Lowest Weights in Cohomology of Variations of Hodge Structure*

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Abstract

Let X be an irreducible complex analytic space with $j : U \hookrightarrow X$ an immersion of a smooth Zariski open subset, and let $\mathbb{V}$ be a variation of Hodge structure of weight n over U. Assume X is compact Kähler. Then $IH^k(X, \mathbb{V})$ is known to carry a pure Hodge structure of weight $k + n$, while $H^k(U, \mathbb{V})$ carries a mixed Hodge structure of weight $\geq k + n$. In this note it is shown that the image of the natural map $IH^k(X, \mathbb{V}) \to H^k(U, \mathbb{V})$ is the lowest weight part of this mixed Hodge structure. In the algebraic case this easily follows from the formalism of mixed sheaves, but the analytic case is rather complicated especially when the complement $X - U$ is not a hypersurface.

Introduction

For a compact Kähler complex manifold X the decomposition of complex valued C^∞ differential k-forms into types induces the Hodge decomposition for the De Rham group $H^k(X, \mathbb{C})$ equipping this group with a pure weight k Hodge structure. For singular or non-compact complex analytic spaces this is no longer true in general. For instance $H^1(\mathbb{C}^*)$ has rank 1 while it should have even rank if it would carry a weight 1 Hodge structure.

Cohomology groups of *algebraic* varieties instead carry a canonical **mixed Hodge structure**, i.e there is a rationally defined increasing weight filtration so that the k-th graded pieces carry a weight k Hodge structure. In the

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example there is only one weight, namely 2 and $H^1(\mathbb{C}^*)$ is pure of weight $(1,1)$. In fact, Deligne [Del71, Del74] constructed a good functorial theory for the cohomology of algebraic varieties. For a smooth variety U the weight filtration can be seen on the level of forms as follows. First choose a smooth compactification X such that $D = X - U$ is a divisor with normal crossings. Cohomology of U can now be calculated De Rham-style using rational forms having at most poles along D. The weight keeps track of the number of branches of D along which one has actual poles. Lowest weight corresponds to forms that extend regularly across D. Indeed, $H^k(U)$ carries a mixed Hodge structure with $W_{k-1}H^k(U) = 0$ and where $W_k H^k(U)$ is the image of the restriction $H^k(X) \to H^k(U)$. In the analytic case, a similar assertion holds provided a Kähler compactification X of U exists. However, the mixed Hodge structure one obtains may depend on the bimeromorphic equivalence class of the compactification.

If one studies how cohomology behaves under morphisms $f : Y \to X$ between compact Kähler complex analytic spaces, the Leray spectral sequence tells one to look at $H^q(X, R^p f_* \underline{\mathbb{Q}}_Y)$. So cohomology groups with varying coefficient system come up naturally. Assume that there is a non-empty Zariski-open subset $U \subset X$ over which Y and f are smooth so that the sheaf $R^p f_* \underline{\mathbb{Q}}_Y|_U$ is locally constant and the fibers carry a weight p Hodge structure. In fact these can be assembled to give the prototype of what is called a **variation of weight p Hodge structure** (cf. for instance [CSP]).

So it is natural to look at $H^k(U, \mathbb{V})$ where $\mathbb{V}$ is a local system which carries a weight n variation of Hodge structure. It generalizes the previous case where $\mathbb{V} = \underline{\mathbb{Q}}_U$ and as in that situation, it is known that there is a mixed Hodge structure on the cohomology group $H^k(U, \mathbb{V})$ provided the local monodromy operators around infinity are quasi-unipotent, a condition which is known to hold as long as the local system is defined over $\mathbb{Z}$ [Schm]. The goal of this note is to show that also in this setting, the lowest weight "comes from the compactification".

One should perhaps clarify what is meant by "coming from the compactification" because this is subtler than in the case of constant coefficients. According to [Zuc] if $\dim X = 1$ this can be explained as follows, Let $j : U \hookrightarrow X$ be the embedding of U into its compactification. The sheaf $j_*\mathbb{V}$ is quasi-isomorphic to the complex of holomorphic forms with values in $\mathbb{V}$ and with L^2 growth conditions at the boundary (with respect to the Poincaré metric). Forgetting the growth conditions gives a complex which computes the cohomology of $\mathbb{V}$ on U; whence a natural restriction map $L^2H^k(U, \mathbb{V}) \to H^k(U, \mathbb{V})$. One of the main results from [Zuc] states that the source has a pure Hodge structure of weight $(k+n)$ which maps to the lowest weight part of a functorial mixed Hodge structure on the target. As claimed before, in the general situation this remains true but it turns out to be easier to replace the approach using L^2-forms by a purely topological approach, namely via the intersection complex.

In fact, for higher dimensional base we shall work with $IH^k(X, \mathbb{V})$. See Remark 3.4 where the two are compared. That $IH^k(X, \mathbb{V})$ carries a pure Hodge structure in the general situation (generalizing the curve case [Zuc]) is much harder. To tackle this problem the second author [Sa88, Sa90] came up with a construction of mixed Hodge modules in which (filtered) D-modules and perverse sheaves play an equally important role: in a mixed Hodge module they come glued in pairs related via the Riemann-Hilbert correspondence. More precisely, by means of the perverse component one defines the rational structure of a mixed Hodge module while the filtered D-module component defines the Hodge filtration. In the algebraic case we can define the functors f_* etc. between the derived categories of mixed Hodge modules to get as much information as possible. In the analytic case, however, we can usually define only the cohomological functors $H^k f_*$, etc. between the abelian categories of mixed Hodge modules since the (global) Zariski topology on analytic spaces is too coarse, and this makes the proof of our main theorem quite complicated.

The goal of this note is to prove the assertion about the lowest weights. This is the content of Theorem 3.5. In the algebraic case the proof uses an argument which resembles the one from [Ha-Sa, Remarks 2.2. i)] used in the l-adic situation and for constant coefficients (actually this works as long as the formalism of mixed sheaves [Sa91] is satisfied). Originally the first named author gave a different proof which works only in the case $X - U$ is a locally principal divisor and resembles the arguments in [Sa90, 4.5.7 and 4.5.9] and [Mor, 3.1.4]. See Remark 3.6. Note that our main theorem in the algebraic case does not follow from the mixed Hodge version of [Mor, 3.1.4] unless j is an affine morphism since the t-structure in loc. cit. is *not* associated to the mixed complexes of weight $\leq k$ in the usual sense, see [Mor, 3.1.2] (and Remark 3.7 below).

In the analytic case the argument is much more complicated unless the complement is a hypersurface. We need a theory of mixed Hodge complexes on analytic spaces, and have to take a blowing-up of X along the complement of U to construct a mixed Hodge structure on $H^k(U, \mathbb{V})$. Then we have to study the direct image of the mixed Hodge complex under the blowing-down map to compare this with the Hodge structure on $IH^k(X, \mathbb{V})$. (If the reader prefers, he may assume as in Remark 2.2. 4) that the variations of Hodge structures and their minimal extensions are of geometric origin in the analytic case.)

In order to make the proof as self-contained as possible, we start with a brief summary of the necessary results from the theory of perverse sheaves and mixed Hodge modules.

The first named author wants to thank Stefan Müller-Stach for asking this question and urging him to write down a proof.

1 Perverse sheaves

We only give a minimal exposition of the theory of perverse sheaves to explain the properties which will be used below. We shall only be working with the so-called middle perversity which respects Poincaré duality. Full details can be found in [B-B-D].

Let X be a complex analytic space. The category of perverse "sheaves" of $\mathbb{Q}$-vector spaces on X, denoted by $\mathrm{Perv}(X;\mathbb{Q})$, is an *abelian category*. The fact that it is abelian follows from its very construction as a core with respect to a t-structure. While the details of this are not so relevant for what follows, one needs to know that the starting point is formed by the *constructible sheaves of* $\mathbb{Q}$*-vector spaces* on X. By definition these are sheaves of finite dimensional $\mathbb{Q}$-vector spaces which are locally constant on the strata of some analytic stratification of X. We assume that the stratification is *algebraic* in the algebraic case. The simplest examples of such sheaves are the locally constant sheaves on X itself, or those which are locally constant on some locally Zariski closed subset Z of X but zero elsewhere.

A core is defined with respect to a so-called t-structure and in the perverse situation the t-structure is defined by certain cohomological conditions, the so called support and co-support conditions. Indeed, instead of starting from complexes of constructible sheaves on X one departs from

$$D_c^{\mathrm{b}}(X;\mathbb{Q}): \quad \begin{array}{l}\text{the derived category of bounded complexes}\\ \text{of sheaves of } \mathbb{Q}\text{-vector spaces on } X \text{ with}\\ \text{constructible cohomology sheaves.}\end{array} \tag{1}$$

By definition a perverse sheaf is such a complex F which obeys the support and co-support conditions:

$$\dim \operatorname{supp} H^p(F) \leq -p, \quad \dim \operatorname{supp} H^p(\mathbb{D}F) \leq -p,$$

where $\mathbb{D}F := \mathbb{R}\underline{\mathrm{Hom}}(F, \mathbb{D}_X)$ is the Verdier dual of F and $\mathbb{D}_X$ is the dualizing complex. For X smooth and d-dimensional latter is just $\mathbb{Q}_X(d)[2d]$. The support condition implies that $H^p(F) = 0$ for $p > 0$ while the co-support condition implies $H^p(F) = 0$ for $p < -d$ (where $d = \dim X$): perverse sheaves are complexes "concentrated in degrees between $-d$ and 0".

On a complex manifold a (finite rank) local system of $\mathbb{Q}$-vector spaces $\mathbb{V}$ can be made perverse by placing it in degree $-d$: the complex $\mathbb{V}[d]$ is a perverse sheaf. If X is no longer smooth this complex has to be replaced by the so-called intersection complex. Indeed, if $U \subset X$ is a dense Zariski-open subset of X which consists of smooth points and $\mathbb{V}$ is any (finite rank) local system of $\mathbb{Q}$-vector spaces on U the **intersection complex** $\mathcal{IC}_X(\mathbb{V}[d])$[1] can be constructed as in [B-B-D] (and 1.1 below). (It is also called the minimal

[1] Some people write $\mathcal{IC}_X(\mathbb{V})$ instead of $\mathcal{IC}_X(\mathbb{V}[d])$

extension.) By definition, its hypercohomology groups are the intersection cohomology groups:

$$IH^k(X, \mathbb{V}[d]) := \mathbb{H}^k(\mathcal{IC}_X(\mathbb{V}[d])). \tag{2}$$

Remark. Even if X itself is smooth an intersection complex on X need not be of the form $\widetilde{\mathbb{V}}[d]$ for some local system $\widetilde{\mathbb{V}}$ defined on X because of non-trivial monodromy "around infinity" $X - U$.

The following two results explain the role of these intersection complexes.

Theorem 1.1 ([Bor84, Chap.V, 4])**.** *Let X be a d-dimensional irreducible complex analytic space and let U be a smooth dense Zariski-open subset of X on which there is a local system $\mathbb{V}$ of finite dimensional $\mathbb{Q}$ vector spaces. The intersection complex $\mathcal{IC}_X(\mathbb{V}[d])$ is up to an isomorphism in the derived category the unique complex of sheaves of $\mathbb{Q}$-vector spaces on X which is perverse on X, which restricts over U to $\mathbb{V}[d]$ and which has no non-trivial perverse sub or quotient objects supported on $X - U$.*

Theorem 1.2 ([B-B-D])**.** *If X is compact or algebraic,* $\mathrm{Perv}(X; \mathbb{Q})$ *is Artinian and Noetherian. Its simple objects are the intersection complexes $F = \mathcal{IC}_Z(\mathbb{V}[\dim Z])$ supported on an irreducible subspace $Z \subset X$ and where $\mathbb{V}$ is associated to an irreducible representation of $\pi_1(U)$, $U \subset Z$ the largest open subset of Z over which F is locally constant.*

We also need *filtered* objects in the abelian category $\mathrm{Perv}(X; \mathbb{Q})$. A priori these are not represented by filtered complexes in the usual sense of the word, since the morphisms are in a derived category: they are "fractions" $[f]/[s] : K \to L$ where the bracket stands for the corresponding homotopy class, $f : K \to N$ is a morphism of complexes and $N \overset{s}{\longleftarrow} L$ is a quasi-isomorphism. However, the category of sheaves on X with constructible cohomology has enough injectives and replacing L by a complex L' of injective objects, the quasi-isomorphism s becomes invertible up to homotopy and so $[f]/[s]$ can be represented by a true morphism $K \to L'$. Next, recall:

Lemma 1.3. *For any morphism of complexes $v : A \to B$, the morphism in the derived category defined by it can be represented by an injective morphism $A \to B'$ of complexes.*

Proof: Take $B' := \mathrm{Cone}(-\mathrm{id} \oplus v : A \to A \oplus B)$. Then A is a subcomplex of B' and we get an injective morphism $A \to B'$ which is identified with v by the quasi-isomorphism $(0, v, \mathrm{id}) : B' \to B$. □

Corollary 1.4. *Let $K \in$* $\mathrm{Perv}(X; \mathbb{Q})$*. Any* finite *filtration on K can be represented by a filtered complex in* $\mathrm{Perv}(X; \mathbb{Q})$.

Proof: Induction on the length of the filtration, assumed to be an increasing filtration W. The above discussion shows that the morphism $W_i \to W_{i+1}$ in $\mathrm{Perv}(X; \mathbb{Q})$ can be represented by a morphism of complexes to which Lemma. 1.3 can be applied. □

2 Mixed Hodge Modules

In this section we put together some properties of mixed Hodge modules which will be used in the sequel. These properties are proven in [Sa88] and [Sa90]. See also the exposition [PS, Cha. 14] where mixed Hodge Modules are introduced axiomatically.

Let X be a complex algebraic variety or a complex analytic space. There exists an abelian category $\mathsf{MHM}(X)$, the category of **mixed Hodge modules** on X.

Remark. Note that for nonproper complex algebraic varieties X we always have $\mathsf{MHM}(X) \neq \mathsf{MHM}(X^{\mathrm{an}})$ because of the difference between algebraic and analytic stratifications. Note also that a mixed Hodge module on an algebraic variety is always *assumed* to be extendable under an open immersion. The last property cannot be well-formulated in the analytic case due to the defect of the Zariski topology on analytic spaces, e.g. Zariski-open immersions are not stable by composition and closed subspaces are not intersections of hypersurfaces Zariski-locally.

Properties 2.1. A) There is a functor

$$\mathrm{rat}_X : D^{\mathrm{b}}\mathsf{MHM}(X) \to D^{\mathrm{b}}_c(X;\mathbb{Q}). \tag{3}$$

such that $\mathsf{MHM}(X)$ is sent to $\mathrm{Perv}(X;\mathbb{Q})$. One says that $\mathrm{rat}_X M$ is the underlying rational perverse sheaf of M. Moreover, we say that

$$M \in \mathsf{MHM}(X) \textbf{ is supported on } Z \iff \mathrm{rat}_X M \text{ is supported on } Z.$$

B) The category of mixed Hodge modules supported on a point is the category of graded polarizable rational mixed Hodge structures; the functor "rat" associates to the mixed Hodge structure the underlying rational vector space.

C) Each object M in $\mathsf{MHM}(X)$ admits a **weight filtration** W such that

- morphisms preserve the weight filtration strictly;
- the object $\mathrm{Gr}^W_k M$ is semisimple in $\mathsf{MHM}(X)$;
- if X is a point the W-filtration is the usual weight filtration for the mixed Hodge structure.

Since $\mathsf{MHM}(X)$ is an abelian category, the cohomology groups of any complex of mixed Hodge modules on X are again mixed Hodge modules on X. With this in mind, we say that for complex $M \in D^{\mathrm{b}}\mathsf{MHM}(X)$ the **weight** satisfies

$$\mathrm{weight}[M] \begin{cases} \leq n, \\ \geq n \end{cases} \iff \mathrm{Gr}^W_k H^i(M) = 0 \begin{cases} \text{for } k > i+n \\ \text{for } k < i+n. \end{cases}$$

We observe that if we consider the weight filtration on the mixed Hodge modules which constitute a complex $M \in D^{\mathrm{b}}\mathsf{MHM}(X)$ of mixed Hodge modules we get a filtered complex in this category.

D) (i) For each morphism $f : X \to Y$ between *complex algebraic varieties*, there are induced functors $f_*, f_! : D^{\mathrm{b}}\mathsf{MHM}(X) \to D^{\mathrm{b}}\mathsf{MHM}(Y)$ and $f^*, f^! : D^{\mathrm{b}}\mathsf{MHM}(Y) \to D^{\mathrm{b}}\mathsf{MHM}(X)$ which lift functors Rf_*, $f_!$ and $f^{-1}, f^!$ respectively; the latter functors are defined on the level of complexes with constructible cohomology sheaves.

D)(ii) In the *analytic case* this is no longer necessarily true but we have:
— for $f : X \to Y$ projective or if X is compact Kähler and $Y = \mathrm{pt}$, there are *cohomological* functors $H^i f_* = H^i f_! : \mathsf{MHM}(X) \to \mathsf{MHM}(Y)$ which lift the perverse cohomological functor ${}^pR^i f_* = {}^pR^i f_!$;
— for any f there are cohomological functors $H^i f^*, H^i f^! : \mathsf{MHM}(Y) \to \mathsf{MHM}(X)$ which lift ${}^pH^i f^{-1}$, ${}^pH^i f^!$ respectively;

E) The functors $f^*, f_!$ do not increase weights in the sense that if M has weights $\leq n$, the same is true for f^*M and $f_!M$.

F) The functors $f_*, f^!$ do not decrease weights in the sense that if M has weights $\geq n$, the same is true for f_*M and $f^!M$.

G) If f is proper, f_* preserves weights, i.e. f_* neither increases nor decreases weights.

Remarks 2.2. 1) Despite the fact that the functors f^* etc. do not exist in the analytic setting, properties E), F), G) still have a meaning as in [Sa90, 2.26] since the weight is defined in terms of cohomology only.
2) Since in the analytic setting Zariski-open immersions are not stable by composition $H^i f_*M$, $H^i f_!M$ do not necessarily exist for analytic morphisms f. This explains why in the analytic case D) not all morphisms are allowed.
3) The reader may interpret the Kähler condition on X in Property D) as the existence of a projective morphism g from a Kähler *manifold* X' onto X. Indeed, the construction of $H^i f_*M$ for $f : X \to \mathrm{pt}$, where M is a pure Hodge module, is reduced to the assertion for X': use the decomposition theorem for g applied to a pure Hodge module on X' which is a subquotient of the pullback of M by g. Then it follows from [CKS], [KK86], [KK87], [KK89]. For the mixed case we can use the weight spectral sequence.
4) It is still unclear whether $H^i f_*M$ exists for proper Kähler morphisms f unless M is constant, see [Sa90b]. If the reader prefers, he may assume that the polarizable Hodge modules in this paper are direct factors of the cohomological direct images of the constant sheaf by smooth Kähler morphisms so that the above assertion follows from the decomposition theorem for the direct image of the constant sheaf by proper Kähler morphisms [Sa90b].

The above properties readily imply various basic properties of mixed Hodge modules. For example, if M is a complex of mixed Hodge modules on X its cohomology $H^q M$ is a mixed Hodge module on X. Properties B) and D) imply:

Lemma 2.3. *Let* $a_X : X \to \mathrm{pt}$ *be the constant map to the point. Assume* X *is algebraic or compact Kähler. Then for any complex* M *of mixed Hodge modules on* X

$$\mathbb{H}^p(X, M) := H^p((a_X)_* M) \tag{4}$$

is a mixed Hodge structure.

For the proof of the main theorem one needs the following two technical constructions. The first is the adjunction construction:

Construction 2.4. Consider a morphism $f : X \to Y$ of *algebraic varieties* and a mixed Hodge module M on Y. The adjunction morphism $f^\# : M \to f_* f^* M$ is a morphism of complexes of mixed Hodge modules. For any bounded complex K of mixed Hodge modules on X, the identity $a_X = a_Y \circ f$ induces a canonical identification $\mathbb{H}^n(Y, f_* K) = \mathbb{H}^n(X, K)$. In particular this holds for $K = f^* M$. Adjunction thus induces a morphism of mixed Hodge structures

$$H^k f^\# : \mathbb{H}^k(Y, M) \to \mathbb{H}^k(X, f^* M). \tag{5}$$

In the *analytic case* this construction remains valid for an open immersion j whose complement is a hypersurface (defined locally by a function g). Indeed, then $j_* j^* M$ is a mixed Hodge module whose underlying D-module comes from localization by g. More generally, consider the complement U of an intersection Z of global hypersurfaces. Then $j_* j^* M$ is a complex of mixed Hodge modules due to a second construction:

Construction 2.5 ([Sa90, 2.19, 2.20])**.** Let g_i, $i = 1, \dots, r$ be holomorphic functions on Y, let $Z = \bigcap_{i=1}^r g_i^{-1}(0)_{\mathrm{red}}$ and $U = Y - Z$. We set $Y_i = Y - g_i^{-1}(0)$ and and for $I \subset \{1, \dots, r\}$ we set $Y_I = \bigcap_{i \in I} Y_i$. Let $i : Z \hookrightarrow Y$, $j : U \hookrightarrow Y$ and $j_I : Y_I \hookrightarrow Y$ be the natural inclusions. Let M be a mixed Hodge module on Y; then also $j^* M$, the restriction of M to U, is a mixed Hodge module on U and there are quasi-isomorphisms in the category $D^{\mathrm{b}}\mathsf{MHM}(Y)$

$$\begin{aligned} i_* i^! M &\xrightarrow{\sim} [\cdots 0 \to M \to B_1 \to B_2 \cdots B_r \to 0], \quad B_k = \bigoplus_{|I|=k} (j_I)_* j_I^* M \\ j_* j^* M &\xrightarrow{\sim} [\cdots 0 \to B_1 \to B_2 \to B_3 \cdots \to B_r \to 0], \quad B_k \text{ in degree } k-1. \end{aligned}$$

The above construction leads to

Lemma 2.6 ([Sa90, (4.4.1)]). *Let $i : Z \subset Y$ be a closed immersion and $j : U = Y - Z \hookrightarrow Y$ be the inclusion of the complement. Assume Y, Z are algebraic or, alternatively, that Z is an intersection of global hypersurfaces of Y. Let M be a mixed Hodge module on Y. There is a distinguished triangle* [2]

$$\begin{array}{ccc} i_*i^!M & \longrightarrow & M \\ & \nwarrow^{[1]} \quad \swarrow^{\alpha} & \\ & j_*j^*M & \end{array} \tag{6}$$

in the bounded derived category of mixed Hodge modules lifting the analogous triangle for complexes with constructible cohomology sheaves. The morphism α induces the adjunction morphism $H^k j^\# : \mathbb{H}^k(Y, M) \to \mathbb{H}^k(U, j^*M)$ for j (see (5)*).*

Proof: In the algebraic setting the local constructions 2.5 for a suitable affine cover glue together to give globally defined quasi-isomorphisms for $i_*i^!M$ and j_*j^*M. The local construction shows the existence of the distinguished triangle. See [Sa90, 4.4.1] for details.

The same argument applies in the analytic case under the assumption that Z is a global complete intersection. See the proof of [Sa90, 2.19]. □

Remark 2.7. The reader may wonder what happens in the general setting of analytic spaces. The problem is that Construction 2.5 can not be globalized to complexes of mixed Hodge modules. However, the *cohomology sheaves* of the complexes do make sense globally and are indeed mixed Hodge modules. Hence also the long exact sequence in cohomology associated to (6) exists in the category of mixed Hodge modules.

3 Polarizable variations of Hodge structure and the main theorem

In this section X is an *irreducible* compact Kähler complex analytic space of dimension d. Let $j : U \hookrightarrow X$ is the inclusion of a dense Zariski-open subset for which we make the crucial
Assumption 1) *U is smooth.*
Recall the basic result linking variation of Hodge structures and polarizable Hodge modules [Sa88, Th. 5.4.3]:

Theorem 3.1. *Suppose that $\mathbb{V}$ is a polarizable variation on U of weight n. If assumption 1) holds, there is a polarizable Hodge module V^{Hdg} of weight $n + d$ on U whose underlying perverse component is $\mathbb{V}[d]$.*

[2]We shall write triangles also as $M' \to M \to M'' \to [1]$

Polarizable Hodge modules in the analytic case in loc. cit. and *pure* Hodge modules in the algebraic case in [Sa90] are slightly different: for the former no condition at infinity is imposed while the condition formulated as Assumption 2) below is supposed to hold for the latter since the mixed Hodge modules are always extendable under open immersions in the algebraic case.

Anyway both form semi-simple categories (this is implied by the polarizability condition, see also 2.1.C) and both satisfy moreover the strict support condition:

Property 3.2. A polarizable weight n Hodge module M is a direct sum of polarizable weight n Hodge modules M_Z which have strict support Z where Z are irreducible subvarieties of X,[3] and the same assertion holds for pure Hodge modules.

The condition at infinity alluded to before is reduced in this case to the following assumption on the local system $\mathbb{V}$:
Assumption 2) *For a smooth compactification* $\bar{U}$ *of* U *with* $D = \bar{U} - U$ *a divisor with normal crossings, the local monodromy operators around* D *are quasi-unipotent.* (This is independent of the choice of a compactification.)
Then, by [Sa90, 3.20, 3.21] one has:

Theorem 3.3. *If assumptions 1) and 2) hold, and if* $\mathbb{V}$ *underlies a polarized variation of Hodge structures of weight* n *on* U*, then there is a unique pure Hodge module* V_X^{Hdg} *of weight* $n+d$ *on* X *having strict support in* X *and which restricts over* U *to* V^{Hdg}.

Remark. Note that this checks with the assertion in Theorem 1.1 which holds for the rational component of the mixed Hodge modules.

We next claim that $IH^k(X,\mathbb{V})$ carries a pure Hodge structure of weight $k+n$. To see this, note that the hypercohomology groups $\mathbb{H}^k(X,\mathcal{IC}_X(\mathbb{V}[d]))$ carry pure Hodge structures of weight $k+d+n$. This follows from the properties mentioned in 2: consider the proper map $a_X : X \to \mathrm{pt}$; then $[a_X]_* V^{\mathrm{Hdg}}$ is a *complex* of pure mixed Hodge modules of weight $n+d$ over a point and its k-th cohomology - which is exactly $\mathbb{H}^k(X,\mathcal{IC}_X(\mathbb{V}[d]))$ - has weight $k+d+n$. Since by (2) one has $IH^k(X,\mathbb{V}) = \mathbb{H}^k(X,\mathcal{IC}_X\,\mathbb{V})$, it follows that $IH^k(X,\mathbb{V})$ indeed carries a pure Hodge structure of weight $k+n$.

Remark 3.4. Suppose that in addition X is smooth and $X-U$ is a divisor with normal crossings. Then, by [CKS, Theorem 1.5], [KK86], [KK87], [KK89] $IH^k(X,\mathbb{V})$ can be identified with $L^2H^k(U,\mathbb{V})$ provided one measures integrability with respect to the Poincaré metric around infinity (one is in the normal crossing situation, so locally around infinity you have a product of disks and punctured disks). Summarizing:

$$\mathbb{H}^k(\mathcal{IC}_X(\mathbb{V})) = IH^k(X,\mathbb{V}) = L^2H^k(U,\mathbb{V})$$

[3] M is said to have **strict support** Z if it is supported on Z but no quotient or sub object of M has support on a proper subvariety of Z.

has a pure Hodge structure of weight $k+n$.

Next one wants to relate intersection cohomology and ordinary cohomology. This is the content of the main theorem:

Theorem 3.5. *Assume (X,U) satisfies the assumptions 1) and 2) above. Then*
a) *There is a morphism of mixed Hodge structures*

$$H^k j^\# : IH^k(X,\mathbb{V}) \to H^k(U,\mathbb{V}); \tag{7}$$

b) *the image of $H^k j^\#$ under* (7) *is exactly the lowest weight part of $H^k(U,\mathbb{V})$.*

Some remarks are in order. In the algebraic case formula (5) for the inclusion $j : U \hookrightarrow X$ and the mixed Hodge module $M := V_X^{\mathrm{Hdg}}$ show that (7) is indeed a morphism of mixed Hodge structures. In the analytic case Lemma 2.6 implies this result when $Z = X - U$ is a hypersurface.

The general case is much subtler. See § 5 where the problems in the analytic case will be fully dealt with. Now we pass to the

Proof in the algebraic case; strategy of the proof in the analytic setting. Let $i : Z = X - U \hookrightarrow X$ be the inclusion. To start, suppose that we are in the algebraic setting. Set $M = V_X^{\mathrm{Hdg}}$, $M' = j_* V^{\mathrm{Hdg}} = j_* j^* V_X^{\mathrm{Hdg}}$ and $M'' = i_* i^! V_X^{\mathrm{Hdg}}$. Form the distinguished triangle (6). Portion of its associated long exact sequence in hypercohomology reads

$$\begin{array}{ccccc} \cdots \to IH^k(X,\mathbb{V}) & \xrightarrow{H^k j^\#} & H^k(U,\mathbb{V}) & & \\ \| & & \| & & \\ \mathbb{H}^{k-d}(X,M) & \longrightarrow & \mathbb{H}^{k-d}(X,M') & \to & \mathbb{H}^{k-d+1}(X,M'') \to \dots \end{array} \tag{8}$$

By Theorem 3.3 $M = V_X^{\mathrm{Hdg}}$ is pure of weight $n+d$, and so by Property 2.1.F the complex $i^! V_X^{\mathrm{Hdg}}$ has weight $\geq n+d$. By Property 2.1.G this also holds for the complex $M'' = i_* i^! V_X^{\mathrm{Hdg}}$. Applying once more Property 2.1.G to the functor $(a_X)_*$ one sees that $\mathbb{H}^{k-d+1}(X,M'')$ has weights $\geq k+n+1$ and hence the image of the map (7) is exactly the weight $(k+n)$-part of $H^k(U,\mathbb{V})$.

Now assume that we are in the analytic setting but assume moreover that Z is a hypersurface. By Lemma 2.6 the same proof works. In the general situation one has to perform a suitable blow-up $\pi : X' \to X$ which is the identity in U and such that $Z' = X' - U$ is a divisor. Now we would like to apply the functor π_*. The problem is that this functor does not exist in the derived categories of mixed Hodge modules. So we have to find a substitute for this which still preserves enough of the Properties 2.1 so that we can complete the proof as in the algebraic case. It turns out that the correct category to use is the one of mixed Hodge complexes. See § 4. □

Remark 3.6. In the algebraic setting the following claim is easily shown to imply the main result as well and can be seen as a refinement of it.

Claim. Suppose Z is a locally principal divisor or j is an affine morphism. Then the adjunction morphism $j^{\#} : V_X^{\mathrm{Hdg}} \to j_*j^*V_X^{\mathrm{Hdg}}$ is injective and identifies V_X^{Hdg} with the lowest weight part of $j_*j^*V_X^{\mathrm{Hdg}} = j_*V^{\mathrm{Hdg}}$

Indeed, the extra hypothesis on j implies (see Construction 2.5 and [Sa90, 2.11]) that j_*V^{Hdg} is a mixed Hodge module (not just a *complex* of mixed Hodge modules) and the main theorem then follows easily from the Claim. The latter can be proved using the long exact sequence $0 \to H^0 i_* i^! V_X^{\mathrm{Hdg}} \to V_X^{\mathrm{Hdg}} \to j_*V^{\mathrm{Hdg}} \to H^1 i_* i^! V_X^{\mathrm{Hdg}}$ where $H^0 i_* i^! V_X^{\mathrm{Hdg}} = 0$ by the strict support condition.

The above claim can alternatively be shown using adjunction. This was how the first named author originally proved the main result.

Remark 3.7. It would not be difficult to construct a mixed Hodge version of [Mor, 3.1.4]. However, this would not immediately imply our main theorem unless j is an affine morphism. Indeed, the t-structure in loc. cit. is defined by the condition that ${}^pH^iK$ has weight $\le k$ and not $\le i + k$ as in the case of mixed Hodge complexes of weight $\le k$, see [Mor, 3.1.2]. It does not seem that there exists a t-structure associated to mixed complexes of weight $\le k$ since the weight filtration is not strict and the weight spectral sequence does not degenerate at E_1 (see also Section 5 on the proof of Theorem 3.5 in the analytic case where mixed Hodge complexes in the Hodge setting are used).

4 Mixed Hodge complexes on analytic spaces

For the proof of Theorem 3.5 in the analytic case we need a theory mixed Hodge complexes on analytic spaces which refines Deligne's theory [Del71] of cohomological mixed Hodge complexes. We present it here in a rather simplified manner which has the defect that the mapping cones are not well-defined. However, this does not cause a problem for the proof of Theorem 3.5 since all we need is the existence of the long exact sequence (12). See [Sa00] for a a more elaborate formulation taking care of the problem with the cones.

Notation. — $MFW(D_X)$: the category of filtered D_X-modules (M, F) with a finite filtration W. For singular X this can be defined by using closed embeddings of open subsets of X into complex manifolds, see [Sa88, 2.1.20].
— $D_h^{\mathrm{b}}FW(D_X)$: the derived category of bounded complexes (M, F, W) such that 1) the sheaves $\bigoplus_p H^i F_p \operatorname{Gr}_k^W M$ are coherent over the sheaf $\bigoplus_p F_p D_X$ and 2) the sheaves $H^i \operatorname{Gr}_k^W M$ are holonomic D_X-modules.
— $D_c^{\mathrm{b}}W(X, \mathbb{Q})$: the derived category of of bounded filtered complexes (K, W) such that W is finite and $\operatorname{Gr}_k^W K \in D_c^{\mathrm{b}}(X, \mathbb{Q})$ for any k: we define $D_c^{\mathrm{b}}W(X, \mathbb{C})$ similarly.

— $D^{\mathrm{b}}_h FW(D_X,\mathbb{Q})$: the "fibre" product of $D^{\mathrm{b}}_h FW(D_X)$ and $D^{\mathrm{b}}_c W(X,\mathbb{Q})$ over $D^{\mathrm{b}}_c W(X,\mathbb{C})$ where the functor $\mathrm{DR} : D^{\mathrm{b}}_h FW(D_X) \to D^{\mathrm{b}}_c W(X,\mathbb{C})$ induced by the De Rham functor is used to glue the two categories. More precisely, its objects are triples

$$\mathcal{M} = ((M,F,W),(K,W),\alpha)$$

where $(M,F,W) \in D^{\mathrm{b}}_h FW(D_X)$, $(K,W) \in D^{\mathrm{b}}_c W(X,\mathbb{Q})$ and

$$\alpha : \mathrm{DR}(M,W) \cong (K,W) \otimes_{\mathbb{Q}} \mathbb{C} \quad \text{in } D^{\mathrm{b}}_c W(X,\mathbb{C})$$

and morphisms in the category are pairs of morphisms of $D^{\mathrm{b}}_h FW(D_X)$ and $D^{\mathrm{b}}_c W(X,\mathbb{Q})$ compatible with α. Forgetting the filtration W we can define $D^{\mathrm{b}}_h F(D_X)$, $D^{\mathrm{b}}_c(X,\mathbb{Q})$ and $D^{\mathrm{b}}_h F(D_X,\mathbb{Q})$ similarly.

— $$\mathrm{Gr}^W_k \mathcal{M} = (\mathrm{Gr}^W_k(M,F), \mathrm{Gr}^W_k K, \mathrm{Gr}^W_k \alpha) \in D^{\mathrm{b}}_h F(D_X,\mathbb{Q}).$$

Definition 4.1. 1) The category of mixed Hodge complexes $\mathsf{MHC}(X)$ is the full subcategory of $D^{\mathrm{b}}_h FW(D_X,\mathbb{Q})$ consisting of $\mathcal{M} = ((M,F,W),(K,W),\alpha)$ satisfying the following conditions for $\mathrm{Gr}^W_k \mathcal{M}$ for any k,i:
(i) The $\mathrm{Gr}^W_k(M,F)$ are strict and we have a decomposition

$$\mathrm{Gr}^W_k \mathcal{M} \cong \bigoplus_j (H^j \mathrm{Gr}^W_k \mathcal{M})[-j]. \tag{9}$$

(ii) The $H^i \mathrm{Gr}^W_k \mathcal{M}$ are polarizable Hodge modules of weight $k+i$.
2) Let $\mathsf{MHW}(X)$ denote the category of weakly mixed Hodge modules, i.e. its objects have a weight filtration W fow which the gradeds Gr^W_k are polarizable Hodge modules of weight k, but there is no condition on the extension between the graded pieces.
3) We say that $\mathcal{M} \xrightarrow{u} \mathcal{M}' \xrightarrow{v} \mathcal{M}'' \xrightarrow{w} \mathcal{M}[1]$ is a *weakly distinguished triangle* in $\mathsf{MHC}(X)$ if u,v,w are morphisms of $\mathsf{MHC}(X)$ and its underlying triangle of complexes of sheaves of $\mathbb{Q}$-vector spaces is distinguished. Here the weight filtration W on $\mathcal{M}[1]$ is shifted by 1 so that $\mathcal{M}[1]$ is a mixed Hodge complex.

Remarks 4.2. 1) Note that the stability by direct images asserted in Corollary 4.8 does not follow from Theorem 4.7 if we replace $k+i$ by k in condition (ii) in the above definition of $\mathsf{MHC}(X)$. (This causes the shift of the filtration W in Proposition 4.4 below.)
2) In the case $X =$pt , we do not have to assume the decomposition (9) in condition (i) of Definition 4.1,1). One reason is that this is only needed to prove the stability by the direct image under a morphism from X. Another reason is that this decomposition actually follows from the other conditions in this case since the category of vector spaces over a field is semisimple.

We have by [Sa88, 5.1.14]

Proposition 4.3. *The category* $\mathsf{MHW}(X)$ *is an abelian category whose morphisms are strictly compatible with* (F, W)*.*

For a mixed Hodge complex $\mathcal{M}$, set

$$H^i\mathcal{M} = (H^i(M, F), {}^pH^i(K), {}^pH^i\alpha).$$

We put a weight filtration on it by letting W_k be the image of $H^iW_{k-i}\mathcal{M}$ (or, equivalently, the one induced by the filtration $\operatorname{Dec} W$ for the underlying D-module (cf. Proposition 4.4 below). This shift of the filtration W comes from condition (ii) in the above definition of $\mathsf{MHC}(X)$.

Using [Sa88, 1.3.6 and 5.1.11], etc. we have

Proposition 4.4. *With the weight filtration* W *defined above, the* $H^i\mathcal{M}$ *are weakly mixed Hodge modules. There is a weight spectral sequence in the abelian category of weakly mixed Hodge modules* $\mathsf{MHW}(X)$

$$E_1^{p,q} = H^{p+q}\operatorname{Gr}^W_{-p}\mathcal{M} \Rightarrow H^{p+q}\mathcal{M}, \tag{10}$$

which degenerates at E_2*, and whose abutting filtration on* $H^{p+q}\mathcal{M}$ *coincides with the weight filtration of weakly mixed Hodge modules shifted by* $p+q$ *as above, i.e.*

$$E_\infty^{p,q} = \operatorname{Gr}^W_q H^{p+q}\mathcal{M} \tag{11}$$

Moreover, $(M, F, \operatorname{Dec} W)$ *is bistrict, and the weight filtration on* $H^{p+q}M$ *is induced by* $\operatorname{Dec} W$ *where* M *is the underlying* D*-module of* $\mathcal{M}$ *and*

$$(\operatorname{Dec} W)_k M^i := \operatorname{Ker}(d : W_{k-i}M^i \to \operatorname{Gr}^W_{k-i} M^{i+1}).$$

Combining this with Proposition 4.3 we get

Proposition 4.5. *A weakly distinguished triangle as in Definition 4.1, 3) induces a long exact sequence in the abelian category* $\mathsf{MHW}(X)$

$$\to H^i\mathcal{M} \overset{u}{\to} H^i\mathcal{M}' \overset{v}{\to} H^i\mathcal{M}'' \overset{w}{\to} H^{i+1}\mathcal{M} \to . \tag{12}$$

For a morphism of mixed Hodge complexes $u : \mathcal{M} \to \mathcal{M}'$, there is a *mapping cone* $\mathcal{M}'' := \operatorname{Cone}(u : \mathcal{M} \to \mathcal{M}')$ in the usual way. Here the weight filtration W on $\mathcal{M}[1]$ is shifted by 1 so that $\operatorname{Gr}^W u$ in the graded pieces of the differential of $\mathcal{M}''$ vanishes and hence conditions (i) and (ii) above are satisfied. However, $\mathcal{M}''$ is *not unique* up to a non-canonical isomorphism because of a problem of homotopy. So we cannot get a triangulated category although there is a weakly distinguished triangle $\mathcal{M} \to \mathcal{M}' \to \mathcal{M}'' \to [1]$ which by Proposition 4.5 induces the long exact sequence (12) in the category $\mathsf{MHW}(X)$.

Since the weight filtration on the perverse component of a weakly mixed Hodge module can be represented by an honest filtered complex (Cor. 1.4) we have:

Proposition 4.6. *Considering a weakly mixed Hodge module as a mixed Hodge complex concentrated in degree* 0 *we get a functor*

$$\iota_X : \mathsf{MHW}(X) \to \mathsf{MHC}(X).$$

By the decomposition theorem for polarizable Hodge modules [Sa88, 5.3.1] together with the uniqueness of the decomposition [Del94], we have

Theorem 4.7. *Let $f : X \to Y$ be a projective morphism, and $\mathcal{M}$ be the image of a polarizable Hodge module by ι_X. Then we have a decomposition*

$$f_*\mathcal{M} \cong \bigoplus_i (H^i f_*\mathcal{M})[-i] \quad \text{in } D^{\mathrm{b}}_h F(D_X, \mathbb{Q}).$$

Combining this with Properties 2.1.D) (ii) we get

Corollary 4.8. *Mixed Hodge complexes and weakly distinguished triangles are stable by the direct image under $f : X \to Y$ if f is projective or if X is compact Kähler and $Y = pt$.*

5 Proof of Theorem 3.5 in the analytic case

Let $\pi : X' \to X$ be a bimeromorphic projective morphism inducing the identity over U and such that $X' - U$ is a hypersurface (defined locally by a function). Let $j' : U \to X'$ denote the inclusion. Then $\mathbb{R}j'_*\mathbb{V}[d]$ is a perverse sheaf, and underlies a mixed Hodge module j'_*V^{Hdg}, see [Sa90, 2.17]. By Proposition 4.6 this gives a mixed Hodge complex concentrated in degree 0

$$\mathcal{M}' = ((M', F, W), (K', W), \alpha) := \iota_{X'}(j'_*V^{\mathrm{Hdg}}), \tag{13}$$

such that $K' = \mathbb{R}j'_*\mathbb{V}[d]$ and $\mathcal{M}'|_U$ is identified with V^{Hdg}. We denote the direct image of $\mathcal{M}'$ by

$$\mathcal{M} = ((M, F, W), (K, W), \alpha) := \pi_*\mathcal{M}' = (\pi_*(M', F, W), \pi_*(K', W), \pi_*\alpha).$$

By Corollary 4.8 this is a mixed Hodge complex since π is projective.

Proposition 5.1. *We have $\mathrm{Gr}^W_{d+n} H^0\mathcal{M} = \iota_X(V_X^{\mathrm{Hdg}})$, and $\mathrm{Gr}^W_k H^i\mathcal{M} = 0$ if $k = d+n+i, i \neq 0$ or if $k < d+n+i$.*

Proof: It suffices to show the assertion for the underlying complex of D-modules M since the condition on strict support in Theorem 3.3 is detected by its underlying D-module. Moreover we may restrict to a sufficiently small open subset Y of X enabling us to apply Construction 2.5.

So let $g_1, \dots, g_r$ be functions on Y such that $Z \cap Y = \bigcap_i g_i^{-1}(0)$. Set $Y_i = Y - g_i^{-1}(0)$. Abusing notation, let $i : Y \cap Z \to Y$, $j : Y - Z \to Y$ denote the inclusions. By Lemma 2.6 there is a distinguished triangle

$$i_*i^!(V_X^{\mathrm{Hdg}}|_Y) \to V_X^{\mathrm{Hdg}}|_Y \to j_*j^*(V_X^{\mathrm{Hdg}}|_Y) \to [1],$$

inducing a long exact sequence of cohomology.

Claim. The underlying bifiltered D-modules of $\iota_Y(H^i j_* j^*(V_X^{\mathrm{Hdg}}|_Y))$ and $H^i\mathcal{M}|_Y$ are isomorphic to each other.

Suppose that the Claim has been shown. Then the same argument as in the proof of Theorem 3.5 in the algebraic case gives the result. Indeed, we have the exact sequence

$$H^i(V_X^{\mathrm{Hdg}}|_Y) \to H^i j_* j^*(V_X^{\mathrm{Hdg}}|_Y) \to H^{i+1} i_* i^!(V_X^{\mathrm{Hdg}}|_Y),$$

and $H^{i+1} i_* i^!(V_X^{\mathrm{Hdg}}|_Y)$ has weights $\geq d+n+i+1$ by Properties 2.1.F and G. This gives the assertion for $i=0$ since $V_X^{\mathrm{Hdg}}|Y$ is pure of weight $d+n$. For $i \neq 0$ we have $H^i(V_X^{\mathrm{Hdg}}|Y) = 0$ and hence the last morphism of the exact sequence is injective so that the assertion follows.
Proof of the Claim. Let $Y' = \pi^{-1}(Y)$, $Y'_i = \pi^{-1}(Y_i)$, and $g'_i = \pi^* g_i$. By Construction 2.5 the associated Čech complex gives a resolution of $j'_* V^{\mathrm{Hdg}}$. The components of this Čech complex are direct sums of $(j'_I)_*(V^{\mathrm{Hdg}}|Y'_I)$ where $Y'_I = \bigcap_{i\in I} Y'_i$ with the inclusion $j'_I : Y'_I \to Y$. By the uniqueness of the open direct image in [Sa90, 2.11] we have moreover

$$\pi_*(j'_I)_*(V^{\mathrm{Hdg}}|Y'_I) = (j_I)_*(V^{\mathrm{Hdg}}|Y_I),$$

where $j_I : Y_I := \bigcap_{i\in I} Y_i \to Y$. So we get the desired isomorphism (using the filtration $\mathrm{Dec}\, W$ from Proposition 4.4), and Proposition 5.1 follows. □

We return to the proof of Theorem 3.5 in the analytic case. Applying Proposition 5.1 to $\mathcal{M}'$, we get

$$\mathrm{Gr}^W_{d+n}\mathcal{M}' = \iota_{X'}(\mathrm{Gr}^W_{d+n} j'_* V^{\mathrm{Hdg}}) = \iota_{X'}(V_{X'}^{\mathrm{Hdg}}).$$

This implies that we get a morphism $u' : \iota_{X'}(V_{X'}^{\mathrm{Hdg}}) \to \mathcal{M}'$ to which we apply π_*. The decomposition Theorem 4.7 together with the semisimplicity of polarizable Hodge modules imply that V_X^{Hdg} is a direct factor of $\pi_* V_{X'}^{\mathrm{Hdg}}$. So we get a morphism

$$u : \iota_X(V_X^{\mathrm{Hdg}}) \to \mathcal{M}.$$

It is not clear whether u is uniquely defined (since the decomposition is not unique). However, its underlying morphism of $\mathbb{Q}$-complexes coincides with the canonically defined adjunction morphism $j^\#$ so that it induces the desired morphism of mixed Hodge structures

$$H^i j^\# : IH^i(X,\mathbb{V}) \to H^i(U,\mathbb{V}).$$

Let $\mathcal{M}''$ be a mapping cone of $u : \iota_X(V_X^{\mathrm{Hdg}}) \to \mathcal{M}$ as defined in Section 4. Remember (13) that $\mathcal{M}$ comes from $j'_* V^{\mathrm{Hdg}}$, a mixed Hodge *module* of weight $\geq n+d$ (by Properties 2.1. F)) and hence $\mathrm{Gr}^W_k \mathcal{M} = 0$ for $k < d+n$. Then, by definition of the cone, one has

$$\mathrm{Gr}^W_k \mathcal{M}'' = \mathrm{Gr}^W_k \mathcal{M} = 0 \quad \text{for } k < d+n. \tag{14}$$

Using Proposition 5.1 (e.g. $\iota_X(V_X^{\mathrm{Hdg}}) = \mathrm{Gr}^W_{d+n} H^0\mathcal{M}$) together with the long exact sequence (12) we get moreover

$$\mathrm{Gr}^W_k H^i\mathcal{M}'' = 0 \quad \text{for } k \le i+d+n.$$

Since by (11) we have $E_\infty^{i,k} = \mathrm{Gr}^W_k H^{i+k}\mathcal{M}'$, the weight spectral sequence (10) implies the surjectivity of

$$\begin{array}{ccc} E_1^{-d-n-1,d+n+1+j} & \xrightarrow{d_1} & E_1^{-d-n,d+n+j+1} \\ \| & & \| \\ H^j \mathrm{Gr}^W_{d+n+1}\mathcal{M}'' & \to & H^{j+1}\mathrm{Gr}^W_{d+n}\mathcal{M}' \end{array}$$

for all j, and this map splits by the semisimplicity of polarizable Hodge modules. So we get the surjectivity of

$$H^i(a_X)_* H^j \mathrm{Gr}^W_{d+n+1}\mathcal{M}'' \to H^i(a_X)_* H^{j+1}\mathrm{Gr}^W_{d+n}\mathcal{M}'' \quad \text{for any } i,j.$$

Claim. This implies the surjectivity of

$$H^i(a_X)_* \mathrm{Gr}^W_{d+n+1}\mathcal{M}'' \to H^{i+1}(a_X)_*\mathrm{Gr}^W_{d+n}\mathcal{M}'' \quad \text{for any } i.$$

Proof of the claim. The truncation $\tau_{\le j}$ on $\mathrm{Gr}^W_k M''$ splits by the definition of mixed Hodge complexes so that $\mathrm{Gr}^W_k M'' \simeq \bigoplus_i H^i(\mathrm{Gr}^W_k M'')[-i]$ where M'' is the underlying D-module of $\mathcal{M}''$. Now the truncation induces a filtration τ' on $H^i(a_X)_*\mathrm{Gr}^W_k M''$ and the preceding splitting for $\mathrm{Gr}^W_k M''$ coming from the truncation induces a splitting for $H^i(a_X)_*\mathrm{Gr}^W_{d+n+1} M''$ coming from τ'. Its factors are isomorphic to $H^{i-j}(a_X)_* H^j \mathrm{Gr}^W_{d+n+1}\mathcal{M}''$ and this factor maps surjectively to the factor of $H^{i+1}(a_X)_*\mathrm{Gr}^W_{d+n}$ isomorphic to $H^{i-j}(a_X)_* H^{j+1}\mathrm{Gr}^W_{d+n}\mathcal{M}''$. □

Again using $\mathrm{Gr}^W_k\mathcal{M}'' = 0$ for $k < d+n$ (14) it follows from the weight spectral sequence for $(a_X)_*\mathcal{M}''$ that

$$\mathrm{Gr}^W_k H^i(a_X)_*\mathcal{M}'' = 0 \quad \text{for } k \le d+n+i.$$

So we get the desired assertion as in the algebraic case (using Corollary 4.8) since the long exact sequence (12) for the direct image of the weakly distinguished triangle of the cone for u under $a_X : X \to$pt reads

$$\cdots H^i(a_X)_*\mathcal{M}'' \to IH^i(X,\mathbb{V}) \xrightarrow{H^i(j^\#)} H^i(U,\mathbb{V}) \to H^{i+1}(a_X)_*\mathcal{M}''.$$

This completes the proof of Theorem 3.5 in the analytic case.

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