} \otimes K_V & \longrightarrow & 0 \end{array}$$

the result then follows from 4.1 and 4.2.

Proposition 5.3. *The local Torelli problem for V is equivalent to the next assertion:*

Let $F_k \in \sigma_{n_k}$ $(k=1, \ldots, s)$. If for all $P \in \sigma_\lambda$ we have:

$$\begin{pmatrix} F_1 \\ \vdots \\ F_s \end{pmatrix} \cdot P \equiv \begin{pmatrix} \partial_0 \phi_1 \ldots \partial_n \phi_1 \\ \vdots \qquad \vdots \\ \partial_0 \phi_s \ldots \partial_n \phi_s \end{pmatrix} \begin{pmatrix} G_0 \\ \vdots \\ G_n \end{pmatrix} \mod \mathfrak{p}$$

then

$$\begin{pmatrix} F_1 \\ \vdots \\ F_s \end{pmatrix} \equiv \begin{pmatrix} \partial_0 \phi_1 \ldots \partial_n \phi_1 \\ \vdots \qquad \vdots \\ \partial_0 \phi_s \ldots \partial_n \phi_s \end{pmatrix} \cdot \begin{pmatrix} L_0 \\ \vdots \\ L_n \end{pmatrix} \mod \mathfrak{p}.$$

Proof. (i) Suppose that this assertion holds. Let $v \in \operatorname{Im} j_*$. Take some $(F_1, \ldots, F_s) \in \bigoplus_k \sigma_{n_k}$ such that $q_1(F_1, \ldots, F_s) = v$ [notation of diagram (I)]. Let $s \in H^0(K_V)$ be arbitrary and $S \in \sigma_{n_\lambda}$ such that $j_\lambda(S) = s$. Then

$$v \cup s = q_1(F_1, \ldots, F_s) \cup j_\lambda(S) = q_2(F_1 S, \ldots, F_s S) \in \operatorname{Im}(j \otimes 1)_*,$$

so by diagram (II) $(F_1 S, \ldots, F_s S) \in (\operatorname{Im} J_2 + \mathfrak{p})$. Hence:

$$\begin{pmatrix} F_1 \\ \vdots \\ F_s \end{pmatrix} \cdot S \equiv \begin{pmatrix} \partial_0 \phi_1 \ldots \partial_n \phi_1 \\ \vdots \qquad \vdots \\ \partial_0 \phi_s \ldots \partial_n \phi_s \end{pmatrix} \begin{pmatrix} G_0 \\ \vdots \\ G_n \end{pmatrix} \text{ mod } \mathfrak{p} \text{ by the definition of } J_2;$$

so the assertion gives $F_k = \sum_{i=0}^n L_i \partial_i \phi_k$ mod $\mathfrak{p}$. But then by diagram (I): $q_1(F_1, \ldots, F_s) \in \operatorname{Im} j_*$.

(ii) The proof of the converse is similar and will be deleted.

In the next section we shall prove (Theorem 6.3) in case all $n_k \geqq 2$:

(*) *If for all* $v = 0, \ldots, n$ *we have:*

$$\begin{pmatrix} F_1 \\ \vdots \\ F_s \end{pmatrix} \xi_v \equiv \begin{pmatrix} \partial_0 \phi_1 \ldots \partial_n \phi_1 \\ \vdots \qquad \vdots \\ \partial_0 \phi_s \ldots \partial_n \phi_s \end{pmatrix} \cdot \begin{pmatrix} A_0 \\ \vdots \\ A_n \end{pmatrix} \text{ mod } \mathfrak{p}$$

where $\deg A_k \leqq \lambda + 1$ *then*

$$\begin{pmatrix} F_1 \\ \vdots \\ F_s \end{pmatrix} \equiv \begin{pmatrix} \partial_0 \phi_1 \ldots \partial_n \phi_1 \\ \vdots \qquad \vdots \\ \partial_0 \phi_s \ldots \partial_n \phi_s \end{pmatrix} \cdot \begin{pmatrix} L_0 \\ \vdots \\ L_n \end{pmatrix} \text{mod } \mathfrak{p}.$$

It is easy to see that (*) implies the second assertion of 5.3 (for instance by induction on deg S) Hence we obtain our main result:

Theorem 5.4. *Let V be a complete intersection of hypersurfaces of degree n_k ($k = 1, \ldots, s$) in $\mathbb{P}_n$ such that $\sum n_k > n+1$, then the local Torelli-theorem for V holds with respect to the periods of the holomorphic $n-s$ forms.*

§ 6. Polynomial Ideals Related to Complete Intersections

In this section we let $\mathfrak{R}$ be the polynomial ring $\mathbb{C}[\xi_0, \ldots, \xi_n]$. Suppose $\mathfrak{a} \subset \mathfrak{R}$ is a homogeneous ideal. The ideal $\mathfrak{a}$ determines a set $V(\mathfrak{a})$ in $\mathbb{P}_n$, i.e. the set of all $x \in \mathbb{P}_n$, such that $f(x) = 0$ for all $f \in \mathfrak{a}$. The ideal $\mathfrak{a}$ has an irredundant decomposition into homogeneous primary ideals, $\mathfrak{a} = \bigcap_{i=1}^r \mathfrak{q}_i$ and this decomposition corresponds to a decomposition of $V(\mathfrak{a})$ into irreducible constituents: $V(\mathfrak{a}) = \bigcup_{i=1}^r V(\mathfrak{q}_i)$.

We recall [15]:

(I) Let $g \in \mathfrak{R}$, then $(\mathfrak{a} : g) = \mathfrak{a} \Leftrightarrow g \notin \sqrt{\mathfrak{q}_i}$ $(i = 1, \ldots, r)$

Because:

(II) $V(\mathfrak{a}) \subseteqq V(\mathfrak{b}) \Leftrightarrow \sqrt{\mathfrak{a}} \supseteqq \sqrt{\mathfrak{b}}$,

we may reformulate (I) as:

(III) Let $g \in \mathfrak{R}$, then $(\mathfrak{a} : g) = \mathfrak{a} \Leftrightarrow V(\mathfrak{q}_i) \not\subset V(g)$ for all $i = 1, \ldots, r$.

We need from [2]:

Lemma 6.1. *If $g_0, \ldots, g_n$ are non-constant homogeneous forms in $\mathfrak{R}$ with no common zero in $\mathbb{P}_n$ and if $P_i \in \mathfrak{R}$ $(i=0, \ldots, n)$ such that $\sum_{i=0}^n P_i g_i = 0$ then there exists a set $\eta_{ij} \in \mathfrak{R}$ such that $P_i = \sum_{j=0}^n \eta_{ij} g_j$ $(i=0, \ldots, n)$. We have $\eta_{ij} = -\eta_{ji}$ and if the P_i are homogeneous such that $\deg(P_i g_i) = M$ is independent of i we may assume η_{ij} to be homogeneous of degree $M - \deg(g_i g_j)$.*

In case $\mathfrak{a} = (g_0, \ldots, g_n)$ and $V(\mathfrak{a}) = \emptyset$, we say $\mathfrak{a}$ is of *the principle class*. We shall apply 6.1 in the following situation:

Let $V(1, \ldots, s)$ be a non-singular complete intersection of $V(\phi_i)$ $(i=1, \ldots, s)$ in $\mathbb{P}_n$ such that for any subset $A \subset \{1, \ldots, s\}$ the set $\bigcap_{i \in A} V(\phi_i)$ is non-singular. Let $\mathfrak{p} = (\phi_1, \ldots, \phi_s)$, $\mathfrak{q} = (\phi_1, \ldots, \phi_{s-1})$.

If $B \subset \{0, \ldots, n\}$ consists of s elements we define $\partial_B \phi_{1,\ldots,s}$ to be the jacobian determinant of $(\partial_\nu \phi_\mu)$ $(\nu \in B, \mu = 1, \ldots, s)$. Let C be a subset of $\{0, \ldots, n\}$ consisting of $s-1$ elements and consider the ideal $\mathfrak{h}$ generated by $\partial_{iC} \phi_{1,\ldots,s}$ $(i \in \{0, \ldots, n\} \setminus C)$.

Lemma 6.2. $(\mathfrak{q}, \mathfrak{h})$ *is of the principle class.*

Proof. We need to see only that $V(\mathfrak{q}, \mathfrak{h}) = \emptyset$. Now $V(\mathfrak{p})$ is non-singular and this means that $V(\mathfrak{p}, \mathfrak{h}') = \emptyset$, where $\mathfrak{h}'$ is generated by all k-th order jacobians $\partial_B \phi_{1,\ldots,s}$. By elementary linear algebra we see that rank $(\partial_\nu \phi_\mu)$ $(\nu = 0, \ldots, n;$ $\mu = 1, \ldots, s)$ is less than s as soon as $\partial_{iC} \phi_{1,\ldots,s} = 0$ for all $i \in \{0, \ldots, n\} \setminus C$. So $V(\mathfrak{p}, \mathfrak{h}) = V(\mathfrak{p}, \mathfrak{h}') = \emptyset$. Next we need an important consequence of *Euler's relation* $[\sum_{a=0}^n \xi_a \partial_a \psi = (\deg \psi) \cdot \psi$ for any homogeneous $\psi \in \mathfrak{R}]$ namely:

$(\mathrm{IV})_C$ $\sum_{a=0}^n \xi_a \partial_{aC} \phi_{1,\ldots,s} \equiv (\deg \phi_s) \cdot \phi_s \partial_C \phi_{1,\ldots,s-1}$ $(\mathfrak{q})$

Suppose $p \in V(\mathfrak{q}, \mathfrak{h})$, then by $(\mathrm{IV})_C$ we have that $\phi_s(p) \partial_C \phi_{1,\ldots,s-1}(p) \neq 0$. By assumption $V(\mathfrak{q})$ is non-singular, so there is a subset C of $\{0, \ldots, n\}$ such that $\partial_C \phi_{1,\ldots,s-1}(p) \neq 0$, so then $p \in V(\mathfrak{p}, \mathfrak{h})$, a contradiction, so $V(\mathfrak{q}, \mathfrak{h}) = \phi$ and this suffices to prove the lemma.

Next we state our main result:

Theorem 6.3. *Suppose for all $\nu = 0, \ldots, n$ we have:*

$$(\mathrm{V})_\nu \quad \begin{pmatrix} F_1 \\ \vdots \\ F_s \end{pmatrix} \xi_\nu \equiv \begin{pmatrix} \partial_0 \phi_1 \ldots \partial_n \phi_1 \\ \vdots \qquad \vdots \\ \partial_0 \phi_s \ldots \partial_n \phi_s \end{pmatrix} \cdot \begin{pmatrix} A_{\nu 0} \\ \vdots \\ A_{\nu n} \end{pmatrix} \bmod \mathfrak{p}$$

where $\deg A_{\nu j} \leqq \lambda + 1$, *then*

$$\begin{pmatrix} F_1 \\ \vdots \\ F_s \end{pmatrix} \equiv \begin{pmatrix} \partial_0 \phi_1 \ldots \partial_n \phi_1 \\ \vdots \qquad \vdots \\ \partial_0 \phi_s \ldots \partial_n \phi_s \end{pmatrix} \cdot \begin{pmatrix} L_0 \\ \vdots \\ L_n \end{pmatrix} \bmod \mathfrak{p}\,.$$

This theorem will follow from the next two lemmas to be proved later on.

Lemma 6.4. *Suppose that*

(α) *$V(1, \ldots, s) \cap V(\xi_i)$ is irreducible for $i = 0, \ldots, n$.*

(β) *$V(1, \ldots, s-q) \cap V(\xi_i)$ is irreducible for $i = 0, \ldots, n$.*

(γ) *given homogeneous A_b of degree $\leqq \lambda + 2$ $(b = 0, \ldots, n)$ such that:*

$\sum_{b=0}^n A_b \partial_b \phi_\beta = 0 \bmod \mathfrak{p}$ for $\beta = 1, \ldots, s$.

then $A_b \equiv B \xi_b \bmod \mathfrak{p}$.

Lemma 6.5. *Suppose that*

(δ) *If* $k \leqq n-3$, $V(1, \ldots, k) \cap V(\xi_i, \xi_j)$ *is irreducible for all pairs* $(i, j) \subset \{0, \ldots, n\}$. *If* $k=n-2$ $V(1, \ldots, k) \cap V(\xi_i, \xi_j, \xi_k) = \emptyset$ *for all triples* $(i, j, k) \subset \{0, \ldots, n\}$.

(ε) *Let* $A_{\nu b}$ *and* $B_{\nu\mu}$ *be homogeneous polynomials* $(\nu, \mu, b = 0, \ldots, n)$ *such that:*

$$A_{\nu b}\xi_\mu - A_{\mu b}\xi_\nu = B_{\mu\nu}\xi_b \bmod \mathfrak{p}$$

then there are polynomials $B'_{\nu b}$ *and* C_ν *such that* $A_{\nu b} = B'_{\nu b}\xi_\nu + C_\nu \xi_b \bmod \mathfrak{p}$.

Proof of 6.3. Conditions (α), (β), and (δ) of 6.4 and 6.5 can assumed to be satisfied by taking an appropriate system of homogeneous coordinates. Now multiply $(V)_\nu$ with ξ_μ and $(V)_\mu$ with ξ_ν and subtract. We see that condition (γ) of 6.4 is satisfied with $A_b := A_{\nu b}\xi_\mu - A_{\mu b}\xi_\nu$, so we obtain the formula $A_{\nu b}\xi_\mu - A_{\mu b}\xi_\nu \equiv B_{\mu\nu}\xi_b \bmod \mathfrak{p}$. Hence we can apply 6.5 to get $A_{\nu b} = B'_{\nu b}\xi_\nu + C_\nu\xi_b \bmod \mathfrak{p}$. Substituting this in $(V)_\nu$ we find:

$$\left\{\begin{pmatrix} F_1 \\ \vdots \\ F_s \end{pmatrix} - \begin{pmatrix} \partial_0\phi_1 \ldots \partial_n\phi_1 \\ \vdots \qquad \vdots \\ \partial_0\phi_s \ldots \partial_n\phi_s \end{pmatrix} \begin{pmatrix} B'_{\nu 0} \\ \vdots \\ B'_{\nu n} \end{pmatrix}\right\} \xi_\nu \equiv \begin{pmatrix} \partial_0\phi_1 \ldots \partial_n\phi_1 \\ \vdots \qquad \vdots \\ \partial_0\phi_s \ldots \partial_n\phi_s \end{pmatrix} \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_n \end{pmatrix} C_\nu (\mathfrak{p}) .$$

By Euler's relation the right hand side is zero mod $\mathfrak{p}$. Now $V(\mathfrak{p})$ is not contained in any hyperplane (all $n_i \geqq 2$), so by (III) we divide out ξ_ν to obtain the desired expression.

Proof of 6.4. We let

$(\mathrm{VI})_b$ $\sum_{a=0}^{n} A_a \partial_a \phi_b \equiv B_b \phi_s \bmod \mathfrak{q}$.

Expand $\partial_B \phi_{1, \ldots, s}$ in subdeterminants as follows. If $B = a \cup C$, then

$$\partial_B \phi_{1, \ldots, s} = \sum_{b=1}^{s-1} (-)^b \, \partial_a \phi_b \cdot \partial_C \phi_{1, \ldots, \hat{b}, \ldots, s} \, .$$

Now multiply $(\mathrm{VI})_b$ with $(-)^b \partial_C \phi_{1, \ldots, \hat{b}, \ldots s}$ and sum up over all $b \in \{1, \ldots, s-1\}$ to obtain with help of this expression:

$(\mathrm{VII})_C$ $\sum_{a=0}^{n} A_a \partial_{aC} \phi_{1, \ldots, s} = \sum_{b=1}^{s-1} (-)^b B_b \phi_s \partial_C \phi_{1, \ldots, \hat{b}, \ldots, s} \bmod \mathfrak{p}$.

Multiply this with $(\deg \phi_s) \cdot \partial_C \phi_{1, \ldots, s-1}$ and use $(\mathrm{IV})_C$:

$$\sum_a \{A_a \deg \phi_s \partial_C \phi_{1, \ldots, s-1} + [\textstyle\sum_{b=1}^{s-1} (-)^{b+1} B_b \partial_C \phi_{1, \ldots, \hat{b}, \ldots, s}] \xi_a\} \cdot \partial_{aC} \phi_{1, \ldots, s}$$
$$\equiv 0 \bmod \mathfrak{p} \, .$$

Because $(\mathfrak{q}, \mathfrak{h})$ is of the principal class (6.2), we may apply 6.1 to it and we obtain:

$$A_a \deg \phi_s \partial_C \phi_{1, \ldots, s-1} + [\textstyle\sum_{b=1}^{s-1} (-)^{b+1} B_b \partial_C \phi_{1, \ldots, \hat{b}, \ldots s}] \, \xi_a$$
$$\equiv \textstyle\sum_{c=0}^{n} D_{ac} \partial_{cC} \phi_{1, \ldots, s} \bmod \mathfrak{q} \quad \text{if} \quad a \in \{0, \ldots, n\} \setminus B \, .$$

Multiply this with ϕ_s and use $(\mathrm{IV})_C$ again:

$$\sum_d \xi_d \partial_{dC} \phi_{1, \ldots, s} A_a + [\textstyle\sum_{b=1}^{s-1} (-)^{b+1} B_b \partial_{bC} \phi_{1, \ldots, \hat{b}, \ldots s}] \phi_s \xi_a$$
$$\equiv [\textstyle\sum_c D_{ac} \partial_{cC} \phi_{1, \ldots, s}] \phi_s \bmod \mathfrak{q} \, .$$

Use $(\mathrm{VII})_C$ to obtain:

$$\sum_d [\xi_d A_a - \xi_a A_d - D_{ad} \phi_s] \partial_{dC} \phi_{1, \ldots, s} \equiv 0 \bmod \mathfrak{q}$$

and again by 6.1 applied to $(\mathfrak{q}, \mathfrak{h})$ we see:

$$(*)\ \xi_d A_a - \xi_a A_d \equiv \sum_t F_{adt} \partial_{tC} \phi_{1,\ldots,s} \bmod \mathfrak{p}\,.$$

Now $\deg F_{adt} = \deg A_a + 1 - \sum_{i=1}^{s} (\deg \phi_i - 1) \leqq s + 2 - n$.

So we must consider two cases:

Case 1. $s < n-2$, then $F_{adt} = 0$ and we obtain:

$$(**)\ \xi_d A_a - \xi_a A_d \equiv 0 \bmod \mathfrak{p}\,.$$

So $\xi_d A_a \equiv 0 \bmod (\mathfrak{p}, \xi_a)$. Now $V(\mathfrak{p}, \xi_a) \not\subset V(\xi_d)$ for $d \neq a$, so by (III) we have $A_a \equiv 0 \bmod (\mathfrak{p}, \xi_a)$, say $A_a = \xi_a D_a \bmod \mathfrak{p}$.

Substituting back in $(**)$ we see D_a may be assumed to be independent of a. This proves the lemma in this case.

Case 2. $s = n-2$. Then the F_{adt} are constants. We want to see that these are in fact zero constants. We then are in the situation of $(**)$.

Multiply $(*)$ with $\partial_a \phi_b$ and sum up over all a. Use $(\mathrm{VI})_b$ to get:

$$\sum_t \left[\sum_a F_{adt} \partial_a \phi_b\right] \partial_{tC} \phi_{1,\ldots,s} \equiv 0 \bmod \mathfrak{p} \quad \text{for all } b\,.$$

Later on we shall prove:

Assertion. If $\sum_t F_t \partial_{tC} \phi_{1,\ldots,s} \equiv 0 \bmod \mathfrak{p}$, then $F_t = \xi_t G_C \bmod \mathfrak{p}$. Here $\deg F_t < \max_k n_k$.

From this we get:

$$\sum_a F_{adt} \partial_a \phi_b \equiv \xi_t G_C \bmod \mathfrak{p} \quad \text{for all} \quad b = 1, \ldots, s\,.$$

And so:

$$\sum_a (F_{adt} \xi_m - F_{adm} \xi_t) \partial_a \phi_b \equiv 0 \bmod \mathfrak{p}$$

and a similar reasoning as the one leading to $(*)$, now applied to the linear forms $L_{ktm} := F_{kdt} \xi_m - F_{kdm} \xi_t$ shows:

$$\begin{aligned} \xi_s(L_{jtm}) - \xi_j(L_{stm}) &\equiv 0 \bmod (\mathfrak{p}, \mathfrak{h})\,, \quad \text{so by degree considerations} \\ &\equiv 0 \bmod \mathfrak{p}\,. \end{aligned}$$

Hence as before $L_{jtm} \equiv 0 \bmod (\mathfrak{p}, \xi_j)$ and $L_{jtm} = \xi_j M_{tm} \bmod \mathfrak{p}$. This means:

$$F_{kdt} \xi_m - F_{kdm} \xi_t - \xi_k M_{tm} \equiv 0 \bmod \mathfrak{p}\,.$$

But $V(\mathfrak{p})$ lies in no hyperplane, so this last expression must be zero, i.e. all $F_{adt} = 0$ and we are in the situation of $(**)$.

Proof of the assertion. Suppose

$$(\mathrm{VIII})_C\ \sum_t F_t \partial_{tC} \phi_{1,\ldots,s} \equiv \deg \phi_s G_C \phi_s \bmod \mathfrak{q}\,.$$

Let $D \subset \{0, \ldots, n\}$ consist of $s-1$ elements. Multiply $(\mathrm{VIII})_C$ with $\partial_D \phi_{1,\ldots,s-1}$ and $(\mathrm{VIII})_D$ with $\partial_C \phi_{1,\ldots,s-1}$ and apply $(\mathrm{IV})_D$ to the difference:

$$\sum_t (F_t \partial_D \phi_{1,\ldots,s-1} - G_s \xi_t) \cdot \partial_{tC} \phi_{1,\ldots,s} \equiv 0 \bmod \mathfrak{q}\,.$$

Again apply (6.1) to $(\mathfrak{q}, \mathfrak{h})$ and the obvious degree consideration to see:

$$(***)\ F_t \partial_D \phi_{1,\ldots,s-1} - G_C \xi_t \equiv 0 \bmod \mathfrak{q}\,.$$

So $F_t \partial_D \phi_{1,\ldots,s-1} = 0 \bmod (\mathfrak{q}, \xi_t)$, now by (β) $(\mathfrak{q}, \xi_t)$ is primary and because $V(1, \ldots, k-1)$ is non-singular $V(\mathfrak{q}, \xi_t) \subset V(\mathfrak{q}) \not\subset V(\partial_D \phi_{1,\ldots,s-1})$ for some D. Then by (III) $F_t \equiv 0 \bmod (\mathfrak{q}, \xi_t)$, say $F_t \equiv \xi_t H_t$. Substituting back in (***) gives that $H_t \equiv G_C \bmod \mathfrak{q}$. This finishes the proof of the assertion.

Proof of 6.5. $B_{\mu\nu}\xi_b \equiv 0\,(\mathfrak{p}, \xi_\nu, \xi_\mu)$. Because of (δ) in both cases mentioned $V(\mathfrak{p}, \xi_\nu, \xi_\mu) \not\subset V(\xi_b)$, so by (III) $B_{\mu\nu} \equiv 0\,(\mathfrak{p}, \xi_\nu, \xi_\mu)$. Suppose that $B_{\mu\nu} \equiv -P_{\nu\mu}\xi_\mu + Q_{\mu\nu}\xi_\nu \bmod \mathfrak{p}$. Substituting back we find eventually: $A_{\nu b} \equiv -P_{\mu\nu}\xi_b + R_{\mu\nu b}$ for all $\mu \neq \nu$. So take a fixed $\mu \neq \nu$ and put $C_\nu = -P_{\mu\nu}$ and $B'_{\nu b} = R_{\mu\nu b}$.

Appendix

Recall the Dolbeault-Serre way of computing the cohomology-groups $H^p(V, F)$, where F is a holomorphic vector bundle on a *CAM* V (cf. [8]). Put $A^{p,q}(F)$ = the $\mathbb{C}$-module of global $\mathscr{C}^\infty$-forms of type (p, q) with coefficients in F. We denote its sheaf of sections by the same symbol.

Because F is holomorphic the usual $\bar\partial$-operator induces a sheaf-homomorphism:

$$\bar\partial : A^{p,q}(F) \to A^{p,q+1}(F).$$

Because of the Dolbeault lemma this gives an exact sheaf sequence:

$$0 \to F \otimes \Omega_V^p \to A^{p,0}(F) \xrightarrow[\bar\partial]{} A^{p,1}(F) \xrightarrow[\bar\partial]{} A^{p,2}(F) \xrightarrow[\bar\partial]{} \ldots \qquad \text{A(1)}$$

which forms a fine resolution of the sheaf $F \otimes \Omega_V^p$. So we have:

$H^{p,q}(V, F) := H^q(V, F \otimes \Omega_V^p)$ is isomorphic to the q-th cohomology group of the resolution A(1), i.e. $H^{p,q}(V, F) \simeq Z^{p,q}(V, F)/\bar\partial A^{p,q-1}(F)$, where $Z^{p,q}(F)$ is the submodule of $A^{p,q}(V, F)$ which vanishes under $\bar\partial$.

There is a natural pairing:

$$A^{p,q}(F) \otimes A^{n-p,n-q}(F^*) \to A^{n,n}$$

$$\alpha \otimes \beta \mapsto \alpha \wedge \beta$$

Integrating $\alpha \wedge \beta$ over V we obtain a complex number, and it is easily seen that this induces a pairing:

$$H^{p,q}(F) \otimes H^{n-p,n-q}(F^*) \to \mathbb{C}. \qquad \text{A(2)}$$

Serre proved that A(2) is a non-singular pairing, establishing a duality between the two modules. In fact this is called *Serre-duality*.

More generally the pairing:

$$A^{p,q}(F) \otimes A^{r,s}(G) \to A^{p+r,q+s}(F \otimes G) \quad (F \text{ and } G \text{ holomorphic vector bundles})$$

$$\alpha \otimes \beta \mapsto \alpha \wedge \beta$$

induces a pairing:

$$H^{p,q}(F) \otimes H^{r,s}(G) \to H^{p+r,q+s}(F \otimes G) \qquad \text{A(3)}$$

this is the *cup-product*. In case $p = r = 0$ we thus have a map:

$$\cup : H^q(F) \otimes H^s(G) \to H^{q+s}(F \otimes G)$$

inducing the map:

$$t: H^q(F) \to \mathrm{Hom}(H^s(G), H^{q+s}(F \otimes G))$$
$$a \mapsto \{b \to a \cup b\}$$

dually this map becomes:

$$t^*: H^{q+s}(F \otimes G)^* \otimes H^s(G) \to H^q(F)^*$$
$$g \otimes b \mapsto \{f \to g(f \cup b)\}.$$

Lemma. *The map t^* is up to the sign the cup-product if we identify the dual spaces with their Serre-dual cohomology-modules.*

Proof. Denoting by κ^* the Serre dual of any κ we have:

$$t^*(g \otimes b)(f) = g(f \cup b) = \int_V f \wedge b \wedge g^* = \pm \int_V f \wedge (g^* \wedge b) \qquad \text{for all } f.$$

Hence $t^*(g \otimes b) = \pm (g^* \cup b)^*$.

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