Deformations and Rigidity for mixed period maps

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Abstract

We prove a rigidity criterion for period maps of admissible variations of graded-polarizable mixed Hodge structure, and establish rigidity in a number of cases, including families of quasi-projective curves, projective curves with ordinary double points, the complement of the canonical curve in families of Kynev–Todorov surfaces, period maps attached to the fundamental groups of smooth varieties and normal functions.

1. Introduction

1.1 Historical background

The rigidity concept the title refers to, concerns a Hodge theoretic variant of a rigidity property that S. Arakelov [Ara71] discovered. He showed that one cannot deform families $\{C_s\}_{s\in\bar{S}}$ of curves of genus $g \geqslant 2$ parametrized by a smooth projective curve $\bar{S}$ with varying moduli, keeping $\bar{S}$ fixed as well as the set, say Σ, over which singular fibers occur. In terms of $\mathcal{M}_g$, the moduli space of curves of genus g, this result states that if the moduli map $\mu : S = \bar{S} - \Sigma \to \mathcal{M}_g$ is not constant, it is rigid, keeping source and target fixed.[1] In the remainder of this introduction ***we shall only consider deformations of maps keeping source and target fixed***.

The cohomology groups $H^1(C_s, \mathbb{Z})$ admit a canonical polarizable weight one Hodge structure. These are classified by a period domain, in this case the generalized Siegel upper half-space $\mathfrak{h}_g$. Since the group of integral automorphisms preserving the polarization is the symplectic group $\mathrm{Sp}_{\mathbb{Z}}(g)$, the period map in this case is a holomorphic map $F : S \to \mathcal{A}_g := \mathrm{Sp}_{\mathbb{Z}}(g)\backslash\mathfrak{h}_g$ which factors through the morphism $\mathcal{M}_g \to \mathcal{A}_g$. The latter morphism is an embedding (this is Torelli's theorem).

It might be the case that, although μ is rigid keeping $(\bar{S}, \Sigma)$ fixed, this need not be the case for F. Geometrically interpreted, polarized weight one Hodge structures are polarized Abelian varieties and G. Faltings, in [Fal83] investigated the analog of Arakelov rigidity in this situation. Let us recall his result in Hodge theoretic terms. The period domain $\mathfrak{h}_g$ classifies (polarized) weight 1 Hodge structure on a free $\mathbb{Z}$-module H. Such a Hodge structure induces Hodge structures of weight 0 on $\mathrm{End}(H)$ as well on its subspace $\mathrm{End}(H, Q)$ of the Q-endomorphisms, that is those $u \in \mathrm{End}\, H$ for which $Q(ux, y) + Q(x, uy) = 0$ for all $x, y \in H$. By means of the period map $F : S \to \mathcal{A}_g$, Hodge structures $F(s)$ of weight one are put on H. The group Γ acts on H as well

2020 Mathematics Subject Classification 32G20; 14D07; 14D05
Keywords: mixed Hodge structure, period map, rigidity, Kynev–Todorov surfaces

[1] For an approach using Teichmüller theory see [IS88].

as $\mathrm{End}(H, Q)$. In particular, its commutant

$$\mathrm{End}^{\Gamma}(H_{\mathbb{C}}, Q) := \{u \in \mathrm{End}(H_{\mathbb{C}}, Q) \mid \gamma \circ u \circ \gamma^{-1} = u\}$$

inherits natural Hodge structures as well. By W. Schmid's result [Sch73, Corollary 7.23], these Hodge structures are independent of s. In technical terms, the period map defines a local system on S carrying a variation of Hodge structure inducing one on the endomorphism bundles and the Hodge decomposition extends as a flat decomposition, hence is independent of s. See Section 2.1. G. Faltings' result is as follows:

Theorem [Fal83, Theorem 2]. *The space of infinitesimal deformations of a period map $F : S \to \mathcal{A}_g$ over a curve S can be canonically identified with the direct summand of* $\mathrm{End}^{\Gamma}(H_{\mathbb{C}}, Q)$ *of Hodge type* $(-1, 1)$. *Consequently, if F is not constant, F is rigid if and only if* $\mathrm{End}^{\Gamma}(H, Q)$ *is pure of type* $(0, 0)$.

Faltings gave an example with $g = 8$ for which $\mathrm{End}^{\Gamma}(H, Q)^{-1,1} \neq 0$ and so this gives a non-rigid (non-isotrivial) family of 8-dimensional Abelian varieties. M.-H Saito [Sai93] made a systematic study and classified these in any dimension.

The Hodge-theoretic rigidity question for higher weight and over any quasi-projective smooth base was first consider by the second author in [Pet90] and it turns out that G. Faltings' result is in essence valid for all weights. There are a couple of differences. Of course, since S is allowed to be higher-dimensional, one has to impose the condition that the period map is generically an immersion instead of being non-constant. Secondly, on a more fundamental level, one should incorporate "Griffiths' transversality" (cf. [Gri68]) an infinitesimal property of variations of geometric origin which is automatic for weight 1 but gives a constraint for most types of higher weight variations. Geometrically this condition means that tangents to the image of the period map belong to the so-called horizontal tangent bundle. This is encapsulated in the statement that period maps are horizontal. It is natural to demand that deformations preserve this property. The result from loc. cit. indeed takes this into account:

Theorem [Pet90, Theorem 3.4]. *Let S be smooth and quasi-projective and $F : S \to \Gamma\backslash D$ a period map. The space of infinitesimal deformations of F remaining horizontal can be canonically identified with* $\mathrm{End}^{\Gamma}(H_{\mathbb{C}}, Q)^{-1,1}$.

The proof of this result is reviewed in Section 2.

1.2 Main results on deformations of mixed period maps

For the purpose of this introduction, a free $\mathbb{Z}$-module H is said to carry a mixed Hodge structure, if $H_{\mathbb{Q}} = H \otimes \mathbb{Q}$ carries an increasing finite filtration W, the weight filtration and $H_{\mathbb{C}} = H \otimes \mathbb{C}$ carries a decreasing filtration F, the Hodge filtration which induces a pure Hodge structure of weight k on $\mathrm{Gr}^W_k H$. If, moreover, each of those are polarized by Q_k, we write Q for the collection of the Q_k and say that (H, W, F, Q) is a graded polarized mixed Hodge structure.

Motivated by geometry, for classifying purposes we keep the weight filtration and the polarization fixed. So on a fixed triple (H, W, Q) we allow only the Hodge filtration to vary. The associated period domains and period maps have been studied in [Usu83, Pea00, Pea01, Pea06].

There are several important differences with the pure situation. First of all, $H_{\mathbb{C}}$ does not have a "mixed" Hodge decomposition, but instead, a canonical decomposition, introduced by Deligne [Del71], the Deligne-decomposition $H_{\mathbb{C}} = \oplus I^{p,q}$ where $I^{p,q}$ has the same dimension as the (p,q)-component of the Hodge structure on Gr^W_{p+q}, but it is no longer the case that $I^{q,p}$ is the complex conjugate of $I^{p,q}$.

Secondly, although, as in the pure case the period domain D is homogeneous under a Lie group G, say $D = G/G^F$, the isotropy group G^Fneed not be compact. Moreover, the group G has in general no real structure: it generally strictly contains $G_{\mathbb{R}}$, the automorphism group of $(H_{\mathbb{R}}, W, Q)$.

As in the pure case the polarization induces a Hodge metric on the tangent bundle to D, which is equivariant with respect to $G_{\mathbb{R}}$, but not the full group G. Period maps are holomorphic, there is a notion of Griffiths' transversality and a concept of horizontal tangent bundle. Period maps F have tangents in the latter bundle. As before, through the period map one gets mixed Hodge structures on H depending on s, i.e., the holomorphic vector bundle $\mathcal{H}$ on S with fibers $\simeq H_{\mathbb{C}}$ receives a variation of mixed Hodge structure (VMHS). The induced varying mixed Hodge structures on the Lie algebra

$$\mathfrak{g}_{\mathbb{R}} = \mathrm{End}^W(H_{\mathbb{R}}, Q)$$

of endomorphisms which preserve W and act by infinitesimal isometries on Gr^W defines a VMHS on the holomorphic vector bundle

$$\mathfrak{g}(\mathcal{H}) = \mathrm{End}^W(\mathcal{H}, Q)$$

over S and, again by [Sch73, Corollary 7.23], the Deligne decomposition on the space of global Γ-equivariant sections of $\mathfrak{g}(\mathcal{H})$ is a flat decomposition, that is, "constant in $s \in S$". The horizontality constraint implies that we restrict our attention to

$$\mathcal{U}^{-1}\mathfrak{g}(\mathcal{H}) = \bigoplus_{q \leqslant 1} \mathfrak{g}^{-1,q}(\mathcal{H}),$$

the ***horizontal endomorphism bundle***. The main result can now be stated as follows:

THEOREM =Theorem 6.2.1. *Let S be quasi-projective and $F : S \to \Gamma\backslash D$ a horizontal holomorphic map to a mixed domain D parametrizing mixed Hodge structures on (H, W, Q) such that the corresponding VMHS is admissible. Suppose that $v \in \mathcal{U}^{-1}\mathfrak{g}(\mathcal{H})$ is Hodge-harmonic, that moreover, v is equivariant with respect to the monodromy group Γ and that the Hodge norm $\|v\|$ is bounded near infinity.*

Then infinitesimal deformations of F that stay horizontal correspond one-to-one to Γ-equivariant horizontal endomorphisms of $\mathfrak{g}(\mathcal{H})$. The space of such deformations is smooth at F.

The statement requires some explanation. Let $v(s)$ be a section of the bundle $\mathcal{U}^{-1}\mathfrak{g}(\mathcal{H})$ on S of the horizontal endomorphisms of $\mathfrak{g}(\mathcal{H})$. In the pure case, as shown in the proof of [Pet90, Theorem 3.2], negativity of the bisectional curvature in horizontal directions implies that its Hodge norm gives rise to a plurisubharmonic function $s \mapsto \|v(s)\|$. One can do with a slightly weaker condition which is more suitable in the mixed situation. This weaker condition is the plurisubharmonicity of an endomorphism v of $\mathfrak{g}(\mathcal{H})$ and will be explained in Section 4.2. In the pure case it indeed implies plurisubharmonicity of the Hodge norm $\|v\|$, and we show that this is also true for several types of mixed Hodge structures of geometric interest. As is well known (see for example [Lel68]), bounded plurisubharmonic functions on a quasi-projective manifold are constant. To make use of this, it suffices to show that $\|v(s)\|$ is bounded near infinity whenever v is preserved by the local monodromy at infinity. This is indeed the case for pure Hodge structures as follows from W. Schmid's norm estimates in [Sch73]. Unfortunately, as Section 5.11 shows, the desired estimates do not hold for mixed variations in general, not even for admissible variations.

However, for several cases of geometric interest, boundedness still holds as shown in the remainder of Section 5.

Remark 1.2.1. Although we only consider period maps to "classical" mixed period domains, the same methods apply to variations with extra structure corresponding to period maps to mixed ***Mumford–Tate domains***. To explain this, first of all, the differential geometric input based on curvature calculations only uses Lie-theoretic calculations involving the mixed Hodge metric and the Deligne types and these calculations remain the same. Indeed, a Mumford–Tate domain is a homogeneous space of the form M/M^F where M is a subgroup of a group G acting transitively on some mixed domain D and $M^F = M \cap G^F$ so that the Hodge metric is the one from D restricted to M/M^F, and the Deligne types are the same as the ones for the mixed Hodge structure of the Lie algebra of G. See also [PP19, Remarks 1.1, 2.4].

Secondly, the calculations for boundedness of the mixed Hodge metric are based on the SL_2-orbit theorem. Its proof uses Lie theory within a given group and one can show that these calculations stay within $M \subset G$. See [KPR19, Section 4] where the pure case is treated. For the mixed situation the arguments are the same.

1.3 Boundedness results

Although for our purposes we only need a one-variable boundedness result, there is one situation where we prove a multivariable version which may be of independent interest:

Theorem =5.3.1. *A flat section of an admissible Hodge–Tate variation $\mathcal{H}$ with unipotent monodromy has bounded Hodge norm with respect to the mixed Hodge metric. Likewise, for a flat sections of* $\operatorname{End}\mathcal{H}$.

Recall from [Pea06] that a variation is of type (I) if there exists an integer k such that the Hodge numbers $h^{p,q}$ are zero unless $p+q = k,\ k-1$ (i.e. Gr^W has exactly two non-zero weight graded-quotients which are adjacent) and it is of type (II) if there is an integer k such that $h^{p,q} = 0$ unless $(p,q) = (k,k)$, $(k-1,k-1)$ or $p+q = 2k-1$ and $h^{k,k}$, $h^{k-1,k-1}$ are non-zero. Using this terminology, we prove the following 1-variable results, similarly of independent interest:

Theorem =Theorems 5.5.4, 5.9.1, Corollaries 5.8.4,5.10.3. *Let $\mathcal{H}$ be a 1-variable admissible variation with unipotent monodromy of one of the following types:*
1. *of unipotent type;*
2. *of type* (I) *or* (II)*;*
3. *a variation whose sole weight graded quotients are* $\mathrm{Gr}^W_0 \cong \mathbb{R}(0)$ *and* Gr^W_{-2}*;*
4. *a variation whose sole weight graded quotients are* $\mathrm{Gr}^W_0 \cong \mathbb{Z}(0)$, Gr^W_{-2} *and* $\mathrm{Gr}^W_{-4} \cong \mathbb{Z}(2)$.

Then a flat section of $\mathcal{H}$ or of $\mathfrak{g}(\mathcal{H})$ has bounded mixed Hodge norm.

We also show that for variations whose sole weight graded quotients are $\mathrm{Gr}^W_0 = \mathbb{Z}$ and Gr^W_{-k} for $k > 2$, the norm estimates required to obtain rigidity need not hold. See Lemma 5.11.1.

1.4 Geometric applications

The first application concerns families of ***quasi-projective smooth curves*** of genus g. In Example 7.1.2 we show that if the monodromy acts irreducibly on cohomology, the family is rigid in the $(-1,0)$-directions provided the curves can be completed by adding $< 2g$ points. The mixed Hodge structures on projective curves with k double points are in a certain sense dual to the ones on a quasi-projective curves which can be compactified to a smooth projective curves upon adding k points. Indeed, there is a dual result for families of ***curves with*** $> 2g$ ***double points*** (see Example 7.2.2). Perhaps worth mentioning here is the use of the rather recent concept of pure ***variations having maximal Higgs field***, a concept introduced by E. Viehweg and exploited in [VZ03]. For instance in Proposition 2.4.2 we state and prove that having a weight one maximal Higgs field implies rigidity. Hence, for the preceding examples maximal Higgs leads to period maps rigid in all horizontal directions.

Next we mention families of ***Kynev and Todorov surfaces***. V. Kynev [Kyn77] has given a construction of surfaces of general type with invariants $h^{1,0} = 0$, $h^{2,0} = 1$, $K^2 = 1$ that violate the infinitesimal Torelli theorem. Other counterexamples to infinitesimal Torelli were given by A. Todorov [Tod81]. His surfaces have the same invariants $h^{1,0} = 0$, $h^{2,0} = 1$, but $2 \leqslant K^2 \leqslant 8$. The period domain of both types of surfaces resemble that of a K3 surface. Like a K3 surface, there is an up to scaling unique holomorphic 2-form but here it vanishes along the canonical curve which is smooth for a generic such surface. Removing this curve gives an open surface intrinsically associated to a Kynev or Todorov surface. Its cohomology then provides an example of a mixed Hodge structure. The Todorov surfaces with $K^2 = 2, \dots, 8$ generalize Kynev surfaces that were previously also investigated in detail by F. Catanese [Cat80] and A. Todorov [Tod80] and so we shall call these ***CKT-surfaces***. We show (cf. Proposition 7.1.5) that a modular family of open CKT-surfaces or of Todorov surfaces is rigid, as is any sufficiently generic subfamily.

We shall also consider deformations of certain unipotent variations. Firstly ***Hodge–Tate variations*** (Section 4.3, Example (2)) and, secondly, ***variations associated to the fundamental group*** of an algebraic manifold (Section 4.3. Example (3)). For the latter, an explicit rigidity criterion is stated later as Proposition 7.4.1. It involves the geometry of the exterior algebra of the 1- and 2-forms of S.

Deformations of other types of algebraic families are investigated in Example 6.2.3 and, more elaborately, in Section 7. These include ***normal functions***, certain ***higher normal functions*** and ***biextensions*** coming from higher Chow groups.

1.5 Structure of the paper

In Section 2 we recall in detail the pure case and the proof of the main result from [Pet90]. The proof presented here differs slightly from the one given in loc. cit. since we want to highlight where problems arise for the mixed case. Further basic developments have been taken place since the publication of [Pet90] which we recall in Section 2.4. Several of these newer examples serve as building blocks in the mixed situation to which we turn in later sections.

In Section 3 we recall some basic material concerning mixed period maps.

One of the main ingredients in the proof of our results is the curvature calculation from [PP19]. We explained in loc. cit. that, unlike in the pure case, the holomorphic sectional curvature is not in general $\leqslant 0$ in horizontal directions and so this is a fortiori true for the holomorphic bisectional curvature. The latter plays a central role in the proof of [Pet90] and our original strategy was to list classes of types of mixed Hodge structure for which this is also true. In Section 4 we come back to the calculations of [PP19] and show that instead of focusing on bisectional curvature, it is better to use a new property, that of plurisubharmonicity of certain global endomorphisms of the Hodge bundle.

The second main ingredient, the norm estimates for the Hodge metric are given in Section 5. The techniques employed in this section are of an entirely different, mainly Lie-theoretic nature.

The proper topic of this paper, deformation theory in the mixed situation, is treated in Section 6 where we prove the main theorem, Theorem 6.2.1 and give criteria for rigidity. The main technical result, Proposition 4.2.4, leads to the geometric examples which are treated in detail in Section 7.

In Appendix A the notion of admissibility is reviewed, and in Appendix B we show that, like in the pure case, the monodromy action on the period domain coming from a mixed variation with an integral structure is properly discontinuous so that the quotient has the structure of an analytic space.

2. The pure case

Although most material in this section concerns published results, we decided to include it for several reasons. Firstly, the new results on deformations in the mixed case build upon the ones from [Pet90] valid in the pure case, but since the mixed case is considerably more involved, it is instructive to explain the pure case in a way that is geared towards techniques we use in the mixed case. Secondly, as explained in the introduction, besides results from [Pet90] and [Pet92] we want to include new results in the pure case since these lead to more examples in the mixed setting.

2.1 Basics on period domains and period maps

Recall that a ***period domain*** parametrizes polarized Hodge structures of weight k on a finite dimensional real vector space $H_{\mathbb{R}}$ with given Hodge numbers $\{h^{p,q}\}$, polarized by a non-degenerate bilinear form Q of parity $(-1)^k$. Such a domain D is homogeneous under the real Lie group $G_{\mathbb{R}} \subset \mathrm{GL}(H_{\mathbb{R}})$ of automorphisms of the polarization Q. The isotropy groups $G_{\mathbb{R}}^F$, $F \in D$ are compact. The domain D is an open set in the compact dual $\check{D}$ upon which the complexification $G_{\mathbb{C}}$ of $G_{\mathbb{R}}$ acts transitively:

$$G_{\mathbb{R}}/G_{\mathbb{R}}^F = D \subset \check{D} = G_{\mathbb{C}}/G_{\mathbb{C}}^F.$$

The Hodge structure on $H_{\mathbb{R}}$ given by F induces a Hodge structure on the Lie algebra of $G_{\mathbb{R}}$ as a sub-Hodge structure of $\operatorname{End} H_{\mathbb{R}}$. It has weight zero with Hodge decomposition $\mathfrak{g}_{\mathbb{C}} = \bigoplus_p \mathfrak{g}^{p,-p}$ where $\mathfrak{g}^{p,-p}$ consists of those endomorphisms that send $H^{s,t}$ to $H^{s+p,t-p}$.

A point $F \in \check{D}$ can be considered as a filtration on $H_{\mathbb{C}}$. Then $F^0\mathfrak{g}_{\mathbb{C}}$ is the Lie algebra of the stabilizer of F in $G_{\mathbb{C}}$. Hence the tangent space $T_F\check{D}$ of $\check{D}$ at F is isomorphic to $\mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}}$. Accordingly, since $F^p\mathfrak{g}_{\mathbb{C}} = \bigoplus_{a\geqslant p} \mathfrak{g}^{a,-a}$, it follows that

$$T_F D = \mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} \simeq \bigoplus_{a>0} \mathfrak{g}^{-a,a}. \tag{1}$$

Every Hodge structure $F \in D$ defines the ***Hodge metric*** on $H_{\mathbb{C}}$ which is given by

$$h_F(x,y) := Q(C_F x, \bar{y}), x, y \in H_{\mathbb{C}}, \tag{2}$$

where $C_F|H^{p,q} = \mathbf{i}^{p-q}$ is the Weil-operator. The Hodge metric is a hermitian metric relative to which the Hodge decomposition of $H_{\mathbb{C}}$ is orthogonal. The induced metric on $\mathfrak{g}_{\mathbb{C}}$ satisfies $\mathfrak{g}^{a,-a} \perp \mathfrak{g}^{b,-b}$ unless $a = b$. In particular, via the isomorphism $T_F D \simeq \bigoplus_{a>0} \mathfrak{g}^{-a,a}$, we obtain a Hodge metric on $T_F D$. Moreover, since

$$h_{gF}(x,y) = h_F(g^{-1}x, g^{-1}y), \quad g \in G_{\mathbb{R}}.$$

it follows that the Hodge metric defines a $G_{\mathbb{R}}$-invariant metric on the tangent bundle of D.

A (real) ***variation of polarized Hodge structure*** over a complex manifold S consists of a local system $\underline{H}_{\mathbb{R}}$ of finite dimensional real vector spaces equipped with a weight k Hodge structure polarized by a $(-1)^k$-symmetric form Q such that

- the Hodge filtrations glue to a holomorphic filtration $\mathcal{F}$ of the holomorphic bundle $\mathcal{H} = \underline{H}_{\mathbb{R}} \otimes \mathcal{O}_S$;
- (Griffiths' transversality) the natural flat connection ∇ induces a vector bundle map $\mathcal{F}^p \to \mathcal{F}^{p+1} \otimes \Omega^1_S$.

Remark. The motivation of this concept is geometric: if $f : X \to S$ is a smooth, proper morphism between complex algebraic varieties, then, by the work of P. Griffiths [Gri68], the associated local system $\underline{H}_{\mathbb{R}} = R^k f^* \mathbb{R}_X$ underlies a variation of pure Hodge structure of weight k. It comes equipped with a natural polarization induced by the cup-product and the Lefschetz decomposition in cohomology. In fact, we may instead consider cohomology with rational coefficients and consider polarizations defined by ample classes. In this way we obtain a rational variation of polarized Hodge structure. There is even a canonical flat integral structure equipped with a polarizing form.

By its very definition, locally in a simply connected open neighborhood U of $s \in S$, the assignment $s \mapsto \mathcal{F}_s$ gives a holomorphic period map $U \to D$. To make sense of this globally, one needs to incorporate the effect of the fundamental group at s: giving a local system $\underline{H}$ is equivalent to giving a representation on H, the fiber of $\underline{H}$ at s. This representation preserves Q and so the image of the fundamental group is a subgroup Γ of G, the ***monodromy group*** of the variation. For variations coming from geometry this subgroup belongs to $G_{\mathbb{Z}}$, the subgroup preserving the integral structure coming from integral cohomology. The monodromy group being closed and discrete, acts properly discontinuously on D. It follows that the quotient $\Gamma\backslash D$ is an analytic space. The ***period map*** in its global incarnation is the holomorphic map

$$F : S \to \Gamma\backslash D.$$

The Griffiths' transversality property is equivalent to the statement that the derivative of the tangent map at s lands in

$$T^{\rm hor}_{F(s)} D = F^{-1}\mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g}^{-1,1},$$

the ***horizontal tangent space*** at F. The corresponding vector bundle is the ***horizontal tangent bundle***

$$T^{\rm hor} D = \mathcal{F}^{-1}\mathfrak{g}(\mathcal{H})/\mathcal{F}^0\mathfrak{g}(\mathcal{H}) \simeq \mathfrak{g}(\mathcal{H})^{-1,1}, \tag{3}$$

where the isomorphism is in the category of C^∞ hermitian vector bundles. Conversely, a holomorphic map from a complex manifold to a quotient of a period domain D by a discrete closed subgroup of G is a period map provided it is locally liftable to D as a horizontal holomorphic map.

2.2 Curvature properties

By [GS69, Theorem 9.1] the holomorphic sectional curvature of D along horizontal tangents is negative and bounded away from zero. As shown in [Pet90], the full curvature tensor along a $(1,0)$-tangent vector of $u \in \mathfrak{g}^{-1,1}$ is given by $R(u,\bar{u}) = -\operatorname{ad}([u,\bar{u}])$ so that the bisectional curvature in the (u,v) unit-norm direction becomes

$$K_F(u,v) = h_F(R(u,\bar{u})v,v) = -h_F([[u,\bar{u}]v],v) = \|[u,v]\|^2_F - \|[\bar{u},v]\|^2_F.$$

As recalled below, in geometric situations u and v commute, which implies $K_F(u,v) \leqslant 0$. We shall outline how this implies that for a global section η of $\mathfrak{g}(\mathcal{H})$ which is of Hodge type $(-1,1)$, the function $F \mapsto \|\eta(F)\|^2$ is plurisubharmonic on S.

This phenomenon occurs more generally for sections η of any holomorphic vector bundle $\mathcal{E}$ equipped with a hermitian metric h. Recall that there is a unique metric $(1,0)$-connection D, the ***Chern connection*** for $(\mathcal{E}, h)$. The bisectional curvature appears in a Bochner type formula [CMSP17, Prop. 11.1.5], a special case of which reads

$$\begin{aligned} \partial_u \partial_{\bar{u}} \|v\|^2 &= \|\mathsf{D}_u v\|^2 - h(R_{\mathsf{D}}(u,\bar{u})v, v) \\ & \quad u \in T_s^{1,0} S, \quad v = \eta(s). \end{aligned} \tag{4}$$

Recall that a real C^2-function f on an open subset U of $\mathbb{C}^n$ is plurisubharmonic if $\mathbf{i}\partial\bar{\partial} f$ is a positive definite $(1,1)$-form. This is equivalent to $\partial_u \partial_{\bar{u}} f \geqslant 0$ for all type $(1,0)$-tangent vectors on U. If $h(R_{\mathsf{D}}(u,\bar{u})v,v) \leqslant 0$, formula (4) shows that $s \mapsto \|\eta(s)\|$ is a plurisubharmonic function on S.

We apply this to our situation with $\mathcal{E}$ the bundle $\mathfrak{g}(\mathcal{H})$ on S. A holomorphic section η of this bundle is invariant under the global monodromy and so in particular invariant under local monodromy at infinity. We now invoke:

Proposition 2.2.1 [Sch73, Cor. 6.7']. *Let there be given a polarized variation over the punctured disk. Then an invariant holomorphic section of the Hodge bundle remains bounded.*

Quasi-projective manifolds do not admit bounded plurisubharmonic functions except constants (cf. [Lel68]). Consequently, in the present situation the Hodge norm $\|\eta\|$ is constant along curves in S and hence on all of S. The bundles on S are pull backs under the period map F of bundles on D and the calculation takes place on D. In particular, tangent vectors from $\xi \in T_s S$ of type $(1,0)$ are pushed to $u = F_*\xi \in F_*(T_s S) \subset T^{\rm hor}_{F(s)} D = \mathfrak{g}^{-1,1}_{F(s)}$. Summarizing the discussion so far we have shown:

Lemma 2.2.2. *Let there be given a polarized variation of Hodge structure $(\mathcal{H}, Q, \mathcal{F})$ over a quasi-projective complex manifold S. Let η be a holomorphic section of the endomorphism bundle $\mathfrak{g}(\mathcal{H})$ which is of Hodge type $(-1,1)$.*

Suppose that for all $u \in T^{\rm hor}_F D$ tangent to the image of the period map at $F = F(s)$, $s \in S$, one has $[u,v] = 0$, $v = \eta(s)$. Then $\|\eta\|$ is a plurisubharmonic bounded (and hence constant) function, $\mathsf{D}\eta = 0$ and $[\bar{u}, v] = 0$.

The next step is to relate the Chern connection and the Gauss–Manin connection ∇ as explained in [CMSP17, Prop. 13.1.1]. It uses the ***Higgs bundle structure*** on the Hodge bundle $\mathcal{H} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$. To explain this, note that the Hodge decomposition is only a C^∞-decomposition. However, $\mathcal{H}^{p,q}$ receives a complex structure through the isomorphism $\mathcal{H}^{p,q} \simeq \mathcal{F}^p/\mathcal{F}^{p+1}$. There is a corresponding operator $\bar{\partial} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{E}^{0,1}_S$ with the property that local sections v of $\mathcal{H}^{p,q}$ are holomorphic if and only if $\bar{\partial} v = 0$. The Gauss–Manin connection ∇ can be decomposed as follows:

$$\nabla = \sigma + \underbrace{\bar{\partial} + \partial}_{\mathsf{D}} + \sigma^*. \tag{5}$$

Here $\partial : \mathcal{H} \to \mathcal{H} \otimes \mathcal{E}^{1,0}_S$ is a differential operator which preserves Hodge type. The operator

$$\sigma : \mathcal{H} \to \mathcal{H} \otimes \mathcal{E}^{1,0}_S,$$

an endomorphism of $\mathcal{H}$ of Hodge type $(-1,1)$ with values in the $(1,0)$-forms, is called the ***Higgs field***. Its adjoint with respect to the Hodge metric is the linear operator $\sigma^* \in \mathfrak{g}(\mathcal{H})^{1,-1} \otimes \mathcal{E}_S^{0,1}$.

By functoriality, a similar decomposition holds for the bundle $\mathfrak{g}(\mathcal{H})$. Since the tangent bundle comes from the adjoint representation of G on the endomorphism bundle, it follows from [CMSP17, Prop. 11.4.3] that for any horizontal tangent vector u of type $(1,0)$ at $F \in D$ we have:

$$\nabla_u = \partial_u + \mathrm{ad}\,(u),$$
$$\nabla_{\bar{u}} = \bar{\partial}_u + \mathrm{ad}\,(\bar{u}).$$

Assuming, as before, that $\mathrm{ad}\,(u)v = [u,v] = 0$, by Lemma 2.2.2, $\partial_u v = \bar{\partial}_u v = 0$ and $[\bar{u}, v] = 0$. Invoking Lemma 2.2.2, we may summarize the above discussion as follows:

Proposition 2.2.3. *Let a η be a holomorphic section of $\mathfrak{g}(\mathcal{H})$ of type $(-1,1)$. At a point F in the image of the period map, set $v = \eta(F)$ and assume that $[u,v] = 0$ for all vectors $u \in T_F D$, tangent to the period map. Then η is parallel with respect to the Gauss–Manin connection. Moreover, one has $[\bar{u}, v] = 0$.*

2.3 Deformations of period maps

The kind of deformations we are interested in are deformations of holomorphic maps $\varphi : X \to Y$ between complex spaces X and Y that keep X and Y fixed. By definition, these are given by complex-analytic maps $\Phi : X \times T \to Y \times T$ with $(T, 0)$ a germ of an analytic space centered at 0 such that

- $\Phi(x,t) = (\varphi_t(x), t)$;
- $\Phi(x,0) = \varphi(x)$.

Such deformations are in one-to-one correspondence to deformations of the graph of φ and as in [Ser06, § 3.4.1] the tangent space at φ of such deformations is given by the space $H^0(X, \varphi^* T(Y))$, the space of infinitesimal deformations of φ keeping X and Y fixed. Here $T(Y)$ is the tangent sheaf of Y, i.e., the dual of the cotangent sheaf of Y.

We apply this to period maps $F : S \to \Gamma\backslash D$. In geometric situations we are interested in deformations of families of varieties and the corresponding deformations $S \times T \to \Gamma\backslash D$ of period maps F that stay locally liftable and horizontal. We pass to the smallest unramified cover of S over which there is no monodromy and lift the period map accordingly, say to $\tilde{F} : \tilde{S} \to D$ and then the space of infinitesimal deformations in which we are interested is the subspace of $H^0(\tilde{F}^* T^{\mathrm{hor}}(D))$ consisting of sections commuting with the monodromy action. In view of the isomorphism (3), such a deformation lifts to a holomorphic section of $\mathfrak{g}(\mathcal{H})$ which at any given point $F \in D$ in the image of the period map projects to $\mathfrak{g}^{-1,1}$. In this situation we can apply Proposition 2.2.3 since the condition $[u,v] = 0$ follows from horizontality (see [CMSP17, Prop. 5.5.1]) and we conclude:

Theorem 2.3.1. *Let S be smooth and quasi-projective and $F : S \to \Gamma\backslash D$ a period map. The space of infinitesimal deformations of F remaining horizontal is isomorphic to the space of flat sections of type $(-1,1)$ of the bundle $\mathfrak{g}(\mathcal{H})$. Moreover, at a point F in the image of the period map, setting $v = \eta(F)$, one has $[\bar{u}, v] = 0$, $v = \eta(F)$, for all tangent vectors at F tangent to the period map.*

Complementing this result we remark that according to an argument generalizing the one given by Faltings [Fal83] for weight 1, the corresponding deformation space is smooth at F (See also the proof of Theorem 6.2.1. (2)):

Proposition 2.3.2. *The space of deformations of a period map F which keep source and target fixed, and stay locally liftable and horizontal, is smooth at F.*

It follows that F is locally rigid precisely when $\mathrm{End}^{\Gamma}(H_{\mathbb{C}}, Q)^{-1,1} = 0$. This gives criteria for rigidity. From the last property, $[\bar{u}, v] = 0$, we see that the concept of a regularly tangent period map as introduced in [Pet90] comes up naturally:

Definition 2.3.3. The period map F is called ***regularly tangent*** at $s \in S$ if the only vector $v \in \mathfrak{g}_{F(s)}^{-1,1}$ with $[\ \bar{u}, v] = 0$ for all $u \in F_* T_s S$ is the zero vector. If this is the case for all s we speak of a period map which is regularly tangent along S.

Corollary 2.3.4. *A period map $F : S \to \Gamma\backslash D$ is rigid (as a period map) if one of the following two conditions hold:*

- *The only flat endomorphism of the underlying local system which is of Hodge type $(-1, 1)$ is the zero endomorphism.*
- *F is everywhere regularly tangent.*

2.4 Examples of rigid period maps

We first recall the following concept:

Definition 2.4.1. (1) A polarized real variation of weight k has Higgs field of ***Hodge–Lefschetz type*** a if

- the Hodge depth is a, that is the only non-zero Hodge numbers are in the range $(a, k - a), \ldots, (k - a, a)$;
- the Higgs field in some, or equivalently, in a generic direction has components $u^j : \mathcal{H}_s^{k-j,j} \to \mathcal{H}_s^{k-j-1,j+1}$, $j = a, \ldots, k - a$ which are all isomorphisms.

This implies that the Hodge depth is exactly a and all non-zero Hodge numbers are equal.
(2) A polarized pure variation has (strictly) ***maximal Higgs field*** if it is a direct sum of variations with Higgs field of Hodge–Lefschetz type, the ***strands*** of the field.[2]

Proposition 2.4.2. *A pure variation which has maximal Higgs field is regularly tangent and hence rigid.*

Proof. Let $\xi \in T_s S$ be generic so that the components of $u = F_* \xi$ are isomorphisms on each Hodge–Lefschetz strand of the variation. Assume $[\bar{u}, v] = 0$ which at an extremal Hodge component means either $\bar{u} \circ v = 0$ or $v \circ \bar{u} = 0$. But since the Hodge components u and its adjoints are isomorphisms on each strand, this implies that the extremal components of v vanish and hence, by induction, all components. $\square$

The preceding result confirms the result [VZ03, Lemma 4.3] for strictly maximal Higgs fields over curves. In loc. cit. several examples are given of families $\{X_s\}_{s \in S}$ of d-dimensional Calabi–Yau's over a curve for which the middle dimensional cohomology gives variation with strictly maximal Higgs field.

We can now enumerate some examples.

[2] For complex variations of Hodge structure, the definition as given in [VZ03] is more complicated, but for real variations it reduces to the one given here.

(1) Maximal Higgs fields, weight 1. Let $(H_{\mathbb{R}}, Q)$ be a weight one polarized Hodge structure and set $V = H^{1,0}$. Consider the hermitian inner product $h(x,y) = \mathbf{i}Q(x,\bar{y})$ on V. The anti-complex linear map $\bar{x} \mapsto h(-,x)$ induces an identification $\overline{V} = V^*$. The Higgs field becomes a Q-symmetric endomorphism $u \in \mathrm{Hom}(V, V^*)$ and hence can be identified with $P_u \in S^2V^*$, a quadratic homogeneous polynomial function on V. Under this identification, u is an isomorphism precisely when P_u has maximal rank. Hence a polarized weight one variation has maximal Higgs field if and only if the corresponding quadratic polynomial has generically maximal rank.

(2) Maximal Higgs fields, weight two. We recall some general properties of weight two polarized Hodge structures $(H_{\mathbb{R}}, Q)$, say $V = H^{2,0}$, $W = H^{1,1}$. The hermitian product $h(x,y) = Q(x,\bar{y})$ restricts non-degenerately on V and under the anti-linear map $\bar{x} \mapsto h(-,x)$ there are identifications $\overline{V} = V^*$ and $\overline{W} = W^*$. If $A : V \to W$ is linear, the anti-linear dual is denoted $\widehat{A} : \overline{W} \to \overline{V}$. Suppose that $u \in \mathrm{End}\, H, Q$ is horizontal, that is, of type $(-1,1)$. Then we have $V \xrightarrow{u_1} W = \overline{W} \xrightarrow{u_2} \overline{V}$ with $u_2 = \widehat{u_1}$.

One easily sees that $Z := \mathrm{Im}(u_1) = [\mathrm{Ker}(u_2)]^{\perp}$ and that $u^* = (u_1^*, u_2^*)$ is such that $u_1^* = 0$ on $Z^{\perp}$ and $u_2^* : \overline{V} \to Z \subset W = \overline{W}$. Applying this to a weight two variation we see that subvariation associated to $(H^{2,0}, Z, H^{0,2})$ is of Hodge–Lefschetz type if and only if u_1 is an injection. Note that the Higgs field is zero on $Z^{\perp}$ and so it can only be of Hodge–Lefschetz type if it vanishes. Concluding, we can only have a maximal Higgs field if $Z^{\perp} = 0$ and then $h^{2,0} = h^{1,1} = h^{0,2}$.

(3) Irreducible modules. If (H,Q) is the typical stalk of a variation of pure polarized Hodge structure on S and $H_{\mathbb{C}}$ is irreducible as a $\pi = \pi_1(S)$-module, $\mathrm{End}^{\pi}_{\mathbb{C}}(H,Q)$ is 1-dimensional and since it has a pure Hodge structure, it has type $(0,0)$. Consequently we have $\mathrm{End}^{\pi,\mathrm{hor}}(H,Q) = 0$ and so, by Corollary 2.3.4, such a variation is rigid.

As a geometric example we may consider a **Lefschetz pencil of complete intersections** in projective space. By S. Lefschetz' theory of the variable cohomology (cf. e.g. [CMSP17, Section 4.2.]) the latter is always absolutely irreducible under the action of the monodromy group. The period map for the family is an immersion except for a cubic surface or an even dimensional intersection of two quadrics (see e.g. [Fle86, Thm. 2.1.]). Hence the Lefschetz pencil itself is rigid as well.

(4) Abelian varieties (or polarizable weight one variations). Ma. Saito [Sai93] gives a complete classification of the non-rigid families $\{A_s\}_{s\in S}$ of g-dimensional abelian varieties A_s. From this it follows that rigid families occur in abundance as we now show. We can decompose the variation into irreducible factors. Assume that none of these factors are isotrivial. Then the family is rigid if one of the following situations occur:

- $g \leqslant 7$;
- the variation is irreducible and g is prime;
- S is non-compact, the variation is irreducible and some local monodromy operator at the boundary has infinite order.

Observe that any weight one variation coming from curves is irreducible since the polarization comes from the irreducible theta-divisor. So non-isotrivial families of genus g curves have rigid period map if for example g is an odd prime number, or if the family has infinite order local monodromy at infinity.

(5) K3-type variations. A variation of Hodge structure on a local system is of ***K3-type***, if it has weight 2 and $h^{2,0} = 1$. In general such a system splits as $S \oplus \underline{T}$ where S is locally constant. If $\underline{T} \neq 0$ it is an irreducible variation, again of K3-type. Geometric examples come from the primitive 2-cohomology of a projective algebraic K3 surface X which splits as

$$H^2_{\text{prim}}(X) = S(X) \oplus T(X),$$

where $S(X)$ is spanned by the classes of the algebraic cycles and $T(X) = S(X)^\perp$. In a family of K3 surfaces, $\{X_s\}_{s\in S}$, there is a maximal locally constant part S of the variation given by algebraic cycles classes, and so the variation splits as $\mathsf{S} \oplus \underline{T}$. In [SZ91] $\underline{T}$ is called the essential variation. The subset $D(\mathsf{S})$ of the period domain corresponding to projective K3 surfaces for which the Picard lattice contains S has dimension $20 - \rho$ where $\rho = \operatorname{rank}(\mathsf{S})$. The generic K3 with period point in $D(\mathsf{S})$ has Picard lattice isomorphic to S. The period map of an essential variation has $D(\mathsf{S})$ as its target. As a special case of the results of [SZ91] we mention:

Proposition 2.4.3. *An essential K3-type variation of rank k on a quasi-projective variety S with immersive period map is rigid in each of the following cases:*

(i) *k is not divisible by 4;*

(ii) *S is not compact and some local monodromy operator at infinity has maximal order of unipotency 3.*

(4—5 bis) In addition to the results mentioned under (4) and (5) the rigidity results from [Pet92] in particular concern abelian varieties and K3-variations:

Proposition 2.4.4. *Families of Abelian varieties and K3-type variations having period maps of rank $\geqslant 2$ are rigid.*

(6) Calabi–Yau manifolds. Proposition 2.4.3 (2) generalizes to Calabi–Yau's:

Theorem. *[Pet10, Cor. 3.5] Let $f : X \to S$ be a non-isotrivial family of k-dimensional Calabi-Yau's over a non-compact curve S and suppose that there is a point at infinity where the local monodromy operator for H^k has maximal order of unipotency $k+1$. Then f is rigid.*

3. Mixed period domains and Hodge metrics

We recall some material from [Del71, Pea00, Pea01, Pea06, Usu83] on mixed Hodge structures and related period domains.

3.1 Basics on mixed Hodge structure

Fix a finite dimensional $\mathbb{Q}$-vector space $H_\mathbb{Q}$ endowed with a finite increasing weight filtration W whose graded pieces Gr^W_k are equipped with non-degenerate $(-1)^k$-symmetric real-valued bilinear forms Q_k. These data are denoted $(H, W, \{Q_k\})_\mathbb{Q}$. Associated to these data the following groups are relevant: the real Lie group

$$G_\mathbb{R} = \{g \in \operatorname{GL}(H_\mathbb{R}) \mid g(W_k) \subset W_k, \operatorname{Gr}^W(g) \in \operatorname{Aut}(\operatorname{Gr}^W(H_\mathbb{R}, Q))\}$$

and its complexification $G_\mathbb{C}$ as well as an intermediate group

$$G = \{g \in G_\mathbb{C} \mid g \text{ induces a real transformation on } \operatorname{Gr}^W(H)\}. \tag{6}$$

A decreasing filtration F on $H_\mathbb{C}$ together with the data $(H, W, \{Q_k\})_\mathbb{R}$ define a ***graded polarized mixed Hodge structure*** if F induces a pure weight-k Hodge structure on Gr^W_k polarized by Q_k. A basic tool is the ***Deligne splitting (or bigrading)*** [Del71] for the mixed Hodge structure, a unique functorial bigrading,

$$H = H_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q} \tag{7}$$

such that $F^p = \bigoplus_{a \geqslant p} I^{a,b}$, $W_k \otimes \mathbb{C} = \bigoplus_{a+b \leqslant k} I^{a,b}$ and

$$\overline{I^{p,q}} = I^{q,p} \mod \bigoplus_{a<p, b<q} I^{a,b}.$$

The graded polarized mixed Hodge structures $(H, W, \{Q_k\})_{\mathbb{R}}$ with fixed Hodge numbers $h^{p,q} = \dim I^{p,q}$ are parametrized by a mixed period domain which we always denote D.

Remark 3.1.1. A mixed Hodge structure is ***split over*** $\mathbb{R}$ if $\overline{I^{p,q}} = I^{q,p}$. Examples occur if the weight filtration has only two adjacent weights. Consider for instance mixed Hodge structures with $h^{0,0} = h^{-1,-1} = 1$, an example of a Hodge–Tate structure. The corresponding mixed domain is $\mathbb{C}$ (while the extension data are isomorphic to $\mathrm{Ext}(\mathbb{Z}(0), \mathbb{Z}(1)) = \mathbb{C}^*$).

In analogy with the pure case, D is a complex manifold. Moreover, D is a homogeneous domain under the group G defined by (6) and so

$$D = G/G^F, \quad G^F = \text{ stabilizer of } F \text{ in } G.$$

There are important differences with the pure case since the group G^F is in general not compact in contrast to $G^F_{\mathbb{R}}$. The real Lie group $G_{\mathbb{R}} = G \cap \mathrm{GL}(H_{\mathbb{R}})$ acts only transitively on the locus of split mixed Hodge structures which need not be a complex manifold. However, if D parametrizes split mixed Hodge structures, $D = G_{\mathbb{R}}/G^F_{\mathbb{R}} = G/G^F$, although in general we have $G \neq G_{\mathbb{R}}$ and, while $G^F_{\mathbb{R}}$ is compact, G^F need not be compact. See [Usu83] for the case of adjacent weights.

As in the pure case, there is a "compact dual" of D,

$$\check{D} = G_{\mathbb{C}}/G^F_{\mathbb{C}}. \tag{8}$$

By functoriality, any point $F \in D$ induces a mixed Hodge structure on $\mathrm{End}(H)$ with Deligne splitting

$$\begin{array}{rcl} \mathrm{End}(H) & = & \bigoplus_{p,q} \mathrm{End}^{p,q}(H), \\ \mathrm{End}^{p,q}(H) & = & \{u \in \mathrm{End}(H) \mid u(I^{r,s}) \subset I^{r+p,s+q} \text{ for all } r,s\}. \end{array} \tag{9}$$

and also on the space $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G_{\mathbb{C}}) = \mathrm{End}(H, W, Q)_{\mathbb{C}}$ of endomorphisms preserving Q:

$$\mathfrak{g}^{r,s} = \mathfrak{g}_{\mathbb{C}} \cap \mathrm{End}^{r,s}(H) \quad r+s \leqslant 0.$$

The restriction on the bigrading comes from the weight preserving property of elements of $G_{\mathbb{C}}$.

There is also an analog of (1). To see this, first observe that the exponential map $u \mapsto \mathrm{e}^u$ maps a neighborhood U of 0 biholomorphically to an open neighborhood of $G_{\mathbb{C}}$ and so, composing with the orbit map yields a biholomorphic map

$$\begin{array}{c} \varphi : U \cap \mathfrak{q}_F \xrightarrow{\simeq} \mathrm{Im}(\varphi) \subset D \\ u \mapsto \mathrm{e}^u .F\,. \end{array} \tag{10}$$

Since the Lie algebra of $G^F_{\mathbb{C}}$ equals $F^0\mathfrak{g}_{\mathbb{C}} = \bigoplus_{r \geqslant 0} \mathfrak{g}^{r,s}$, the subspace

$$\mathfrak{q}_F = \bigoplus_{r<0} \mathfrak{g}^{r,s} \tag{11}$$

is a vector space complement to $F^0\mathfrak{g}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Accordingly, $d\varphi(0)$ induces a natural isomorphism of complex vector spaces

$$T_F(D) \simeq \mathfrak{q}_F. \tag{12}$$

3.2 Period maps for variations of mixed Hodge structure

Similarly to a pure variation, one can speak of a ***variation of graded polarized mixed Hodge structure*** on S. The only difference with the pure case is the presence of a weight filtration with the property that on its k-graded quotients the Hodge filtration induces a pure polarized variation of weight k. Such variations are in one to one correspondence with period maps to the mixed period domain D for the graded polarized mixed Hodge structure on a typical fiber. The map sending s to the point $F(s) \in D$ corresponding to the mixed Hodge structure on the fiber over s of the local system is well defined locally, for instance if S is a polydisc or, more generally, a simply connected manifold. We say that we have a ***local period map*** $S \to D$, $s \mapsto F(s)$. As in the pure case, there is a monodromy group Γ and we get a well defined (global) ***period map***

$$F : S \to \Gamma\backslash D.$$

Again, as in the pure case, variations coming from geometry have an underlying integral structure. In particular, this implies that Γ acts properly discontinuously on D and so $\Gamma\backslash D$ is an analytic space. For lack of a good reference, we provide a proof of this fact in Appendix B.

The period map is horizontal, meaning that the derivative at $s \in S$ sends T_sS to the subspace of the tangent space $T_{F(s)}D$ given by

$$\mathrm{Gr}^{-1}_{\mathcal{F}}\mathfrak{g}(\mathcal{H})_s = \bigoplus_{q\leqslant 1} \mathfrak{g}^{-1,q}_{F(s)}.$$

This is a consequence of Griffiths' transversality. Since one only uses the Hodge filtration to describe of the tangent bundle as well as the horizontal tangent bundle the description in the mixed case parallels the one in the pure case. For later reference we make this more explicit. Using the induced Hodge filtration on the endomorphism bundle, we have a surjective map of holomorphic vector bundles on $\check{D}$

$$\mathcal{F}^{-1}\mathfrak{g}(\mathcal{H}) \xrightarrow{\pi^{\mathrm{hor}}} T^{\mathrm{hor}}(\check{D}) = \mathrm{Gr}^{-1}_{\mathcal{F}}\mathfrak{g}(\mathcal{H}). \tag{13}$$

Mixed period maps of geometric origin have all of the above properties. See e.g. [SZ85, Usu83].

To close this section, we observe that the same argument used in the pure case shows:

Lemma 3.2.1. *For a local period map $F : S \to D$, the image of the tangent space at s is an abelian subalgebra of $\mathfrak{g}_{\mathbb{C}}$ contained in $U^{-1}\mathfrak{g}_{F(s)} = \bigoplus_{q\leqslant 1} \mathfrak{g}^{-1,q}_{F(s)}$.*

3.3 Mixed Hodge metrics

The mixed Hodge metric $h_{(F,W)}$ on H is defined as follows. We first declare the splitting (7) to be orthogonal and then define the metric on $I^{p,q}$ making use of the graded polarization on $\mathrm{Gr}^W H$ as follows. The summand $I^{p,q}$ maps isomorphically onto the subspace $H^{p,q}$ of Gr^W_{p+q}. So on classes $[z]$ of elements $z \in I^{p,q} \subset W_{p+q}$ modulo W_{p+q-1} the metric $h_{F,W}$ can be defined by setting:

$$h_{(F,W)}(x,y) = (\mathrm{Gr}h)_F([x],[y]), \quad x,y \in I^{p,q}. \tag{14}$$

Let $*$ denote the adjoint with respect to the metric h_F. Then,

$$* : \mathrm{End}^{p,q}(H) \to \mathrm{End}^{-p,-q}(H). \tag{15}$$

The Hodge metric induces a metric on $\mathrm{End}\, H$ given by

$$h_F(\alpha,\beta) = \mathrm{Tr}(\alpha\beta^*) \tag{16}$$

where β^* is the adjoint of β with respect to $h_{F,W}$. The Deligne splitting (9) of $\operatorname{End} H$ is then orthogonal with respect to the associated metric. The induced Hodge metric on the holomorphic tangent space $T_{D,F}$ of D at F comes from the natural identification (12).

In the sequel, we make use of the following orthogonal splittings.

$$\begin{array}{lcl} \mathfrak{g}_{\mathbb{C}} & = & \underbrace{\mathfrak{n}_+ \oplus \mathfrak{g}_F^{0,0}}_{\mathrm{Lie}(G_{\mathbb{C}}^F)} \oplus \underbrace{\mathfrak{n}_- \oplus \Lambda_F^{-1,-1}}_{\mathfrak{q}_F}, \\ \text{where} & & \\ \mathfrak{n}_+ & = & \bigoplus_{a\geqslant 0, b<0} \mathfrak{g}_F^{a,b}, \\ \mathfrak{n}_- & = & \bigoplus_{a<0, b\geqslant 0} \mathfrak{g}_F^{a,b}, \\ \Lambda_F^{-1,-1} & = & \bigoplus_{a\leqslant -1, b\leqslant -1} \mathfrak{g}_F^{a,b}. \end{array}$$

See Figure 1 below.

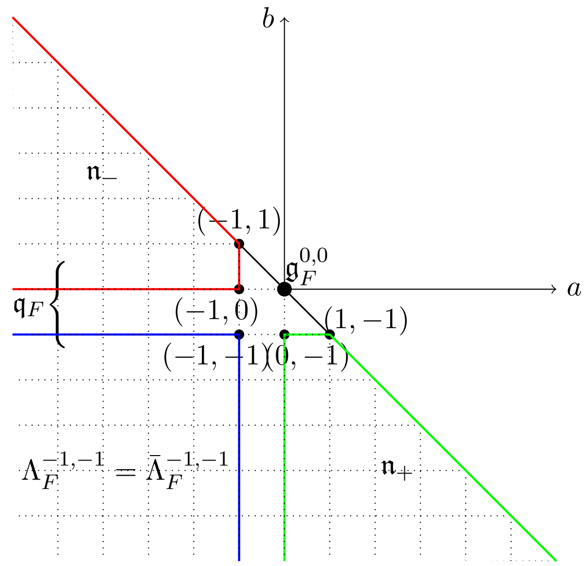


Figure 1. Decomposition of $\mathfrak{g}_{\mathbb{C}}$

The orthogonal decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}_+ \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{n}_- \oplus \Lambda_F^{-1,-1}$ defines respective orthogonal projectors

$$\begin{array}{rcl} \pi_{\pm} & : \mathfrak{g}_{\mathbb{C}} \longrightarrow & \mathfrak{n}_{\pm}, \\ \pi_0 & : \mathfrak{g}_{\mathbb{C}} \longrightarrow & \mathfrak{g}^{0,0}, \\ \pi_{\Lambda^{-1,-1}} & : \mathfrak{g}_{\mathbb{C}} \longrightarrow & \Lambda_F^{-1,-1}, \\ \pi_{\mathfrak{q}} & : \mathfrak{g}_{\mathbb{C}} \longrightarrow & \mathfrak{q}_F. \end{array} \tag{17}$$

3.4 Higgs bundles in the mixed setting

As in the pure case, the Hodge filtration $\mathcal{F}^\bullet$ defines a Higgs bundle structure on a variation of mixed Hodge structure over S. Using the Deligne splitting (7) on the fiber F, the role of $H^{p,q}$ is played by

$$U_F^p := \bigoplus_q I_F^{p,q} \simeq F^p/F^{p+1},$$

which cuts out $H^{p,q}$ on $\mathrm{Gr}^W_{p+q}H$. These form the fibers of the $\mathcal{C}^\infty$ bundles

$$\mathcal{U}^p = \bigoplus_q \mathcal{I}^{p,q} \simeq \mathcal{F}^p/\mathcal{F}^{p+1}. \tag{18}$$

The Higgs structure is slightly more involved than in the pure case: by [Pea00], the Gauss–Manin connection of $\mathcal{H}$ decomposes as

$$\nabla = \tau_0 + \bar\partial + \partial + \theta. \tag{19}$$

Here $\bar\partial$ and ∂ are differential operators of type $(0,1)$ and $(1,0)$ which preserve $\mathcal{U}^p$. The first, $\bar\partial$, gives the holomorphic structure induced by the $\mathcal{C}^\infty$-isomorphism $\mathcal{U}^p \simeq \mathcal{F}^p/\mathcal{F}^{p+1}$ which defines the Higgs bundle $\mathcal{U}_{\mathrm{Higgs}}$. The ***Higgs field*** in this setting is the operator θ, an endomorphism of $\mathcal{H}$ sending $\mathcal{U}^p$ to $\mathcal{U}^{p-1}$ with values in the $(1,0)$-forms and τ_0 is an endomorphism sending $\mathcal{U}^p$ to $\mathcal{U}^{p+1}$ with values in the $(0,1)$-forms. The Higgs field has a geometric interpretation which directly follows from its construction:

Lemma 3.4.1. *Let $F : S \to D$ be a local period map. Under the correspondence* (12)*, the Higgs field in a tangent direction $\xi \in T_sS$ can be identified with $F_*\xi$ viewed as a degree -1 endomorphism of $\mathcal{U}_{\mathrm{Higgs}}$:*

$$\theta^{1,0}_\xi = F_*\xi : \mathcal{U}_{\mathrm{Higgs},s} \to \mathcal{U}_{\mathrm{Higgs},s}, \quad F_*\xi \in \mathfrak{q}^{\mathrm{hor}}_{F(s)}.$$

In particular, the period map is injective, if and only if for all non-zero directions ξ the map $\theta^{1,0}_\xi$ is not the zero-map.

By functoriality all this applies to the endomorphism bundle $\mathfrak{g}(\mathcal{H})$ with induced variation of mixed Hodge structure. In the latter set-up we have:

Lemma 3.4.2. *Let η be a local holomorphic section of $\mathcal{U}^{-1}\mathfrak{g}(\mathcal{H})$ at $s \in S$ and $\xi \in T_s(S)$ a tangent vector of type $(1,0)$ at s. Set $v = \eta(s), u = F_*\xi \in \mathfrak{g}^{\mathrm{hor}}_{F(s)}$. Then*

$$\nabla_\xi v = \partial_\xi v + \mathrm{ad}\,(u)v, \tag{20}$$

$$\nabla_{\bar\xi} v = \pi^{(0)}\,\mathrm{ad}\,(\pi_+\bar u)v. \tag{21}$$

Here the bundle map $\pi^{(0)}$ stands for the orthogonal projection onto $\mathcal{U}^0$.

Proof. First consider the general case of a mixed variation $\mathcal{H}$ and $u \in \mathcal{U}^p$. The operator $\bar\partial$ in (19) breaks up in a component of bi-degree $(0,0)$ and a component τ_- of bi-degrees $(0,-1)+(0,-2)+\dots$. Comparing with [Pea00, Lemma 5.11], letting $\pi^{(p)}$ stands for the orthogonal projection onto $\mathcal{U}^p$, we see

$$\pi^{(p)}\pi_+(\bar u) = \tau_-, \quad \pi^{(p+1)}\pi_+(\bar u) = \tau_0.$$

Since the action of $\mathfrak{g}_{F(0)}$ on $\mathrm{End}(\mathcal{H}_{F(0)})$ is through the adjoint action, setting $p=-1$ we see that τ_0 gives rise to $\pi^{(0)}\,\mathrm{ad}\,(\pi_+(\bar u))v$. Since η is holomorphic, $\bar\partial\eta = 0$. As to θ, comparing with equation (5.20) in loc. cit. we see that θ gives $\mathrm{ad}\,(u)v$. This proves the result. □

4. Differential geometry

4.1 The Chern connection on the endomorphism bundle

Let D be the Chern connection on the endomorphism bundle. In [PP19, § 5] we calculated it for the bundles $\mathcal{U}^{(p)}$ and found

$$\mathsf{D} = \bar\partial + \partial - \tau_-^*,$$

where τ_-^* is the adjoint of the operator τ_- with respect to the mixed Hodge metric. We already calculated $\tau_- = \pi^{(p)}\pi_+(\bar{u})$ and so $\tau_-^* = \pi^{(p)}(\pi_+(\bar{u}))^*$. By functoriality this holds also for the endomorphism bundle using the adjoint action, where we apply it for $\mathcal{U}^{(-1)}$. Since in this situation $\pi^{(-1)}$ is the same as projection onto $\mathfrak{q}$, we get:

$$\mathsf{D}_\xi\eta = \partial_\xi v - \pi_{\mathfrak{q}}[(\pi_+\bar{u})^*, v], \quad u = F_*\xi,\ v = \eta(s). \tag{22}$$

4.2 Curvature and plurisubharmonicity of Hodge norms

In contrast to the pure case, the biholomorphic bisectional curvature of the horizontal tangent bundle is not always semi-negative as expressed by the following theorem.

PROPOSITION 4.2.1. *The bisectional curvature of the Hodge metric in unit directions $u, v \in U^{-1}\mathfrak{g}_F$ equals*

$$K(u,v) = \|[u^{-1,1}, v]\|^2 + \|\pi_{\mathfrak{q}}[(\pi_+\bar{u})^*, v]\|^2 - \|[\pi_+\bar{u}, v]\|^2 - \operatorname{Re} h(\pi_{\mathfrak{q}}[\pi_+[u,\bar{u}], v]).$$

Proof. We use the curvature tensor for the Hodge metric h as given in [PP19, Theorem 3.4]:

$$\begin{aligned} R_h(u,\bar{u}) &= R_1 + R_2 + R_3 \\ R_1 &= -[\pi_{\mathfrak{q}} \operatorname{ad}((\pi_+\bar{u})^*), \pi_{\mathfrak{q}} \operatorname{ad}((\pi_+\bar{u})] \\ R_2 &= -\operatorname{ad}(\pi_0[u,\bar{u}]) \\ R_3 &= \pi_{\mathfrak{q}}(\operatorname{ad}(\pi_+[\bar{u}^*, u])) + \pi_{\mathfrak{q}}(\operatorname{ad}(\pi_+[\bar{u}, u])). \end{aligned}$$

To calculate $K(u,v)$ from this, we follow the proof of [PP19, Theorem 4.1] and calculate the terms $h(R_j v, v)$ for $j = 1,2,3$ of the biholomorphic sectional curvature. With $\|-\| = \|-\|_F$ the Hodge norm on $\operatorname{End} H$ we have

$$\begin{aligned} h(R_1 v, v) &= \underbrace{-\|\pi_{\mathfrak{q}}[\pi_+(\bar{u}), v]\|^2}_{A_1} + \underbrace{\|\pi_{\mathfrak{q}}[(\pi_+\bar{u})^*, v]\|^2}_{A_2} \\ h(R_2 v, v) &= -h([\pi_0[u,\bar{u}], v], v) = A_3 \\ h(R_3 v, v) &= -\operatorname{Re} h(\pi_{\mathfrak{q}}[\pi_+[u,\bar{u}], v], v). \end{aligned}$$

To calculate A_3 remark that $\pi_{\mathfrak{q}}(\operatorname{ad}(\pi_0[u,\bar{u}])) = \operatorname{ad}([u^{-1,1}, (u^{-1,1})^*])$ and so

$$A_3 = h(R_2 v, v) = \|[u^{-1,1}, v]\|^2 - \|[(u^{-1,1})^*, v]\|^2.$$

Next, observe that $[(u^{-1,1})^*, v] \in U_F^0$ and $\pi_{\mathfrak{q}}[\pi_+\bar{u}, v] \in U_F^{-1}$ have different bidegrees and hence are mutually orthogonal with sum equal to $[\pi_+\bar{u}, v]$. Consequently,

$$-\|[(u^{-1,1})^*, v]\|^2 \underbrace{-\|\pi_{\mathfrak{q}}[\pi_+\bar{u}, v]\|^2}_{A_1} = -\|[\pi_+\bar{u}, v]\|^2.$$

The result follows. $\square$

We consider now Eqn. (4) in the present situation. As a consequence of Eqn. (22) and Proposition 4.2.1, we have

$$\left\{ \begin{array}{rcl} \partial_u\partial_{\bar{u}}\|v\|^2 &=& \|\mathsf{D}_u\eta\|^2 - K(u,v) \\ &=& \|\partial_u v(s) + \pi_{\mathfrak{q}}[\pi_+\bar{u}^*, v]\|^2 - \|\pi_{\mathfrak{q}}[\pi_+\bar{u}^*, v]\|^2 + \|[\pi_+\bar{u}, v]\|^2 \\ && -\|[u^{-1,1}, v]\|^2 + \operatorname{Re} h(\pi_{\mathfrak{q}}[\pi_+[u,\bar{u}], v], v). \end{array} \right.$$

If $\pi_{\mathfrak{q}}[\pi_+\bar{u}^*, v]$ and v are orthogonal, this simplifies to give

$$\begin{aligned} \partial_u\partial_{\bar{u}}\|v\|^2 &= \|\partial_\xi v(s)\|^2 + \|[\pi_+\bar{u}, v]\|^2 \\ &\quad - \big([u^{-1,1}, v]\|^2 + \operatorname{Re} h(\pi_{\mathfrak{q}}[\pi_+[u,\bar{u}], v]), v\big)\,. \end{aligned} \tag{23}$$

A direct consequence of (23) and Lemma 3.4.2 gives the following variant of Proposition 2.2.3 in the mixed case:

Proposition 4.2.2. *Let there be given a graded polarized mixed variation of Hodge structure $(\mathcal{H}, Q, \mathcal{F})$ over a quasi-projective complex manifold S. Let η a holomorphic section of $\mathcal{U}^{-1}(\mathfrak{g}(\mathcal{H}))$. For $s \in S$, let $v = \eta(s)$, viewed as a horizontal tangent vector at $F = F(s) \in D$. Suppose that for all $u \in T_F^{\mathrm{hor}} D$ tangent to the image of the period map at all images $F \in D$ of the period map, one has*

$$[u^{-1,1}, v] = 0 \tag{24}$$

$$h(\pi_{\mathfrak{q}}[\pi_+\bar{u}^*, v], v) = 0 \tag{25}$$

$$\operatorname{Re} h(\pi_{\mathfrak{q}}[\pi_+[u,\bar{u}], v], v) = 0. \tag{26}$$

Then the function $\|v\|^2$ is plurisubharmonic and, if bounded (and hence constant), we have

$$\partial_\xi v = [\pi_+\bar{u}, v] = 0. \tag{27}$$

If, moreover, $[u, v] = 0$, η is a flat section.

Conversely, if η is flat, then $\|v(s)\|$ is constant and (27) holds.

Remark 4.2.3. (1) In the cases of interest to us, flat sections are bounded in the mixed Hodge norm. See Section 5, although this is not the case in general as shown in Subsection 5.11.
(2) In the pure case the conditions $[u, v] = 0$ and $[u^{1,1}, v] = 0$ are equivalent and the two remaining conditions hold for type reasons.

For easy reference, a section η with the property that for all tangent vectors u along S the conditions (24)–(25) hold, is called ***a pluri-subharmonic endomorphism***.

To give geometric examples where this phenomenon occurs, we first prove:

Proposition 4.2.4. *In the situation of Proposition 4.2.2, assuming that $[u, v] = 0$, the endomorphism η is plurisubharmonic in the following cases:*

(i) *the pure case;*
(ii) *$\mathbb{R}$-split variations (e.g. two adjacent weights) in directions $v = v^{-1,0}$;*
(iii) *in the setting of unipotent variations (i.e. $u^{-1,1} = 0$) provided either $\Lambda^{-1,-1} = 0$ and $v = v^{-1,0}$, or $u \in \Lambda^{-1,-1}$ and $v^{-1,1} = 0$.*
(iv) *variations with $u = u^{-1,1} + u^{-1,-1}$ in directions $v = v^{-1,-1}$.*
(v) *two non-adjacent weights, say $0, k$, $|k| \geqslant 2$ with $h^{0,0} = 1$, $h^{p,-p} = 0$ for $p \neq 0$, in directions $v = v^{-1,-k+1}$.*
(vi) *A variation of type*

$$\begin{tikzcd} & I^{0,0} \arrow[d, "u^{-1,-1}"'] \arrow[dr, "u^{0,-2}"] & \\ I^{-2,0} \arrow[dr, "u^{0,-2}"'] & I^{-1,-1} \arrow[l, "u^{-1,1}"'] \arrow[d, "u^{-1,-1}"] & I^{0,-2} \arrow[l, "u^{-1,1}"'] \\ & I^{-2,-2} & \end{tikzcd}$$

in directions $v = v^{-1,-1}$.

In cases (1), (4) and (5), one has $K(u,v) \leqslant 0$.

In all cases, if $\|\eta\|$ is bounded, then η is parallel for the Gauss–Manin connection.

Proof. The pure case is Lemma 2.2.2. In the remaining cases we consider the conditions for v to be pseudo-plurisubharmonic separately. Condition (24) follows either trivially since $u^{-1,1} = 0$, or it follows from $[u, v] = 0$ since u has two Hodge types while $v^{-1,1} = 0$.

For conditions (25) and (26) we write

$$u = \alpha + \beta + \lambda, \quad \alpha = u^{-1,1}, \quad \beta = u^{-1,0}, \quad \lambda = \pi_\Lambda^{-1,-1} u.$$

Observe that

$$\begin{aligned} \bar{u} &= \alpha^* + \epsilon + \delta, \\ \epsilon &= \pi^{(0)}(\bar{u}) \in \bigoplus_{q\geqslant 2} \mathfrak{g}^{0,-q}, \\ \delta &= \pi_+(\bar{u}^{-1,0}) \in \mathfrak{g}^{0,-1} \end{aligned}$$

and so

$$\begin{aligned} \pi_{\mathfrak{q}}[\pi_+[u,\bar{u}], v] &= [[\beta, \alpha^*], v] + [[\lambda, \alpha^*], v], \\ [\beta, \alpha^*] &\in \mathfrak{g}^{0,-1}, \\ [\lambda, \alpha^*] &\in \bigoplus_{k\geqslant 2} \mathfrak{g}^{0,-k}. \end{aligned}$$

First consider condition (26). In case (3) one has $\pi_+[u,\bar{u}] = 0$. In case (2), $\pi_+[u,\bar{u}] = [\beta, \alpha^*]$ has bi-degree $(0,-1)$ and so sends $v = v^{-1,0}$ to 0. In case (4) and (6), $\pi_+[u,\bar{u}] = [\lambda, \alpha^*]$ has bi-degree $(0,-2)$ and so sends $v = v^{-1,-1}$ to zero. In case (5) $\pi_+[u,\bar{u}]$ has bi-degree $(0,-k)$ and so sends $v = v^{-1,1-|k|}$to zero.

Next, consider (25) and remark that

$$\begin{aligned} (\pi_+\bar{u})^* &= \alpha + \epsilon^* + \delta^*, \\ \epsilon^* &\in \bigoplus_{q\geqslant 2} \mathrm{End}^{0,q}, \quad \delta^* \in \mathrm{End}^{0,1}\,. \end{aligned}$$

(i) In the $\mathbb{R}$-split case, $\epsilon = 0$. In $\pi_{\mathfrak{q}}[(\pi_+\bar{u})^*, v]$ the terms of bi-degree $(-1,1)$ come from $[\delta^*, v^{-1,0}] + [\epsilon^*, \pi_{\Lambda^{-1,-1}} v]$. This proves (25) since then $\pi_{\mathfrak{q}}[(\pi_+\bar{u})^*, v] = [u^{-1,1}, v^{-1,0}] + [\delta^*, v^{-1,0}]$ has bi-degree $(-2,1) + (-1,1)$ and hence is orthogonal to $\partial_\xi v$ since it has bi-degree $(-1,0)$.

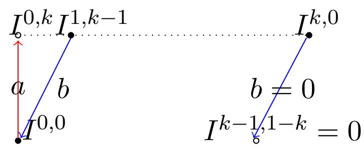
(ii) In the unipotent situation we also have $\epsilon = 0$ and now $\pi_+\bar{u}^* = \delta^*$ which vanishes if $u \in \Lambda^{-1,-1}$ and else has pure type $(0,1)$. But then $\pi_{\mathfrak{q}}[(\pi_+\bar{u})^*, v^{-1,0}]$ has bi-degree $(-1,1)$ and so is orthogonal to $v = v^{-1,0} + v_{\Lambda^{-1,-1}}$.

(iii) In this case $\epsilon = 0$ and $\delta = 0$, we find that $(\pi_+\bar{u})^* = \alpha$ so that $\pi_{\mathfrak{q}}[(\pi_+\bar{u})^*, v] = [u^{-1,1}, v] = 0$ which is condition (24) and we just proved it.

(iv) Here we show that $\pi_{\mathfrak{q}}[(\pi_+\bar{u})^*, v] = 0$ using:

LEMMA 4.2.5. *Let $a \in \mathrm{End}^{0,k}$, $b \in \mathfrak{g}^{-1,1-k}$ and let $c = \pi_{\mathfrak{q}}(a \circ b) \in \mathfrak{g}^{-1,1}$. Suppose that $h^{p,q} = 0$ unless $p + q = k \geqslant 1$ or $p = q = 0$. Then $c = 0$.*

Proof. Let $x \in I^{1,k-1}$. Then $c(x) \in I^{0,k}$. To show that $c(x) = 0$ it suffices to show that it is orthogonal to $I^{0,k}$. Observe that every element $y \in I^{0,k}$ is of the form $y = \bar{z}$ for some $z \in$



$I^{k,0}$ which is the case because of the assumption on the Hodge numbers. But $\pm \mathbf{i}^k h(c(x), y) = Q(c(x), z) = -Q(x, c(z)) = 0$ since $b(z) = 0$. □

We apply this lemma with $a = \pi_+ \bar{u}^* = \epsilon^* \in \mathrm{End}^{0,k}$, $b = v \in \mathfrak{g}^{-1,1-k}$.

(v) The last case is clear from type considerations.

For the assertion about the curvature, observe that the only term in the expression for $K(u, v)$ given in Proposition 4.2.1 that causes trouble is $\pi_{\mathfrak{q}}[(\pi_+\bar{u})^*, v]$ which, as we showed above, vanishes in cases (4) and (5). □

4.3 Horizontal plurisubharmonic endomorphisms: geometric examples

We indicate how some of the geometric examples mentioned in the introduction fit in with the cases exhibited in Proposition 4.2.4.

(i) ***Normal functions.*** We explain how to interpret a classical normal function as a variation of $\mathbb{Z}$-mixed Hodge structure. Suppose that $X = X_o$ is a smooth projective variety. A homologically trivial algebraic p-cycle Z in X canonically determines an extension

$$\nu_Z \in \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(0), H_{2p+1}(X, \mathbb{Z}(-p))).$$

in the category of $\mathbb{Z}$-mixed Hodge structures by pulling back the exact sequence

$$0 \to H_{2p+1}(X, \mathbb{Z}(-p)) \to H_{2p+1}(X, Z, \mathbb{Z}(-p)) \to H_{2p}(Z, \mathbb{Z}(-p)) \to \cdots$$

along the inclusion $\mathbb{Z}(0) \hookrightarrow H_{2p}(Z, \mathbb{Z}(-p))$ sending 1 to the class of Z. It is well known (cf. [Car87]) that

$$\mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(0), H_{2p+1}(X, \mathbb{Z}(-p))) \simeq J^p(X),$$

the intermediate Jacobian of X. The point in $JH_{2p+1}(X, \mathbb{Z}(p))$ corresponding to the cycle Z under this isomorphism is $\int_\Gamma$, where Γ is a real $2p+1$ chain that satisfies $\partial\Gamma = Z$.
If $X = X_o$ varies in a smooth family X_s with smooth base S, say $\pi : X \to S$, the groups $H_{2p+1}(X_s, \mathbb{Z}(-p))$ form a local system $\underline{H}_{2p+1}(-p)$ defining a variation of Hodge structure. The intermediate Jacobians vary holomorphically, and glue together to give the relative intermediate Jacobian $J^p(X/S)$.
Suppose that Z is an algebraic cycle in X which is proper over S of relative dimension p and such that Z_s the fiber over $s \in S$ is homologous to zero. Then Z_s defines a point ν_{Z_s} in the intermediate Jacobian $J^p(X_s)$. These give a holomorphic section ν_Z of $J^p(X/S)$, and this is the classical normal function. It can be viewed as an extension

$$\mathrm{Ext}^1_{\mathsf{VMHS}}(\mathbb{Z}(0), \underline{H}_{2p+1}(-p))$$

in the category of variations of mixed Hodge structures. Such a variation has two adjacent weights $0, -1$ and by case (2) of Proposition 4.2.4, $\|v^{-1,0}\|$ is plurisubharmonic. For this example the term $\pi_{\mathfrak{q}}[(\pi_+\bar{u})^*, v]$ need not vanish and so we cannot conclude from Proposition 4.2.1

that $K(u,v) \leqslant 0$. However, a more sophisticated argument as in [PP19, proof of Prop. 6.2] reveals that $K(u^{-1,0}, v^{-1,0}) \leqslant 0$.

(ii) ***Hodge–Tate variations.*** Only extension data can be deformed. These are deformations with $v = v^{-1,-1}$. Case (3) of Proposition 4.2.4 shows that $\|v^{-1,-1}\|$ is harmonic. Of course the biholomorphic curvature is 0 since D is flat. As a simple example of a 1-parameter variation, suppose $h^{-1,-1} = 2, h^{0,0} = 1$ and let $\{e_1, e_2, e_3\}$ be a basis of the lattice $H_{\mathbb{Z}}$. Let F_o denote the reference filtration

$$I^{0,0}_{(F_o,W)} = \mathbb{C}e_1, \qquad I^{-1,-1}_{(F_o,W)} = \mathbb{C}e_2 \oplus \mathbb{C}e_3.$$

Then the period domain $D = G/G^{F_o}$ is isomorphic to the unipotent group $U_{\mathbb{C}}$ consisting of the matrices

$$g_{a,b} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}, \qquad a, b \in \mathbb{C}$$

via the action of $U_{\mathbb{C}}$ on F_o. Consider the period map $\mathbb{C}^* \to \Gamma\backslash\mathbb{C}^2$ given by $u \mapsto g_{\log u, 0}.F_o$ and with monodromy group Γ the unipotent group consisting of elements $g_{a,0} \in G$, $a \in \mathbb{Z}$. This variation clearly has a deformation leading to a variation over $\mathbb{C}^* \times \mathbb{C}$ given by the map $(u,v) \mapsto g_{\log u, v}.F_o$.

Contrast this with the following example of a biextension of Hodge Tate type with Hodge numbers $h^{0,0} = h^{-1,-1} = h^{-2,-2} = 1$. Let $\{e_1, e_2, e_3\}$ be a basis of $H_{\mathbb{Z}}$. Let F_o denote the reference filtration such that

$$I^{-2,-2}_{(F_o,W)} = \mathbb{C}e_3, \qquad I^{-1,-1}_{(F_o,W)} = \mathbb{C}e_2, \qquad I^{0,0}_{(F_o,W)} = \mathbb{C}e_1$$

The period domain $D = G/G^F$ is isomorphic to the unipotent group $U_{\mathbb{C}}$ consisting of matrices of the form

$$g_{a,b,c} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}, \quad a, b, c \in \mathbb{C}$$

by the action of $U_{\mathbb{C}}$ on F_o. Let E_{ij} denote the 3×3 matrix whose only non-zero entry is 1 in row i and column j. Then, the Lie algebra of $U_{\mathbb{C}}$ is equal to $\mathfrak{g}^{-1,-1} \oplus \mathfrak{g}^{-2,-2}$ where $\mathfrak{g}^{-1,-1}$ is spanned by $u_0 = E_{21}$ and $u_1 = E_{32}$ while $\mathfrak{g}^{-2,-2}$ is spanned by $u_2 = E_{31} = [u_1, u_0]$. Now a period map can be given locally as $z \mapsto \exp(\Gamma(z)).F_o$ where

$$\Gamma(z) = f_0 u_0 + f_1 u_1 + f_2 u_2 \implies \exp(\Gamma(z)) = \begin{pmatrix} 1 & 0 & 0 \\ f_0 & 1 & 0 \\ f_2 + \frac{1}{2} f_0 f_1 & f_1 & 1 \end{pmatrix}.$$

If it is injective we may assume that $f_0 = z$. The commutativity condition for horizontal directions gives $df_0 \wedge df_1 = 0$ and so $f_1 = \varphi(z)$ for some function φ with $\varphi(0) = 0$. The horizontality condition gives $df_2 + \frac{1}{2}(f_1 df_0 + f_0 df_1) = 0$ and so $f_2 = \psi(z)$ for some function ψ with $\psi(0) = 0$. This implies that a non-trivial injective period map has a curve as its image and hence must be rigid.

As a concrete example we take the Hodge–Tate variation associated to the dilogarithm. Here $S = \mathbb{P}^1 - \{0, 1, \infty\}$ with global coordinate s. The period map

$$\mathbb{P}^1 - \{0,1,\infty\} \to U_{\mathbb{Z}}\backslash D$$

is then given by the functions $f_0(s) = -(\log 2 + \log(1-s))$ and $f_1(s) = \log 2 + \log s$ which vanish at $s = \frac{1}{2}$. The horizontality condition gives

$$f_2(s) = -\frac{1}{2}\int_{\frac{1}{2}}^{s}\left(\frac{\log t}{1-t} + \frac{\log(1-t)}{t}\right) dt = -\frac{1}{2}\mathrm{Li}_2 s + \frac{1}{2}\mathrm{Li}_2(1-s).$$

(iii) ***Variations of mixed Hodge structures attached to fundamental groups.***
Let us briefly explain which variations we are considering. Let X be a smooth algebraic variety and let J_x be the kernel of the ring homomorphism $\mathbb{Z}\pi_1(X,x) \to \mathbb{Z}$ given by $\sum n_\gamma \gamma \mapsto \sum n_\gamma$, $\gamma \in \pi_1(X,x)$. There are mixed Hodge structures on J_x/J_x^n which depend on the base point $x \in X$. For $n = 3$ these can be explicitly described, following [Hai87, Section 6]: the mixed Hodge structure on the dual, $\mathrm{Hom}_{\mathbb{Z}}(J_x/J_x^3, \mathbb{C})$ is an extension

$$0 \to H^1(X) \to \mathrm{Hom}_{\mathbb{Z}}(J_x/J_x^3, \mathbb{C}) \xrightarrow{p} \mathrm{Ker}(H^1(X) \otimes H^1(X) \to H^2(X)) \to 0,$$

provided $H_1(X)$ is torsion free. Here we want pure Hodge structures and this forces H^1 to be of pure weight $\ell = 1$ or 2 and weight $H^2 = 2\ell$. Geometric examples include X smooth projective or $X = \mathbb{P}^1 - \Sigma$, Σ a finite set of points. The extension depends on x, but the two pure Hodge structures remain fixed so that $u^{-1,1} = 0$ and we are in the unipotent situation with $v = v^{-1,-\ell+1}$, $u = u^{-1,-\ell+1}$. If $\ell = 1$ we have $\Lambda^{-1,-1} = 0, v = v^{-1,0}$ and if $\ell = 2$, $u, v \in \Lambda^{-1,-1}$ and so case (3) of Proposition 4.2.4 shows that $\|v^{-1,-\ell+1}\|$ is plurisubharmonic. One can directly verify that also $K(u^{-1,-\ell+1}, v^{-1,-\ell+1}) \leqslant 0$.

(iv) ***Nilpotent orbits associated to Kähler classes.*** As explained in the introduction, these variations have Hodge types $(-1,1)$ and $(-1,-1)$. However, v can a priori have any type $(-1,k)$, $k \leqslant 1$. By case (4) of Proposition 4.2.4, endomorphisms for which $v = v^{-1,-1}$ are plurisubharmonic and then $K(u,v) \leqslant 0$. Note that for a family of projective manifolds over a quasi-compact base S we can assume that we have a variation of integral Hodge structures polarized by a family of independent flat integral Kähler classes (corresponding to ample divisors).

(v) ***Higher normal functions.***
Let $\pi : X \to S$ be a smooth projective family. Recall (see the introduction) that a higher normal function is an extension in

$$\mathrm{Ext}^1_{\mathsf{VMHS}}(\mathbb{Q}(0), R^{p-1}\pi_*\mathbb{Q}(q)), \quad w = p - 2q - 1 < 0.$$

Case (5) of Proposition 4.2.4 tells us that $\|v^{-1,w+1}\|$ is plurisubharmonic and $K(u, v^{-1,w+1}) \leqslant 0$.

(vi) ***Biextensions of bidegrees*** $(0,0), (-2,-2), (-2,0), (-1,-1), (0,-2)$***.*** Case (6) of Proposition 4.2.4. shows that $\|v^{-1,-1}\|$ is plurisubharmonic. Geometric examples arise as a special case of a more general construction given by J. Burgos Gill, S. Goswami and the first author in [BGGP22], two higher Chow cycles in $\mathcal{Z}^p(X,1)$ on a d-dimensional variety X with $p+q = d+2$ determine in a canonical way a special type of mixed Hodge structure. For a family of surfaces, we have $d = 2$ and the resulting variation is of biextension type with bidegrees $(0,0), (-2,-2), (-2,0), (-1,-1), (0,-2)$. For more details on this example see Section 5.10.

5. Norm Estimates for admissible variations

Let $\mathcal{H} \to \Delta^*$ be an admissible variation of graded-polarized mixed Hodge structure over the punctured disk Δ^* with unipotent monodromy $T = e^N$. Recall that $\mathfrak{g}(\mathcal{H}) \subset \mathcal{H} \otimes \mathcal{H}^*$ is the sub-variation of mixed Hodge structure generated by local sections which preserve W and induce infinitesimal isometries on Gr^W. In this section, we show that in the cases enumerated below, the mixed Hodge norm of a monodromy invariant section of $\mathfrak{g}(\mathcal{H})$ is bounded. In section (5.11), we show that $\|N\|$ can be unbounded on Δ^* for higher normal functions.

In section 12 of [KNU08], K. Kato, C. Nakayama and S. Usui prove mixed Hodge norm estimates using their SL_2-orbit theorem. However, the metric used in [KNU08] involves an artificial twisting of the Hodge metric on each Gr^W, and hence is different than the metric used in this paper. In [HP15], The first author and Tatsuki Hayama construct an intrinsic "twisted metric" on D which which gives the same norm estimates as [KNU08] for admissible variations for which the limit MHS is not split over $\mathbb{R}$. The twisted metric considered in [HP15] is only invariant under $G_{\mathbb{R}}$, i.e. $g \in \exp(\Lambda^{-1,-1}_{(F,W)})$ need not induce an isometry $L_{g*} : T_F(D) \to T_{g.F}(D)$ by left translation. For this reason, the curvature computations of [PP19] do not apply to this metric on D.

The material in this section assumes familiarity with the definition and basic theory of admissible variations of mixed Hodge structure as outlined in Appendix (A).

Before continuing, we emphasize that if (F, W) is a graded-polarized mixed Hodge structure with underlying vector space V and

$$g \in G_{\mathbb{R}} \cup \exp(\Lambda^{-1,-1}_{(F,W)}), \qquad \alpha \in \mathrm{gl}(V), \tag{28}$$

then

$$\|\alpha\|_{(g.F,W)} = \|g^{-1}.\alpha\|_{(F,W)} \tag{29}$$

where $g.\alpha = \mathrm{Ad}(g)\alpha$. Indeed, if $\{v_j\}$ is an unitary frame with the respect to the mixed Hodge metric $h_{(F,W)}$ then $\{gv_j\}$ is a unitary frame for $h_{(g.F,W)}$. Therefore,

$$\begin{aligned}
\|\alpha\|^2_{(g.F,W)} &= \sum_j h_{(g.F,W)}(\alpha(gv_j), \alpha(gv_j)) \\
&= \sum_j h_{(F,W)}(g^{-1}\alpha(gv_j), g^{-1}\alpha(gv_j)) \\
&= \sum_j h_{(F,W)}((g^{-1}.\alpha)(v_j), (g^{-1}.\alpha)(v_j)) = \|(g^{-1}.\alpha)\|_{(F,W)}
\end{aligned}$$

In particular, if $\mathcal{H}$ is a variation of type I or II as defined in section (5.8) it will not be the case that $\mathfrak{g}(\mathcal{H})$ is type I or type II. Nonetheless, all of the calculations in section (5.8) depend only on (29) and a version of the SL_2-orbit theorem for nilpotent orbits of type I or II.

By way of notation $z = x + \mathbf{i}y$ throughout this section. In the several variable case $z_j = x_j + \mathbf{i}y_j$.

5.1 Variations of Pure Hodge Structure

Let $\mathcal{H} \to \Delta^*$ be a variation of pure Hodge structure over the punctured disk with unipotent local monodromy. By Corollary (6.7) of [Sch73], the Hodge norm of an invariant class is bounded. This result is a consequence of Schmid's SL_2-orbit theorem [Sch73]. If $\mathcal{H}$ is a variation of pure Hodge structure then so is $\mathcal{H} \otimes \mathcal{H}^*$, and hence if $\alpha \in \mathcal{H} \otimes \mathcal{H}^*$ is monodromy invariant then $\|\alpha\|$ has bounded Hodge norm.

For future use, we recall that by the Monodromy Theorem (see Theorem (6.1), [Sch73]) if $\mathcal{H} \to \Delta^*$ is a variation of pure Hodge structure with unipotent monodromy $T = e^N$, then $N^\ell = 0$ where ℓ is the maximum number of successive non-zero Hodge summands of $\mathcal{H}$ (e.g. for a family of curves of positive genus, $\ell = 2$ since $\mathcal{H} = H^{1,0} \oplus H^{0,1}$).

Applied to a variation of graded-polarized mixed Hodge structure $\mathcal{H} \to \Delta^*$ with unipotent monodromy $T = e^N$, it follows that if $\mathrm{Gr}^W_{2p}(\mathcal{H})$ is pure of type (p,p) then N acts trivially on Gr^W_{2p}.

5.2 Local Normal Form

In connection with deformations of mixed period maps and the derivation of the norm estimates below, we recall the following (eq. (2.5) [CK89], eq. (6.8) [Pea00]).

Let $(s_1, ..., s_{a+b})$ be holomorphic coordinates on the polydisk Δ^{a+b} and $\Delta^{*a} \times \Delta^b$ denote the complement of the divisor $s_1 \cdots s_a = 0$. Let U^a denote the a-fold product of the upper half-plane with Cartesian coordinates $(z_1, \ldots, z_a)$ and covering map $U^a \times \Delta^b \to \Delta^{*a} \times \Delta^b \subset \Delta^a \times \Delta^b$ given by the formula

$$(z_1, \ldots, z_a; s_{a+1}, \ldots, s_b) \mapsto (e^{2\pi \mathbf{i} z_1}, \ldots, e^{2\pi \mathbf{i} z_a}, s_{a+1}, \ldots, s_b)$$

Let $\mathcal{H}$ be an admissible variation of graded-polarized mixed Hodge structure over $\Delta^{*a} \times \Delta^b$ with unipotent monodromy $T_j = e^{N_j}$ about $s_j = 0$. Then, (cf. (82)), admissibility implies that the period map of $\mathcal{H}$ can be lifted to a holomorphic, horizontal map of the form

$$F(z_1, \ldots, z_a; s_{a+1}, \ldots, s_b) = e^{\sum_j z_j N_j}.\psi(s) \tag{30}$$

where $\psi(s)$ is a holomorphic map $\Delta^{a+b} \to \check{D}$ with $\psi(0) = F_\infty$.

To continue, define:

$$\mathcal{C} = \{ \sum_j \lambda_j N_j \mid \lambda_1, \ldots, \lambda_a > 0 \} \tag{31}$$

Then, by admissibility, there exists an increasing filtration $M(\mathcal{C}, W)$ such that if $N \in \mathcal{C}$ then $M(N, W)$ equals $M(\mathcal{C}, W)$. The results of Kashiwara show that if $\mathcal{H}$ is admissible then (F_∞, M) is a mixed Hodge structure relative to which each N_j is a $(-1,-1)$-morphism. Moreover, if $\mathfrak{g}_\mathbb{C}$ is the Lie algebra attached to the period map (30) of $\mathcal{H}$, then (F_∞, M) induces a graded-polarizable mixed Hodge structure on $\mathfrak{g}_\mathbb{C}$.

In particular, if $\mathfrak{g}_\mathbb{C} = \oplus_{p,q} \mathfrak{g}^{p,q}$ is the Deligne bigrading induced by (F_∞, M), then

$$\mathfrak{q} = \bigoplus_{p<0} \mathfrak{g}^{p,q} \tag{32}$$

is a vector space complement to the stabilizer $\mathfrak{g}_\mathbb{C}^{F_\infty}$ in $\mathfrak{g}_\mathbb{C}$. Therefore, after shrinking Δ^{a+b} as needed, it follows that there exists a unique $\mathfrak{q}$-valued holomorphic function $\Gamma(s)$ which vanishes at 0 such that

$$\psi(s) = e^{\Gamma(s)}.F_\infty \tag{33}$$

Let $\Gamma_{-1} = \sum_q \Gamma^{-1,q}(s)$. By equation (6.14) and Theorem (6.16) of [Pea00], the function $\Gamma_{-1}(s)$ satisfies the following integrability condition

$$\left[N_j + 2\pi \mathbf{i} s_j \frac{\partial \Gamma_{-1}}{\partial s_j}, N_k + 2\pi \mathbf{i} s_k \frac{\partial \Gamma_{-1}}{\partial s_k} \right] = 0 \tag{34}$$

for all j and k (with $N_\ell = 0$ for $\ell > a$).

Conversely, given an admissible nilpotent orbit

$$\theta(z_1, \ldots, z_a) = e^{\sum_j z_j N_j}.F_\infty$$

and a holomorphic function $\Gamma_{-1} : \Delta^{a+b} \to \oplus_q \mathfrak{g}_\mathbb{C}^{-1,q}$ which vanishes at zero and satisfies the integrability condition (34), there exists a unique holomorphic function $\Gamma : \Delta^{a+b} \to \mathfrak{q}$ which vanishes at 0 such that

$$F(z_1, \ldots, z_a; s_{a+1}, \ldots, s_b) = e^{\sum_j z_j N_j} e^{\Gamma(s)}.F_\infty \tag{35}$$

arises from the period map of a variation of graded-polarized mixed Hodge structure defined for $\Im(z_1),\dots,\Im(z_a) \gg 0$ and $s_{a+1},\dots,s_b \sim 0$ with $\Gamma_{-1} = \sum_q \Gamma^{-1,q}$. We call (35) the ***local normal form*** of the period map.

Remark 5.2.1. A published version of (34) and (35) for variations of pure Hodge structures appears in [CF01]. The key point is that the reconstruction of Γ from Γ_{-1} is really a statement about the horizontal distribution, and hence applies equally well to the mixed case. The full mixed case appears in [Pea00].

5.3 Hodge–Tate Variations

In section (5.5) it will be shown that if $\mathcal{H} \to \Delta^*$ is a unipotent variation of mixed Hodge structure in the sense of R. Hain and S. Zucker then any flat section of $\mathcal{H}$ has bounded mixed Hodge norm. In this section, we prove the following several variable result:

Theorem 5.3.1. *Let $\mathcal{H} \to \Delta^{*a} \times \Delta^b$ be an admissible Hodge–Tate variation with unipotent monodromy $T_j = e^{N_j}$ about $s_j = 0$. Let v be a flat section of $\mathcal{H}$. Then, v has bounded Hodge norm $\|v\|$ with respect to the mixed Hodge metric of $\mathcal{H}$. Likewise, if α is a flat section of $\mathcal{H}\otimes\mathcal{H}^*$ has bounded mixed Hodge norm $\|\alpha\|$.*

Proof. By Prop. (2.14), [SZ85] if N acts trivially on Gr^W then $M = M(N,W)$ exists if and only if $N(W_\ell) \subseteq W_{\ell-2}$ for all ℓ, wherefrom $M = W$.

To continue, recall $\mathcal{H}$ is Hodge–Tate means $\mathcal{H}^{p,q} = 0$ if $p \neq q$. Therefore, by the Monodromy theorem discussed at the end of (5.1), it follows that N acts trivially on Gr^W.

In particular, it follows from the previous paragraph that each $N_j = \log(T_j)$ acts trivially on Gr^W. By admissibility, it follows there exists a fixed increasing filtration $M = M(\mathcal{C},W)$ such that $M = M(N,W)$ for each $N \in \mathcal{C}$. As each $N \in \mathcal{C}$ acts trivially on Gr^W it follows that $M = W$ and $N(W_\ell) \subset W_{\ell-2}$ for each ℓ. Accordingly, since $\mathcal{C}$ consists of arbitrary positive linear combination of $N_1,\dots,N_a$ it follows that $N_a(W_\ell) \subset W_{\ell-2}$ for each index ℓ.

Since $M = W$, it follows that $F_\infty \in D$ and hence $\psi(s)$ also takes values in D. Moreover, since D classifies Hodge–Tate structures it follows that for any $F \in D$,

$$W_{-2}\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p<0} \mathfrak{g}^{p,p}_{(F,W)} = \Lambda^{-1,-1}_{(F,W)} \tag{36}$$

and hence $N_1,\dots,N_a \in \Lambda^{-1,-1}_{(F,W)}$.

Relative to the fixed reference fiber H of $\mathcal{H}$ used to define the period map into D, a flat section of $\mathcal{H}$ corresponds to an element of H which is contained in $\bigcap_j \mathrm{Ker}(N_j)$. Thus, by equation (36),

$$\begin{aligned}\|v\|_{(F(z;s),W)} &= \|v\|_{(e^{\sum_j z_j N_j} e^{\Gamma(s)}.F_\infty,W)} \\ &= \|e^{-\sum_j z_j N_j} v\|_{(e^{\Gamma(s)}.F_\infty,W)} \\ &= \|v\|_{(e^{\Gamma(s)}.F_\infty,W)}\end{aligned}$$

where the middle step is justified by the fact that $\sum_j z_j N_j$ belongs to $W_{-2}\mathfrak{g}_{\mathbb{C}}$ and equation (36). As $e^{\Gamma(s)}.F$ takes values in a compact subset of D, it follows from the last line of the equation that $\|v\|_{(F(z;s),W)}$ is bounded. The proof for the case of $\mathcal{H}\otimes\mathcal{H}^*$ is identical, except that $\alpha \in \cap_j \mathrm{Ker}(\mathrm{ad}\,(N)_j)$ and $e^{-\sum_j z_j N_j} v$ is replaced by $e^{-\sum_j z_j\,\mathrm{ad}\,(N)_j}\alpha$. $\square$

5.4 Gradings and Splittings of Mixed Hodge Structures

Let V be finite dimensional vector space over a field of characteristic zero. Then, a grading of V is a semisimple endomorphism Y of V with integral eigenvalues. In particular, a grading of V determines an increasing filtration

$$W_k(Y) = \bigoplus_{\ell\in\lambda(Y),\ell\leqslant k} E_\ell(Y) \tag{37}$$

where $\lambda(Y)$ is the set of eigenvalues of Y and $E_\ell(Y)$ is the ℓ-eigenspace of Y. If W is an increasing filtration of V we say Y grades W if $W(Y) = W$. In particular, a mixed Hodge structure (F, W) determines a grading $Y_{(F,W)}$ which acts a multiplication by $p+q$ on each non-zero summand $I^{p,q}$ of the Deligne bigrading of (F, W).

As discussed in Remark (3.1.1), the mixed Hodge structure (F, W) is split over $\mathbb{R}$ if $\overline{I^{p,q}} = I^{q,p}$. Equivalently, (F, W) is split over $\mathbb{R}$ if $\overline{Y_{(F,W)}} = Y_{(F,W)}$. In general, we say a grading Y is defined over $\mathbb{R}$ if $\overline{Y} = Y$.

By Proposition (2.20) of [CKS86], given a mixed Hodge structure (F, W) there exists a unique, real element

$$\delta \in \Lambda^{-1,-1}_{(F,W)} \tag{38}$$

such that $(\hat{F}, W) = (e^{-i\delta}.F, W)$ is an $\mathbb{R}$-split mixed Hodge structure. Moreover, δ commutes with all morphisms of (F, W) [3]. We henceforth call $(\hat{F}, W)$ the *Deligne δ-splitting* of (F, W).

Suppose now that if $\theta(z) = e^{zN}.F$ is a nilpotent orbit of pure Hodge structure of weight k polarized by Q. Let $W = W(N)[-k]$ and $(\hat{F}, W) = (e^{-i\delta}.F, W)$ be the Deligne δ-splitting of (F, W). Then, by equation (3.11) in [CKS86], δ is an infinitesimal isometry of Q. Likewise if $\theta(z) = e^{zN}.F$ is an admissible nilpotent orbit of mixed Hodge structure the Deligne δ-splitting of the limit mixed Hodge structure (F, M) is given by an element $\delta \in \mathfrak{g}_{\mathbb{R}}$. The proof of this last statement boils down to showing the compatibility of Deligne's construction with passage to Gr^W.

If Y is a grading of W, $y > 0$ and $\alpha \in \mathbb{R}$ we define

$$y^{\alpha Y} = \exp(\alpha \log(y) Y)$$

wherefrom $y^{\alpha Y}$ acts on Gr^W_k as multiplication by $y^{\alpha k}$. Accordingly, if γ belongs to the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ attached to a classifying space D with weight filtration W then

$$y^{\alpha Y}.\gamma = \mathrm{Ad}(y^{\alpha Y})\gamma = y^{\alpha Y}\gamma y^{-\alpha Y}$$

induces the same action on Gr^W as γ. Therefore $y^{\alpha Y}.\gamma \in \mathfrak{g}_{\mathbb{C}}$. If Y is defined over $\mathbb{R}$ then the adjoint action of $y^{\alpha Y}$ preserves $\mathfrak{g}_{\mathbb{R}}$.

5.5 Unipotent Variations of Mixed Hodge Structure

Let $\mathcal{H}$ be a variation of graded-polarizable mixed Hodge structure over a smooth, complex algebraic variety S. Then, $\mathcal{H}$ is said to be unipotent [HZ87] if the global monodromy representation of $\mathcal{H}$ is unipotent. Equivalently, the variations of Hodge structure induced by $\mathcal{H}$ on Gr^W are constant ((1.4), [HZ87]). The global structure of admissible unipotent variations of mixed Hodge structure on S is governed by mixed Hodge theoretic representations of the fundamental group of S (Thm. (1.6), [HZ87]).

[3] This means $(-k, -k)$-morphisms for any integer k.

For the remainder of this section we assume $\mathcal{H} \to \Delta^*$ is admissible and unipotent in the sense of Hain and Zucker. We prove that the mixed Hodge norm of a flat section of $\mathcal{H}$ is bounded.

To begin, we note that in this case, we again have $M = M(N, W)$ equals W (see Prop. (2.14), [SZ85] or (1.5), [HZ87]). Thus, as in (33) and (82), we can write the lift of the period map of $\mathcal{H}$ to the upper half-plane in the form

$$F(z) = e^{zN} e^{\Gamma(s)}.F_\infty \tag{39}$$

where $\Gamma(s)$ is a $\mathfrak{q}$-valued function which vanishes at $s = 0$.

Remark 5.5.1. In the unipotent case, the function $\Gamma(s)$ takes values in the subalgebra $W_{-1}\mathfrak{q}$ of $\mathfrak{q}$ consisting of elements which act trivially on Gr^W.

To continue, let $(\hat{F}_\infty, M) = (e^{-i\delta}.F_\infty, M)$ be Deligne's δ-splitting (38) of (F_∞, M), keeping in mind that $M = W$. Let

$$Y = Y_{(\hat{F}_\infty, M)} \tag{40}$$

and note that $\overline{Y} = Y$ since $(\hat{F}_\infty, M)$ is split over $\mathbb{R}$. Note that since $[Y, N] = -2N$ we have

$$y^{-Y/2} e^{iN} y^{Y/2} = e^{iyN} \tag{41}$$

Define

$$\begin{aligned} e^{\Gamma(s,y)} &= \mathrm{Ad}\,(\, e^{iN})\, \mathrm{Ad}\,(\, y^{Y/2}) e^{\Gamma(s)} \\ e^{i\delta(y)} &= \mathrm{Ad}\,(\, y^{Y/2}) e^{i\delta} \end{aligned} \tag{42}$$

Lemma 5.5.2. *If $\mathcal{H} \to \Delta^*$ is unipotent in the sense of Hain and Zucker then (we are using the linear action and not the adjoint action here)*

$$F(z) = e^{xN} y^{-Y/2} e^{\Gamma(s,y)} e^{i\delta(y)} e^{iN}.\hat{F}_\infty \tag{43}$$

Proof. Starting from (39), we have

$$\begin{aligned} F(z) &= e^{xN} e^{iyN} e^{\Gamma(s)}.F_\infty \\ &= e^{xN} e^{iyN} e^{\Gamma(s)} e^{i\delta}.\hat{F}_\infty \\ &= e^{xN} y^{-Y/2} e^{iN} y^{Y/2} e^{\Gamma(s)} e^{i\delta}.\hat{F}_\infty \end{aligned} \tag{44}$$

To further refine (44), we note that $[N, \delta] = 0$ as N is a $(-1,-1)$-morphism of $(\hat{F}_\infty, M)$ and $Y(\hat{F}^p_\infty) \subseteq \hat{F}^p_\infty$ since $Y = Y_{(\hat{F}_\infty, M)}$. Therefore,

$$\begin{aligned} e^{i\delta}.\hat{F}_\infty &= y^{-Y/2} y^{Y/2} e^{-iyN} e^{iyN} e^{i\delta}.\hat{F}_\infty \\ &= y^{-Y/2} e^{-iN} y^{Y/2} e^{iyN} e^{i\delta}.\hat{F}_\infty \\ &= y^{-Y/2} e^{-iN} y^{Y/2} e^{i\delta} e^{iyN}.\hat{F}_\infty \\ &= y^{-Y/2} e^{-iN} e^{i\delta(y)} y^{Y/2} e^{iyN}.\hat{F}_\infty \\ &= y^{-Y/2} e^{-iN} e^{i\delta(y)} e^{iN} y^{Y/2}.\hat{F}_\infty \\ &= y^{-Y/2} e^{-iN} e^{i\delta(y)} e^{iN}.\hat{F}_\infty \end{aligned} \tag{45}$$

Inserting (45) into (44) and simplifying gives (43). $\square$

Fix a norm $|*|$ on $\mathfrak{g}_\mathbb{C}$. Observe that since $\Gamma(0) = 0$ it follows that $|\Gamma(s, y)|$ can be bounded by a constant multiple of $|s|(-\log|s|)^b$. Likewise, since

$$\delta \in \Lambda^{-1,-1}_{(F_\infty, M)} = \Lambda^{-1,-1}_{(\hat{F}_\infty, M)}$$

it follows that δ decomposes as $\delta = \delta_{-2} + \delta_{-3} + \cdots$ relative to $\mathrm{ad}\,(Y)$ since $Y = Y_{(\hat{F}_\infty, M)}$. Accordingly, by equation (42) it follows that $|\delta(y)|$ can be bounded by a multiple of $1/y$.

To continue, let $F \in D$ and $\mathfrak{q}_F = \oplus_{p<0}\, \mathfrak{g}^{p,q}_{(F,W)}$. Then, since $\mathfrak{q}_F$ is a vector space complement to $\mathfrak{g}^F_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ it follows from the inverse function theorem that there exists a neighborhood $\mathcal{N}$ of 0 in $\mathfrak{g}_{\mathbb{C}}$ and unique holomorphic functions $v : \mathcal{N} \to \mathfrak{q}_F$, $\phi^\dagger : \mathcal{N} \to \mathfrak{g}^F_{\mathbb{C}}$ such that

$$u \in \mathcal{N} \implies e^u = e^{v(u)} e^{\phi^\dagger(u)} \tag{46}$$

In particular, by uniqueness $v(0) = \phi^\dagger(0) = 0$.

On the other hand, by [Pea00] there exists a neighborhood $\mathcal{Q}$ of 0 in $\mathfrak{q}_F$ and distinguished real analytic functions $\tilde{\gamma} : \mathcal{Q} \to \mathfrak{g}_{\mathbb{R}}$, $\tilde{\lambda} : \mathcal{Q} \to \Lambda^{-1,-1}_{(F,W)}$ and $\tilde{\phi} : \mathcal{Q} \to \mathfrak{g}^F_{\mathbb{C}}$ such that

$$v \in \mathcal{Q} \implies e^v = e^{\tilde{\gamma}(v)} e^{\tilde{\lambda}(v)} e^{\tilde{\phi}(v)} \tag{47}$$

Combining (46) and (47) it follows that after shrinking $\mathcal{N}$, we have a real-analytic decomposition

$$e^u = e^{\gamma(u)} e^{\lambda(u)} e^{\phi(u)} \tag{48}$$

upon setting $\gamma(u) = \tilde{\gamma}(v(u))$, $\lambda(u) = \tilde{\lambda}(v(u))$ and $\phi(u) = \tilde{\phi}(v(u))\phi^\dagger(u)$.

Denote the dependence of the functions appearing in (48) on F by γ_F, λ_F and ϕ_F. Then, since the decomposition

$$V = \bigoplus_{p,q} I^{p,q}_{(F,W)}$$

is C^∞ with respect to $F \in D$, it follows that γ_F, λ_F and ϕ_F also have a C^∞ dependence on F. Accordingly, a soft analysis argument shows that given $F_o \in D$ there exists a compact set $K \subset D$ containing F_o and constants ρ and C such that

$$F \in K, \quad |u| < \rho \implies |\gamma_F(u)|, \quad |\lambda_F(u)|, \quad |\phi_F(u)| < C|u| \tag{49}$$

For the remainder of this section, we will drop the subscript F from γ, λ and ϕ.

Let $F_o = e^{iN}.\hat{F}_\infty \in D$ with corresponding compact set K and constants ρ and C as in (49). Let $F(y) = e^{i\delta(y)}.F_o$. Then, by our previous estimates of $\delta(y)$ and $\Gamma(s, y)$ it follows that there exists a constant $a > 0$ such that $|s| = e^{-2\pi y} < e^{-2\pi a}$ implies $F(y) \in K$ and $|\Gamma(s, y)| < \rho$. Therefore, by (49)

$$e^{\Gamma(s,y)} = e^{\gamma(s,y)} e^{\lambda(s,y)} e^{\phi(s,y)} \tag{50}$$

relative to $F(y)$.

Remark 5.5.3. Since $\Gamma(s)$ takes values in $W_{-1}\mathfrak{g}_{\mathbb{C}}$, so does $\Gamma(s, y)$. Therefore, $\gamma(s, y)$, $\lambda(s, y)$ and $\phi(s, y)$ take values in the subalgebra $W_{-1}\mathfrak{g}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ consisting of elements which act trivially on Gr^W.

Theorem 5.5.4. *Let $\mathcal{H} \to \Delta^*$ be a unipotent variation of mixed Hodge structure and v be a flat section of $\mathcal{H}$. Then, the mixed hodge norm $\|v\|$ is bounded.*

Proof. To reduce notation, we write the mixed Hodge norm with respect to (F, W) as $\| * \|_F$ since W is fixed throughout the proof. Returning to equation (43), we have

$$\begin{aligned}
\|v\|_{F(z)} &= \|v\|_{e^{xN} y^{-Y/2} e^{\Gamma(s,y)} e^{i\delta(y)} e^{iN}.\hat{F}_\infty} \\
&= \|e^{-xN} v\|_{y^{-Y/2} e^{\Gamma(s,y)} e^{i\delta(y)} e^{iN}.\hat{F}_\infty} \\
&= \|v\|_{y^{-Y/2} e^{\Gamma(s,y)} e^{i\delta(y)} e^{iN}.\hat{F}_\infty} \\
&= \|v\|_{y^{-Y/2} e^{\Gamma(s,y)}.F(y)}
\end{aligned} \tag{51}$$

because $e^{xN} \in G_{\mathbb{R}}$, $v \in \operatorname{Ker}(N)$ and $F(y) = e^{i\delta(y)}e^{iN}.\hat{F}_\infty$. Inserting (50) into (51) and noting that $e^{\phi(s,y)}$ preserves $F(y)$ it follows that

$$\begin{aligned} \|v\|_{F(z)} &= \|v\|_{y^{-Y/2}e^{\gamma(s,y)}e^{\lambda(s,y)}.F(y)} \\ &= \|v\|_{y^{-Y/2}e^{\gamma(s,y)}y^{Y/2}y^{-Y/2}e^{\lambda(s,y)}.F(y)} \\ &= \|y^{+Y/2}e^{-\gamma(s,y)}y^{-Y/2}v\|_{y^{-Y/2}e^{\lambda(s,y)}.F(y)} \\ &= \|\exp(-\operatorname{Ad}(y^{+Y/2})\gamma(s,y))v\|_{y^{-Y/2}e^{\lambda(s,y)}.F(y)} \end{aligned} \tag{52}$$

since $\operatorname{Ad}(y^{-Y/2})$ preserves $\mathfrak{g}_{\mathbb{R}}$ and $\gamma(s,y)$ takes values in $\mathfrak{g}_{\mathbb{R}}$. As noted after Lemma (5.5.2), $|\Gamma(s,y)|$ can be bounded by a constant multiple of $|s|(-\log|s|)^b$. By (49), at the price of adjusting the constant multiplier, the same is true of $|\gamma(s,y)|$ and $|\lambda(s,y)|$. Likewise, as Y is semisimple with a finite number of eigenvalues, $|\operatorname{Ad}(y^{+Y/2})\gamma(s,y)|$ can be bounded by $c.|s|(-\log|s|)^m$, $c \in \mathbb{C}$. Since $\operatorname{Ad}(y^{+Y/2})\gamma(s,y)$ takes values in a nilpotent Lie algebra $W_{-1}\mathfrak{g}_{\mathbb{C}}$, it follows that

$$\exp(-\operatorname{Ad}(y^{+Y/2})\gamma(s,y)) = 1\!\!1 + \epsilon(s,y)$$

where $|\epsilon(s,y)|$ can be bounded by a constant multiple of $|s|(-\log|s|)^m$ for $|s|$ sufficiently small.

To continue, we note that since $\lambda(s,y)$ takes values in $\Lambda^{-1,-1}_{(F(y),W)}$, $Y = \bar{Y}$ and $\operatorname{Ad}(y^{-Y/2})$ acts on $\mathfrak{g}_{\mathbb{C}}$ it follows that

$$\operatorname{Ad}(y^{-Y/2})\lambda(s,y) \in \Lambda^{-1,-1}_{(y^{-Y/2}.F(y),W)}$$

By construction,

$$\begin{aligned} y^{-Y/2}.F(y) &= y^{-Y/2}e^{i\delta(y)}e^{iN}.\hat{F}_\infty \\ &= y^{-Y/2}y^{Y/2}e^{i\delta}y^{-Y/2}e^{iN}.\hat{F}_\infty \\ &= e^{i\delta}y^{-Y/2}e^{iN}y^{Y/2}y^{-Y/2}.\hat{F}_\infty \\ &= e^{i\delta}e^{iyN}.\hat{F}_\infty \end{aligned}$$

since Y preserves $\hat{F}_\infty$. Accordingly, since δ commutes with N we have

$$y^{-Y/2}.F(y) = e^{iyN}.F_\infty$$

Putting the last three equations together, we have

$$y^{-Y/2}e^{\lambda(s,y)}.F(y) = \exp(\operatorname{Ad}(y^{-Y/2})\lambda(s,y))e^{iyN}.F_\infty$$

where $\operatorname{Ad}(y^{-Y/2})\lambda(s,y) \in \Lambda^{-1,-1}_{(e^{iyN}.F_\infty,W)}$.

Returning to (52), we have

$$\begin{aligned} \||v\|_{F(z)} &= \|(1+\epsilon(s,y))v\|_{y^{-Y/2}e^{\lambda(s,y)}.F(y)} \\ &= \|(1+\epsilon(s,y))v\|_{\exp(\operatorname{Ad}(y^{-Y/2})\lambda(s,y))e^{iyN}.F_\infty} \\ &= \|\exp(-\operatorname{Ad}(y^{Y/2})\lambda(s,y))(1+\epsilon(s,y))v\|_{e^{iyN}.F_\infty} \end{aligned}$$

since $\operatorname{Ad}(y^{-Y/2})\lambda(s,y) \in \Lambda^{-1,-1}_{(e^{iyN}.F_\infty,W)}$.

As above, $|\operatorname{Ad}(y^{Y/2})\lambda(s,y)|$ is bounded by a constant multiple of $|s|(-\log|s|)^{m'}$. Therefore,

$$\exp(-\operatorname{Ad}(y^{Y/2})\lambda(s,y)) = 1\!\!1 + \mu(s,y)$$

where $|\mu(s,y)|$ can be bounded by a multiple of $|s|(-\log|s|)^{m'}$ for $|s|$ sufficiently small. Thus,

$$\|v\|_{F(z)} = \|(1+\mu(s,y))(1+\epsilon(s,y))v\|_{e^{iyN}.F_\infty}$$

Finally, since $N \in \Lambda^{-1,-1}_{(F,W)}$ and $v \in \operatorname{Ker}(N)$ we have

$$\begin{aligned}\|v\|_{F(z)} &= \|e^{-iyN}(1+\mu(s,y))(1+\epsilon(s,y))v\|_{F_\infty} \\ &= \|e^{-iyN}(1+\mu(s,y))(1+\epsilon(s,y))e^{iyN}v\|_{F_\infty}\end{aligned}$$

Therefore, since $|y| = \frac{-1}{2\pi}\log|s|$ it follows that

$$e^{-iyN}(1+\mu(s,y))(1+\epsilon(s,y))e^{iyN} \to \nVdash$$

as $y \to \infty$. Thus, $\|v\|_{F(z)}$ is bounded. □

Remark 5.5.5. If $\mathcal{A}$ and $\mathcal{B}$ are unipotent variations of mixed Hodge structure then so is $\mathcal{A} \otimes \mathcal{B}$. In particular, we can apply the previous theorem to flat sections of $\mathcal{H} \otimes \mathcal{H}^*$.

5.6 Hodge theory of sl_2-pairs

By equation (3.11) in [CKS86], if $\theta(z) = e^{zN}.F$ is a nilpotent orbit of pure Hodge structure of weight k polarized by Q, $W = W(N)[-k]$ and $Y = Y_{(F,W)}$ then $H = Y - k\nVdash$ is belongs to the complex Lie algebra of infinitesimal isometries of Q. Likewise, if $(\hat{F}, W)$ is the Deligne δ-splitting of (F, W) then $\hat{H} = Y_{(\hat{F},W)} - k\nVdash$ belongs to the Lie algebra of real infinitesimal isometries of Q.

As discussed in § 2 of [CKS86], given a nilpotent element $N \in \mathrm{gl}(V)$, there is a bijective correspondence between gradings H of $W(N)$ such that $[H, N] = -2N$ and representations $\rho : \mathrm{sl}_2 \to \mathrm{gl}(V)$ such that

$$\rho\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = N, \qquad \rho\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = H \tag{53}$$

We call such a pair (N, H) an sl_2-pair, and (N, H, N^+) the associated sl_2-triple, where

$$N^+ = \rho\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

If N and H are infinitesimal isometries of Q then so is N^+. Thus, by the previous paragraph, a nilpotent orbit θ of pure, polarized Hodge structure of weight k determines a representation of $\mathrm{sl}_2(\mathbb{C})$ into the complex Lie algebra of infinitesimal isometries of the polarization via the sl_2-pair

$$(N, Y_{(F,W)} - k\nVdash) \tag{54}$$

Likewise, the sl_2-pair $(N, Y_{(\hat{F},W)} - k\nVdash)$ defines a representation of $\mathrm{sl}_2(\mathbb{R})$ into the Lie algebra of real, infinitesimal isometries of the polarization.

5.7 Two theorems of P. Deligne

Let W be an increasing filtration of a finite dimensional vector space V over a field of characteristic zero. Let $\mathrm{End}^W(V)$ denote the subspace of $\mathrm{End}(V)$ consisting of elements which preserve W.

Let $\mathrm{Gr}^W = \oplus_k \mathrm{Gr}^W_k$ and $\mathbf{Y}$ be the grading of Gr^W which acts on Gr^W_k as multiplication by k. For clarity, given an element $A \in \mathrm{End}^W(V)$ we let $\mathrm{Gr}^W(A)$ denote the induced action of A on Gr^W. Then, an element $\alpha \in \mathrm{End}(\mathrm{Gr}^W)$ commutes with $\mathbb{Y}$ if and only if there exists an element $A \in \mathrm{End}^W(V)$ such that $\mathrm{Gr}^W(A) = \alpha$.

More precisely, given a grading Y' of W and element $A \in \mathrm{End}^W(V)$ we have a decomposition

$$A = \sum_{k \geqslant 0} A_{-k}, \qquad [Y', A_{-k}] = -kA_{-k} \tag{55}$$

of A into eigencomponents with respect to $\mathrm{ad}(Y')$. Moreover, $\mathrm{Gr}^W(A_0) = \mathrm{Gr}^W(A)$. Therefore, given $\alpha \in \mathrm{End}(\mathrm{Gr}^W)$ which commutes with $\mathbf{Y}$ and a grading Y' of W there exists a unique element $\alpha_0 \in \mathrm{End}^W(V)$ which commutes with Y' such that $\mathrm{Gr}^W(\alpha_0) = \alpha$. We call α_0 the lift of α with respect to Y'.

Suppose now that $(e^{zN}.F, W)$ is an admissible nilpotent orbit and let $M = M(N, W)$. Then, $Y_M = Y_{(F,M)}$ is a grading of M which preserves W and satisfies $[Y_M, N] = -2N$. In [Del93], P. Deligne constructs a grading $Y = Y(N, Y_M)$ of W and an associated sl_2-pair $(N_0, Y_M - Y)$ which generalizes the construction (54) as follows: Let N be a nilpotent element of $\mathrm{End}^W(V)$ such that $M = M(N, W)$ exists. Let Y_M be a grading of M which preserves W and satisfies $[Y_M, N] = -2N$. Then, it follows from the definition of the relative weight filtration that

$$(\mathrm{Gr}^W(N), \mathrm{Gr}^W(Y_M) - \mathbf{Y})$$

is an sl_2-pair which commutes with $\mathbf{Y}$. Let $\tilde{N}^+$ be the third element of the associated sl_2-triple. By construction, $\tilde{N}^+$ commutes with $\mathbb{Y}$. Given a choice of grading Y' of W let N_0 and N_0^+ the corresponding lifts of $\mathrm{Gr}^W(N)$ and $\tilde{N}^+$. Note the lift of $\mathrm{Gr}^W(Y_M) - \mathbf{Y}$ is just $H = Y_M - Y'$.

THEOREM 5.7.1. *(P. Deligne, [Del93]) Let Y_M be a grading of M which preserves W, i.e. $Y_M(W_k) \subseteq W_k$ for all k, such that $[Y_M, N] = -2N$. Then, there exists a unique grading $Y = Y(N, Y_M)$ of W such that $[Y, Y_M] = 0$ and*

$$[N - N_0, N_0^+] = 0$$

Another way of stating this result is that there exists a unique choice of grading Y of W which commutes with Y_M such that $(N_0, Y_M - Y)$ is an sl_2-pair with the following property: If

$$N = \sum_{k \geqslant 0} N_{-k}, \qquad [Y, N_{-k}] = -kN_{-k}$$

is the decomposition of N with respect to $\mathrm{ad}(Y)$ then for each positive integer k, N_{-k} is either zero or a vector of highest weight $k-2$ for the associated adjoint representation of sl_2. In particular, N_{-1} is either zero or a vector of highest weight -1 with respect to the sl_2-triple $(N_0, Y_M - Y, N_0^+)$. Therefore, $N_{-1} = 0$. Likewise, N_{-2} commutes with N_0, $Y_M - Y$ and N_0^+ since it is a vector of highest weight zero for the adjoint representation.

LEMMA 5.7.2. *If $(e^{zN}.F, W)$ is an admissible nilpotent orbit, $M = M(N, W)$ and $Y = Y(F, Y_{(F,M)})$ is the grading of Theorem (5.7.1) then Y preserves F.*

Proof. See Theorem 4.15, [Pea01]. □

COROLLARY 5.7.3. *Let $(e^{zN}.F, W)$ be an admissible nilpotent orbit with limit mixed Hodge structure (F, M) split over $\mathbb{R}$. Let $Y = Y(N, Y_{(F,M)})$ and $N = N_0 + N_{-2} + \cdots$ be the corresponding decomposition of N with respect to $\mathrm{ad}(Y)$. Then, $Y = \overline{Y}$ and*

$$Y_{(e^{zN_0}.F,W)} = Y$$

for $\Im(z) > 0$.

Proof. By definition $\overline{N} = N$, whereas $\overline{Y_{(F,M)}} = Y_{(F,M)}$ since (F, M) is split over $\mathbb{R}$. Therefore, by virtue of the linear algebraic nature of Deligne's construction, $Y = \overline{Y}$. By the previous Lemma, Y preserves $e^{zN_0}.F$ and hence $Y = Y_{(e^{zN_0}.F,W)}$ since $\overline{Y} = Y$. □

One important consequence of W. Schmid's SL_2-orbit theorem [Sch73] is the construction of another splitting operation $(F, W) \mapsto (e^{-\xi}.F, W)$, which we call the sl_2-*splitting*, on the category of mixed Hodge structures. If $(F,W)\mapsto(e^{-i\delta}.F,W)$ is Deligne's δ-splitting then ξ (resp. δ) can be expressed as universal Lie polynomials in the Hodge components of δ (resp. ξ) relative to (F,W).

Theorem 5.7.4. *(P. Deligne, [Del93]) Let $(e^{zN}.F,W)$ be an admissible nilpotent orbit with limit mixed Hodge structure (F,M) split over $\mathbb{R}$. Let $Y=Y(N,Y_{(F,M)}$ and $N=N_0+N_{-2}+\cdots$ be the decomposition of N into eigencomponents with respect to $\operatorname{ad}(Y)$. Then, $(e^{zN_0}.F,W)$ is the sl_2-splitting of $(e^{zN}.F,W)$ and $e^{\xi}=e^{zN}e^{-zN_0}$.*

Proof. See [BP13]. For the simpler statement that

$$e^{iyN}e^{-iyN_0}\in\exp(\Lambda^{-1,-1}_{(e^{iyN_0}.F,W)}) \tag{56}$$

see the last section of [KP03]. □

Remark. For proofs an extensive discussion of these results and their history, see [BP13], [BPR17] and references therein.

5.8 Normal Functions and Biextensions

Recall (cf. [Pea06]) that a variation is type (I) if there exists an integer k such that its Hodge numbers $h^{p,q}$ are zero unless $p+q=k$, $k-1$ (i.e. Gr^W has exactly two non-zero weight graded-quotients which are adjacent). We say that a variation is type (II) if there is an integer k such that $h^{p,q}=0$ unless $(p,q)=(k,k)$, $(k-1,k-1)$ or $p+q=2k-1$ and $h^{k,k}$, $h^{k-1,k-1}$ are non-zero.

To continue, given a classifying space D for period maps of type (I) or (II), with ambient vector space V (contrasting previous usage of H), we let H be the subgroup of G consisting of elements which induce real automorphisms on W_k/W_{k-2} for each index k. In the case where D is classifying space of type (I), $H=G_{\mathbb{R}}$. When D is of type (II), H will also contain the complex subgroup $\exp(W_{-2}(\mathfrak{g}_{\mathbb{C}}))$. Moreover, by the form of the Hodge diamond of a type (II) mixed Hodge structure, it follows that

$$\Lambda^{-1,-1}_{(F,W)}=W_{-2}(\mathfrak{g}_{\mathbb{C}}) \tag{57}$$

for any element $F\in D$. For this reason (see Theorem (2.19), [Pea06]), it follows that H acts by isometries on D. Set $\mathfrak{h}=\mathrm{Lie}(H)$.

Theorem 5.8.1. *(see [Pea06, Theorem 4.2]) Let $e^{zN}.F$ be an admissible nilpotent orbit of type (I) or (II), with relative weight filtration $M=M(N,W)$ and δ-splitting $(F,M)=(e^{i\delta}.\hat{F},M)$. Let (N_0,H,N_0^+) denotes the sl_2 triple attached to the nilpotent orbit $e^{zN}.\hat{F}$ by Theorem (5.7.4), and $N=N_0+N_{-2}$ denote the corresponding decomposition of N with respect to $\operatorname{ad}Y$ where $H=Y_{(\hat{F}_\infty,M)}-Y$.[4] Then, there exists an element*

$$\zeta\in\mathfrak{h}\cap\mathrm{Ker}(N)\cap\Lambda^{-1,-1}_{(\hat{F},M)}$$

and distinguished real analytic function $g:(a,\infty)\to H$ such that

(a) $e^{iyN}.F=g(y)e^{iyN}.\hat{F}$;

(b) $g(y)$ *and* $g^{-1}(y)$ *have convergent series expansions about* ∞ *of the form*

$$g(y)=e^{\zeta}(1+g_1y^{-1}+g_2y^{-2}+\cdots)$$
$$g^{-1}(y)=(1+f_1y^{-1}+f_2y^{-2}+\cdots)e^{-\zeta}$$

[4] Of course $N_{-2}=0$ for variations of type (I).

with $g_k,\ f_k \in \operatorname{Ker}((\operatorname{ad} N_0)^{k+1}) \cap \operatorname{Ker}(\operatorname{ad} N_{-2})$.

COROLLARY 5.8.2. *(see, Corollary 4.3, [Pea06]) Let* $\mathcal{H} \to \Delta^*$ *be an admissible variation of type* (I) *or* (II), *with period map* $F(z) : U \to D$ *and nilpotent orbit* $e^{zN}.F$. *Then, adopting the notation of Theorem* (5.8.1), *there exists a distinguished, real–analytic function* $\gamma(z)$ *with values in* $\mathfrak{h}$ *such that, for* $\Im(z)$ *sufficiently large,*

(i) $F(z) = e^{xN} g(y) e^{iyN_{-2}} y^{-H/2} e^{\gamma(z)}.F_o$;
(ii) $|\gamma(z)| = O(\Im(z)^{\beta} e^{-2\pi\Im(z)})$ *as* $y \to \infty$ *and* x *restricted to a finite subinterval of* $\mathbb{R}$, *for some constant* $\beta \in \mathbb{R}$.

where $F_o = e^{iN_0}.\hat{F}$.

LEMMA 5.8.3. *If* $\mathcal{H}$ *is a variation of type* (II) *then* $\alpha \in \mathfrak{g}_{\mathbb{C}} \cap \operatorname{Ker}(\operatorname{ad} N)$ *if and only if* $\alpha \in \mathfrak{g}_{\mathbb{C}} \cap \operatorname{Ker}(\operatorname{ad} N_0) \cap \operatorname{Ker}(\operatorname{ad} N_{-2})$.

Proof. Since $N = N_0 + N_{-2}$, clearly $\operatorname{Ker}(\operatorname{ad} N_0) \cap \operatorname{Ker}(\operatorname{ad} N_{-2}) \subseteq \operatorname{Ker}(\operatorname{ad} N)$. Conversely, suppose $\alpha \in \mathfrak{g}_{\mathbb{C}} \cap \operatorname{Ker}(\operatorname{ad} N)$. The non-zero weight graded-quotients of $gl(V)^W$ are

$$\operatorname{Gr}^W_{\ell}(V \otimes V^*) \cong \bigoplus_{j+k=\ell} \operatorname{Gr}^W_j(V) \otimes \operatorname{Gr}^W_k(V^*), \qquad \ell \leqslant 0$$

from which it follows that the only non-zero weight graded quotients of $gl(V)^W$ occur in weights 0, -1 and -2. Using $\operatorname{ad}(Y)$ we can write $\alpha = \alpha_0 + \alpha_{-1} + \alpha_{-2}$. Then,

$$0 = [N, \alpha] = [N_0 + N_{-2}, \alpha_0 + \alpha_{-1} + \alpha_{-2}]$$

and hence $[N_0, \alpha_0] = 0$, $[N_0, \alpha_{-1}] = 0$, $[N_{-2}, \alpha_{-1}] = 0$, $[N_{-2}, \alpha_{-2}] = 0$ and

$$[N_0, \alpha_{-2}] + [N_{-2}, \alpha_0] = 0$$

By the Monodromy theorem discussed at the end of (5.1), it follows that N acts trivially on Gr^W_0 and Gr^W_{-2}. Therefore, $N_0(V) \subseteq W_{-1}$ and hence $\alpha_{-2}(N_0(V)) = 0$. Likewise, $\alpha_{-2}(V) \subseteq W_{-2}$ and $\operatorname{Gr}^W_{-2} = W_{-2}/\{0\}$. As such, $N_0(\alpha_{-2}(V)) = 0$. This shows, $[N_0, \alpha_{-2}] = 0$ and hence $[N_{-2}, \alpha_0] = 0$ as well by the previous equation. □

COROLLARY 5.8.4 cf. Theorem 4.7, [Pea06]. *Let* $\mathcal{H} \to \Delta^*$ *be an admissible variation of type* (I) *or* (II) *with unipotent monodromy* $T = e^N$. *Let* $\alpha \in \mathfrak{g}(V)$ *be a flat, global section, which acts by infinitesimal isometries of the graded-polarizations. Then,* α *has bounded mixed Hodge norm.*

Proof. In the notation of Corollary (5.8.2), the statement boils down to computing the asymptotic behavior of

$$\|\alpha\|_{F(z)} = \|\alpha\|_{e^{xN} g(y) e^{iyN_{-2}} y^{-H/2} e^{\gamma(z)}.F_o}$$

for $\alpha \in$ Ker(ad N) in some vertical strip of width 1 in the upper half-plane. By part (*b*) of Theorem (5.8.1), it follows that $g(y)$ and $e^{iyN_{-2}}$ commute. Accordingly, by (57) and (29) and the fact that $g(y)$ takes values in $G_{\mathbb{R}}$, it follows that

$$\begin{aligned}
\|\alpha\|_{F(z)} &= \|e^{-xN}.\alpha\|_{g(y)e^{iyN_{-2}}y^{-H/2}e^{\gamma(z)}.F_o} \\
&= \|\alpha\|_{g(y)e^{iyN_{-2}}y^{-H/2}e^{\gamma(z)}.F_o} \\
&= \|e^{-iyN_{-2}}g^{-1}(y).\alpha\|_{y^{-H/2}e^{\gamma(z)}.F_o} \\
&= \|g^{-1}(y)e^{-iyN_{-2}}.\alpha\|_{y^{-H/2}e^{\gamma(z)}.F_o}
\end{aligned}$$

Again for emphasis here $G_{\mathbb{C}}$ acts linearly on filtrations while it acts by the adjoint action on $G_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$. By the previous Lemma, $\alpha \in \mathrm{Ker}(\mathrm{ad}(N_0)) \cap \mathrm{Ker}(\mathrm{ad}(N_{-2}))$, and so the preceding equation simplifies to

$$\|\alpha\|_{F(z)} = \|g^{-1}(y).\alpha\|_{y^{-H/2}e^{\gamma(z)}.F_o}$$

Returning to part (b) of Theorem (5.8.1), it follows that upon decomposing f_k into isotypical components with respect to (N_0, H, N_0^+) that f_k occurs in components of highest weight $\leqslant k$ since $f_k \in \mathrm{Ker}((\mathrm{ad}\, N_0)^{k+1})$. Therefore, since $\zeta \in \mathrm{Ker}(\mathrm{ad}\,(N))$ and f_k is the coefficient of y^{-k} in the expansion of $f(y) = g^{-1}(y)$ it follows that

$$\widetilde{g}^{-1}(\infty) = \lim_{y\to\infty} \mathrm{Ad}(y^{H/2})g^{-1}(y) \tag{58}$$

exists as an element of the Lie group H[5]. Thus,

$$\|\alpha\|_{F(z)} = \|\tilde{g}^{-1}(y)y^{H/2}.\alpha\|_{e^{\gamma(z)}.F_o}$$

where $\widetilde{g}^{-1}(y) = \mathrm{Ad}\,(y^{H/2})g^{-1}(y)$. Finally, since $\alpha \in \mathrm{Ker}(\mathrm{ad}\, N_0)$ it follows that $y^{H/2}\alpha$ converges as $y \to \infty$. As $\gamma(z) \to 0$ as $y \to \infty$ and x constrained to a finite interval, the proof is now complete. $\square$

5.9 $\mathbf{Ext}^1(\mathbb{R}(0), \mathbf{weight\ -2})$

Let $\mathcal{A}$ and $\mathcal{B}$ be variations of pure Hodge structure of respective weights a and b. Assume that $a = b + 2$. Then,

$$\mathrm{Ext}^1_{\mathrm{AVMHS}}(\mathcal{A}, \mathcal{B}) \cong \mathrm{Ext}^1_{\mathrm{AVMHS}}(R, \mathcal{A}^* \otimes B)$$

where $R = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ and AVMHS is the category of admissible variations of graded-polarizable mixed Hodge structure. Accordingly, for the remainder of this section, we will consider a variation of Hodge structure $\mathcal{H} \to \Delta^*$ of weight -2 and an admissible variation $\mathcal{H} \in \mathrm{Ext}^1_{\mathrm{AVMHS}}(\mathbb{R}(0), \mathcal{H})$, with unipotent monodromy $T = e^N$.

Theorem 5.9.1. *If v is a flat section of $\mathcal{H}$ then $\|v\|$ is bounded.*

Proof. Let (F, M) denote the δ-splitting of the limit mixed Hodge structure of $\mathcal{H}$. Let $Y = Y(N, Y_{(F,M)})$ be the grading of W constructed in Theorem (5.7.1). Then, by virtue of the short length of the weight filtration W of $\mathcal{H}$,

$$N = N_0 + N_{-2}$$

with respect to $\mathrm{ad}(Y)$.

If $N = N_{-2}$ the variation is unipotent in the sense of R. Hain and S. Zucker, and the result follows from section (5.5). If $N = N_0$ the result follows from section (5.12) below. It remains to consider the case where $N = N_0 + N_{-2}$ with both N_0 and N_{-2} non-zero. In this case, we will show that v is a section of $W_{-2}(\mathcal{H}) = \mathcal{H}$, and hence the result follows from W. Schmid's SL_2-orbit theorem. $\square$

To complete the proof, we recall that $[N_0, N_{-2}] = 0$, $\bar{Y} = Y$ and Y preserves F by Lemma (5.7.2). For the remainder of this section we assume that both N_0 and N_{-2} are non-zero. From this, we will derive a contradiction unless $v \in W_{-2}$.

[5] There is a typo at the end of the proof of Theorem 4.7 in[Pea06], $\mathrm{Ad}\,(Y^{H/2})f_k y^{-k}$ is a polynomial without constant term in $y^{-1/2}$.

By the monodromy theorem, N acts trivially on Gr^W_0 and hence N_0 acts trivially on $E_0(Y) \cong \mathrm{Gr}^W_0$. By Corollary (5.7.3), $Y = Y_{(e^{iN_0}.F,W)}$, and hence if e_0 is a generator of $I^{0,0}_{(e^{iN_0}.F,W)}$ then $N_0(e_0) = 0$ and

$$e_0 = e^{-iN_0}(e_0) \in F^0$$

Since $[Y, Y_{(\hat{F},M)}] = 0$ it follows that $(N_0, Y_{(F,M)} - Y)$ restricts to a trivial sl_2-pair on $E_0(Y)$. Therefore, $e_0 \in M_0$. As such,

$$e_0 \in F^0 \cap \overline{F^0} \cap M_0 = I^{0,0}_{(F,M)}$$

Accordingly $N_{-2}(e_0) \in I^{-1,-1}_{(F,M)}$. Moreover, since $[N_0, N_{-2}] = 0$ and $N_0(e_0) = 0$ it follows that

$$N_0 N_{-2}(e_0) = N_0 N_{-2}(e_0) - N_{-2} N_0(e_0) = [N_0, N_{-2}](e_0) = 0$$

Thus, $N_{-2}(e_0) \in \mathrm{Ker}(N_0) \cap I^{-1,-1}_{(F,M)} \cap W_{-2}$. Moreover, if $N_{-2}(e_0) = 0$ then $N = N_0$ due to the short length. By assumption, $N_{-2} \neq 0$, and hence $N_{-2}(e_0) \neq 0$.

Suppose now that $v \in \mathrm{Ker}(N)$ and $v = v_0 + v_{-2}$ with $v_j \in E_j(Y)$. If $v_0 = 0$ we are done. Otherwise, after rescaling, we can assume that $v_0 = e_0$. To continue, observe that $N_{-2}(v_{-2}) = 0$ by the short length of W. Therefore, since $N_0(e_0) = 0$,

$$N(v) = N_{-2}(e_0) + N_0(v_{-2}) = 0$$

and hence

$$N_{-2}(e_0) \in \mathrm{Ker}(N_0) \cap \mathrm{Im}(N_0) \cap I^{-1,-1}_{(F,M)} \cap W_{-2} \tag{59}$$

As we must also have $N_{-2}(e_0) \neq 0$, the following Lemma completes the proof:

Lemma 5.9.2. *For (F, M) as above,* $\mathrm{Ker}(N_0) \cap \mathrm{Im}(N_0) \cap I^{-1,-1}_{(F,M)} \cap W_{-2} = 0$.

Proof. This is a statement about the SL_2-orbits of pure Hodge structure induced by $(e^{zN}.F, W)$ on W_{-2}. By [Sch73], these are classified as follows: Let,

(a) $\mathbb{C}^2 = \mathrm{span}(e, f)$ with $e = \bar{e}$ type $(1, 1)$ and $f = \bar{f}$ type $(0, 0)$ with respect to the limit mixed Hodge structure, and

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis $\{e, f\}$. The resulting nilpotent orbit is pure of weight 1.

(b) $E(p, q) = \mathrm{span}(e, f)$ with $p > q$, N acting trivially, $e = \bar{e}$, $f = \bar{f}$ and $e + if$ of type (p, q) with respect to the limit mixed Hodge structure;

(c) $\mathbb{R}(p)$ is rank 1 of pure of type $(-p, -p)$ and N acting trivially.

Then, every SL_2-orbit of pure Hodge structures is a direct sum of factors which are tensor products of the form $\mathrm{Sym}^m(\mathbb{C}^2) \otimes \mathbb{R}(p)$ and $\mathrm{Sym}^n(\mathbb{C}^2) \otimes E(p, q)$ where m and $n \geqslant 0$ and $\mathrm{Sym}^0(\mathbb{C}^2) = \mathbb{R}(0)$.

To continue, we observe that in the language of the orbit types (a)–(c) the Lemma asserts that

$$\mathrm{Ker}(N) \cap \mathrm{Im}(N) \cap I^{-1,-1} = 0 \tag{60}$$

(relative to the limit mixed Hodge structure) as N_0 becomes just N for the induced orbit on W_{-2}.

Next, we note that the factor $\mathrm{Sym}^n(\mathbb{C}^2) \otimes E(p, q)$ never contributes any Tate classes to the limit mixed Hodge structure, so we need only consider factors of the form $\mathrm{Sym}^m(\mathbb{C}^2) \otimes \mathbb{R}(p)$.

Moreover, since $\mathrm{Sym}^m(\mathbb{C}^2)$ underlies a nilpotent orbit of weight m, we must have $p = m+1$ in order to obtain an nilpotent orbit of pure Hodge structure of weight -2.

To finish the proof of the lemma, observe that on the factor $\mathrm{Sym}^m(\mathbb{C}^2)$,

$$\mathrm{Ker}(N) \cap \mathrm{Im}(N) = \mathbb{C}f^m$$

where m must be > 0 (in order to have a non-trivial N action). Moreover, f^m belongs to $I^{0,0}$ of the limit mixed Hodge structure of $\mathrm{Sym}^m(\mathbb{C}^2)$. Accordingly, $\mathrm{Ker}(N) \cap \mathrm{Im}(N)$ is contained in $I^{-m-1,-m-1}$ of the limit mixed Hodge structure of $\mathrm{Sym}^m(\mathbb{C}^2) \otimes \mathbb{R}(m+1)$. As $m > 0$, equation (60) holds. $\square$

We now consider the variation $\mathfrak{g}(\mathcal{H})$ where $\mathcal{H} \in \mathrm{Ext}^1_{\mathrm{AVMHS}}(\mathbb{R}(0), \mathcal{H})$ with $\mathcal{H}$ pure of weight -2. Since $\mathcal{H}$ only has weights 0 and -2 whereas $\mathcal{H}^*$ has weights 0 and 2 it follows that $\mathcal{H} \otimes \mathcal{H}^*$ has weights -2, 0 and 2. Therefore, $\mathfrak{g}(\mathcal{H})$ only has weights 0 and -2 since $\mathfrak{g}(\mathcal{H})$ is the subvariation consisting of elements which preserve the weight filtration and induce infinitesimal isometries of the graded-polarizations. Therefore, Theorem (5.9.1) applies to $\mathfrak{g}(\mathcal{H})$ upon viewing it as an extension of $\mathbb{R}(0)$ by a variation of pure Hodge structure of weight -2.

5.10 Biextensions arising from higher height pairings

Let X be a smooth, complex projective variety of dimension d. Following the notation of [BGGP22], let $Z \in \mathcal{Z}^p(X,1)_{00}$ and $W \in \mathcal{Z}^q(X,1)_{00}$ be higher cycles representing elements of $\mathsf{CH}^p(X,1)$ and $\mathsf{CH}^q(X,1)$ respectively. Then:

Theorem 5.10.1. *([BGGP22, Theorem A]) Assume that*

(i) $p + q = d + 2$;
(ii) $\delta Z = \delta W = 0$
(iii) the intersection of Z and W satisfies some extra technical conditions.

Then, there is a canonical mixed Hodge structure $B_{Z,W}$ attached to Z and W from which one can extract a Hodge theoretical height pairing $\langle Z, W\rangle_{\mathrm{Hodge}}$. Moreover, if Z and W both have real regulator zero then

$$\langle Z, W\rangle_{\mathrm{Hodge}} = \langle Z, W\rangle_{\mathrm{Arch}}$$

where $\langle Z, W\rangle_{\mathrm{Arch}}$ is the Archimedean part of an intersection pairing on arithmetic Chow groups.

The mixed Hodge structure $B_{Z,W}$ has weight graded-quotients $\mathrm{Gr}^W_0 \cong \mathbb{Z}(0)$, Gr^W_{-2} and $\mathrm{Gr}^W_{-4} \cong \mathbb{Z}(2)$. Let $X \to S$ be a family of smooth complex projective varieties and Z, W be a flat family of higher cycles over S such that $\langle Z_s, W_s\rangle$ is defined over a Zariski dense open subset of S. In this way, the construction of Theorem 5.10.1 produces an admissible variation of mixed Hodge structure $\mathcal{H}$ over a Zariski dense open set of S with weight graded quotients $\mathrm{Gr}^W_0(\mathcal{H}) \cong \mathbb{Z}(0)$, $\mathrm{Gr}^W_{-2}(\mathcal{H})$ and $\mathrm{Gr}^W_{-4}(\mathcal{H}) \cong \mathbb{Z}(2)$.

Lemma 5.10.2. *Let (F, W) be a mixed Hodge structure with underlying vector space V and weight graded quotients $\mathrm{Gr}^W_0 \cong \mathbb{Z}(0)$, Gr^W_{-2} and $\mathrm{Gr}^W_{-4} \cong \mathbb{Z}(2)$. Let $\mathfrak{g}_{\mathbb{C}}(U)$ denote the Lie algebra of elements of $gl(U)$ which preserve $W(U)$ and induce infinitesimal isometries of $\mathrm{Gr}^{W(U)}$ where $U = W_{-2}(V)$, V or V/W_{-4}. Then, since elements of $\mathfrak{g}_{\mathbb{C}}(V)$ preserve W, we have an induced map*

$$q : \mathfrak{g}_{\mathbb{C}}(V) \to \mathfrak{g}_{\mathbb{C}}(V/W_{-4})$$

and a restriction map

$$r : \mathfrak{g}_{\mathbb{C}}(V) \to \mathfrak{g}_{\mathbb{C}}(W_{-2})$$

By abuse of notation, let $q(F)$ and $r(F)$ denote the mixed Hodge structure induced by (F, W) on $\mathfrak{g}_{\mathbb{C}}(V/W_{-4})$ and $\mathfrak{g}_{\mathbb{C}}(W_{-2})$. Let $\beta \in \mathfrak{g}_{\mathbb{C}}$ be horizontal with respect to F. Then,

$$\|\beta\|_F \leqslant \|q(\beta)\|_{q(F)} + \|r(\beta)\|_{r(F)} \tag{61}$$

Proof. The key point is that $W_{-4}\mathfrak{g}(V)$ is pure of type $(-2,-2)$ and $W_{-3}\mathfrak{g}(V) = W_{-4}\mathfrak{g}(V)$. Therefore, if $\beta = \sum_{p,q} \beta^{p,q}$ denote the decomposition of β into Hodge components with respect to (F, W) then $\beta^{p,q} = 0$ unless $p \geqslant -1$. As such $\beta^{p,q} = 0$ unless $p + q = 0$ or $p + q = -2$. Thus, (61) captures the mixed Hodge norm of $\sum_{p+q=-2} \beta^{p,q}$ accurately and double counts the mixed Hodge norm of $\sum_{p+q=0} \beta^{p,q}$. □

Suppose now that α is a horizontal section of $\mathfrak{g}(\mathcal{H})$ then pointwise application of the previous Lemma shows that

$$\|\alpha\|_{\mathcal{H}} \leqslant \|q(\alpha)\|_{\mathfrak{g}(\mathcal{H}/W_{-4}\mathcal{H})} + \|r(\alpha)\|_{\mathfrak{g}(W_{-2}\mathcal{H})} \tag{62}$$

Corollary 5.10.3. *Let $\mathcal{H} \to \Delta^*$ be an admissible variation of graded-polarized mixed Hodge structure over the punctured disk with unipotent monodromy. Assume that $\mathcal{H}$ has weight graded quotients $\mathrm{Gr}^W_0 \cong \mathbb{Z}(0)$, Gr^W_{-2} and $\mathrm{Gr}^W_{-4} \cong \mathbb{Z}(0)$. Let α be a flat, horizontal section of $\mathfrak{g}(\mathcal{H})$. Then, α has bounded mixed Hodge norm.*

Proof. By (62), $\|\alpha\|_{\mathcal{H}}$ is bounded by $\|q(\alpha)\|$ and $\|r(\alpha)\|$. Moreover, $q(\alpha)$ and $r(\alpha)$ are flat since α is flat and W is flat. Therefore, the result follows from Theorem (5.9.1) and the last paragraph of section (5.9). □

5.11 A case where norm estimates fail

In this section, we show via admissible nilpotent orbits that in the case of a higher normal function with weight graded quotients $\mathrm{Gr}^W_0 = \mathbb{Z}$ and Gr^W_{-k} for $k > 2$, the norm estimates required to obtain rigidity need not hold.

Lemma 5.11.1. *Let $(e^{zN}.F, W)$ be an admissible nilpotent orbit with limit mixed Hodge structure (F, M) split over $\mathbb{R}$. Let $Y = Y(N, Y_{(F,M)})$ and $N = N_0 + \cdots + N_{-k}$ relative to $\mathrm{ad}(Y)$ with $N_{-k} \neq 0$. Then, $\|N\|_{(e^{zN}.F,W)} = \|N\|_{(e^{iyN}.F,W)}$ and there a non-zero constant K such that*

$$\lim_{y\to\infty} y^{(2-k)/2}\|N\|_{(e^{iyN}.F,W)} = K$$

In particular, $\|N\|_{(e^{zN}.F,W)}$ is bounded for $k = 2$ and unbounded for $k > 2$.

Proof. Note that N and $H = Y_{(F,M)} - Y$ are elements of $\mathfrak{g}_{\mathbb{R}}$. Since $\overline{Y} = Y$ it follows that $\overline{N_0} = N_0$. As $\mathrm{Gr}^W(N_0) = \mathrm{Gr}^W(N)$ it follows that $N_0 \in \mathfrak{g}_{\mathbb{R}}$. Thus, omitting W from the mixed Hodge norm as in (51), we have

$$\begin{aligned}\|N\|_{e^{zN}.F} &= \|N\|_{e^{xN}e^{iyN}.F} = \|e^{-xN}.N\|_{e^{iyN}.F} \\ &= \|N\|_{e^{iyN}.F} = \|N\|_{e^{iyN}e^{-iyN_0}e^{iyN_0}.F} \\ &= \|e^{iyN_0}e^{-iyN}.N\|_{e^{iyN_0}.F}\end{aligned}$$

where the last step is justified by equation (56). To continue, we note that since $[H, N_0] = -2N_0$ we have

$$e^{iyN_0} = y^{-H/2}e^{iN_0}y^{H/2} = y^{-H/2}.e^{iN_0} \tag{63}$$

wherefrom

$$e^{iyN_0}.F = y^{-H/2}e^{iN_0}y^{H/2}.F = y^{-H/2}e^{iN_0}.F$$

since H preserve F. Moreover, as a consequence of the SL_2-orbit theorem in the pure case, $F_o = e^{iN_0}.F \in D$. Therefore,

$$\begin{aligned}
\|N\|_{e^{zN}.F} &= \|e^{iyN_0}e^{-iyN}.N\|_{e^{iyN_0}.F} \\
&= \|e^{iyN_0}.N\|_{e^{iyN_0}.F} \\
&= \|e^{iyN_0}.N\|_{y^{-H/2}.F_o} \\
&= \|y^{H/2}e^{iyN_0}.N\|_{F_o} \\
&= \|e^{iN_0}y^{H/2}.(N_0 + \cdots + N_{-k})\|_{F_o}
\end{aligned}$$

where the last step is justified by (63). Accordingly, as $[H, N_{-j}] = (j-2)N_{-j}$ for $j = 0, \ldots, k$ it follows that $\|N\|_{e^{zN}.F}$ is asymptotic to a constant multiple of $y^{(k-2)/2}$ for large y. □

5.12 The case $N = N_0$

Let $\mathcal{H} \to \Delta^*$ be an admissible nilpotent orbit with unipotent monodromy $T = e^N$. Let (F_∞, M) be the limit mixed Hodge structure of $\mathcal{H}$ with δ-splitting

$$(\hat{F}_\infty, M) = (e^{-i\delta}.F_\infty, M) \tag{64}$$

Let $Y_M = Y_{(\hat{F}_\infty, M)}$ and $Y = Y(N, Y_M)$. Let

$$N = N_0 + N_{-2} + \cdots \tag{65}$$

denote the decomposition of N into eigencomponents for $\operatorname{ad} Y$. Let

$$(N_0, H, N_0^+), \qquad H = Y_M - Y \tag{66}$$

be the associated representation of $\mathrm{sl}_2(\mathbb{R})$ of Theorem (5.7.1).

In this section we prove the following result, by essentially modifying the unipotent case accordingly:

Theorem 5.12.1. *If $N = N_0$ and v is a flat, global section of $\mathcal{H}$ then v has bounded mixed Hodge norm.*

As the first step towards the proof of Theorem (5.12.1), we note that since $N = N_0$, (N_0, H) is an sl_2-pair and $[N, \delta] = 0$, it follows that δ is a sum of lowest weight vectors for (N_0, H, N_0^+). Therefore,

$$\delta = \delta_0 + \delta_{-1} + \cdots, \qquad [H, \delta_{-j}] = -j\delta_{-j} \tag{67}$$

relative to the eigenvalues of $\operatorname{ad} H$. Let

$$\delta(y) = \operatorname{Ad}(y^{H/2})\delta = y^{H/2}.\delta = \sum_{k \geqslant 0} \delta_{-k} y^{-k/2} \tag{68}$$

Lemma 5.12.2. *In the notation of (64)–(68), if $N = N_0$ then*

$$\delta(\infty) := \lim_{y \to \infty} \delta(y) = \delta_0 \tag{69}$$

and $e^{i\delta(\infty)}e^{iN}.\hat{F}_\infty \in D$.

Proof. Equation (69) follows directly from equation (68). To prove that the point $e^{i\delta(\infty)}e^{iN}.\hat{F}_\infty$ belongs to D, observe that it is sufficient to consider only the pure case, since the property of being a MHS is only about the induced filtrations on Gr^W. Accordingly, for the remainder of this proof only, we assume $e^{zN}.F_\infty$ is a nilpotent orbit of pure Hodge structure. By W. Schmid's SL_2-orbit theorem, we have

$$y^{H/2}e^{i\delta}e^{iyN}.\hat{F}_\infty = y^{H/2}g(y)e^{iyN}.\hat{F}_\infty$$

which we can rewrite as

$$e^{i\delta(y)}e^{iN}.\hat{F}_\infty = y^{H/2}g(y)y^{-H/2}e^{iN}.\hat{F}_\infty \tag{70}$$

using $e^{iyN}.\hat{F}_\infty = y^{-H/2}e^{iN}.\hat{F}_\infty$. Mutatis mutandis, the argument of equation (58) shows that

$$\widetilde{g}(\infty) = \lim_{y\to\infty} y^{H/2}g(y)y^{-H/2}$$

exists, and is an element of $G_\mathbb{R}$ (since we are in the pure case). By W. Schmid's SL_2-orbit theorem, $e^{iN}.\hat{F}_\infty \in D$. Taking the limit of (70) as $y \to \infty$, it follows that

$$e^{i\delta(\infty)}e^{iN}.\hat{F}_\infty = \widetilde{g}(\infty)e^{iN}.\hat{F}_\infty \in D$$

as required. □

To continue, let $F(z) = e^{zN}e^{\Gamma(s)}.F_\infty$ be the local normal form of the period map of $\mathcal{H}$. Then, in analogy with equation (44) we have

$$\begin{aligned} F(z) &= e^{xN}e^{iyN}e^{\Gamma(s)}.F_\infty \\ &= e^{xN}e^{iyN}e^{\Gamma(s)}e^{i\delta}.\hat{F}_\infty \\ &= e^{xN}y^{-H/2}e^{iN}y^{H/2}e^{\Gamma(s)}e^{i\delta}.\hat{F}_\infty \end{aligned} \tag{71}$$

In analogy with the derivation of (45), since $[H,N] = -2N$ and H preserves $\hat{F}_\infty$, we have

$$\begin{aligned} e^{i\delta}.\hat{F}_\infty &= y^{-H/2}y^{H/2}e^{-iyN}e^{iyN}e^{i\delta}.\hat{F}_\infty \\ &= y^{-H/2}e^{-iN}y^{H/2}e^{iyN}e^{i\delta}.\hat{F}_\infty \\ &= y^{-H/2}e^{-iN}y^{H/2}e^{i\delta}e^{iyN}.\hat{F}_\infty \\ &= y^{-H/2}e^{-iN}e^{i\delta(y)}y^{H/2}e^{iyN}.\hat{F}_\infty \\ &= y^{-H/2}e^{-iN}e^{i\delta(y)}e^{iN}y^{H/2}.\hat{F}_\infty \\ &= y^{-H/2}e^{-iN}e^{i\delta(y)}e^{iN}.\hat{F}_\infty \end{aligned} \tag{72}$$

Inserting (72) into (71) yields

$$\begin{aligned} F(z) &= e^{xN}y^{-H/2}e^{iN}y^{H/2}e^{\Gamma(s)}y^{-H/2}e^{-iN}e^{i\delta(y)}e^{iN}.\hat{F}_\infty \\ &= e^{xN}y^{-H/2}e^{\Gamma(s,y)}e^{i\delta(y)}e^{iN}.\hat{F}_\infty \end{aligned} \tag{73}$$

where

$$e^{\Gamma(s,y)} = e^{iN}y^{H/2}e^{\Gamma(s)}y^{-H/2}e^{-iN} = \exp(e^{iN}y^{H/2}.\Gamma(s)) \tag{74}$$

In particular, since $\Gamma(0) = 0$ and $|s| = e^{-2\pi y}$, there exist positive constants C, k and a such that

$$|s| < a \implies |\Gamma(s,y)| < C|s|(-\log|s|)^k \tag{75}$$

with respect to a choice of fixed norm $|*|$ on $\mathfrak{g}_\mathbb{C}$.

Proof of Theorem (5.12.1). Since $v \in \mathrm{Ker}(N)$ it follows from (73) that

$$\begin{aligned} \|v\|_{F(z)} &= \|v\|_{e^{xN}y^{-H/2}e^{\Gamma(s,y)}e^{i\delta(y)}e^{iN}.\hat{F}_\infty} \\ &= \|e^{-xN}.v\|_{y^{-H/2}e^{\Gamma(s,y)}e^{i\delta(y)}e^{iN}.\hat{F}_\infty} \\ &= \|v\|_{y^{-H/2}e^{\Gamma(s,y)}e^{i\delta(y)}e^{iN}.\hat{F}_\infty} \end{aligned} \tag{76}$$

since $N \in \mathfrak{g}_{\mathbb{R}}$. To continue, observe that $v \in \mathrm{Ker}(N)$ implies that

$$v = \sum_{k \geqslant 0} v_{-k}, \qquad H.v_{-k} = -kv_{-k}$$

where the number of non-zero terms is finite since $\mathcal{H}$ has finite rank. Therefore,

$$v(y) := y^{H/2}.v = \sum_{k \geqslant 0} v_{-k} y^{-k/2} \tag{77}$$

is a vector-valued polynomial in $y^{-1/2}$, and hence

$$\|v\|_{F(z)} = \|y^{H/2}.v\|_{e^{\Gamma(s,y)}e^{i\delta(y)}e^{iN}.\hat{F}_\infty} = \|v(y)\|_{e^{\Gamma(s,y)}e^{i\delta(y)}e^{iN}.\hat{F}_\infty} \tag{78}$$

To finish the proof, recall that D is an open subset of $\check{D}$ in the complex analytic topology, and $G_{\mathbb{C}}$ acts transitively on $\check{D}$ by biholomorphisms. In particular, since $e^{i\delta(\infty)}e^{iN_0}.\hat{F} \in D$ by Lemma (5.12.2), it follows from equation (68) and the estimate (75) that there exists a constant $b > 0$ such that

$$K = \{\, e^{\Gamma(s,y)}e^{i\delta(y)}e^{iN}.\hat{F} \mid y \geqslant b, \;\; |s| \leqslant e^{-2\pi y} \,\}$$

is a compact subset of D. Therefore, since $\lim_{y\to\infty} v(y) = v(0)$ it follows from equation (76) that $\|v\|_{F(z)}$ is bounded as $y \to \infty$ and x is constrained to a finite interval. $\square$

Remark. (1) If $N = N_0$ for $\mathcal{H}$ then $N = N_0$ for $\mathcal{H} \otimes \mathcal{H}^*$.
(2) The results in this section cover the case of variations of pure Hodge structure and variations of type (I).

6. Deformations of admissible mixed period maps

6.1 General set-up

The set-up is similar to the one in the pure case. More precisely, we only consider deformations of a period map $F : S \to \Gamma\backslash D$ such that

- S, D and Γ remain fixed.
- the deformation remains locally liftable and horizontal.

However, there is an additional requirement "at infinity": we want the variation to be admissible. This concept is recalled in Appendix A. Note that our convention of admissibility includes as a requirement that the monodromy operators around the boundary are quasi-unipotent. This is for instance the case if the variation has an underlying $\mathbb{Z}$-structure such as the ones coming from geometry. Pure variations are automatically admissible, and this is also the case for mixed variations of geometric origin (cf. [SP08, Def. 14.49]).

Mixed period maps of admissible variations will be called ***admissible period maps***. Admissibility is preserved under small deformations:

Lemma 6.1.1. *If F is an admissible period map, sufficiently small deformations of F that stay horizontal, also stay admissible.*

Proof. We can test admissibility on curves and so we may replace S with a curve. We employ the test given in [Pea00].

For a neighborhood of $p \in \bar{S} - \partial S$ we take a small disc Δ centered at p with coordinate s and monodromy T around the origin. We may assume that T is unipotent. Set $N = -\log T$. If $s \in \Delta - \{0\}$, we may put $s = \mathrm{e}^{2\pi \mathbf{i} z}$. Then the untwisted period map $\mathrm{e}^{-zN}.F(z)$ extends over the origin as a holomorphic map $\Delta \to \check{D}$ where its value at $s = 0$ is traditionally denoted $F(\infty) \in \check{D}$ (since it corresponds to a limit for $s \to \infty$). The canonical extension to Δ of the local system (with weight filtration and rational structure) over Δ^* puts a weight filtration and rational structure on the "central" fiber H over 0. Admissibility implies that there is a relative weight filtration M on the central fiber H and $(H, M, F(\infty))$ is a mixed Hodge structure, the "limit mixed Hodge structure". Hence we have a Deligne decomposition and we can speak of horizontal endomorphisms with respect to the limit mixed Hodge structure. We shall call these "limit-horizontal" and denote these as $\mathfrak{q}^{\text{hor}}_{F(\infty)}$. In this case the local normal form (35) reads

$$F(s) = \exp\left(\frac{\log(s)}{2\pi\mathbf{i}} N\right) \exp(\Gamma(s)).F(\infty),\ \Gamma(0) = 0,$$

and where

$$\Gamma(s) = 1 + \Gamma_{-1}(s) + \Gamma_{-2}(s) + \dots, \quad \Gamma_{-k}(s) \in U^{-k}_{F(\infty)}$$

is uniquely determinable from $\Gamma_{-1} \in \mathfrak{q}^{\text{hor}}_{F(\infty)}$. Let

$$F(s,t) = \exp\left(\frac{\log(s)}{2\pi\mathbf{i}} N\right) \exp(\Gamma(s,t)).F(\infty), \quad \Gamma(s,t) \in \mathfrak{q}_{F(\infty)}.$$

be a deformation of $F(s)$ as a period map. This is nothing but a 2-parameter period map $\Delta^* \times \Delta \to \Gamma\backslash D$ with trivial monodromy in the second factor. If now

$$\exp(\Gamma(s,t)) = 1 + \tilde{\Gamma}_{-1}(s,t) + \tilde{\Gamma}_{-2}(s,t) + \dots, \tilde{\Gamma}_{-k}(s,t) \in U^{-k}_{F(\infty)},$$

the initial value constraint reads $\tilde{\Gamma}_{-1}(0,0) = \Gamma_1(0) = 0$ and the "Higgs bundle constraint" holds since $F(s,t)$ is assumed to be horizontal. Indeed, the Higgs bundle constraint is equivalent to the image at any point of the tangent space under the period map being an abelian subspace of $\mathfrak{g}_{\mathbb{C}}$ which is the case, cf. Lemma 3.2.1. But then, by loc. cit., $F(s,t)$ is an admissible nilpotent orbit with the same relative weight filtration M and limit mixed Hodge structure $F(\infty)$ as before. $\square$

In view of the above, we call deformations of admissible period maps that stay locally liftable and horizontal (and hence admissible) simply ***admissible deformations***.

Remark 6.1.2. Recall the commutative diagram (13) which provides a surjection

$$\mathcal{F}^{-1}\mathfrak{g}(\mathcal{H}) \xrightarrow{\pi^{\text{hor}}} F^*T^{\text{hor}}(\Gamma\backslash D).$$

Choosing a lift for this map at some point $s \in S$ determines a unique global lift. This is a consequence of the rigidity theorem for variations of admissible mixed Hodge structures (cf. [SZ85, Theorem 4.20] for S a curve and the remarks in [BZ90, § 9] for the general case). But at a given point s, there is a natural identification of $T^{\text{hor}}_{F(s)}D$ with the subspace $U^{-1}\mathfrak{g}_{F(s)}$ of $\mathfrak{g}_{F(s)}$ and so we have a unique global lift. This lift can be used to identify infinitesimal deformations of an admissible variation with a subspace of the space of sections of $\mathcal{U}^{-1}\mathfrak{g}(\mathcal{H}) \subset \mathcal{F}^{-1}\mathfrak{g}(\mathcal{H})$.

6.2 Main results

Theorem 6.2.1 Main Theorem I. *Let S be quasi-projective and $F : S \to \Gamma\backslash D$ a horizontal holomorphic map to a mixed domain D parametrizing mixed Hodge structures on $(H, W, Q)_{\mathbb{R}}$ and assume that the variation of mixed Hodge structure $\mathcal{H}$ corresponding to F is admissible.*

(i) Let η be a global holomorphic section of $\mathfrak{g}(\mathcal{H})$ corresponding to an admissible infinitesimal deformation of F with bounded Hodge norm. If the section η is plurisubharmonic along S,

then η is a flat section of $\mathfrak{g}(\mathcal{H})$ which is moreover horizontal, i.e., a section of $\mathcal{U}^{-1}\mathfrak{g}(\mathcal{H}) = \bigoplus_{k\leqslant 1}\mathfrak{g}^{-1,k}(\mathcal{H})$.
Equivalently, at any point $s \in S$, $\eta(s)$ is a horizontal endomorphism of $\mathfrak{g}(\mathcal{H}_s)$ which commutes with the action of the fundamental group $\pi_1(S,s)$.

(ii) Conversely, let $\eta(s)$ be a flat horizontal section of $\mathcal{U}^{-1}\mathfrak{g}(\mathcal{H})$ such that $\eta(s)$ commutes with every element in $F_*T_{S,s} \subset U^{-1}\mathfrak{g}_{F(s)}$. If the (constant) Hodge norm $\|\eta(s)\|$ is small enough, then η defines a deformation of F keeping source and target fixed and which remains a period map.

Proof. (1) This is a direct application of Proposition 4.2.2. The condition $[u,v]=0$ follows as in the pure case, since we are considering deformations which stay horizontal (see e.g. [CMSP17, Prop. 5.5.1]).
(2) We use an argument due to Faltings (for weight 1) [Fal83]. Let η a parallel horizontal section of $\mathfrak{g}(\mathcal{H})$ and define the filtration $F_\eta(s)$ by setting

$$F_\eta(s) = \mathrm{e}^{\eta(s)}\, F(s), \quad s \in S.$$

On the weight graded parts $Q(f(s), f(s)) = 0$, $f(s) \in F_\eta(s)$. The map $s \mapsto F_\eta(s)$ is holomorphic but might land in the compact dual $\check{D}$ (cf. formula (8)).

We claim

- the second Riemann condition holds if $\|\eta(s)\|$ is small enough so that this filtration gives a point inside the period domain D.
- The commuting property guarantees horizontality.

To prove these claims, first note that since η is parallel, its Hodge norm is constant and hence also the auxiliary operators

$$\begin{aligned} w_{k,\ell} &= (\eta^*)^\ell \circ \eta^k + (\eta^*)^k \circ \eta^\ell, \quad k \neq \ell \\ w_{k,k} &= (\eta^*)^k \circ \eta^k, \end{aligned}$$

have constant Hodge norm. These operators, being self-adjoint, have real eigenvalues (which might be negative). Let the smallest of these be $m_{k,\ell}$. Suppose that the nilpotent operator η has index of nilpotency M and set

$$\mu = \sum_{1\leqslant k\leqslant \ell\leqslant M} \frac{m_{k,\ell}}{k!\ell!}.$$

Then for all $f(s) \in F_s$ we have

$$\|\,\mathrm{e}^{\eta(s)}\, f(s)\|^2_{F(s)} = h_{F(s)}\left([\mathrm{id} + \sum_{1\leqslant k\leqslant \ell\leqslant M} \frac{1}{k!\ell!} w_{k,\ell}] f(s), f(s)\right) \geqslant (1-|\mu|)\|f(s)\|^2$$

and so if $|\mu| < 1$ (which is the case if η is close to zero) we have

$$\|\,\mathrm{e}^{\eta(s)}\, f(s)\|^2_{F(s)} \geqslant (1-|\mu|)\|f(s)\|^2_{F(s)} > 0.$$

Hence, as we claimed, Q polarizes the induced Hodge structures on weight graded parts so that the deformed period map $F_\eta = \mathrm{e}^\eta F$ gives a holomorphic map $S \to \Gamma\backslash D$. If, moreover, for all $s \in S$ and all tangents $\xi \in T_sS$ one has $[\eta(s), F_*\xi] = 0$, the deformation F_η satisfies Griffiths' transversality condition since this commutativity implies

$$\begin{aligned}\nabla_\xi F^p_\eta(s) &= \left(F_*\xi.\, \mathrm{e}^{\eta(s)}\right).F^p(s)\\ &= \left(\mathrm{e}^{\eta(s)}.F_*\xi\right).F^p(s)\\ &= \mathrm{e}^{\eta(s)}\,\nabla_\xi F^p(s) \subset \mathrm{e}^{\eta(s)}\,F(s)^{p-1} = F^{p-1}_\eta(s).\end{aligned}$$

Here we use (cf. (20)) that ∇_ξ acts as $\partial_\xi + \mathrm{ad}\,(F_*\xi)$ on $\mathfrak{g}_{F(s)}$, and that $\partial_\xi\eta = 0$, because η is locally constant. □

Remark 6.2.2 Smoothness. The second part of the theorem is equivalent to the relevant deformation space being smooth at F since it shows that the latter space is isomorphic to a small ball in the vector space of flat horizontal sections of the bundle $\mathfrak{g}(\mathcal{H})$. In particular, F is rigid if and only if this component is a non-reduced point.

Examples 6.2.3 Non-rigid examples. ***(1) Hodge–Tate variations.*** As we have remarked in Section 4.3 (2), one can easily construct variations that can be deformed in suitable $(-1,-1)$-directions.
(2) Nilpotent orbit associated to Kähler classes. We come back to Example (4) in Section 4.3. The variation we started with is the $\mathbb{R}$-split variation defined by the total cohomology of a family of Kähler manifolds. The nilpotent orbit construction gives a deformation of the associated period map as in the second part of Theorem 6.2.1. The role of $\eta(s)$ is played by $\sum_{j=1}^k u_j N_j(s)$ where the N_j are coming from independent ample classes which gives a multi-parameter deformation. Suppose that the dimension of the Kähler cone in $H^{1,1}(X) \cap H^2(X,\mathbb{R})$ equals κ. Then $k \leqslant \kappa$. If this inequality is strict, the variation is not rigid in at least one $(-1,-1)$-direction.
(3) Biextensions coming from higher Chow cycles on surfaces. We studied these in Section 4.3 (6). Observe that as in the previous example, a flat infinitesimal deformation v in a $(-1,-1)$-direction gives rise to a nilpotent orbit of deformations and so these deformations are never rigid in such directions. An example of such a flat v can be constructed as follows. Let X_s, $s \in S$ be family of surfaces embedded in a product $\mathbb{P}^a \times \mathbb{P}^b$ of projective spaces, A, A' hyperplane sections coming from $\mathbb{P}^a$ giving rise to biextension variation $\mathcal{H}_s$ over S and C, D hyperplane sections coming from $\mathbb{P}^b$. Then (C,D) defines an independent flat infinitesimal variation v of biextension type and hence $\exp(tv)\mathcal{H}_s$ is a deformation of $\mathcal{H}_s$.

In order to formulate the second main result, we recall that Proposition 4.2.2 states that for a plurisubharmonic horizontal endomorphism η and for all all tangents u to the period map, one has $\pi_\mathfrak{q}[\pi_+\bar{u}, v] = 0$, $v = \eta(s)$. Moreover, this property is equivalent to v being parallel.

In analogy with the pure case (cf. Definition 2.3.3) we introduce the following concept:

Definition 6.2.4. Fix a subspace $\mathfrak{a} \subset \mathfrak{g}^{\mathrm{hor}}_\mathbb{C}$. The period map F is called ***regularly tangent*** at $s \in S$, respectively ***regularly tangent in the*** $\mathfrak{a}$***-directions***, if the only vector $v \in \mathfrak{g}^{\mathrm{hor}}_{F(s)}$, respectively $v \in \mathfrak{a}$, with $\pi_\mathfrak{q}[\pi_+\bar{u}, v] = 0$ for all $u \in F_*T_sS$ is the zero vector.

Remark 6.2.5. Because of type reasons, a period map can only be regularly tangent if there are non-zero $(-1,1)+(-1,0)$-directions. Moreover, if F is regularly tangent in the $(-1,1)$-directions as well as in $(-1,0$-directions, then F is regularly tangent in all directions.

Theorem 6.2.6 Main Theorem II. *Fix a subspace* $\mathfrak{a} \subset \mathfrak{g}^{\mathrm{hor}}_\mathbb{C}$*. Suppose we are in one of the following situations:*

(i) *the pure case with* $\mathfrak{a} = \mathfrak{g}^{-1,1}$*;*

(ii) *we have two adjacent weights and* $\mathfrak{a} = \mathfrak{g}^{-1,0}$*;*

(iii) *in the setting of unipotent variations , i.e.* $u^{-1,1} = 0$ *provided either* $\Lambda^{-1,-1} = 0$ *and* $\mathfrak{a} = \mathfrak{g}^{-1,0}$*, or* $u \in \Lambda^{-1,-1}$ *and* $\mathfrak{a} = \oplus_{k \leqslant 0} \mathfrak{g}^{-1,k}$*.*

(iv) $u = u^{-1,1} + u^{-1,-1}$*, and* $\mathfrak{a} = \mathfrak{g}^{-1,-1}$*.*

(v) *two non-adjacent weights, say* $0, k$*,* $|k| \geqslant 2$ *with* $h^{0,0} = 1$*,* $h^{p,-p} = 0$ *for* $p \neq 0$*, and* $\mathfrak{a} = \mathfrak{g}^{-1,-k+1}$*. Moreover, we assume that* $\|v\|$ *is bounded near infinity.* [6]

(vi) *A variation of type*

$$\begin{tikzcd} & I^{0,0} \arrow[d, "u^{-1,-1}"'] \arrow[dr, "u^{0,-2}"] & \\ I^{-2,0} \arrow[dr, "u^{0,-2}"'] & I^{-1,-1} \arrow[l, "u^{-1,1}"'] \arrow[d, "u^{-1,-1}"] & I^{0,-2} \arrow[l, "u^{-1,1}"'] \\ & I^{-2,-2} & \end{tikzcd}$$

and $\mathfrak{a} = \mathfrak{g}^{-1,-1}$*.*

Then deformations of F *in the* $\mathfrak{a}$*-directions are in one-to-one correspondence with those endomorphisms of* (H, Q) *that belong to* $\mathfrak{a}$ *and which intertwine the action of the monodromy.*

In particular, the following properties are equivalent:

– F *has no horizontal deformations in* $\mathfrak{a}$*-directions;*

– (H, Q) *has no endomorphisms in* $\mathfrak{a}$*-directions intertwining the action of the monodromy.*

These properties hold in particular, if F *is regularly tangent in the* $\mathfrak{a}$*-directions at* $o \in S$ *(and hence along* S*).*

Proof. In each of the above cases, by Proposition 4.2.4, a holomorphic horizontal endomorphism is plurisubharmonic and so its Hodge norm is plurisubharmonic. By the results of Section 5, this function is bounded. Now apply Theorem 6.2.1. □

6.3 Conditions implying rigidity

Suppose we have a variation with two weights 0 and 1 and of Hodge width are a, respectively b. Suppose that $a > b$ and the weight 0 variation is a direct sum of two variations, one having maximal Higgs field and a piece Z' of pure type $(0, 0)$ with trivial Higgs field. We claim that this implies that the mixed variation is then is regularly tangent in the $(-1, 0)$-directions. To illustrate the set-up, we take $a = 2$ and $b = 1$:

$$\begin{tikzcd} H^{1,-1} & H^{0,0} \arrow[l, "(u^{-1,1})^*"] & H^{-1,1} \arrow[l, "(u^{-1,1})^*"] \\ & H^{1,0} \arrow[u, "v"'] & H^{0,1} \arrow[l, "(u^{-1,1})^*"] \arrow[u, "v"'] \end{tikzcd}.$$

Indeed, by assumption, the upper row splits into two strands at most, that is $H^{0,0} = Z \oplus Z'$ such that the upper right component of $(u^{-1,1})^*$ maps isomorphically to Z which in turn is mapped isomorphically to $H^{-1,1}$ by the relevant component of $(u^{-1,1})^*$.

[6]This is always the case for $|k| = 2$ by Section 5.

To test regularity, suppose that $[(u^{-1,1})^*, v] = 0$. The commutative diagram implies that the image of $v : H^{1,0} \to H^{0,0}$ lands in Z and so, if $(u^{-1,1})^* \circ v = 0$ on $H^{1,0}$, we must have $v|_{H^{1,0}} = 0$. Since then $(u^{-1,1})^* \circ v = 0$ on $H^{0,1}$, a similar argument shows that $v = 0$ on $H^{0,1}$ as well.

Now remark that if $h^{1,-1} = 1$, the Higgs field in the $u^{-1,1}$-direction is maximal precisely if it is non-zero. By Lemma 3.4.1 this is the case if and only if the tangent map to the weight zero period map in that direction is non-zero. We have shown:

PROPOSITION 6.3.1. *Suppose we have a mixed period map F for a variation of adjacent weights 0 and 1. For the pure weight 0 variation we assume*

- *the only non-zero Hodge numbers are $h^{-1,1} = h^{1,-1} = 1$ and $h^{0,0} \geqslant 1$.*
- *its period map is non-constant.*

Then F is regularly tangent in the $(-1,0)$-directions and hence admits no deformation in these directions.

We finish this section by giving a criterion for rigidity using the monodromy action. It uses the following general result.

LEMMA 6.3.2. *Let π be a group, k a field and V, V_1, V_2 finite dimensional k-vector spaces,*

$$0 \to V_1 \xrightarrow{i} V \xrightarrow{p} V_2 \to 0$$

an exact sequence of π-modules and $\varphi \in \operatorname{End}^\pi V$, i.e., an endomorphism of V intertwining the π-action. Suppose

- *φ induces the zero map on V_1 and V_2.*
- *V_1 is an irreducible π-module.*
- *$\dim V_1 > \dim V_2$.*

Then $\varphi = 0$.

Proof. We claim that the assumptions imply that the map φ induces a π-equivariant morphism $\bar\varphi : V_2 \to V_1$ and if it is the zero-map then $\varphi = 0$. Let us prove this claim. First we define $\bar\varphi$. Lift $\bar x \in V_2$ to an element $x \in V$. Then $\varphi(x) \in i(V_1)$ since φ is π-equivariant and induces the zero map on V_2. So $\varphi(x) = i(y)$. Then set $\bar\varphi(\bar x) = y$. This is independent of the lift since φ induces the zero map on V_1. By construction $\bar\varphi = 0$ if and only $\varphi = 0$.

Since V_1 is irreducible as a π-module, by Schur's lemma, either $\bar\varphi = 0$ or $\bar\varphi(V_2) = V_1$. In the latter case we would have $\dim V_2 \geqslant \dim V_1$, contrary to the third assumption and hence $\varphi = 0$. □

COROLLARY 6.3.3. *Consider a period map $F : S \to \Gamma\backslash D$ associated to a two-step weight filtration $0 \subset W_1 \subset W_2 = H$. If the weight graded quotients have distinct dimensions and the one of largest dimension is an irreducible Γ-module, then F is rigid in the $(-1,0)$-directions. So, if in addition the induced period maps for the weight-graded pure variations of Hodge structure on S are rigid, then F is rigid as a period map.*

Proof. By duality we may assume that $\dim W_1 > \dim \operatorname{Gr}_2^W$. We apply Lemma 6.3.2 with $v \in \mathfrak{g}^{-1,0}$ playing the role of φ. So $v = 0$ and hence, by Theorem 6.2.6, F is rigid. □

7. Examples of rigid mixed period maps

7.1 Complements of smooth divisors

Let X be a smooth compact variety of dimension $d+1$ and $Y \subset X$ a smooth divisor. We let $i : Y \hookrightarrow X$ be the inclusion and $j : U = X - Y \hookrightarrow X$ the inclusion of the complement. Then we have an exact sequence (in rational cohomology)

$$0 \to \mathrm{Coker}(H^{k-2}(Y(-1)) \xrightarrow{i_*} H^k(X)) \xrightarrow{j^*} H^k(U) \xrightarrow{r} \mathrm{Ker}(H^{k-1}(Y)(-1) \xrightarrow{i_*} H^{k+1}(X)) \to 0,$$

an extension of a weight $k+1$ Hodge structure by a weight k Hodge structure. Since the category of pure polarized Hodge structures is abelian, there are splittings $H^r(X) = \mathrm{Im}(i_*) \oplus P^r(X)$, and $H^r(Y)(-1) = \mathrm{Ker}(i_*) \oplus V^{r+2}(Y)$ so the sequence reduces to

$$0 \to P^k(X) \xrightarrow{j^*} H^k(U) \xrightarrow{r} V^{k+1}(Y) \to 0.$$

If Y is an ample divisor, this sequence is only interesting in the middle dimensions $d, d+1$ and simplifies to

$$0 \to H^{d+1}_{\mathrm{prim}}(X) \xrightarrow{j^*} H^{d+1}(U) \xrightarrow{r} H^d_{\mathrm{var}}(Y)(-1) \to 0. \tag{79}$$

Suppose that we have a family of such pairs (X_s, Y_s), $s \in S$, with S quasi-projective and smooth. We give some applications of Eqn. (79).

First we invoke Corollary 6.3.3 and deduce:

Proposition 7.1.1. *The period map for $H^{d+1}(U)$ is rigid in the $(-1,0)$-directions if the following two conditions hold simultaneously:*

- *the monodromy representation on $H^{d+1}_{\mathrm{prim}}(X)$ is irreducible;*
- $\dim H^{d+1}_{\mathrm{prim}}(X) > \dim H^d_{\mathrm{var}}(Y)$.

If, in addition, the period maps associated to $H^{d+1}_{\mathrm{prim}}(X)$ and $H^d_{\mathrm{var}}(Y)$ are rigid, then the period map is rigid in all horizontal directions.

Example 7.1.2. The obvious example is a family $\{C_s - \Sigma_s\}$ of ***quasi-projective smooth curves***. If the monodromy acts irreducibly on $H^1(C_s)_{\mathbb{C}}$ and if also $\#\Sigma < 2g$, the mixed period map is rigid in the $(-1,0)$-directions.

More generally, we can consider the Hodge structure on $H^1(X)$ for X of any dimension (and for $H^0(Y)$). For instance take any rigid family of abelian varieties (see Section 2.4 Example (6)) and leave out a smooth, possibly reducible divisor. If the monodromy action is irreducible and Y does not have too many components, the mixed period map will again be rigid.

We next we use Eqn. (79) in conjunction with Proposition 6.3.1. So we start from a ***K3-type*** Hodge structure that is, we recall, a weight two Hodge structure with $h^{2,0} = h^{0,2} = 1$ and $h^{p,q} = 0$ for $p < 0$ or $q < 0$. As a consequence of Proposition 6.3.1 we have:

Proposition 7.1.3. *Suppose that $H^2_{\mathrm{prim}}(X_s)$ is a non-constant variation of K3-type Hodge structure. Then the mixed period map for $H^2(X_s - Y_s)$ is rigid in the directions of type $(-1,0)$. The above holds in particular for X_s a K3 surface.*

Remark 7.1.4. To obtain examples with rigidity in all horizontal directions, one can consult the examples in Section 2.4, in particular Proposition 2.4.3.

One can handle many more geometric examples based on the remark that surfaces with $h^{2,0}=1$ have K3 type Hodge structure on H^2 and $H^2_{\rm prim}$. Let us especially consider the case of regular surfaces, that is surfaces with $b_1=0$, that are moreover minimal and of general type. By [BHPdV04, Thm. VII.2.1] one then has $K^2=1,\dots,8$ and one finds $h^{1,1}=20-K^2$ so that the period domain for the primitive cohomology has dimension

$$d(H^2_{\rm prim})=19-K^2.$$

Since $p_g=1$, there is a unique canonical curve K of arithmetic genus K^2+1.

For the purpose of this article we say that X is a ***Catanese–Kynev–Todorov surface*** or ***CKT-surface*** if X is a simply connected Galois $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ cover of the plane with Hodge numbers $h^{2,0}(X)=1$, $h^{1,1}(X)=19$ (and so $K_X^2=1$). These were first constructed by V. Kynev [Kyn77] and investigated in detail by F. Catanese [Cat80] and A. Todorov [Tod80]. Let us recall (loc. cit.) some of their properties. The quotient by one of the involutions is a double cover of $\mathbb{P}^2$ which is branched in the union of two cubics meeting transversely. This is a K3 surface Z with 9 ordinary double points. The family of such Z depends on 10 effective parameters and the period domain is a linear section $D_2\cap L$ of codimension 9 of the period domain D_2 for K3 surfaces with a degree 2 polarization. In other words, the K3 family has a period map which is generically one-to-one onto a suitable quotient of $D_2\cap L$. Over a general line lies a smooth genus 2 curve C in Y. Branching in C and the 9 ordinary double points produces the desired surface of general type. Since there is an ample divisor and 9 smooth rational curves of self-intersection -2 on the K3 surface, this shows that the Picard number of the general member is at least $1+9=10$. Equality follows from the surjectivity of the period map for Z. In constructing the second double cover, the choice of the line gives 2 extra parameters which do not vary with the Hodge structure and so for those surfaces the period map has fibers of dimension 2. The resulting surfaces of general type depend on $10+2=12$ moduli.

A. Todorov [Tod81] has generalized the above construction to give surfaces of general type with $b_1=0$, $h^{2,0}=1$ and $K^2=2,\dots,8$. We call these ***Todorov surfaces***. These are birational to double covers of a classical Kummer surface, branched in a quadratic section passing through $8-K^2$ double points plus the remaining $8+K^2$ double points. These last double points resolve to -2 curves on the K3 surface and the resulting family has $19-(8+K^2)=11-K^2$ moduli. The choice of the quadric section adds K^2+1 parameters which do not vary with the Hodge structure and so, as before, we get in total 12 parameters and the period map has fibers of dimension K^2+1. To calculate the generic Picard number, note that the $8-K^2$ double points through which the curve passes give just as many (-2) curves and there are 3 more independent divisors on the Kummer surface we started with. The results have been summarized in Table 1. In the table $d(H^2_{\rm prim})$ stands for the dimension of the period domain for the weight two K3-variation with period map F_2, "moduli" stands for the number of moduli of the CKT- and Todorov-surfaces[7], ρ is the generic Picard number of the K3 surface, $\dim W_2$ is the dimension of the essential part of the variation and $\dim(W_3/W_2)=2g(K_s)=\dim H^1(K_s)$.

The main result about these surfaces is as follows:

Proposition 7.1.5. *Let $\{X_s\}_{s\in S}$ be a family of CKT-surfaces or of Todorov surfaces and let $K_s\subset X_s$ be the canonical curve. The family $\{X_s-K_s\}$ is rigid if all of the following conditions hold:*

[7] The full moduli space for surfaces with these invariants is expected to be (much) larger. See for example [Cat80, CD89] for $K^2=1,2$.

Table 1. Invariants for open CKT and Todorov modular families

K^2	$d(H^2_{\text{prim}})$	moduli	ρ	fiber dim. of F_2	$(\dim W_2, \dim W_3/W_2)$
1	18	12	10	2	(11, 4)
2	17	12	9	3	(11, 6)
3	16	12	8	4	(11, 8)
4	15	12	7	5	(11, 10)
5	14	12	6	6	(11, 12)
6	13	12	5	7	(11, 14)
7	12	12	4	8	(11, 16)
8	11	12	3	9	(11, 18)

(i) the family $\{K_s\}_{s\in S}$ of curves is rigid.

(ii) The essential part of the K3 variation is non-constant and rigid.

(iii) The mixed period map is an immersion.

These conditions are all satisfied for a family for which rank of the essential variation is not divisible by 4 or whose period map has rank $\geqslant 2$. This holds in particular for a modular family, that is, a family with 12 effective parameters as well as subfamilies of a modular family having $\geqslant 2$ effective parameters.

Proof. First of all, since by (2) the K3 variation is non-constant, Proposition 7.1.3 implies that the variation is rigid in $(-1,0)$-directions. It is rigid in $(-1,1)$ directions if this is the case for the pure variations coming from the curves as well as for the K3 variation. Assumption (1) covers the curve case (since we are interested in the variations coming from the geometry of the open surfaces) and (2) covers the K3 variation. Condition (3) then implies that the family of open surfaces is itself rigid whenever the mixed variation is rigid.

Condition (1) holds as soon as the period map for the curves is an immersion. This is a consequence of Arakelov's theorem, recalled in Section 1.1. For a modular family this is the case. Indeed, for a modular family the period for the fibers of F_2 is injective. By Proposition 2.4.3, the second condition is satisfied if the rank of the essential variation is not divisible by 4 and Proposition 2.4.4 shows rigidiy for period has of rank $\geqslant 2$. From the table we see that this is the case for a modular family.

The third condition is a bit more involved since the pure K3 variation does not determine the family because of the failure of infinitesimal Torelli. Indeed, this is exactly the reason they were constructed! The failure of infinitesimal Torelli is caused by the non-trivial kernel of the tangent map to the K3 period map. Since $T_X \simeq \Omega^1_X \otimes K_X^{-1}$, the tangent to the period map is the map

$$u_S^{(2)} : T_S \to H^1(T_X) \to \operatorname{Hom}\left(H^0(K_X) \to H^1(\Omega^1_X) \simeq H^1(\Omega^1_X)\right),$$

where the resulting morphism on the right,

$$\mu_X : H^1(T_X) = H^1(\Omega^1_X(-K)) \to H^1(\Omega^1_X),$$

is induced by multiplication by a non-zero section of K_X vanishing along the canonical curve K. From the exact sequence

$$0 = H^0(\Omega^1_X) \to H^0(\Omega^1_X|K) \to H^1(\Omega^1_X(-K)) \to H^1(\Omega^1_X),$$

one sees that the kernel of μ_X is isomorphic to $H^0(\Omega^1_X|K)$. To interpret this space, recall that, as observed by A. Todorov [Tod80, proof of Prop. 4.1] and F. Catanese [Cat84, p. 150] the involution τ on X that produces the K3-quotient induces a splitting of the exact sequence

$$0 \to \mathcal{O}_K(-K) \to \Omega^1_X|_K \to \Omega^1_K \to 0.$$

Indeed, local coordinates (x, y) centered at a point P of K can be chosen in such a way that $x = 0$ gives the canonical curve K and $\tau^*x = -x, \tau^*y = y$. Then the eigenspace decomposition of $\Omega^1_{X,P}$ is just $\mathbb{C}(dx)_P \oplus \mathbb{C}(dy)_P$ and this gives a global splitting along K with the first factor giving $\mathcal{O}_K(K)$ and the second Ω^1_K. For a modular family $T_sS \simeq H^1(T_X)$ and then the split sequence shows that the kernel of the Higgs field $u^{(2)}_S$ is isomorphic to $H^0(\Omega^1_K)$. Its dimension, $K^2 + 1$, is the genus of the canonical curve K, as indicated in Table 1. This kernel is captured by the cup product

$$\mu_K : H^1(T_K) \to \mathrm{Hom}(H^{1,0}(K), H^{0,1}(K)),$$

which is injective (infinitesimal Torelli) since by [Tod81, Lemma 5.2], the canonical curve is non-hyperelliptic for the Todorov surfaces. The Higgs field $u^{(1)}_S$ for the pure weight 1 variation is the composition of the map $T_sS \to H^1(T_K)$ and μ_K and it is generically injective for a modular family. Combining the two calculations, we have shown that the kernel of the partial mixed Higgs field $u^{(2)}_S + u^{(1)}_S$ is trivial and so the mixed period map is an immersion. Hence $X_s - K_s$ can be locally reconstructed from the period map. For a subfamily this is also the case. $\square$

7.2 Projective varieties singular along a smooth divisor

Let X be a compact variety of dimension $d + 1$ whose singular locus Y is a smooth divisor. We let $\sigma : \tilde{X} \to X$ be the desingularization of X and set $\tilde{Y} = \sigma^{-1}Y$, $i : Y \hookrightarrow X$, $\tilde{\imath} : \tilde{Y} \hookrightarrow \tilde{X}$ be the inclusions. Then by [SP08, § 5.3.2] we have an exact sequence of rational cohomology groups

$$0 \to \mathrm{Coker}\left(H^{k-1}(\tilde{X}) \oplus H^{k-1}(Y) \xrightarrow{\tilde{\imath}^* - \sigma^*} H^{k-1}(\tilde{Y})\right) \longrightarrow H^k(X) \longrightarrow$$
$$\mathrm{Ker}\left(H^k(\tilde{X}) \oplus H^k(Y) \xrightarrow{\tilde{\imath}^* - \sigma^*} H^k(\tilde{Y})\right) \to 0.$$

In this case $\sigma : \tilde{Y} \to Y$ is an unramified double cover, $\mathrm{Coker}\,\sigma^*$ is the anti-invariant part of the cohomology and σ^* is an embedding. Assuming that $\tilde{Y}$ is a hyperplane section (or, more generally, very ample), in the middle dimension, the kernel of of $\tilde{\imath}^*$ is then the variable cohomology. Hence the sequence reduces to

$$0 \to H^d_{\mathrm{prim}}(\tilde{Y})^- \to H^{d+1}(X) \to H^{d+1}_{\mathrm{var}}(\tilde{X}) \to 0.$$

As a consequence of Corollary 6.3.3, we have

Proposition 7.2.1. *Suppose that the monodromy representation on $H^d_{\mathrm{prim}}(\tilde{Y}_s)^-$ is irreducible and that*

$$\mathrm{rank}(H^d_{\mathrm{prim}}(\tilde{Y}_s)^-) > \mathrm{rank}(H^{d+1}_{\mathrm{var}}(\tilde{X}_s)).$$

Then the mixed period map for $H^d(X_s)$ is rigid in the $(-1, 0)$-directions.

Remark. For Kynev–Todorov surfaces one can also use M. Letizia's argument [Let84] showing that the mixed Hodge structure generically determines the pair consisting of the surface and its canonical curve.

Example 7.2.2. Projective curves with δ ordinary double points. Here $d = 0$ and we get

$$0 \to \oplus^\delta \mathbb{Z} \to H^1(X) \to H^1(\tilde{X}) \to 0.$$

The mixed period map is rigid in $(-1,0)$-directions, if the monodromy of the family of curves acts irreducibly on the set of double points of which there are many ($\delta > 2g$). This result is dual to the case of an open curve treated in Example 7.1.2 (a).

By Proposition 2.4.2, rigidity for the pure variation follows if the Higgs field is maximal and for weight one this is the case for "most" period maps.

7.3 Normal functions and higher normal functions

Recall that these are associated to a variation of the form $H_{2p+1}(X_s)(-p)$ where $\{X_s\}_{s\in S}$ is a family of smooth complex projective varieties equipped with a family Z_s of p-dimensional algebraic cycles homologous to zero proving a variation of mixed Hodge structure

$$0 \to H_{2p+1}(X_s)(-p) \to \mathcal{H}^p(Z/S) \to \mathbb{Z}(0) \to 0.$$

As a consequence of Corollary 6.3.3 we have:

Proposition 7.3.1. *If the monodromy acts irreducibly on $H_{2p+1}(X_s)$, the normal function $\mathcal{H}^p(Z/S)$ is rigid in $(-1,0)$-directions. If, moreover, the period map associated to $H_{2p+1}(X_s)$ is rigid, the normal function is rigid in all directions.*

As an example, for p even we have a normal function associated to cycles in a Lefschetz pencil of complete intersections. Also normal functions for certain K3-variations, abelian varieties and Calabi–Yau's give examples of normal functions, rigid in all directions. See the examples in Section 2.4.3.

A similar result holds for higher normal functions

$$0 \to H^{p-1}(X_s)(q) \to \mathcal{H}^{p,q} \to \mathbb{Q}(0) \to 0$$

with $p-2q-1 < 0$. Here we have rigidity for $\mathcal{H}^{p,q}$ in $(-1,k)$-directions with $k = p-2q-1$ provided for these directions boundedness for the Hodge norm at infinity holds.

7.4 Unipotent variations

We consider adjacent weights and rigidity in $(-1,0)$-directions only:

$$H^{p,q} \underset{(u^{-1,0})^*}{\overset{v^{-1,0}}{\rightleftarrows}} H^{p+1,q}.$$

Such a v is regularly tangent if for some u the relation $u^*\circ v = 0$ implies $v = 0$ which is the case if u is surjective (then $v^*\circ u = 0$ is equivalent to $v^* = 0$.) More generally this is the case if for given $x \in H^{p+1,q}$ we can find $u = u_x$ with x in its image since then $v^*(x) = v^*\circ u_x(x') = 0$ by assumption.

In [PP19, Thm. 3.6] we considered the differential geometric aspects of unipotent variations of mixed Hodge structures associated the based fundamental group of X when the base point $x \in X$ varies. The set-up is detailed in Section 4.3, example (6). If we vary the base point in a submanifold $S \subset X$, by [PP19, Lemma 6.8], the Higgs field comes from a map

$$u : \mathrm{Ker}[H^1(X) \otimes H^1(X) \to H^2(X)] \otimes T^{1,0}_s S \to H^1(X)$$

given by

$$\alpha \otimes \beta \otimes \theta \mapsto (\theta \lrcorner \alpha)\beta - (\theta \lrcorner \beta)\alpha.$$

This map is of Hodge type $(-1,0)$ since it sends $I^{2,0} \subset H^{1,0} \otimes H^{1,0}$ to $H^{1,0}$ and $I^{1,1} \subset H^{1,0} \otimes H^{0,1}$ to $H^{0,1}$. Moreover, the restriction to $I^{2,0}$ determines the entire morphism. Note also that u factors through $\mathrm{Ker}[\Lambda^2 H^1(X) \to H^2(X)]$. Let $V = H^{1,0}(X)$, $K = \mathrm{Ker}[\Lambda^2 V \to H^{2,0}(X)]$, $T = T_s S$ and consider the maps $e : T \to V^*$ given by $e_\theta(\omega) = \theta \lrcorner \omega$ and

$$u : K \otimes T \to V, \quad \sum_{i,j,k} (\omega_i \wedge \omega_j) \otimes \theta_k = \sum_{i,j,k} [e_{\theta_k}(\omega_i)\omega_j - e_{\theta_k}(\omega_j)\omega_i]. \tag{80}$$

If this map is surjective, for every $\omega \in V$ we can find $\theta_j \in T$ and $A^j \in K$ such that $\sum_j u(A^j \otimes \theta_j) = \omega$ which suffices to show regular tangency. We formulate the conclusion explicitly:

Proposition 7.4.1. *Let X be a smooth projective variety and consider the variation of mixed Hodge structure on* $\mathrm{Hom}_{\mathbb{Z}}(J_x/J_x^3, \mathbb{C})$ *where x varies over a smooth subvariety $S \subset X$. If the map u from* (80) *is surjective, the variation is rigid,*

To see what this means geometrically, suppose for instance that there is a generic direction θ such that $u_\theta(A) = u(A \otimes \theta) = 0$ imposes $\dim K - \dim V$ independent conditions on K. Then the map u_θ is surjective which implies regular tangency. Since the condition $u_\theta(A) = 0$ amounts to $\dim V$ equations on A, the latter condition can only hold if $\dim K \geqslant 2 \dim V$ and if so, for generic θ these equations are expected to be independent. Depending on the geometry of the cotangent bundle this then is the case or not.

Appendix A. Admissibility

In [SZ85], J. Steenbrink and S. Zucker defined a category of admissible variations of graded-polarizable mixed Hodge structure over a punctured disk Δ^* with unipotent monodromy. This definition can be modified to handle the case of quasi-unipotent monodromy via a covering trick (see § 1.8 of [Kas86]). Given this local model, the category of admissible variations of graded-polarized mixed Hodge structure over a smooth complex algebraic curve C is defined as follows: The curve C has a smooth completion $\bar{C}$ which is unique up to isomorphism. A graded-polarizable variation $\mathcal{H} \to C$ is admissible if and only if for each $p \in \bar{C} - C$ the restriction of $\mathcal{H}$ to a deleted neighborhood of p is admissible.

In higher dimensions, let S be a smooth quasi-projective variety over $\mathbb{C}$ and $j : S \to \bar{S}$ be a smooth partial compactification of $\bar{S}$ such that $\bar{S} - S$ is a normal crossing divisor. In [Kas86], M. Kashiwara showed that one obtains a good category of admissible variations of graded-polarizable mixed Hodge structure on S via a curve test. In particular, the admissibility of $\mathcal{H}$ does not depend on the choice of $j : S \to \bar{S}$.

Implicit in the previous paragraph is the assumption that the local monodromy of $\mathcal{H}$ is quasi-unipotent, which we shall assume throughout this appendix. This is automatic whenever $\mathcal{H}$ carries an integral structure $\mathcal{H}_{\mathbb{Z}}$ (e.g. variations of geometric origin). To continue, we recall that if $f : A \to B$ is a holomorphic map between complex manifolds and $\mathcal{H}$ is a variation of graded-polarizable mixed Hodge structure on B then, $f^*(\mathcal{H})$ is a variation of graded-polarizable mixed Hodge structure on A (see § 1.7, [Kas86]).

We now recall the definition of an admissible variation of mixed Hodge structure over the punctured disk with unipotent monodromy following Steenbrink and Zucker: Let $\Delta = \{ s \in \mathbb{C} \mid |s| < 1 \}$ and $\Delta^* = \Delta - \{0\}$. Let $\mathcal{H} \to \Delta^*$ be a variation of graded-polarizable mixed Hodge structure. Let U denote the upper half-plane $\{ z = x + iy \in \mathbb{C} \mid y > 0 \}$, and $U \to \Delta^*$ be the covering map $s = e^{2\pi \mathbf{i} z}$.

After selecting a choice of graded-polarization (in order to define the classifying space D), the period map of $\mathcal{H}$ fits into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{F} & D \\ {\scriptstyle s}\big\downarrow & & \big\downarrow \\ \Delta^* & \xrightarrow{\varphi} & \langle T\rangle\backslash D \end{array} \implies F(z+1) = T.F(z) \tag{81}$$

where $T = e^N$. Accordingly, the map

$$\tilde{\psi} : U \to \check{D}, \qquad \tilde{\psi}(z) = e^{-zN}.F(z)$$

satisfies $\tilde{\psi}(z+1) = \tilde{\psi}(z)$ and hence descends to a map $\psi : \Delta^* \to \check{D}$.

By Schmid's nilpotent orbit theorem (Thm (4.12), [Sch73]), if $\mathcal{H}$ is pure then

$$\lim_{s\to 0} \psi(s) = F_\infty \in \check{D} \tag{82}$$

exists. Moreover, $N(F_\infty^p) \subseteq F_\infty^{p-1}$ and there exists a constant a such that $\Im(z) > a \implies e^{zN}.F_\infty \in D$. Finally, given a $G_{\mathbb{R}}$-invariant metric on D, there exist constants K and b such that

$$\Im(z) > a \implies d_D(F(z), e^{zN}.F_\infty) < K\Im(z)^b e^{-2\pi\Im(z)}$$

Remark. Schmid's result also covers the case of pure variations of Hodge structure with quasi-unipotent monodromy by passage to a finite cover. If t is another choice of holomorphic coordinate on Δ which vanishes at $0 \in \Delta$ then tracing through the above construction shows that the resulting limit filtration is related to (82) by the action of $e^{\lambda N}$ where λ depends on $(ds/dt)_0$.

In contrast, the mixed period domain D' with Hodge numbers $h^{1,1} = h^{0,0} = 1$ is isomorphic to $\mathbb{C}$ and has trivial infinitesimal period relation. Accordingly, the period map $\varphi : \mathbb{C}^* \to D'$ given by $\varphi(s) = e^{1/s}$ arises from a Hodge–Tate variation with trivial monodromy which does not have limit Hodge filtration.

Let V be a finite dimensional vector space and W be an increasing filtration of V. Then, $W[j]$ is the filtration $W[j]_k = W_{j+k}$. Given a nilpotent endomorphism N of a finite dimensional vector space V, it follows from upon writing N in Jordan canonical form that exists a unique, increasing monodromy weight filtration $W(N)$ of V such that

— $N(W_k) \subseteq W_{k-2}$;
— $N^k : \mathrm{Gr}_k^W \to \mathrm{Gr}_{-k}^W$ is an isomorphism

for each k.

Suppose instead that V is equipped with an increasing filtration W such that $N(W_k) \subseteq W_k$ for each index k. Then, there exists at most one increasing filtration $M = M(N, W)$ of V such that

— $N(M_k) \subset M_{k-2}$;
— M induces on Gr_k^W the filtration $W(N : \mathrm{Gr}_k^W \to \mathrm{Gr}_k^W)[-k]$.

If M exists it is called the relative weight filtration of W with respect to N. In general, $M(N, W)$ does not exist. For example, if W has only two non-trivial weight graded quotients which are adjacent (e.g. Gr_0^W and Gr_{-1}^W) then $M(N, W)$ exists if and only if W has an N-invariant splitting.

Definition A.1. Let $\mathcal{H} \to \Delta^*$ be a variation of graded-polarized mixed Hodge structure with unipotent monodromy $T = e^N$ and weight filtration W. Let $\varphi : \Delta^* \to \langle T\rangle\backslash D$ be the period map of $\mathcal{H}$. Then, $\mathcal{H}$ is admissible if

(a) The limit Hodge filtration (82) exists;

(b) The relative weight filtration $M = W(N, W)$ exists.

A variation of graded-polarized mixed Hodge structure $\mathcal{H} \to \Delta^*$ with quasi-unipotent monodromy is admissible if the pullback $f^*(\mathcal{H})$ to a finite covering of Δ^* with unipotent monodromy is admissible.

Remark. See § 3 of [SZ85] and § 1.8–1.9 of [Kas86] for the definition of admissibility in terms of the canonical extension of $\mathcal{H}$ to a system of holomorphic vector bundles over Δ.

An increasing filtration W of a vector space V is pure of weight k if $\mathrm{Gr}^w_\ell = 0$ for $\ell \neq k$ and $\mathrm{Gr}^W_k \cong V$. Reviewing the definition of $M = M(N, W)$ it follows that if W is pure of weight k then $M = W(N)[-k]$ (Prop. (2.11), [SZ85]).

COROLLARY A.2. *If $\mathcal{H} \to \Delta^*$ is a variation of pure, polarized Hodge structure then $\mathcal{H}$ is admissible.*

Proof. The limit Hodge filtration exists by Schmid's nilpotent orbit theorem, and the relative weight filtration exists by the previous paragraph. □

In the pure case, it follows from Schmid's SL_2-orbit theorem (Thm. (5.13), [Sch73]) that if φ is a the period map of a variation of polarizable Hodge structure $\mathcal{H} \to \Delta^*$ of weight k with unipotent monodromy $T = e^N$ then

$$(F_\infty, W(N)[-k]) \tag{83}$$

is a mixed Hodge structure relative to which N is a $(-1,-1)$-morphism, where F_∞ is the limit Hodge filtration (82). Moreover, it follows from the SL_2-orbit theorem (Thm. (6.6) and Cor. (6.7), [Sch73]) that the Hodge norm of a flat section of $\mathcal{H}$ is bounded.

One of the main results of [SZ85] is that if $\mathcal{H} \to \Delta^*$ is an admissible variation of graded-polarized mixed Hodge structure then (F_∞, M) is a mixed Hodge structure relative to which N is a $(-1,-1)$-morphism. In particular $N(F^p_\infty) \subseteq F^{p-1}_\infty$. Moreover if

$$\theta(z) = e^{zN}.F_\infty, \tag{84}$$

then there exists a constant $a > 0$ such that $\Im(z) > a \implies \theta(z) \in D$. Finally, by [Pea01] it follows that there exists constants K and b such that

$$\Im(z) > a \implies d_D(F(z), \theta(z)) \leqslant K \Im(z)^b e^{-2\pi\Im(z)}.$$

DEFINITION A.3. Let D be a classifying space of graded-polarized mixed Hodge structure with underlying filtration W and associated real Lie algebra $\mathfrak{g}_\mathbb{R}$. Then, the pair (N, F) consisting of an element $N \in \mathfrak{g}_\mathbb{R}$ and $F \in \check{D}$ defines an admissible nilpotent orbit $\theta(z) = e^{zN}.F$ if

(a) $N(F^p) \subseteq F^{p-1}$;

(b) The relative weight filtration $M = M(N, W)$ exists;

(c) There exists a such that $\Im(z) > a \implies \theta(z) \in D$.

The foundations of the theory of admissible nilpotent orbits of graded-polarized mixed Hodge structure is given by Kashiwara in [Kas86], where they are called infinitesimal mixed Hodge modules. In the pure case, a strengthened form of Schmid's several variable nilpotent orbit theorem as well as the several variable SL_2-orbit theorem appear in [CKS86].

Appendix B. Properly Discontinuous Actions on Mixed Period Domains

Let $\mathcal{H} \to S$ be a variation of graded-polarized mixed Hodge structure on a complex manifold S. Let $\rho : \pi_1(S, b) \to G_{\mathbb{R}}$ be the monodromy representation of $\mathcal{H}$ on the reference fiber $V = \mathcal{H}_b$ and $D = G/G^F$ be the classifying space of graded-polarized mixed Hodge structure defined by $\mathcal{H}_b$. Let W denote the weight filtration of V.

Proposition B.1. *If Γ is discrete and closed in $G_{\mathbb{R}}$ then Γ acts properly discontinuously on D, and hence the quotient $\Gamma\backslash D$ is a complex analytic space.*

Proof. In the case where $\mathcal{H}$ is a variation of pure Hodge structure, this result is well known from the work of P. Griffiths, and boils down to the fact that, in the pure case, the stabilizer $G_{\mathbb{R}}^F$ a point $F \in D$ is compact.

Turning to the mixed case, let K and K' be compact subsets of D. The map from D to the graded classifying spaces D_j is continuous, and hence the respective images K_j and K_j' of K and K' in D_j are compact for all j. If Γ does not act properly discontinuously, there exist an infinite set of distinct elements $g_n \in \Gamma$ such that $g_n(K) \cap K'$ is non-empty for all n. Then $((\mathrm{Gr}^W g_n)K_j) \cap K_j'$ is non-empty for all j and n. Since, by P. Griffiths' results, the action of $\mathrm{Gr}^W\Gamma$ on each D_j is properly discontinuous, it follows that the set $\{\mathrm{Gr}^W g_n\}$ contains only finitely many elements. Thus, after partitioning $\{g_n\}$ into a finite collection of subsets, we may assume that there exists $h \in \Gamma$ such that for all n we have $\mathrm{Gr}^W g_n = \mathrm{Gr}^W h$ for an infinite collection $\{g_n\}$. From this we shall derive a contradiction.

To this end, we introduce the complex, unipotent Lie group

$$U_{\mathbb{C}} = \{g \in \mathrm{GL}(V_{\mathbb{C}})) \mid (g - \mathrm{id})W_k \subset W_{k-1}\}$$

and let $U_{\mathbb{R}} = U_{\mathbb{C}} \cap \mathrm{GL}(V_{\mathbb{R}})$. Observe that $u_n := g_n h^{-1} \in U_{\mathbb{R}}$ for each index n, since g_n and h induce the same action on Gr^W.

To continue let $\mathcal{Y}$ denote the set of all (complex) gradings of W (see section 5.4). Then, the group $G_{\mathbb{C}}$ acts continuously on $\mathcal{Y}$ via the adjoint action. Moreover, by (2.2, [CKS86]), the subgroup $U_{\mathbb{C}}$ acts simply transitively on $\mathcal{Y}$. Furthermore, the map

$$Y : D \to \mathcal{Y}, \quad F \mapsto Y(F), \text{ the Deligne grading of } (F, W)$$

is continuous, and hence both $Y(K)$ and $Y(K')$ are compact subset of $\mathcal{Y}$. By construction,

$$Y(g.\tilde{F}) = g.Y(\tilde{F})$$

for any $\tilde{F} \in D$ and $g \in G_{\mathbb{R}}$. Applying this to $g_n, h \in G_{\mathbb{R}}$, we find

$$Y(g_n(K)) = g_n \cdot Y(K) = (u_n h) \cdot Y(K) = u_n(h \cdot Y(K)),$$

with $h \cdot Y(K)$ compact. So our question is: for how many $u_n \in U_{\mathbb{R}}$ can $u_n \cdot h \cdot Y(K)$ intersect $Y(K')$?

Fix $Y_o \in \mathcal{Y}$. Since $U_{\mathbb{C}}$ acts simply transitively upon $\mathcal{Y}$, it follows that there are compact subsets C' and C'' of $U_{\mathbb{C}}$ such that

$$Y(K') = C' \cdot Y_o, \qquad h \cdot Y(K) = C'' \cdot Y_o$$

So, if $u_n \cdot h \cdot Y(K)$ intersects $Y(K')$ then there exist elements $c' \in C'$ and $c'' \in C''$ such that

$$u_n c'' \cdot Y_o = c' \cdot Y_o$$

By simple transitivity, $u_n c'' = c'$ and hence u_n belongs to the compact set $C = C'(C'')^{-1}$. Equivalently, $g_n = u_n h$ belongs to the compact subset $C \cdot h \subset G_{\mathbb{C}}$.

By hypothesis, the image of Γ in $G_{\mathbb{R}}$ (and hence $G_{\mathbb{C}}$) is discrete and closed. As $C \cdot h$ is compact, it can contain only finitely many elements g_n from Γ, which contradicts the supposition that there infinitely many elements $g_n \in \Gamma$ such that $\mathrm{Gr}^W(g_n) = \mathrm{Gr}^W(h)$. Hence Γ acts properly discontinuously on D. □

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