SCIENCE CHINA
Mathematics

Special Issue on Algebraic Geometry
• ARTICLES •

https://doi.org/10.1007/s11425-024-2319-5

A remarkable class of elliptic surfaces of amplitude 1 in weighted projective spaces

In memory of Gang Xiao

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Received January 27, 2024; accepted July 23, 2024; published online October 21, 2024

Abstract Surfaces of amplitude 1 in the ordinary projective space are of the general type, but this need not be the case in weighted projective spaces. Indeed, there are 4 classes of quasi-smooth weighted hypersurfaces in $\mathbb{P}(1,2,a,b)$ of amplitude 1 with an elliptic pencil cut out by hyperplanes. Their moduli spaces are constructed, and the monodromy of their universal families is determined as well as their period maps which turn out to be generally immersive. For those that are not, a mixed Torelli theorem holds. We added an application to certain compactifications of moduli spaces of surfaces of the general type with $K^2 = 1$, $p_g = 2$ and $q = 0$ as a follow up of Gallardo et al. (2022), as well as detailed SageMath-calculations. The appendix written by Wim Nijgh shows that the general member of the types 1 and 2 elliptic family has a "trivial" Picard lattice, i.e., is spanned by fiber components and a multisection.

Keywords elliptic surface, period maps, (mixed) Torelli theorems, Picard lattices, monodromy of universal families, Tate and Mumford-Tate conjectures, KSBA compactifications

MSC(2020) 14J27, 32G20

Citation: Pearlstein G, Peters C, Nijgh W. A remarkable class of elliptic surfaces of amplitude 1 in weighted projective spaces. Sci China Math, 2024, 67, https://doi.org/10.1007/s11425-024-2319-5

1 Introduction

Hypersurfaces and, more generally, complete intersections in weighted projective spaces are basic entries in the geography of algebraic varieties. In particular, Reid [36] gave a list of 95 families of weighted projective K3 hypersurfaces with Gorenstein singularities.

There are several instances where a moduli space of a class of surfaces can be described in terms of weighted complete intersections. We mention the Kunev surfaces [27, 47] which are bidegree $(6,6)$ complete intersections in $\mathbb{P}(1,2,2,3,3)$, and certain Horikawa surfaces studied in [34] which are hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$.

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1. We recall some properties of hypersurfaces in weighted projective spaces and refer to [13, 22] for details. If X is a degree d hypersurface in the weighted projective space $\mathbb{P}(a_0, \dots, a_n)$, its ***type*** is the symbol $(d, [a_0, \dots, a_n])$. Following [22], we see that the integer $\alpha(X) = d - (a_0 + \cdots + a_n)$ is called the ***amplitude*** of X. If $n = 3$ and X would be smooth, $\alpha(X) < 0$, $\alpha(X) = 0$, and $\alpha(X) > 0$ respectively corresponds to X being a rational or ruled surface, a K3-surface, and a surface of general type. Since weighted projective spaces and their hypersurfaces therein, in general, are singular, the amplitude no longer measures their place in the classification. Hunt and Schimmrigk [20] found a striking example of this phenomenon: the degree 66 Fermat-type surface $x_0^{66} + x_1^{11} + x_2^3 + x_3^2 = 0$ in $\mathbb{P}(1, 6, 22, 33)$ of amplitude $66 - (1+6+22+33) = 4$ turns out to be an elliptic K3-surface. In fact, it is isomorphic to the unique K3 surface with the cyclic group of order 66 as its automorphism group described by Inose [23]. Kollár [26, Section 5] found several families of hypersurfaces in the weighted projective space with positive amplitude which have the same rational cohomology as projective space. In the surface case, one then obtains rational surfaces with positive amplitude, for example, the surface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ given by $x_0^{d_0}x_1 + x_1^{d_1}x_2 + x_2^{d_2}x_3 + x_3^{d_3}x_0$, where $(d_0, d_1, d_2, d_3) = (4, 5, 6, 7)$ and $(a_0, a_1, a_2, a_3) = (174, 143, 124, 95)$ which has degree 839, amplitude 303 and Hodge numbers $p_g = 0$ and $h^{1,1} = 2$.

In this paper, we restrict our discussion to families of surfaces in the weighted projective 3-space whose amplitude is 1 and which are not of the general type. The classification of surfaces suggests looking for conditions that give K3 surfaces or elliptic surfaces.

Except in Section 7 where we apply the results of previous sections, we only consider quasi-smooth surfaces, i.e., surfaces whose only singularities occur where the weighted projective space has singularities. This ensures that a surface of amplitude α has a canonical sheaf $\mathcal{O}(\alpha)$ (see Remark 2.3) which simplifies many calculations. Assuming that $\alpha = 1$ and that $p_g = 1$ then leads to degree $d = a + b + 4$ surfaces in $\mathbb{P}(1, 2, a, b)$[1]. This restricts the possibilities to just four cases. Two give properly elliptic surfaces and two give K3 surfaces.

Proposition (= Propositions 2.5 and 4.5). *The only quasi-smooth hypersurfaces X of the type $(d, [1, 2, a, b])$ with a, b co-prime odd integers and such that $d = a + b + 4$ are*

1. $(14, [1, 2, 3, 7])$*, for example,* $x_0^{14} + x_1^7 + x_2^4 x_1 + x_3^2$;
2. $(12, [1, 2, 3, 5])$*, for example,* $x_0^{12} + x_1^6 + x_2^4 + x_1 x_3^2$;
3. $(16, [1, 2, 5, 7])$*, for example,* $x_0^{16} + x_1^8 + x_0 x_2^3 + x_1 x_3^2$;
4. $(22, [1, 2, 7, 11])$*, for example,* $x_0^{22} + x_1^{11} + x_0 x_2^3 + x_3^2$.

The types mentioned in 1 *and* 2 *give properly elliptic surfaces and those of the types* 3 *and* 4 *give K*3 *surfaces.*

The examples figured in the above proposition are preferred members of the given type which will be referred to as the ***basic examples***. The monomials present for each of the basic examples are required for quasi-smoothness. The minimal surfaces they define can be shown to be Delsarte surfaces in Shioda's terminology (see [39, § 13.2.1] for details).

Remark. We show (see Subsection 4.1) that all the type 3 surfaces are birational to surfaces of the type $(9, [1, 1, 3, 4])$ and all the type 4 surfaces are birational to surfaces of the type $(12, [1, 1, 4, 6])$. The first is number 8 in Reid's list of 95 families, and the second is number 14.

2. In the weighted case, the group of projective automorphisms is in general not reductive which causes problems when we want to construct moduli spaces of weighted hypersurfaces. In our situation, we circumvent this problem by giving certain normal forms which give projectively isomorphic surfaces if and only if they are in the orbit of some fixed algebraic torus of projective transformations. In each of the four cases, this gives a quasi-projective moduli space of the expected dimension (see Subsection 3.1).

Remark. A general approach to the construction of geometric quotients under non-reductive group actions has been proposed in [5, 6, 14]. Based on this, Bunnett [8] showed that certain classes of weighted hypersurfaces admit GIT-moduli spaces. In his work, it is crucial that the weights divide the degree (in

[1] Note that a quasi-smooth surface in $\mathbb{P}(1, 1, a, b)$ with amplitude 1 has $p_g \geqslant 2$ since $H^0(\mathcal{O}(1))$ corresponds to the polynomials of degree 1.

order to have Cartier divisors instead of $\mathbb{Q}$-divisors for the linearization). Another crucial assumption concerns the unipotent radical of the group of projective automorphisms of the weighted projective space. Neither one of these holds for our examples.

The collection of degree $d = a + b + 4$ weighted hypersurfaces in $\mathbb{P}(1, 2, a, b)$ in a natural way forms an ordinary projective space $\mathbb{P}^N$ by considering the $N + 1$ coefficients in front of all the possible monomials. The quasi-smooth hypersurfaces belong to a Zariski-open subset $U_{1,2,a,b}$ of this projective space. The tautological family $\mathscr{F}_{a,b}$ of degree d quasi-smooth hypersurfaces over $U_{1,2,a,b}$ is called the corresponding ***universal family***. Using degenerations having an isolated exceptional unimodal Arnol'd-type singularity, we show that the global monodromy group of the universal families in the cases 1–4 is as big as possible:

Proposition (= Proposition 3.10). *Let L be the middle cohomology group of the minimal resolution of singularities of a quasi-smooth member of $\mathscr{F}_{a,b}$, $S \subset L$ be the Picard lattice of a general member and $T = S^{\perp}$ be the transcendental lattice. Then the monodromy group of the universal family of such quasi-smooth hypersurfaces is the subgroup of $O^{\#-}(L)$ preserving T and inducing the identity on S*[2]*.*

3. Our examples all are simply connected and the Hodge structure on the middle cohomology group looks like that of a K3 surface (see Proposition 4.1). In particular, the period domain is of a similar type (see formula (5.2)).

It is well known that for a Kuranishi family of K3 surfaces, the period map is always an immersion and so infinitesimal Torelli holds. In the setting of elliptic surfaces having multiple fibers, this is no longer the case according to an observation of Chakiris [11].

Theorem (See [11]). *Simply connected elliptic surfaces with $p_g > 0$ and having one or at most two multiple fibers* (*with co-prime multiplicities*) *are counterexamples to the Torelli theorem*: *the fiber of the period map for its Kuranishi family is positive-dimensional.*

The proof in loc. cit. is only sketched. We therefore decided to give a (simple) proof in the case of one multiple fiber (the situation occurring in our examples) (see Proposition 5.1).

In our setting, these results need to be used with care since our deformations are restricted to the ones that keep the surface in a fixed weighted projective space. As we show in Appendix A.1, the period map for the Kuranishi family of the basic examples in the sense of Lemma 2.4, as given above, has a 1-dimensional kernel. However, as shown in Appendix A.3, this is not generically the case.

Proposition (= Proposition 5.2). *The period map for the Kuranishi family* (*in the sense of Lemma* 2.4) *for a general types* 1–4 *surface is an immersion.*

An interesting arithmetic consequence of this result has been signaled to us by Moonen, namely, the validity of the Tate conjecture as well as the Mumford-Tate conjecture for each of the present surfaces (see Corollary 5.3).

4. The results in the paper involving the structure of the period domain as well as the behavior of the period map use precise information about the fibers of the genus 1 fibrations on a general surface from each of the four classes. The determination of the fiber types is relatively standard and has been facilitated by calculations in SageMath. Together with the obvious bisection which comes from the resolution of singularities, this gives a sublattice of the Picard lattice but in general, it is hard to determine whether this is the entire Picard lattice.

Given suitable models in our families which are defined over the integers, counting points on reductions at one or two "good" primes gives a by-now standard method to determine the Picard number of a general member of a family. Originally, we applied another trick to show this for type 1 surfaces, but this trick cannot be applied to type 2 surfaces. Thanks to the competence of Wim Nijgh, the type 2 surfaces could be handled by applying the first mentioned method. From the way, this proof was set up, and we only recently found out that a general type 2 surface is birational to a type 1 surface of the sort for which we had shown that the Picard number is generally equal to 2. Since this birational transformation lowers

[2] See below for the notation.

the Picard number by 1, the original surface has generally Picard rank 3 (see Remarks 3.2(1) and 4.6).

Since there are many possible birational transformations, this would not easily have been discovered before having gone through the details of Nijgh's approach. Therefore, it was clear to us that his proof forms a natural companion to our paper and so we placed it in Appendix C. We want to mention that he furthermore proves (see Remark C.25(2)) that a similar but much simpler approach also implies that a general type 1 surface has Picard number 2.

Using the determination of the general Picard lattice for the four types 1–4, in Proposition 5.8, we calculate the transcendental lattice of the generic surfaces.

5. As in the case of the Kunev example, in the properly elliptic case, there is a unique canonical divisor K on the surface X and one may associate to the pair $(X, \mathrm{supp}(K))$ the mixed Hodge structure on $H^2(X \setminus \mathrm{supp}(K))$. We arrive in this way at two further results: first of all, Theorem 6.1, stating that for the associated *mixed period map* in the cases where ordinary infinitesimal Torelli fails, the infinitesimal Torelli theorem does hold for the mixed Hodge structure, and secondly, Corollary 6.5 which states that a wide class of related variations is rigid in the sense of [33], i.e., all the deformations of the mixed period map keeping source and target fixed are trivial.

6. In Section 7, we give an application to certain compactifications of moduli spaces of surfaces of the general type with $K^2 = 1$, $p_g = 2$ and $q = 0$ as a follow up of [17].

Conventions and notation. • A lattice is a free $\mathbb{Z}$-module of finite rank equipped with a non-degenerate symmetric bilinear integral form which is denoted by a dot.

• A rank one lattice $\mathbb{Z}e$ with $e.e = a$ is denoted by $\langle a \rangle$, and orthogonal direct sums by $\textcircled{\perp}$. Other standard lattices are the hyperbolic plane U, and the root-lattices A_n $(n \geqslant 1)$, B_n $(n \geqslant 1)$, D_n $(n \geqslant 4)$ and E_n $(n = 6, 7, 8)$.

• If one replaces the form on the lattice L by m-times the form, $m \in \mathbb{Z}$, this scaled lattice is denoted by $L(m)$.

• $A(L) = L^*/L$ is the discriminant group of a lattice L, and b_L is the discriminant bilinear form. In the case where L is even, q_L denotes the discriminant quadratic form (see Subsection 5.2).

• The orthogonal group of a lattice L is denoted by $O(L)$, $O^{\#}(L)$ is the subgroup of isometries inducing the identity on $A(L)$, and $O^{\#\pm}(L)$ is the subgroup of $O^{\#}(L)$ consisting of isometries with signed spinor norm 1 (see Subsection 3.2).

• We denote weighted projective spaces in the usual fashion by $\mathbb{P}(a_0, \ldots, a_n)$ with weighted homogeneous coordinates, i.e., $x_0, \ldots, x_n$. Let $I \subset \{0, \ldots, n\}$. The weighted subspace obtained by setting the coordinates in $\{0, \ldots, n\} \setminus I$ equal to zero is denoted by P_I so that the coordinate points are $\mathsf{P}_0, \ldots, \mathsf{P}_n$.

A degree d polynomial with such weights has ***symbol*** $(d, [a_0, \ldots, a_n])$. Let F be a polynomial with this symbol. We set

$$\Omega_n = \sum_{j=0}^{n} x_j dx_0 \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

$$J_F = (\partial F/\partial x_0, \ldots, \partial F/\partial x_n) \subset \mathbb{C}[x_0, \ldots, x_n], \text{ the Jacobian ideal of } F,$$

$$R_F = \mathbb{C}[x_0, \ldots, x_n]/J_F, \text{ the Jacobian ring of } F, \ R_F^k \text{ degree } k \text{ part of } R_F.$$

• We often do not write coefficients in front of monomials and so we use the shorthand $\sum_{k_0,\ldots,k_n} x_0^{k_0} \cdots x_n^{k_n}$ instead of $\sum_{k_0,\ldots,k_n} a_{k_0,\ldots,k_n} x_0^{k_0} \cdots x_n^{k_n}$.

2 Weighted projective hypersurfaces

2.1 Generalities

In this subsection, we recall some results from the literature on hypersurfaces in weighted projective spaces (see, e.g., [13, 22, 41]). Recall that $\mathbb{P} := \mathbb{P}(a_0, \ldots, a_n)$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ under the

$\mathbb{C}^*$-action given by $\lambda(x_0,\dots,x_n)=(\lambda^{a_0}x_0,\dots,\lambda^{a_n}x_n)$. We may always assume that $a_0\leqslant a_1\leqslant\cdots\leqslant a_n$. The affine piece $x_k\neq 0$ is the quotient of $\mathbb{C}^n$ with coordinates $(z_0,\dots,\widehat{z_k},\dots,z_n)$ by the action of $\mathbb{Z}/a_k\mathbb{Z}$ given on the coordinate $z_i=x_i/x_k^{(a_i/a_k)}$ by $\rho^{a_i}z_i$, where ρ is a primitive a_k-th root of unity. Observe that in the case $a_0=1$, the coordinates $z_j=x_j/x_0, j=1,\dots,n$ are actual coordinates on the affine set $x_0\neq 0$; there is no need to divide by a finite group action.

In general, $\mathbb{P}$ has cyclic quotient singularities of the transversal type $\frac{1}{h}(b_1,\dots,b_k)$, i.e., these are the image of $0\times\mathbb{C}^\ell\subset\mathbb{C}^k\times\mathbb{C}^\ell$, where $\mathbb{Z}/h\mathbb{Z}$ acts on $\mathbb{C}^k$ by $\zeta(x_1,\dots,x_k)=(\zeta^{b_1}x_1,\dots,\zeta^{b_k}x_k)$, and ζ is a primitive h-th root of unity. More precisely, the simplex $x_{j_1}=\cdots=x_{j_k}=0$ is singular if and only if the set of weights that result after discarding $a_{j_1},\dots,a_{j_k}$ are not co-prime, i.e., with gcd equal to $h_{j_1,\dots,j_k}$, and then transversal to the simplex, one has a singularity of the type

$$\frac{1}{h_{j_1,\dots,j_k}}(a_0,\dots,\widehat{a_{j_1}},\dots,\widehat{a_{j_k}},\dots,a_n).$$

So in the case where any n-tuple from the collection $\{a_0,\dots,a_n\}$ of weights is co-prime, the only possible singularities occur in codimension greater than or equal to 2. We call such weights ***well formed*** and in what follows we assume that this is the case.

A hypersurface $X=\{F=0\}$ in $\mathbb{P}$ is quasi-smooth if the corresponding variety $F=0$ in $\mathbb{C}^{n+1}$ is only singular at the origin. This implies that the possible singularities of quasi-smooth hypersurfaces come from the singularities of $\mathbb{P}$. Such a hypersurface has at most cyclic quotient singularities, i.e., it is a V-variety. A hypersurface of degree d in $\mathbb{P}$ is called ***well formed*** if its weights are well formed and if moreover $h_{ij}=\gcd(a_i,a_j)$ divides d for $0\leqslant i<j\leqslant n$. All our examples are well formed hypersurfaces. To test if $F=0$ is quasi-smooth, one uses the Jacobian criterion: the only solution to $\nabla F(\underline{x})=0$ is $\underline{x}=(x_0,\dots,x_n)=0$.

We quote a result implied by Fletcher's statement [22, Theorem 8.1]. We use it to exclude types that do not give a quasi-smooth weighted hypersurface.

Lemma 2.1. *Given a weighted projective space $\mathbb{P}=\mathbb{P}(a_0,\dots,a_n)$ and an integer d with $d>\max a_j$, let A be the set of weights dividing d and B be the remaining set of weights. Suppose that a quasi-smooth degree d hypersurface in $\mathbb{P}$ exists. Then the set of weights $\{a_0,\dots,a_n\}$ satisfies the following conditions*:

(1) *For each $\beta\in B$, there is a weight γ and some positive integer r such that $d=r\beta+\gamma$.*

(2) *No weight appears more than once as such a remainder γ.*

Example 2.2. We give two examples of surfaces having the type $(d;(1,a_1,a_2,a_3))$ and which will be called ***basic degree d quasi-smooth hypersurfaces*** in $\mathbb{P}(1,a_1,a_2,a_3)$.

(1) Assume that $A=\{1,a_1,a_2\}$ and $d=q_3a_3+a_j$, $a_j\in A$. Then $P=x_0^d+x_1^{d/a_1}+x_2^{d/a_2}+x_{a_j}x_3^{q_3}$ is quasi-smooth as follows from the Jacobian criterion.

(2) Assume that $A=\{1,a_1\}$ and $d=q_2a_2+a_j$, $a_j\in A$, $d=q_3a_3+a_k$, $a_k\in A$ but $k\neq j$. Then $P=x_0^d+x_1^{d/a_1}+x_jx_2^{q_2}+x_kx_3^{q_3}$ is quasi-smooth.

In what follows, especially in Section 4, the following remark will be used tacitly.

Remark 2.3. By [13, Theorem 3.3.4], if X is quasi-smooth of amplitude $\alpha(X)$, then the sheaf $\mathcal{O}_X(\alpha(X))$ is the canonical sheaf of X. However, contrary to what happens in ordinary projective space, if $\alpha\geqslant 1$ the minimal model of X need not be of the general type, as we see in Proposition 4.5.

2.2 On the Hodge decomposition of weighted hypersurfaces

A result by Steenbrink [41] states that the Hodge decomposition for quasi-smooth hypersurfaces X of degree d in weighted projective space $\mathbb{P}(a_0,\dots,a_n)$ can be stated in terms of the Jacobian ring R_F using Griffiths' residue calculus, as in the non-weighted case. The Hodge number $h^{n,0}(X)$ equals $\dim H^0(X,\omega_X)$, where ω_X is the canonical sheaf. This Hodge number can be calculated from the amplitude $\alpha(X)$ since by [13, Theorem 3.3.4], $h^{n,0}(X)=\dim H^0(X,\mathcal{O}(\alpha(X)))$ in the case where X is quasi-smooth.

Since quasi-smooth hypersurfaces in the weighted projective space are V-manifolds, as in the case of the ordinary projective space, we have the following lemma.

Lemma 2.4 (See [48, § 1]). *The subspace* $\mathsf{Def}_{\rm proj}$ *of the Kuranishi space of deformations of* X *within* $\mathbb{P}(a_0,\ldots,a_n)$ *is smooth with tangent space canonically isomorphic to* R_F^d. *The Kuranishi family restricted to* $\mathsf{Def}_{\rm proj}$ *is called the* **Kuranishi family of the type** $([d],(a_0,\ldots,a_n))$.

2.3 The four types of surfaces

We now give the classification of the surfaces we are interested in.

Proposition 2.5. *The only quasi-smooth hypersurfaces* X *of the form* $(d,[1,2,a,b])$ *with* a,b *co-prime odd integers and such that* $d=a+b+4$ *have the following characteristics*:

(1) $(14,[1,2,3,7])$, *basic quasi-smooth example* $x_0^{14}+x_1^7+x_2^4x_1+x_3^2$;

(2) $(12,[1,2,3,5])$, *basic quasi-smooth example* $x_0^{12}+x_1^6+x_2^4+x_1x_3^2$;

(3) $(16,[1,2,5,7])$, *basic quasi-smooth example* $x_0^{16}+x_1^8+x_0x_2^3+x_1x_3^2$;

(4) $(22,[1,2,7,11])$, *basic quasi-smooth example* $x_0^{22}+x_1^{11}+x_0x_2^3+x_3^2$.

Proof. We divide the possible cases according to the partition $\{1,2,a,b\}=A\sqcup B$ of Lemma 2.1. We do not assume that $a<b$ since their roles are symmetric. Indeed, the constraints are $d=a+b+4$, a and b odd, and $\gcd(a,b)=1$. If $a\in A$, according to whether $b\notin A$ (Cases (A) and (B)) or $b\in A$ (Case (C)), we have

$$d=ka=\begin{cases}\text{(A) either } rb+2,\\ \text{(B) or } rb+1,\\ \text{(C) or } rb.\end{cases}$$

Interchanging the roles of a and b, we see that this also covers the case $b\in A$ and so there remains the case $2\in A$, i.e., $2|d$, and then

$$\text{(D) } d=2k=ra+2=sb+1 \text{ or } d=2k=sa+1=rb+2.$$

We first assume that (A) holds. Since $d=ka=a+b+4$, we have

$$b=(k-1)a-4. \tag{2.1}$$

Since a and b are odd, (2.1) implies that k and hence d is even and (A) implies that also r is even. Put $k=2\kappa$ and $r=2\rho$. We rewrite (A) as $2\rho\kappa a-(a+4)\rho-a\kappa+1=0$ and hence

$$(2\rho-1)[a(2\kappa-1)-4]=a+2$$

and we can test low values of a. For $a=3$, this reads $(2\rho-1)(6\kappa-7)=5$ with the only solution $(\rho,\kappa)=(1,2)$ which yields $b=5$ and $d=12$. For $a=5$, one gets $(2\rho-1)(10\kappa-9)=7$ with the solution $(4,1)$ which yields $b=1$ which can be discarded. For $a=7$, one gets $(2\rho-1)(14\kappa-11)=9$ with the solution $(2,1)$ which yields $b=3$ and $d=14$. For $a=9$, one gets $(2\rho-1)(18\kappa-12)=11$ which has no solution. There are no other solutions. To see this, write

$$(\rho(2\kappa-1)-\kappa)a=4\rho-1. \tag{2.2}$$

For $\rho=1$, the equation (2.2) gives $(2\kappa-2)a=6$ which gives back the solution $a=3$, $b=5$ and so we may assume $\rho\geqslant 2$. We may also use that $a\geqslant 11$. By (2.2), this gives $(2\rho\kappa-\rho-\kappa)\cdot 11\leqslant 4\rho-1$, or, multiplying by 2,

$$(2\rho-1)(22\kappa-15)\leqslant 13$$

and so $13\geqslant(2\rho-1)(22\kappa-15)\geqslant 66\kappa-45$ which has no positive integer solution.

Case (B) has the solution $(22,[1,2,7,11])$ with $a=11$, $k=2$, $b=7$ and $r=3$. This follows as in Case (A). Here, we set $k=2\kappa$, $r=2\rho+1$ and obtain

$$2\rho[a(2\kappa-1)-4]=a+3.$$

As before, the smallest value of a with a solution is $a = 11$. There are no solutions with $a \geqslant 13$. To see this, we use the analog of (2.2) which reads $(4\rho\kappa - 2\rho - 1)a = 8\rho - 3$, and from $a \geqslant 13$, we derive

$$\rho(52\kappa - 34) \leqslant 10,$$

which is not possible for positive integers (ρ, κ).

Case (C) implies $ka = rb = r[ka - 4]$, which leads to $(r-1)ka = 4r$ and since k and r must be even (recall that $d = a + b + 4 = ra = kb$ with a and b odd), which leads to a contradiction.

In Case (D), eliminating a and b and substituting in $2k = d = a + b + 4$, we find

$$(2k-4)rs - 2k(r+s) + (2s+r) = 0$$

with r even and s odd which can be rewritten as

$$[(k-2)(r-1) - 1][2(k-2)(s-1) - 3] = (k-1)(2k-1).$$

Since $r \geqslant 2$ and $s \geqslant 3$, $k = 8$ is the smallest value of k with the solution $(r, s) = (2, 3)$, which leads to $(16, [1, 2, 5, 7])$. Since $r \geqslant 3$ and $s \geqslant 2$, we get the inequality $(4k - 11)(k - 3) \leqslant (k - 1)(2k - 1)$ which gives $(k - 2)(k - 8) \leqslant 0$ which has no positive integer solutions > 8. $\square$

3 The universal family: Normal forms, moduli, and global monodromy

The collection of degree d weighted hypersurfaces in $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$ form an ordinary projective space $\mathbb{P}^N$ in a natural way by considering the $N + 1$ coefficients in front of all the possible monomials. The quasi-smooth hypersurfaces form a Zariski-open subset $U_{a_0,\dots,a_n}$ of this projective space. The tautological family $\mathscr{F}_{a_0,\dots,a_n}$ of degree d quasi-smooth hypersurfaces over $U_{a_0,\dots,a_n}$ is called the corresponding ***universal family***. The group $\widetilde{G}$ of substitutions $x_j \mapsto p_j(x_0, \dots, x_j)$, $j = 0, \dots, n$, where p_j is weighted homogeneous of degree a_j, acts on $\mathbb{P}(a_0, \dots, a_n)$. Since $\lambda \in \mathbb{C}^*$ sending $(x_0, \dots, x_n)$ to $(\lambda^{a_0} x_0, \lambda^{a_1} x_1, \dots, \lambda^{a_n} x_n)$ multiplies each weighted homogeneous polynomial F of degree d with λ^d, the group $G = \widetilde{G}/\mathbb{C}^*$ acts effectively on hypersurfaces. The embedding of the subgroup $\mathbb{C}^* \subset \widetilde{G}$ is due to the weights and so will be referred to as the 1-***subtorus for the weights***.

3.1 Normal forms and moduli

We show how to obtain a quasi-projective moduli space as a certain GIT-quotient of $U_{1,2,a,b}$. The draw-back is that the group G of weighted projective substitutions is not in general reductive. We can circumvent this in our case by giving normal forms for the equation of quasi-smooth hypersurfaces in the universal family. On hypersurfaces with their equations in the normal form, a reductive subgroup T of G (in fact a 3-dimensional algebraic torus) acts effectively in such a way that hypersurfaces in the normal form are in the same T-orbit if and only they are in the same G-orbit.

Proposition 3.1. *With $(a, b) \in \mathbb{Z}^2$ as in Proposition 2.5, let $\mathbb{C}[x_0, x_1, x_2, x_3]$ be the homogeneous coordinate ring of $\mathbb{P}(1, 2, a, b)$. Assume that $F \in \mathbb{C}[x_0, x_1, x_2, x_3]$ defines a quasi-smooth surface $(F = 0)$ of degree d in $\mathbb{P}(1, 2, a, b)$ which does not pass through $(0 : 1 : 0 : 0)$ (i.e., the coefficient of $x_1^{d/2}$ is non-zero). For the type 4, i.e., for $(a, b) = (7, 11)$, assume in addition that the coefficient of $x_1^4 x_2^2$ is non-zero.*

Then, there exist polynomials G_j, homogeneous of degree j in the degree 2 monomials x_0^2 and x_1 such that via the automorphism group of $\mathbb{P}(1, 2, a, b)$ the form F can be put in the following normal form:

(1) *In the case $(a, b, d) = (3, 7, 14)$, we have*

$$F = x_1 x_2^4 + G_0 x_0^5 x_2^3 + G_4(x_0^2, x_1) x_2^2 + x_0\, G_5(x_0^2, x_1) x_2 + G_7(x_0^2, x_1) - x_3^2.$$

(2) *In the case $(a, b, d) = (3, 5, 12)$,*

$$F = x_1 x_3^2 + x_0 x_3\, G_2(x_0^3, x_2) + G_0 x_2^4 + G_3(x_0^2, x_1) x_2^2 + x_0\, G_4(x_0^2, x_1) x_2 + G_6(x_0^2, x_1), \quad G_0 \neq 0.$$

(3) *In the case* $(a,b,d)=(5,7,16)$,

$$F = x_1x_3^2 + x_0^4G_1(x_0^5,x_2)x_3 + r_0x_0x_2^3 + G_0x_1^3x_2^2 + x_0\,G_5(x_0^2,x_1)x_2 + G_8(x_0^2,x_1),$$

where r_0 *is a non-zero constant.*

(4) *In the case* $(a,b,d)=(7,11,22)$,

$$F = x_0x_2^3 + G_0x_1^4x_2^2 + x_0x_2G_7(x_0^2,x_1) + G_{11}(x_0^2,x_1) - x_3^2, \quad G_0 \neq 0,$$

where the coefficient of x_0^{22} *in* G_{11} *is zero.*

In each case, the subgroup of the automorphism group T *of* $\mathbb{P}(1,2,a,b)$ *which preserves a normal form of the given type consists of transformations of the form* $x_j \mapsto c_jx_j$ *with* $c_j \in \mathbb{C}^*$, $j=0,1,2,3$ *modulo the* 1*-subtorus for the weights. More concretely, for types* 1 *and* 4*, this can be identified with the subgroup of* $(\mathbb{C}^*)^3$ *consisting of triples* (c_0,c_1,c_2) *for which* $c_1c_2^4=1$ *and* $c_0c_2^3=1$ *respectively, while for types* 2 *and* 3*, this is the subgroup of* $(\mathbb{C}^*)^4$ *consisting of quadruples* (c_0,c_1,c_2,c_3) *for which* $c_1c_3^2=1$.

The stabilizer under T *of a general such* F *in all the cases is the identity.*

The proof of this result is relegated to Appendix B. Note that the supplementary condition on $x_1^{d/2}$ which is not required in loc. cit. but will be used in Proposition 4.5 is stable under the group action.

Remark 3.2. (1) A type 2 surface is birational to a type 1 surface: multiply the normal form with x_1 and perform the change of variables $y_0=x_0$, $y_1=x_1$, $y_2=x_2$ and $y_3=x_1x_2$. This does not yield the normal form for the type 1, but it does after changing y_3 in $y_3+\frac{1}{2}y_0G_2(y_0^3,y_2)$. In the resulting normal form, the term with y_0^{14} is missing, showing that after the birational transformation the type 2 family gives the subfamily of the type 1 where the coefficient of x_0^{14} in the normal form vanishes.

(2) For the type 3, the coefficient of $x_1^3x_2^2$ in the normal form of Proposition 3.1 is non-zero. Since this condition is stable under the action of T, the moduli point of the basic example is on the boundary of $\mathscr{M}_{5,7}$. Likewise, the condition on the coefficient of $x_1^4x_2^2$ for the type 4 implies that the basic example cannot be transformed in the normal form and so the moduli point of the basic example is on the boundary of $\mathscr{M}_{7,11}$.

Corollary 3.3. (1) *In each of the above cases, the points in the Zariski-open subset* $U_{1,2,a,b}$ *of degree* $d=a+b+4$ *quasi-smooth hypersurfaces in* $\mathbb{P}(1,2,a,b)$ *are* G*-stable and* $\mathscr{M}_{a,b}=U_{1,2,a,b}//G$ *is a geometric quotient.*

(2) $\mathscr{M}_{a,b}$ *has dimension* 18, 17, 16 *and* 18 *for types* 1–4*, respectively.*

Proof. (1) Since for types 3 and 4, the basic examples are on the boundary of $\mathscr{M}_{a,b}$, we need to check quasi-smoothness for at least one surface whose moduli-point lies in the interior. This is done in Appendix B. The group T acts effectively on hypersurfaces defined by homogeneous forms in the coefficients of a weighted homogeneous polynomial of degree d. As in ordinary projective space (see [31, Proposition 4.2]), the locus of hypersurfaces that are not quasi-smooth defined in this way a "discriminant form", a T-invariant homogeneous polynomial in the coefficients. By construction, this polynomial is non-zero on $U_{1,2,a,b}$. By definition, all the points in $U_{1,2,a,b}$ are then semi-stable. Since T-orbits are closed in $U_{1,2,a,b}$ and (as in the projective setting) since a weighted hypersurface of types 1–4 by [16, Theorems 2.1 and 3.1] has a finite automorphism group, the points of $U_{1,2,a,b}$ are stable and the GIT-quotient $U_{1,2,a,b}//G$ is a geometric quotient.

(2) One counts the number coefficients of the monomials in the normal form which are not fixed, and subtracts 2 for types 1 and 4, and 3 for the other two types. For example, for the type 2, one finds $3+1+4+5+7-3=17$, and for the type 4, this becomes $1+8+11-2=18$. □

The universal family on $U_{1,2,a,b}$ does not descend to the geometric quotient $\mathscr{M}_{a,b}$. However, it does so over the open subset $U^0_{1,2,a,b} \subset U_{1,2,a,b}$ corresponding to surfaces having no automorphisms except the identity. This is a non-empty set since the stabilizer of T on the general F is the identity as asserted above. Introduce the following notion.

Definition 3.4. A ***modular family*** is an algebraic family over a smooth, quasi-projective base which is locally (in the analytic topology) isomorphic to the Kuranishi family of the type $([d],(1,2,a,b))$ (see Lemma 2.4).

The above discussion can thus be rephrased as follows.

Corollary 3.5. *The family over $U^0_{1,2,a,b}//G$ obtained from the universal family of degree $d = a + b + 4$ weighted hypersurfaces in $\mathbb{P}(1, 2, a, b)$ is a modular family.*

Remark 3.6. (1) The standard holomorphic 2-form ω_F on F given by the residue of $x_0\Omega_3/F$ is not fixed under the double plane involution $x_3 \mapsto -x_3$ for types 1 and 4. One can view $\mathscr{M}_{a,b}$ as the moduli space for the pair (F, ω_F). Alternatively, we could put a non-zero coefficient in front of x_3^2 and replace T with a larger group acting also on this coefficient. However, then the isotropy group at a generic double cover would always contain the double cover involution preventing the existence of a universal family over a Zariski-open subset of $U_{1,2,a,b}$.

(2) The above normal forms only generically give quasi-smooth hypersurfaces.

(3) As for ordinary projective spaces, the universal family of quasi-smooth hypersurfaces of given degree is flat over the base. This is because resolving the singularities of weighted projective space also resolves the singularities of the hypersurfaces. The resolved universal family being flat, also the universal family itself is flat.

3.2 Global monodromy of the universal families

Brief survey of singularity theory. The purpose of this subsection is to investigate certain 1-parameter degenerations X_t of quasi-smooth hypersurfaces in $U_{1,2,a,b}$ in the relation to the global monodromy of the universal family. So we want to find a disc

$$D = \{t \in \mathbb{C} \mid |t| < r\}$$

embedded in $\mathbb{P}^N$ such that (i) $D^* = D\backslash\{0\}$ belongs to $U_{1,2,a,b}$ with $X_t, t \in D^*$ a quasi-smooth hypersurface and (ii) X_0 (corresponding to $0 \in D$) has an isolated singularity at x_0 of some given type suited for the calculation of global monodromy groups. The main object associated with (X_0, x_0) is the ***Milnor fiber*** which is the intersection of X_t ($|t|$ small enough) with a small enough ball with center at x_0.

In what follows, we freely quote results from Ebeling's book [15]. To understand these results, we need to recall some more lattice theory. Recall that a lattice is a free group of finite rank equipped with a symmetric bilinear form which we denote by a dot. A root r in a lattice L is a vector with $r \cdot r = -2$ and it determines a reflection σ_r sending $x \in L$ to $x + (x \cdot r)r$. The group of isometries generated by a set of roots Δ is called its ***Weyl group*** $W(\Delta)$. Associated with Δ being its ***Dynkin diagram***, the vertices correspond to the roots and an edge is drawn between two edges corresponding to roots r and s if $r \cdot s = 1$. In the lattices we consider, only one other type of edge appears, i.e., if $r \cdot s = -2$, one draws two dashed edges between the corresponding vertices.

The middle homology group of the Milnor fiber equipped with the intersection pairing is the ***Milnor lattice*** . Its rank is the ***Milnor number*** $\mu(X_0, x_0)$. Turning once around $0 \in D$ induces the monodromy-operator T on $H^2(X_t, \mathbb{Z})$ as well as on the Milnor lattice. By [15, § 1.6], the Milnor lattice contains a sublattice, its ***vanishing lattice*** $L = L(X_0, x_0)$.

In order to determine the global monodromy group, the graph of a basic vanishing lattice, $\Delta_{\min}$ depicted in Figure 1 is of crucial importance. It intervenes in the notion of a complete vanishing lattice since the notion of vanishing lattice has a complete algebraic description:

Definition 3.7. (1) A ***vanishing lattice*** consists of a pair (L, Δ) of a (possibly degenerate) lattice L and a set of roots Δ spanning L and forming a single orbit under $W(\Delta)$.

(2) A vanishing lattice (L, Δ) ***contains the vanishing lattice*** (L', Δ') if L' is a primitive sublattice of L and $\Delta' \subset \Delta$.

(3) A vanishing lattice (L, Δ) is ***complete*** if it contains $(L_{\min}, \Delta_{\min})$.

The main interest in complete vanishing lattices is that their isometry group is almost equal to its Weyl group. Here, two notions intervene related to a lattice L: the ***spinor norm*** of an isometry of L and the ***discriminant group of*** L.

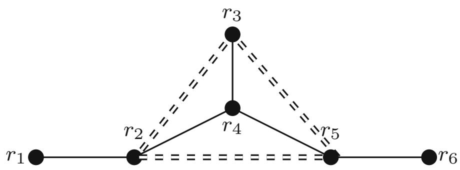

Figure 1 $(L_{\min}, \Delta_{\min})$

To define the former, recall that the Cartan-Dieudonné theorem states that all the isometries of a $\mathbb{Q}$-vector space V with a non-degenerate product are products of reflections, i.e.,

$$\sigma_x : V \to V, \quad \sigma_x(v) = v - [2(x.v)/(x.x)]v.$$

One defines the $\pm$-spinor norm of such a product of reflections as follows:

$$\mathsf{Nm}^{\varepsilon}_{\text{spin}}(\sigma_{x_1} \circ \cdots \circ \sigma_{x_r}) = \begin{cases} 1, & \text{if } \#\{j \in \{1, \ldots, r\} \mid \varepsilon q(x_j) < 0\} \text{ is even,} \\ -1, & \text{otherwise.} \end{cases}$$

The group generated by isometries γ with $\mathsf{Nm}^{\varepsilon}_{\text{spin}}(\gamma) = 1$ is denoted by $O^{\#\varepsilon}(L)$. Here, $\varepsilon = -1$ plays a central role.

The discriminant group makes only sense for non-degenerate lattices L, those for which the map

$$L \to L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$$

given by $x \mapsto (y \mapsto y{\cdot}x)$ is injective. Then the discriminant group by definition is the group $A(L) = L^*/L$.

We can now formulate the main technical result we are going to invoke.

Theorem 3.8 (See [15, Theorem 5.3.5])**.** *Let (L, Δ) be a complete vanishing lattice. Then $W(\Delta)$ is the subgroup $O^{\#-}(L)$ of the orthogonal group $O(L)$ of L consisting of isometries with the $(-)$-spinor norm $+1$ and inducing the identity on the discriminant group.*

Example 3.9. From [15, Proposition 5.3.5], one deduces that a root lattice with a Dynkin diagram of the type $T^1_{p,q,r}$ depicted in Figure 2 has a different root basis making it a complete vanishing lattice. Such Dynkin diagrams come up as vanishing lattices for the 14 exceptional unimodal families of Arnol'd [1]. In Table 1, we describe three of those which play a role in Subsection 3.3 below. The vanishing lattice $T^1_{p,q,r}$ is given in Figure 2 by means of a Dynkin diagram with μ vertices. The "modulus" a is any complex number. This lattice is non-degenerate.

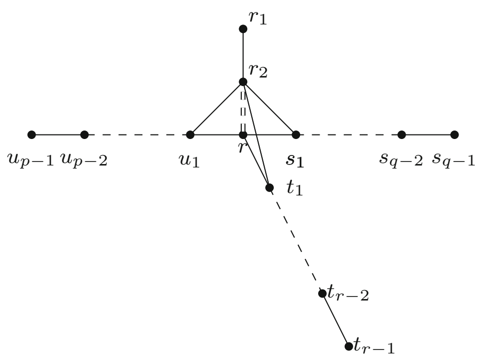

Figure 2 Vanishing lattice given by $T^1_{p,q,r}$ $(p \geqslant 2, q \geqslant 3, r \geqslant 7)$ is complete

Table 1 Three exceptional unimodal singularities

Notation	Normal form	Milnor number μ	Dynkin diagram
K_{12}	$x^3+y^7+z^2+axy^5$	12	$T^1_{2,3,7}$
K_{13}	$x^3+xy^5+z^2+ay^8$	13	$T^1_{2,3,8}$
K_{14}	$x^3+y^8+z^2+axy^6$	14	$T^1_{2,3,9}$

3.3 Applications to monodromy

The embedding of the Milnor fiber into X_t induces a lattice morphism $j_*:\Lambda(X_0,x_0)\to H_2(X_t,\mathbb{Z}) \simeq H^2(X_t,\mathbb{Z})$[3)], which is in general injective nor surjective. We consider the global monodromy of the universal families. Note that monodromy not only preserves the hyperplane class but also the singularities of the weighted projective space. In our case, the quasi-smooth members of the universal families $\mathscr{F}_{a,b}$ have singularities only at some of the isolated singular points of $\mathbb{P}(1,2,a,b)$ and we take the minimal resolution of their singularities. We see (Proposition 4.1) that in two cases, additional exceptional curve configurations are present which are also preserved by the monodromy. For a general member $\widetilde{X}$ of each of the resulting families of smooth (but not always minimal) surfaces, we show that all of the curves just mentioned generate the Picard lattice (Proposition 4.5), and so the transcendental lattice of $\widetilde{X}$ is left invariant.

The main result here is as follows.

Proposition 3.10. *Let $L=H^2(\widetilde{X},\mathbb{Z})$ be the middle cohomology group of the minimal resolution of singularities of a quasi-smooth member of $\mathscr{F}_{a,b}$, let $S\subset L$ be the Picard lattice of a general member and $T=S^{\perp}$ the transcendental lattice. Then the monodromy group of the universal family of such quasi-smooth hypersurfaces is the subgroup of $O^{\#-}(L)$ preserving T and inducing the identity on S.*

Proof. The proof is similar to the proof of Beauville given in [3, Section 2]. The main ingredients are the following:

- The monodromy representation of $\pi(U_{1,2,a,b})$ on L is the same as the representation induced by a Lefschetz pencil.
- The discriminant locus is connected implying that all the vanishing cycles are conjugate under monodromy and so these give a vanishing lattice.
- The vanishing cycles generate the orthogonal complement of S in L which is precisely T.
- There is a weighted degree $d=a+b+4$ hypersurface in $\mathbb{P}(1,2,a,b)$ with an exceptional unimodal isolated singularity from Arnol'd's list. Its vanishing lattice is non-degenerate and so embeds in T. Hence T is a complete vanishing lattice.

Since the first three assertions can be proven as in the ordinary hypersurface case, it suffices to exhibit suitable singularities in each of the four cases. We exhibit such a singularity at $(0,0,0)$ in the affine chart $x_0\neq 0$. We refer to Table 1 for the notation of these singularities.

For the type 1, the form $x_1^7+x_3^2+x_0^5x_2^3+x_0x_1^5x_2$ gives a K_{12}-singularity and so does $x_0x_1^3+x_0^8x_2^7+x_3^2+x_0x_1^5x_3$ for the type 4. For the type 2, one has a K_{13}-singularity given by $x_0x_1x_2^3+x_0^2x_1^5+x_0^2x_3^2+x_0x_1^4x_2$ and finally, $x_0x_2^3+x_1^8+x_0^2x_3^2+x_0x_1^6x_2$ gives a K_{14} for the type 3. □

4 Invariants and elliptic pencils on the four types of surfaces

4.1 Invariants

We start this section by comparing types 3 and 4 with two families from Reid's list of surfaces, which shows directly that these are birational incarnations of K3 surfaces.

For the type 3, we start multiplying the normal form from Proposition 3.1 by x_0^2 yielding

$$F=x_1(x_0x_3)^2+x_0^5G_1(x_0^5,x_2)x_0x_3+r_0(x_0x_2)^2+G_0x_1^3(x_0x_2)^3$$

3) The latter isomorphism is the Poincaré duality.

$$+ x_0^2 G_5(x_0^2, x_1) x_0 x_2 + x_0^2 G_8(x_0^2, x_1).$$

Therefore, we can change variables to $y_0 = x_0^2$, $y_1 = x_1$, $y_2 = x_0x_2$ and $y_3 = x_0x_3$, except for possibly the term

$$x_0^5 G_1(x_0^5, x_2) x_0 x_3 = x_0^5(Ax_0^5 + x_2) x_0 x_3 = (Ax_0^{10} + x_0^5 x_2) x_0 x_3,$$

which can also be rewritten in the variables x_0^2, x_0x_2 and x_0x_3. So after these substitutions, one obtains a surface of the type $(18, [2, 2, 6, 8])$, or, equivalently a surface of the type $(9, [1, 1, 3, 4])$.

For the type 4, first multiply all the monomials by x_0^2. In the new variables $y_0 = x_0^2, y_1 = x_1, y_2 = x_0x_2, y_3 = x_0x_3$ and as for the type 3, one sees that all the type $(22, [1, 2, 7, 11])$ surfaces are birational to surfaces of the type $(12, [1, 1, 4, 6])$.

Next, we give a table of the Hodge numbers and the number of projective moduli resulting from applying Steenbrink's approach outlined in Subsection 2.2 for the four types of surfaces we just found as well as for the two surfaces in the Reid incarnation which we denote by 3^* (resp. 4^*). We observe that the last column of the table corroborates the dimensions of the moduli spaces found in Corollary 3.3. For details, see also Appendix A.

In what follows, we focus on the incarnations 1–4, i.e., we consider the surfaces $X \subset \mathbb{P}(1, 2, a, b)$ having singularities at most at $\mathsf{P}_2 = (0 : 0 : 1 : 0)$ and $\mathsf{P}_3 = (0 : 0 : 0 : 1)$. The minimal resolution $\widetilde{X}$ of the singularities does not necessarily give a minimal surface as we see for types 3 and 4. We let X' be its minimal model[4].

We calculate the invariants for the four types 1–4, making use of the Hodge numbers in Table 2.

Proposition 4.1. *X' is a simply connected surface with invariants $e(X') = 24$, $K_{X'}^2 = 0$, $b_2(X') = 22$, Hodge numbers $(h^{2,0}, h^{1,1}, h^{0,2}) = (1, 20, 1)$ and signature $(3, 19)$. More precisely, we have the following:*

(1) *For the type 1, the general surface X has only one cyclic singularity at P_2 of the type $\frac{1}{3}(1, 1)$ which is resolved by a rational curve of self-intersection (-3).*

(2) *For the type 2, there is generically only one cyclic singularity at P_3 of the type $\frac{1}{5}(1, 3)$ which is resolved by a chain of two transversally intersecting rational curves of self-intersections -2 and -3, respectively.*

(3) *For the type 3, the surface X has generically two singularities: a $\frac{1}{5}(1, 1)$-singularity at P_2 resolved by a single rational curve with self-intersection -5, and a $\frac{1}{7}(1, 5)$-singularity at P_3 resolved by a chain of three rational curves with self-intersections -2, -2 and -3, respectively. The surface X' is obtained by blowing down an exceptional configuration in $\widetilde{X}$ consisting of a chain of two smooth rational curves of self-intersections -1 and -2.*

(4) *For the type 4, the surface X has generically one singularity at P_2 of the type $\frac{1}{7}(1, 2)$ resolved by a chain of two rational curves with self-intersections -2 and -4, respectively. The surface X' is obtained by blowing down an exceptional curve in $\widetilde{X}$.*

Proof. First of all, observe that X and hence $\widetilde{X}$ and X' are all simply connected since all the quasi-smooth hypersurfaces (of dimension > 1) in a weighted projective space are simply connected. In particular, X cannot be a rational or ruled surface, and X' is uniquely determined.

Table 2 Invariants for the classes of elliptic weighted surfaces

Symbol	$h^{2,0} = h^{0,2}$	$h^{1,1}_{\text{prim}}$	No. of projective moduli
1. $(14, [1, 2, 3, 7])$	1	18	18
2. $(12, [1, 2, 3, 5])$	1	17	17
3. $(16, [1, 2, 5, 7])$	1	17	16
3^*. $(9, [1, 1, 3, 4])$	1	16	16
4. $(22, [1, 2, 7, 11])$	1	18	18
4^*. $(12, [1, 1, 4, 6])$	1	18	18

[4] X being rational nor ruled, (see below) the minimal model is unique.

Secondly, all the surfaces have canonical sheaf $\mathcal{O}(1)$ with the 1-dimensional space of sections generated by x_0. Since p_g is a birational invariant, $p_g(X')=1$. For the Hodge numbers, it therefore suffices to show that $e(X')=24$ in all the cases, since then $h^{1,1}(X')=24-4=20$. We now treat the four cases separately.

Type 1. The normal form given in Proposition 3.1(1) shows that the surface always passes through the singular point $\mathsf{P}_2=(0:0:1:0)\in\mathbb{P}(1,2,3,7)$. We use the techniques as described in Section 2. The affine piece $x_2\neq 0$ containing P_2 is the $\mathbb{Z}/3\mathbb{Z}$-quotient of $\mathbb{C}^3$ with coordinates $\{z_0,z_1,z_3\}$ and the surface is the quotient of the smooth surface $g=0$ with $g=z_1+$ higher order terms. Since at the origin $\nabla g=(0,1,0)$, z_0 and z_3 are local coordinates and the $\mathbb{Z}/3\mathbb{Z}$-action is given by $(z_0,z_3)\mapsto(\rho z_0,\rho^7 z_3)=(\rho z_0,\rho z_3)$, ρ is a primitive root of unity. This gives a singularity of the type $\frac{1}{3}(1,1)$ which is resolved by a rational curve of self-intersection (-3). From Table 2, we see that $b_2(X)=2+1+h^{1,1}_{\mathrm{prim}}=21$ and hence $e(X)=23$. Since the singularity is resolved by one rational curve, $e(\widetilde{X})=23-1+2=24$, and so $K^2_{\widetilde{X}}=0$ by Noether's theorem.

For the type 2, this is similar, but now the surface always passes through $\mathsf{P}_3=(0:0:0:1)$. The affine piece $x_3\neq 0$ is the $\mathbb{Z}/5\mathbb{Z}$-quotient of $\mathbb{C}^3$ with coordinates $\{z_0,z_1,z_2\}$ and the surface is the quotient of the smooth surface $g=0$ with $g=z_1+$ higher order terms as we see from the normal form from Proposition 3.1(2). Thus the surface has a singularity of the type $\frac{1}{5}(1,3)$. It is resolved by a chain of two transversally intersecting rational curves of self-intersections -2 and -3, respectively. Table 2 now gives $e(X)=22$. The singularity is resolved by a chain of two rational curves and so $e(\widetilde{X})=22-1+3=24$ and then $K^2_{\widetilde{X}}=0$ by Noether's theorem.

Let us now investigate the type 3. Here, we have two singularities at P_2 and at P_3. As in the previous cases, we find that the former is of the type $\frac{1}{5}(1,1)$, resolved by a single (-5)-curve, while the latter is a $\frac{1}{7}(1,5)$-singularity resolved by a chain of three rational curves of self-intersections -2, -2 and -3, respectively. From Table 2, we find that $e(X)=22$ and so $e(\widetilde{X})=22-2+2+4=26$ implying that $\widetilde{X}$ becomes minimal after twice successively blowing down. The resulting surface X' then has $e=24$ and $K^2_{X'}=0$ as it should.

Finally, let us pass to the type 4. Here, there is one singularity at P_3 of the type $\frac{1}{7}(1,2)$ resolved by a chain of two rational curves with self-intersections -4 and -2. Using Table 2, we find $e(\widetilde{X})=e(X)-1+3=23-1+3=25$ and so $\widetilde{X}$ contains one exceptional curve. Blowing down gives X' with $e(X')=24$ and $K^2_{X'}=0$. □

Remark 4.2. We could also calculate $K^2_{\widetilde{X}}$ using Reid's calculus of discrepancies, i.e., using an expression of the form $K_{\widetilde{X}}=\sigma^*(\omega_X)+\Delta$, where $\sigma:\widetilde{X}\to X$ is the minimal resolution of X and Δ is a $\mathbb{Q}$-divisor with support on the exceptional divisors. For example, in the case (3), denote the exceptional chain at P_2 by E and at P_3 by F_1, F_2 and F_3. Then the discrepancy divisor is $\Delta=-\frac{3}{5}E-\frac{1}{7}(F_1+2F_2+3F_3)$ and $\Delta^2=\frac{78}{35}$. Then

$$K^2_{\widetilde{X}}=(\mathcal{O}(1)+\Delta)^2=\frac{8}{35}-\frac{78}{35}=-2.$$

4.2 Generalities on elliptic surfaces

An elliptic surface is a surface X admitting a holomorphic map $f:X\to C$, where C is a smooth curve and the general fiber of f is a smooth genus 1 curve. Such a fibration f is called a ***genus 1 fibration***[5)]. We assume that X does not contain (-1)-curves as a component of a fiber of f.

The possible singular fibers of an elliptic fibration have been enumerated by Kodaira (see e.g. [2, Chapter V, § 7]). These are the non-multiple fibers of types I_b, $b\geqslant 1$, II, III, IV, $I_b^*, b\geqslant 0$, II^*, III^* and IV^*, where the irreducible fibers are I_1 with one ordinary node and II with one cusp. A type III fiber consists of two -2 curves touching each other in one point, explaining the Euler number $2\times1+1=3$. The multiple fibers are multiples of a smooth fiber or of a singular fiber of the type I_b, $b\geqslant 1$. For the purpose of this article, in Table 3 we also give the type of lattice that occurs after omitting one irreducible component of multiplicity 1.

5) Note that a scheme-theoretic fiber $f^{-1}s$, $s\in S$ of f may be multiple, i.e., $f^{-1}s=mF_0$, F_0 reduced.

Table 3 Non-multiple singular fibers of an elliptic fibration

Kodaira's notation	Lattice component	Euler number
$I_b, b \geqslant 2$	$A_{b-1}(-1)$	b
I_1		1
II		2
III	$A_1(-1)$	3
IV	$A_2(-1)$	4
I_b^*	$D_{4+b}(-1)$	$b+6$
II^*	$E_8(-1)$	10
III^*	$E_7(-1)$	9
IV^*	$E_6(-1)$	8

The Kodaira dimension $\kappa(X)$ of an elliptic surface can be equal to $-\infty$, 0, or 1. This can be determined from the plurigenera $P_m(X) = \dim H^0(X, K_X^{\otimes m})$, $m \geqslant 1$. For example, $\kappa(X) = 1$ if at least one plurigenus is $\geqslant 2$. More generally, one applies the canonical bundle formula.

Proposition 4.3 (See [2, Chapter V, § 12])**.** *Let $f : X \to C$ be a genus 1 fibration and $g = \mathrm{genus}(C)$. Assume that $\{m_i F_i \mid i \in I\}$ is the set of multiple fibers (I is finite but possibly empty). There is a divisor D on C of degree $d := \chi(\mathcal{O}_X) + 2g - 2$ such that the canonical divisor K_X of X is given by*

$$K_X = f^* D + \sum_{i \in I} (m_i - 1) F_i.$$

With $\delta := d + \sum_{i \in I} (1 - m_i^{-1})$, one has

$$\begin{cases} \delta < 0 \Leftrightarrow \kappa(X) = -\infty, \\ \delta = 0 \Leftrightarrow \kappa(X) = 0, \\ \delta > 0 \Leftrightarrow \kappa(X) = 1. \end{cases}$$

Corollary 4.4. *A genus 1 fibration $X \to \mathbb{P}^1$ (on a minimal surface X) with $p_g = 1$ and $q = 0$ and at least one multiple fiber has Kodaira dimension 1.*

4.3 The elliptic fibrations on the four classes of surfaces

We first make the following observation. From the invariants of X' given in Proposition 4.1 coupled with the classification of algebraic surfaces [2], we infer that X' either is a (minimal) K3 surface or a (minimal) properly elliptic surface. In both cases, the canonical divisor $K_{X'}$ has self-intersection 0. We show below that X has a pencil of genus 1 curves. On X', the resulting pencil $|F'|$ then necessarily is fixed point free since $F' \cdot F' = 0$ (by the genus formula) and hence gives a holomorphic map $X' \to \mathbb{P}^1$.

Proposition 4.5. *The rational map on X given by $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0^2 : x_1) \in \mathbb{P}_1$ induces an elliptic fibration $\pi : X' \to \mathbb{P}^1$. For types 1 and 2, the surface $X' = \widetilde{X}$ has the Kodaira dimension 1, and for types 3 and 4, the minimal surface X' is a K3 surface, a surface of Kodaira dimension 0.*

Furthermore, for types 1 and 2, resolving the orbifold singularity yields a rational curve of selfintersection -3 which is a bisection for the elliptic fibration. For types 3 and 4, on the minimal model X', there is a rational curve with selfintersection -2 originating from the orbifold singularities which is a section for the elliptic fibration.

The generic fiber type (i.e., the number and type of singular fibers) has been summarized in Table 4.

Proof. **Step 1.** The general fiber of π in all the cases is a smooth genus 1 curve.

The fiber of π can be viewed as a curve C of degree d in $\mathbb{P}(1, a, b) \subset \mathbb{P}(1, 2, a, b)$ passing through the singular points of $\mathbb{P}(1, a, b)$ which are the same as those of X. Its equation is obtained by eliminating x_1 from the equation of the surface. Note that its amplitude is $a + b + 4 - a - b - 1 = 3$ so that $\omega_C = \mathcal{O}_C(3)$ which has two sections x_0^3 and x_2 for types 1 and 2, and one section x_0^3 for types 3 and 4.

Table 4 The generic fiber types

Cases	$\pi^{-1}(0:1)$ or $\lambda=\infty$	$\pi^{-1}(1:0)$ or $\lambda=0$	Remaining singular fibers
$(14,[1,2,3,7])$	$2I_0$	I_1	$23\times I_1$
$(12,[1,2,3,5])$	$2I_0$	I_2	$22\times I_1$
$(16,[1,2,5,7])$	II	I_3	$19\times I_1$
$(22,[1,2,7,11])$	I_1	I_0	$23\times I_1$

For types 1 and 2, the curve C has one ordinary double point at the unique singularity of X. Resolving the singularity of X separates the branches on $\widetilde{X}$ so that the resulting curve is smooth and has genus 1. For types 3 and 4, the curve C is already a smooth genus 1 curve.

Step 2. Determine the Kodaira dimensions.

We already saw in Subsection 4.1 that quasi-smooth surfaces of types 3 and 4 are K3 surfaces. We next show that for types 1 and 2, the surface has Kodaira dimension 1. Since the pencil is given by $(x_0^2:x_1)$, there is a double curve over $(0:1)$. The weighted plane $x_0=0$ $(=\mathbb{P}(2,a,b))$ cuts the surface in this curve in which it has amplitude 2. It is a quasi-smooth curve on the basic surface (and hence on the general quasi-smooth surface) and passes through the unique orbifold point of X. So on $\widetilde{X}$, this curve is a smooth elliptic curve, the reduction of a double fiber of the type $2I_0$. The surface has a positive Kodaira dimension in both cases since $P_2(\widetilde{X})=2$ (note that $2K_{\widetilde{X}}$ is a fiber of the pencil which moves in a linear system of projective dimension 1).

Step 3. Determine the fiber types.

To verify the fiber types of the following special surfaces we used SageMath to show quasi-smoothness and to calculate some discriminants; quasi-smoothness has also been verified manually[6)].

Type 1. It suffices to establish this for one example, for which we take the quasi-smooth surface $x_0^{14}+x_1^7+x_2^4x_1+x_3^2+x_0^{11}x_2+x_0^5x_2^3=0$. Away from the the plane $x_0=0$, we may assume $x_0=1$ and $x_1=\lambda$ which gives the inhomogeneous equation $\lambda z_2^4+z_2^3+z_2+(1+\lambda^7)+z_3^2=0$. This equation describes a varying double cover of $\mathbb{P}^1$ branched in 4 points. Its singular members are found from the discriminant of the left-hand side with respect to z_2 which is the degree 24 polynomial $-256\lambda^{24}+768\lambda^{17}-192\lambda^{16}-27\lambda^{14}-768\lambda^{10}+384\lambda^9+6\lambda^8+54\lambda^7+256\lambda^3-219\lambda^2-6\lambda-31$ with non-zero discriminant and so there are 24 singular fibers, necessarily of the type I_1, as claimed.

Type 2. The resolution of the unique orbifold point produces a -3 curve (which is a bisection) and a -2 curve which must be part of the reducible fiber over $(1:0)$. To find the generic fiber type, consider the special quasi-smooth surface

$$x_3^2x_1+2x_3x_0(x_2^2+x_0^6)+x_2^4+x_1^3x_2^2+x_0x_1^4x_2+x_1^6=0.$$

As before, set $x_0=1$, $x_1=\lambda$, $x_j=z_j$, $j=2,3$. The elliptic fibration is given by

$$-w^2=(\lambda-1)z_2^4+(\lambda^4-2)z_2^2+\lambda^5z_2+(\lambda^6-1),$$

where $w=\lambda z_3+2(z_2^2+1)$. One can check that in this chart the surface is smooth. The verification on the remaining points of the surface is easy using that for these $x_0=0$.

The discriminant of the left-hand side of the above equation with respect to z_2 is the degree 24 polynomial

$$\begin{aligned}\lambda^2(&144\lambda^{22}-388\lambda^{21}+329\lambda^{20}-58\lambda^{19}+357\lambda^{18}-1160\lambda^{17}+1064\lambda^{16}-816\lambda^{15}\\&+1536\lambda^{14}-1168\lambda^{13}+160\lambda^{12}-896\lambda^{11}+1696\lambda^{10}-1056\lambda^9\\&+896\lambda^8-1408\lambda^7+768\lambda^6+512\lambda^4-512\lambda^3-256\lambda+256),\end{aligned}$$

whose second factor is without multiple factors since the discriminant is a (huge) non-zero integer, which shows the claim.

6) See Appendix A for the SAGE code we used for checking quasi-smoothness.

Type 3. Consider the special example of a quasi-smooth surface given by

$$x_1A + x_0B = 0, \quad A = x_1^7 - x_3^2, \quad B = x_0^8x_3 + x_0^3x_2x_3 + (x_0^{15} + x_0^{10}x_2 + x_0^5x_2^2 + x_2^3)$$

and the genus 1 fibration given by the rational map π. The line P_{23} given by $x_0 = x_1 = 0$ lies on the surface and so is the indeterminacy locus of this map. It contains the two singular points P_2 and P_3. The plane $x_0 = 0$ contains this line as well as the curve C_A given by $A = x_0 = 0$; the plane $x_1 = 0$ also contains the line as well as the curve C_B given by $B = x_1 = 0$. This curve is rational and has a singularity at $(0 : 0 : 1) \in \mathbb{P}(1, 5, 7)$, corresponding to the singularity P_3 on the surface X. On X', this gives rise to an I_3-type fiber at $\lambda = 0$. In fact, on X', the proper transform of the resolution of P_3 consists of three -2-curves, two of which together with the proper transform of C_B yields the I_3-configuration and the remaining (-2)-curve (which comes from a (-3)-curve on $\widetilde{X}$) is a section.

Similarly, one shows that $C_A \subset \mathbb{P}(2, 5, 7)$ is a smooth rational curve passing through $(0 : 1 : 0)$ corresponding to $\mathsf{P}_2 \in X$. On $\widetilde{X}$, this becomes a (-2)-curve F meeting the total transform E of the line L_{01} transversally. One has $E \cdot E = -1$ and $K_{\widetilde{X}} = 2E + F$ so that $c_1^2(\widetilde{X}) = -4 + 4 - 2 = -2$ in agreement with $c_2(\widetilde{X}) = 26$. On X', this gives a type *II*-fiber at $\lambda = 0$ as explained in Figure 3. Note that the original fibration $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0^2 : x_1)$ has a double fiber $x_0^2 = 0$ but it becomes absorbed as a multiplicity 2 component in a fiber which together with another (-1)-curve contracts to the cusp at $\lambda = \infty$.

In the chart $x_0 = 1$, the elliptic fibration is given by $x_1 = \lambda$, which gives the family

$$\lambda^8 - \lambda x_3^2 + x_3 + x_2x_3 + (1 + x_2 + x_2^2 + x_2^3) = 0.$$

Multiplying by λ^3 and making a change of variables $x_2\lambda = x, x_3\lambda^2 = y$, we have

$$y^2 - xy - \lambda y = x^3 + \lambda x^2 = \lambda^2 x + \lambda^3 + \lambda^{11}.$$

The discriminant of this elliptic curve equals

$$t^3\left(- 27t^{19} - 76t^{11} + 30t^{10} - 3t^9 + \frac{1}{16}t^8 - 44t^3 - 28t^2 - 5t + \frac{1}{8}\right),$$

where the degree 19 polynomial in t can be shown to have a non-zero discriminant. This shows that away from $\lambda = 0$ and $\lambda = \infty$, there are 19 irreducible type I_1-fibers as claimed.

Type 4. The plane $x_0 = 0$ intersects the general surface in a rational curve which on the desingularization $\widetilde{X}$ becomes a -1-curve (with multiplicity 2) intersecting the -4-curve in two points and thus on X' this becomes an I_1-type fiber. On a general quasi-smooth type 4 surface, the elliptic fibration has 23 further I_1-type fibers as one sees for example by computing the discriminant with respect to z_2 of the left-hand side of the expression $z_2^3 + \lambda^4 z_2^2 + (\lambda^7 + \lambda^3 + 1)z_2 + \lambda^{11} = z_3^2$, which represents the elliptic fibration for the quasi-smooth surface $x_0x_2^3 + x_1^4x_2^2 + (x_0x_1^7 + x_0^9x_1^3 + x_0^{15})x_2 + x_1^{11} = x_3^2$. This discriminant is the degree 23 polynomial

$$4\lambda^{23} - 8\lambda^{22} - 4\lambda^{21} + 20\lambda^{18} - 12\lambda^{17} + 20\lambda^{15} - 11\lambda^{14} - 12\lambda^{13}$$
$$+ 2\lambda^{11} - 24\lambda^{10} - 4\lambda^9 + a^8 - 12\lambda^7 - 12\lambda^6 - 12\lambda^3 - 4,$$

and it has no double roots which shows that there are indeed 23 type I_1 fibers away from $(0 : 1)$. Moreover, substituting $x_1 = 0$ shows that the fiber at $(1 : 0)$ is smooth elliptic, confirming that $\lambda = 0$ is not a root of the discriminant. □

Remark 4.6. If we check what happens under the birational transformation given in Remark 3.2(1) which transform a type 2 surface in a type 1 surface, only the fibers over $t = 0$ are affected: for the type 2 surface, we have a type *II*-fiber and the bisection meets each component in a single point, while for the type 1 surface the component of the type *II*-fiber coming from the quotient singularity contracts, giving a fiber of the type I_1 whose singularity lies on the bisection.

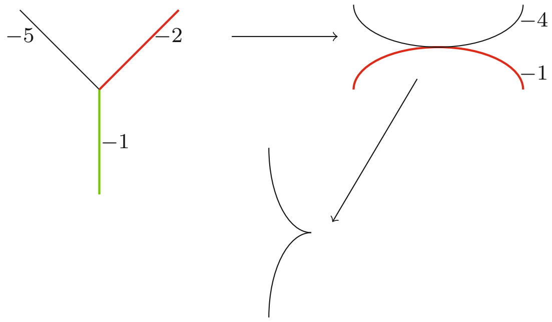

Figure 3 (Color online) Creating a cuspidal fiber

The fiber structure of an elliptic pencil allows us to calculate the so-called ***trivial Picard lattice***, i.e., the lattice spanned by the fibers and one (multi)section. There might be more (multi)sections, enlarging the Picard lattice. This is however not the case, as we show now. Note that the argument is different for all the types 1–4.

Corollary 4.7. *The Picard lattice* $\mathrm{Pic}(X')$ *of the minimal model* X' *of the generic member of the four families coincides with the trivial lattice. In fact, one has*

(1) *for the type* 1, $\mathrm{Pic}(X') \simeq \langle 1\rangle \oplus \langle -1\rangle$;

(2) *for the type* 2, $\mathrm{Pic}(X') \simeq \langle 1\rangle \oplus \langle -1\rangle \oplus \langle -2\rangle$;

(3) *for the type* 3, $\mathrm{Pic}(X') \simeq U \oplus A_2(-1)$;

(4) *for the type* 4, $\mathrm{Pic}(X') \simeq U$.

Proof. For the type 1, the Picard lattice contains a half fiber F_0 (the canonical curve) and the bisection E with $E.E = -3$ coming from blowing up the singularity. Denote their classes by f and s. Then $f.s = 1$, $f.f = 0$ and $s.s = -3$. Passing to the classes $s + 2f$ and $s + f$, one finds that the $\mathbb{Z}$-span of the classes gives a lattice isometric to $S := \langle 1\rangle \oplus \langle -1\rangle$. This lattice is primitive.

For the type 2, the Picard lattice contains F_0, the bisection given by the exceptional curve E with $E.E = -3$, and a reducible fiber $G + G'$ of the type I_2. Denote their classes by f, s, g and g'. Passing to the classes $s + 2f$, $s + f$ and $-f + g$, we see that the $\mathbb{Z}$-span of the classes gives a lattice isometric to $\langle 1\rangle \oplus \langle -1\rangle \oplus \langle -2\rangle$.

Type 3 concerns an elliptic K3 surface with a section and generically one reducible fiber of the type I_3. Hence the trivial Picard lattice is isometric to $U \oplus A_2(-1)$.

Type 4 concerns an elliptic K3 surface with a section and generically no reducible fibers so that the trivial Picard lattice is isometric to U.

To show that generically the Picard lattice equals the trivial lattice, in types 1 and 2, we bound the Picard number. We claim that for the type 1, the Picard group does not have rank $\geqslant 3$. For this, it suffices to find a family admitting a non-symplectic automorphism g of order 11 having at least 1 modulus and for which the period map is non-constant. Indeed, by[7] [21, Corollary 1.14 in Chapter 15], the transcendental lattice of a surface in that family has rank divisible by $10 = \phi(11)$ and so is either 10 or 20. If it were 10 for all the surfaces in the family, the Picard number would be constant and so also the period map would be constant.

The required family is given by $F_a = x_1x_2^4 + ax_1^7 + x_0^{11}x_2 + x_1^4x_2^2 - x_3^2$, $a \in \mathbb{C}$. Its members are generally quasi-smooth, the tangent to the kernel of the period map at each quasi-smooth member is in the (fixed) direction $x_0^9x_1x_2$ which can be shown to be independent of the deformation direction x_1^7 of the family. Finally, F_a admits the automorphism $g(x_0, x_1, x_2, x_3) = (\rho_{11}x_0, x_1, x_2, x_3, x_4)$ which sends the form ω which is the residue along $F_a = 0$ of the form Ω_3/F_a to $\rho_{11}\omega$ and so the action is indeed non-symplectic.

The proof that for the type 2, the Picard group has rank 3 has been relegated to Appendix C. The proof has an arithmetic flavor and uses reduction mod 2 and mod 3.[8]

[7] The proof only uses the K3-type intersection lattice of the surface.

[8] After Appendix C was ready, we realized the existence of the birational transformation of Remark 3.2 which shows that this gives another proof (see Remark 4.6).

Types 3 and 4 concern families of K3 surfaces for which the number of moduli equals the dimension of the period domain. Indeed, for the type 3, one has 16 moduli, and for the type 4, there are 18 moduli (see Table 2). We just calculate the trivial Picard lattice which has rank 4 (resp. 2), and so the dimension of the period domain associated with the transcendental lattice is less than or equal to $22-4-2=16$, (resp. less than or equal to $22-2-2=18$). We prove that the period map for a modular family in both cases generally is an immersion (see Proposition 5.2) and so equality holds which implies that the Picard lattices are as stated. □

Remark 4.8. (1) The elliptic fibrations on X' of types 1 and 2 having a single smooth double fiber $2F_0$ admit an inverse logarithmic transformation (see [2, § V.13]) which leaves the fibration outside the double fiber intact but replaces $2F_0$ by a smooth fiber which is no longer a double fiber. The resulting surface thus is a K3-surface X''. Note that this procedure changes the Kodaira-dimension! Since one can perform a logarithmic transformation on any smooth fiber of the resulting fibration on X'', one can in this way construct elliptic surfaces, i.e., Y_t, $t \in \mathbb{P} \setminus \{(0:1)\}$, that are not obtained from surfaces that like X are weighted hypersurfaces of degree 14 in $\mathbb{P}(1,2,3,7)$. The Picard lattice and the transcendental lattices are the same as for X, the surfaces Y_t are projective and their period point belongs to the same period domain.

(2) The statements for types 3 and 4 can be used to confirm the specific calculations in Belcastro's thesis [4] for the numbers 22 and 23 in Reid's list since these are birational to our types 3 and 4. We find NS $= ⦹^2 D_4(-1) ⦹ U(2)$ (resp. NS $= ⦹ D_5(-1) ⦹ D_4(-1) ⦹ U(2)$). For details on the calculation for the first of these, see the forthcoming book[9].

5 Hodge theoretic aspects: The pure variation

5.1 The period map

The existence of a double fiber in the elliptic fibration causes Torelli to fail everywhere. This was observed already by Chakiris [11, Theorem 2]. We give a simple proof which shows that in these cases, infinitesimal Torelli always fails. We give it here because the geometric proof in [11] is only sketched.

Proposition 5.1. *Let X be an elliptic surface fibered over $\mathbb{P}^1$ with a unique multiple fiber mF_0, $m \geqslant 2$, and such that $K_X \simeq (m-1)F_0$. The period map for a Kuranishi family of such elliptic surfaces has everywhere* 1*-dimensional fibers. This holds in particular for the classes* 1 *and* 2 *from Table* 2[10]*.*

Proof. First, note that $p_g(X)=1$ and so $(m-1)F_0$ is the unique canonical divisor. Also $q(X)=0$, e.g., because of Theorem 4.3. We reason as in [10, p. 150]. The failure of infinitesimal Torelli is caused by the non-trivial kernel of the tangent map to the period map. The latter is the map

$$\mu : H^1(T_X) \to \mathrm{Hom}(H^0(K_X) \to H^1(\Omega^1_X)). \tag{5.1}$$

Since $T_X \simeq \Omega^1_X \otimes K_X^{-1}$, the morphism μ is induced by multiplying $H^1(T_X)$ by a non-zero section ω of K_X vanishing along the canonical divisor $K=(m-1)F_0$. So, from the exact sequence

$$0 = H^0(\Omega^1_X) \to H^0(\Omega^1_X \otimes \mathcal{O}_K) \to H^1(\Omega^1_X(-K_X)) \xrightarrow{\cdot\omega} H^1(\Omega^1_X),$$

one sees that the kernel of μ is isomorphic to $H^0(\Omega^1_X \otimes \mathcal{O}_K)$. The problem now is that K is not reduced as soon as $m \geqslant 3$.

If $m=2$ and the multiple fiber is of the type $2I_0$, the normal bundle sequence for $K \subset X$ reads

$$0 \to \mathcal{O}_K(-K) \xrightarrow{\cdot\omega} \Omega^1_X \otimes \mathcal{O}_K \to \Omega^1_K \to 0$$

and since $\mathcal{O}_K(-K)$ is a torsion line bundle on the elliptic curve K, the exact cohomology sequence shows that $H^0(\Omega^1_X \otimes \mathcal{O}_K) \simeq H^0(\Omega^1_K) \simeq H^0(\mathcal{O}_C)$, which is 1-dimensional. If we have a multiple fiber of the type $2I_b$, $b \geqslant 1$, the argument is essentially the same.

[9] Peters C, Sterk H. Symmetric and quadratic forms, with applications to coding theory, algebraic geometry and topology. To appear

[10] The Kuranishi family is not the same as the modular family from Definition 3.4; the latter has a fixed polarization.

If $m \geqslant 3$, the argument is more involved. We sketch it only for $m = 3$ so that $K_X = 2F_0$. One now uses the so-called decomposition sequence for reducible divisors $D = A + B$ which reads (see [2, Chapter II.1])

$$0 \to \mathcal{O}_A(-B) \to \mathcal{O}_C \xrightarrow{\text{restr}} \mathcal{O}_B \to 0.$$

We apply it to $K = F_0 + F_0$ and tensor it with $\Omega^1_X|F_0$. This introduces the two locally free sheaves $\Omega^1_X|F_0$ and $\Omega^1_X(-F_0)|F_0$ on the elliptic curve F_0. The first sheaf fits into the normal bundle sequence for F_0 in X,

$$0 \to \mathcal{O}_{F_0}(-F_0) \to \Omega^1_X|F_0 \to \Omega^1_{F_0} \to 0$$

and since $\mathcal{O}_{F_0}(-F_0)$ is torsion, the same argument as before gives $H^0(\Omega^1_X|F_0) \simeq H^0(\Omega^1_{F_0}) \simeq \mathbb{C}$. The sheaf $\Omega^1_X(-F_0)|F_0$ fits into the normal bundle sequence twisted by $\mathcal{O}_{F_0}(-F_0)$ which reads

$$0 \to \mathcal{O}_{F_0}(-2F_0) \to \Omega^1_X(-F_0)|F_0 \to \Omega^1_{F_0}(-F_0) \to 0.$$

Hence $H^0(\Omega^1_X(-F_0)|F_0) \simeq H^0(\Omega^1_{F_0}(-F_0)) = 0$ and $H^1(\Omega^1_X(-F_0)|F_0) = 0$. Plugging all this into the exact sequence of the twisted decomposition sequence

$$0 \to \Omega^1_X(-F_0)|F_0 \to \Omega^1_X \otimes \mathcal{O}_K \to \Omega^1_X|F_0 \to 0$$

shows that $H^0(\Omega^1_X \otimes \mathcal{O}_K) \simeq H^0(\Omega^1_X|F_0) \simeq \mathbb{C}$. By induction, one can show the result for all m. □

As stated in the introduction, in our case, the four types of modular families give rise to a polarized variation of the Hodge structure, each with an associated period domain, i.e., $D_{a,b}$, and the associated Kuranishi family (with fixed polarization) gives rise to a (local) period map $M_{a,b} \to D_{a,b}$. In our case, the kernel at $F \in \mathcal{M}_{a,b}$ of the period map is precisely the kernel of the multiplication map $R^d_F r : i \to ght{\cdot}x_0 R^{d+1}_F$. This kernel varies with F. For example, the calculations in Appendix A.1 show that at the point corresponding to the ***basic type***, this kernel has dimension 1 in all the cases. However, at a general point, this kernel has dimension 0, as shown in Appendix A.3, proving the following result.

Proposition 5.2. *The period map for the Kuranishi family for a general types* 1–4 *surface*[11] *is an immersion.*

An interesting arithmetic consequence of this result[12] is the validity of the Tate conjecture as well as the Mumford-Tate conjecture for each of the present surfaces. Note that types 1 and 2, being not of the general type, are not mentioned in loc. cit. These two classes of surfaces give the first properly elliptic surfaces (in characteristic 0) for which these conjectures now are known to be true.

Corollary 5.3. *For each of the surfaces of types* 1–4*, the Tate conjecture for divisor classes and the Mumford-Tate conjecture for rank* 2 *cohomology hold.*

Proof. This follows from [30, Proposition 9.2]. Indeed, since the moduli stacks of the surfaces are irreducible of positive dimension and the condition (P) in loc.cit. follows from the immersive property of the period map, all the conditions are satisfied. □

Remark 5.4. Tu [48] has shown that infinitesimal Torelli for weighted hypersurfaces hinges on the validity of Macaulay's theorem which is only true if the degree of the hypersurface is high enough. Tu's result [48, Theorem 2] indeed does not apply in the present situation, not even in the case where the minimal model of the resolution is a K3 surface obtained from the basic surfaces of types 3 and 4.

5.2 The generic transcendental lattice for the four families

Digression on lattices. The discriminant form of a non-degenerate integral lattice L plays a central role if L is not unimodular. We already introduced the discriminant group $A(L) = L^*/L$ just above

11) Recall our convention on Kuranishi families from Lemma 2.4.

12) For background on the Tate and the Mumford-Tate conjectures, we refer to [30, § 1].

Theorem 3.8. Extending the form on L in a $\mathbb{Q}$-bilinear fashion to $L \otimes \mathbb{Q}$, one obtains a well-defined $\mathbb{Q}/\mathbb{Z}$-valued form b_L on the discriminant group by setting

$$b_L : A(L) \times A(L) \to \mathbb{Q}/\mathbb{Z}, \quad \bar{x}.\bar{y} \mapsto x.y \bmod \mathbb{Z} \ (\textbf{\textit{discriminant bilinear form}}).$$

A lattice L such that $x.x$ is even is called an ***even lattice***. These come with an integral quadratic form q given by $q(x) = \frac{1}{2}x.x$ and for these one considers a finer invariant, the ***discriminant quadratic form***

$$q_L : A(L) \to \mathbb{Q}/\mathbb{Z}, \quad \bar{x} \mapsto q(x) \bmod \mathbb{Z}.$$

The discriminant form is a so-called torsion form and such forms are completely local in the sense that these decompose into p-primary forms where p is a prime dividing the discriminant. More precisely, b_L is the orthogonal direct sum of the discriminant forms of the localizations $L_p = L \otimes \mathbb{Q}_p$ and so it ties in with the ***genus*** of the lattice, i.e., the set of isometry classes $\{L_p\}_{p \text{ prime}}$ together with $L \otimes \mathbb{R}$. The same holds for q_L if L is even.

Example 5.5. Some torsion forms play a role later on. For calculations on the root lattices, see for example [39, Table 2.4].

(1) The lattice $\langle n \rangle$ with n even has discriminant group $\mathbb{Z}/n\mathbb{Z}$ and discriminant quadratic form which on $\bar{x}$ takes the value $\frac{1}{n}\bar{x} \in \mathbb{Q}/\mathbb{Z}$. The form is denoted by $\langle \frac{1}{n} \rangle$.

(2) The discriminant group of the root lattice A_n is the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$. The discriminant quadratic form assumes the value $-n/(n+1) \in \mathbb{Q}/\mathbb{Z}$ on the generator and is denoted by $\langle \frac{-n}{n+1} \rangle$.

A celebrated result of Nikulin [32, Corollary 1.16.3] emphasizes the role of the discriminant form in determining the genus as follows:

Theorem. *The genus of the non-degenerate lattice is completely determined by its type* (*i.e., being even or odd*)*, its rank, index and the discriminant form.*

It is well known that the number of isometry classes in a genus is finite. It is also called the ***class number*** of the genus. We state a criterion for class number 1 due to Kneser [25] and Nikulin [32, 1.13.3].

Theorem 5.6. *Let L be a non-degenerate* ***indefinite*** *even lattice of rank r. Its class number is* 1 *if the discriminant group of L can be generated by at most $r-2$ elements. Hence, in this case, L is uniquely determined by its rank, index and the discriminant quadratic form.*

Example 5.7. Any indefinite odd unimodular lattice of signature (s,t) is unique in its genus and represented by a diagonal lattice of the form $\oplus^s\langle 1 \rangle \oplus \oplus^t\langle -1 \rangle$. Any even unimodular lattice of signature (s,t) is unique in its genus, satisfies $s-t \equiv 0 \bmod 8$ and if $t \geqslant s$, is represented by $\oplus^s U \oplus \oplus^{\frac{1}{8}(t-s)} E_8(-1)$. For example, the intersection lattice of a K3 surface is isometric to $\oplus^3 U \oplus \oplus^2 E_8(-1)$.

Although the lattices we encounter are odd, the preceding result will be applied to certain even sublattices. Here, we use a topological result which we recall now. For any compact orientable 4-dimensional manifold X, the second Stiefel-Whitney class w_2 is a characteristic class for the inner product space $H^2(X, \mathbb{F}_2)$, i.e., $w_2.x + x.x = 0$ for all the classes $x \in H^2(X, \mathbb{F}_2)$. To pass to integral cohomology, one uses the reduction mod 2 map, induced by the natural projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$:

$$\rho_2 : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}/2\mathbb{Z}).$$

Any lift of w_2 under ρ_2 is an integral characteristic element since the intersection pairing is compatible with the reduction modulo 2. In the special case where X is a compact almost complex manifold of complex dimension 2, there is a canonical choice for a lift, i.e., the first Chern class c_1. In our situation, we apply it to lattices orthogonal to the class of a fiber of an elliptic fibration.

Application. Recall that in the projective situation the orthogonal complement of the Picard lattice is the transcendental lattice, the smallest integral sublattice of H^2 whose complexification contains $H^{2,0}$.

Proposition 5.8. *The transcendental lattice T for a generic member of the universal family of quasi-smooth surfaces of degree $d = a+b+4$ in $\mathbb{P}(1,2,a,b)$ is even and*

- *for the type* 1, $T \simeq \oplus^2 U \oplus \oplus^2 E_8(-1)$;

- *for the type* 2, $T \simeq \langle 2\rangle \textcircled{\perp} U \textcircled{\perp}^2 E_8(-1)$;
- *for the type* 3, $T \simeq \textcircled{\perp}^2 E_8(-1) \textcircled{\perp} A_2$;
- *for the type* 4, $T \simeq \textcircled{\perp}^2 U \textcircled{\perp} \textcircled{\perp}^2 E_8(-1)$.

Proof. Recall that in the algebraic situation, the transcendental lattice and the Picard lattice are each other's orthogonal complement and both are primitively embedded in H^2. In order to apply Theorem 5.6, we observe that for a non-degenerate primitive sublattice S of a unimodular lattice L and its orthogonal complement $T = S^{\perp}$, one has $A_S \simeq A_T$ while $q_S \simeq -q_T$.

As previously observed, the transcendental lattice is a birational invariant and so we may and do compute it on the minimal model X' if it differs from $\widetilde{X}$. Proposition 4.1 states that $H^2(X')$ has rank 22 and signature $(3, 19)$. If X' is a K3 surface, the lattice $H^2(X')$ is even. In the other two cases, it is odd since the stated multisections have odd self-intersection number. However, the class $c_1(X) = -K_X = -F_0$ is a characteristic class and so $x.x$ is even for classes x orthogonal to the class of fiber. In particular, the transcendental lattice is even.

We recall that Corollary 4.7 gives the Picard lattices for the minimal model X' of the generic member of each of the four families. The Picard lattice for the type 1 surfaces is isometric to the unimodular lattice $\langle 1\rangle \textcircled{\perp} \langle -1\rangle$. Hence the generic transcendental lattice is even, unimodular and of signature $(2, 18)$. So, using Example 5.7, we see that it is isometric to $\textcircled{\perp}^2 U \textcircled{\perp} \textcircled{\perp}^2 E_8(-1)$ and a similar argument applies for the type 2.

For the type 3, the Picard lattice is isometric to $U \textcircled{\perp} A_2(-1)$. The transcendental lattice is an even lattice of signature $(2, 16)$ and discriminant form the one of A_2. Up to isometry, there is only one lattice, the lattice $A_2 \textcircled{\perp} \textcircled{\perp}^2 E_8(-1)$. For the type 4, the Picard lattice is isometric to U. So by Example 5.7, in this case, the generic transcendental lattice is isometric to $\textcircled{\perp}^2 U \textcircled{\perp} \textcircled{\perp}^2 E_8(-1)$. □

5.3 Lattice polarized variations

Recall that in each of the four cases the Hodge structure on H^2 and on H^2_{prim} is of the K3-type since $h^{2,0} = 1$. So the period domain associated with H^2_{prim} is

$$D(H^2_{\mathrm{prim}}) = \{[\omega] \in \mathbb{P}(H^2_{\mathrm{prim}} \otimes \mathbb{C}) \mid \omega \wedge \omega = 0,\, \omega \wedge \overline{\omega} > 0\},$$

a domain of the K3-type of dimension $b_2 - 3 = h^{1,1}_{\mathrm{prim}}$. Since the modular families (see Definition 3.4) in all the cases have a generic Picard lattice of rank $\geqslant 2$, the associated period map is not surjective. In such cases, one uses the smaller domain

$$D(T) = \{[\omega] \in \mathbb{P}(T \otimes \mathbb{C}) \mid \omega \wedge \omega = 0,\, \omega \wedge \overline{\omega} > 0\}, \tag{5.2}$$

associated with the general transcendental lattice $T \subset H^2(X, \mathbb{Z})_{\mathrm{prim}}$ for the modular family for X. One speaks then of ***lattice polarized families*** and their associated variations of the Hodge structure. More precisely, these are the S-polarized variations, where $S = T^{\perp}$ is the generic Picard lattice. Since the period maps for the generic member of a modular family are immersions (see Proposition 5.2), Proposition 5.8 gives rise to Table 5.

Table 5 Generic ranks

Cases	$\dim D(T)$	Generic rank of the period map
1. $(14, [1, 2, 3, 7])$	18	18
2. $(12, [1, 2, 3, 5])$	17	17
3. $(16, [1, 2, 5, 7])$	16	16
4. $(22, [1, 2, 7, 11])$	18	18

6 Associated variation of the mixed Hodge structure

6.1 Mixed Torelli

We assume now that X is an elliptic surface of the type 1 or type 2.

We first describe a subvariety[13)] inside $\mathscr{M}_{3,7}$ (resp. inside $\mathscr{M}_{3,5}$), where the infinitesimal Torelli fails and to which the basic surfaces belong.

Let $W \subset U_{1,2,3,7}$ be the sublocus of normal forms with $G_0 = G_5 = 0$. It has dimension $13 - 2 = 11$. A quasi-smooth surface $V(g)$ with $g \in W$ is invariant under the involution $\iota(x_0) = -x_0$ and ι acts on the Jacobian ring R. The $+1$ eigenspace R^+_{14} of ι on R_{14} corresponds to first order deformations which are tangent to L and transverse to the T-orbit, confirming that $\dim(W//T) = \dim R^+_{14} = 11$.

The forms $F(\Omega_3/g^2)$, $F \in R^-_{15}$ whose residues belong to the primitive $(1,1)$-part of $H^2(V(f))$, are all ι-invariant and descend to give rational forms on the quotient of $V(g)$ by ι. This is a quasi-smooth degree 14 surface $S \subset \mathbb{P}(2,2,3,7)$ and appears as number 22 on Reid's list of 95 K3 surfaces. In the present case, the primitive part $H^2_{\mathrm{prim}}(S)$ has Hodge numbers $(1,10,1)$. The residue map sending $F \in R^-_{15}$ to its residue on $V(f)$ induces a homomorphism $R^-_{15} \to H^{1,1}_{\mathrm{prim}}(S)$. One checks that it is generically an isomorphism and so it follows that the multiplication map $\mathrm{mult}_{x_0} : R^+_{14} \to R^-_{15}$ is a map from a vector space of dimension 11 to a vector space of dimension 10, and hence always has a 1-dimensional kernel.

In $U_{1,2,3,5}$, we consider the sublocus W' of normal forms given by $G_4 = 0$ and the vanishing of two terms in G_2 (only the one containing $x_0^3 x_2$ is allowed) has dimension 13 and so $\dim(W'//T) = \dim R^+_{12} = 10$.

The quotient by the involution $x_0 \mapsto -x_0$ is a K3 surface in $\mathbb{P}(2,2,3,5)$ of degree 12 which is number 18 in Reid's list. Its primitive cohomology has Hodge numbers $(1,9,1)$. The same argument shows that the multiplication map $\mathrm{mult}_{x_0} : R^+_{12} \to R^-_{13}$ in this case always has a 1-dimensional kernel.

In these cases, $\widetilde{X} = X'$ is fibered over $\mathbb{P}^1$ with a unique double fiber $2F_0$ and $K_{X'} \simeq F_0$. We set $U := X' \setminus F_0$ and we consider the variation of the mixed Hodge structure on $H^2(U)$ when X varies in one of the subfamilies of a modular family coming from the codimension 1 sublocus $W \subset U_{1,2,3,7}$ (resp. $W' \subset U_{1,2,3,5}$), where Torelli fails. For brevity, we call the resulting variation the ***subcanonical modular variation of mixed Hodge structure.***

Theorem 6.1. *The period map of the subcanonical modular variation of the mixed Hodge structure of the type* 1 *or type* 2 *surfaces is an immersion.*

Proof. We first determine the mixed Hodge structure on $H^2(U)$ by means of the exact Gysin sequence

$$0 \to H^0(F_0)(-1) \xrightarrow{i_*} H^2(X') \xrightarrow{j^*} H^2(U) \xrightarrow{\mathrm{res}} H^1(F_0)(-1) \to 0,$$

where $i : F_0 \hookrightarrow X'$, $j : U \hookrightarrow X'$ are the inclusions, and "res" is the residue map. We see that $W_2 H^2(U) \simeq H^2(X')/H^2(F_0)(-1)$ and $\mathrm{Gr}^W_3 H^2(U) \simeq H^1(F_0)(-1)$.

We next consider the variation of mixed Hodge structure given by $H^2(U_F)$, where $U_F = U \setminus F$, $F = F_0 + G$ is a deformation of a quasi-smooth reference surface F_0, and G varies over an open neighborhood of F_0 in a base of a Kuranishi family as described by Corollary 3.5. So tangent directions are identified with polynomials $G \in R^d$. The infinitesimal variation is described by the Higgs fields[14)] $\theta^2_\xi : I^{2,0} \to I^{1,1}$, $\theta^3_\xi : I^{2,1} \to I^{1,2}$ in the direction of ξ. The map θ^2_ξ is induced by the map μ (see (5.1)), and if ξ corresponds to the polynomial G, it is represented by the multiplication

$$R^1_F \xrightarrow{\cdot G} R^{d+1}_F, \quad G \in R^d_F.$$

We consider the following two particular cases (belonging to the locus L (resp. L')):

- Type 1: $F_0 = x^{14} + y^7 + yz^4 + w^2 + x^2y^3z^2$, $\eta = x^{12}y + (1/7)y^4z^2$.
- Type 2: $F_0 = x^{12} + y^6 + z^4 + yw^2 + x^2y^2z^2$, $\eta = x^{10}y + (1/6)y^3z^2$.

In Appendix A.1, it is shown that $V(F_0)$ is quasi-smooth and in Appendix A.3 that θ^2_ξ is injective except in the direction $\xi = \eta$.

13) There might be other subvarieties, where this holds as well.

14) For the background on Higgs fields in the mixed setting, see [33].

To calculate θ^2_η, we consider the family of canonical curves E_t attached to the surfaces $X_t = V(F_0+t\eta)$. In the case (1), E_t is defined by $y^7+yz^4+w^2+t(1/7)y^4z^2$. Moreover, $H^{1,0}(E_0)$ is generated by y while $H^{0,1}(E_0)$ is generated by y^5z^2, and hence multiplication by the tangent direction y^4z^2 is injective. For the type 2, E_t is defined by $y^6+z^4+yw^2+t(1/6)y^3z^2$. Moreover, $H^{1,0}(E_0)$ is generated by y whereas $H^{0,1}(E_0)$ is generated by y^4z^2, and hence multiplication in the tangent direction y^3z^2 is injective.

This takes care of the direction in which θ^2_ξ fails to be injective and shows that the period map not only is injective along $W//T$ (resp. $W'//T$), but—by the lower semi-continuity of the rank function—at the generic point of the moduli spaces $\mathscr{M}_{3,7}$ and $\mathscr{M}_{3,5}$. □

6.2 Rigidity: The pure case

We first consider rigidity for the pure polarized variations of the Hodge structure. To avoid confusion, we explain the rigidity concept we use here. A variation of Hodge structure comes with a period map $f : S \to \Gamma\backslash D$, where D is a period domain classifying the kind of Hodge structures underlying the variation, and Γ is the monodromy group of the variation. Rigidity in this setting is a rather restricted concept.

Definition 6.2. A deformation of a period map $f : S \to \Gamma\backslash D$ consists of a locally liftable horizontal map $F : S \times T \to \Gamma\backslash D$ extending f in the obvious way. If no such deformation exists except $f \times \mathrm{id}$, the map f is called rigid.

Recall that the essential part of a K3-type variation is given by the variation on the generic transcendental lattice. Since we have a weight 2 variation, one may apply [35, Theorem 3]. In our case, this implies the following proposition.

Proposition 6.3. *If the period map associated with the essential part of a K3-type variation over a quasi-projective variety has rank greater than or equal to* 2*, it is rigid in the above sense.*

Taking into account the possible failure of infinitesimal Torelli over certain subvarieties parametrizing modular families, we thus find the following rigidity results.

Proposition 6.4. *The essential part of a variation of types* 1–4 *is rigid if the period map has rank greater than or equal to* 2*. In particular the variation over a quasi-projective subvariety* S *of a modular family of dimension greater than or equal to* 3 *is rigid for types* 1 *and* 2*, and if* $\dim S \geqslant 2$ *for types* 3 *and* 4.

6.3 Rigidity for mixed period maps

Just as in the proof of [33, Proposition 7.2.5], we deduce from the rigidity in the pure case.

Corollary 6.5. *Consider for a family of the type* 1 *or type* 2 *surfaces the complements of the supports of the canonical curve. If the mixed period map of the resulting family has rank greater than or equal to* 3*, the family is rigid.*

Proof. [33, Proposition 7.2.5] states that it is sufficient to show the following three properties of a family as described in the assertion, i.e, $\widetilde{X}_s$, $s \in S$, and S is smooth and quasi-projective.

(1) The family of canonical curves in $\widetilde{X}_s$ is rigid.

(2) The essential part of the K3 variation has a non-constant period map and is rigid.

(3) The mixed period map is an immersion.

(1) follows for the weight one variation from the moving curve F_0 from [35, Theorem 3] since in the course of the proof of Theorem 6.1, we prove that the period map for the canonical curves in $\widetilde{X}_s$ is not constant.

(2) is the statement of Proposition 6.4 and (3) is Theorem 6.1. □

7 An application to KSBA theory

Recall that the 28-dimensional moduli space $\mathbf{M}$ of surfaces of the general type with $K^2 = 1$, $p_g = 2$ and $q = 0$ admits a KSBA compactification $\overline{\mathbf{M}}$ proposed by and named after Kollár, Shepherd-Barron and Alexeev (see [17, 37] for recent results about these surfaces).

In this section, we give an application of the results of this paper to the Hodge theory of some of the boundary divisors of the KSBA compactification.

7.1 Overview of the results of [17]

The generic member of $\mathbf{M}$ has a canonical model which is a quasi-smooth hypersurface in $\mathbb{P}[1, 1, 2, 5]$. After completing the square, such a surface can be put in the form

$$w^2 = f(x, y, z), \quad \deg(x) = \deg(y) = 1, \quad \deg(z) = 2, \quad \deg(w) = 5. \tag{7.1}$$

For each of Arnol'd's exceptional unimodal singularity of the type

$$\Sigma = E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13}, \tag{7.2}$$

a corresponding boundary divisor D_Σ of the KSBA compactification $\overline{\mathbf{M}}$ of $\mathbf{M}$ is constructed.

To construct the boundary divisors D_Σ, we start with a triple (p, q, d) of positive integers and let V denote the homogeneous component of degree 10 of $\mathbb{C}[x, y, z]$ with respect to (7.1). Let ω be the weight function which is defined on the monomials of V by the rule

$$\omega(x^a y^b z^c) = pb + qc - d. \tag{7.3}$$

Then, ω determines a $\mathbb{C}^*$-action on V by the linear extension of the rule:

$$t * (x^a y^b z^c) = \begin{cases} t^{-\omega} x^a y^b z^c, & \omega \leqslant 0, \\ x^a y^b z^c, & \omega > 0. \end{cases}$$

By construction, this action extends continuously to a holomorphic map $\mathbb{C} \times V \to V$.

Given a polynomial $f \in V$, let

$$\mathcal{S}(f) = V(w^2 - t * f) \subseteq \mathbb{P}(1, 1, 2, 5) \times \mathbb{C}$$

and $\pi : \mathcal{S}(f) \to \mathbb{C}$ be the morphism $\pi((x : y : z : w), t) = t$. Let $\mathcal{S}_t(f) = \pi^{-1}(t)$. The fiber $\mathcal{S}_0(f)$ is defined by the monomials of non-negative degree with respect to ω. For a generic polynomial f of degree 10 and a suitable choice of (p, q, d), the branch curve B_0 of $\mathcal{S}_0(f)$ has a unique singularity at $(1 : 0 : 0)$ of the type Σ appearing in (7.2) (see [17] for the details).

In the setting described in the previous paragraphs, the KSBA program compactifies the germ of the family $\mathcal{S}(f) \to \mathbb{C}$ at $t = 0$ by replacing $\mathcal{S}_0(f)$ with a new central fiber $\tilde{Z}_f \cup \tilde{Y}_f$, where

- $\tilde{Z}_f$ is birational to a (p, q)-weighted blow up of $\mathcal{S}_0(f)$ at $(1 : 0 : 0)$; the surface $\tilde{Z}_f$ has at worst rational singularities and $h(\mathcal{O}) = 1$;
- $\tilde{Y}_f$ is an A-D-E K3-surface which is defined by the monomials of non-positive weight; more precisely, in terms of the data (p, q, d), $\tilde{Y}$ is a degree d hypersurface in $\mathbb{P}(1, p, q, d/2)$ when d is even and degree $2d$ in $\mathbb{P}(1, 2p, 2q, d)$ when d is odd;
- $\tilde{Z}_f$ and $\tilde{Y}_f$ are glued together along a common $\mathbb{P}(p, q)$.

Fixing the data (p, q, d) and varying the polynomial f define a divisor D_Σ in the KSBA compactification of $\mathbf{M}$. Moreover, by the results of [17, Section 4], there exists a Zariksi open subset $\mathscr{U} \subseteq D_\Sigma$ over which there exists a flat, proper family $p : \mathscr{S} \to \mathscr{U}$ whose fibers $p(u) = \tilde{Z}_u \cup \tilde{Y}_u$ are surfaces of the type described above.

Remark 7.1. Preliminary calculations show that the framework described above is generally applicable to hypersurface degenerations in weighted projective 3-space, provided that certain numerical conditions hold, i.e., a choice of weight function ω defines a family of surfaces $\mathcal{S}(f) \to \mathbb{C}$ whose central fiber can be modified by adjoining a "tail surface" to obtain a KSBA stable limit. The details will appear in a follow up to [17].

By [42], there exists a Zariski open subset $\mathscr{U}_1 \subseteq \mathscr{U}$ over which $\mathscr{V} = R^2 p_*(\mathbb{Q})$ is the underlying $\mathbb{Q}$-local system of an admissible variation of the graded-polarizable mixed Hodge structure. Therefore, $\mathscr{H} = \mathrm{Gr}_2^W(\mathscr{V})$ is a variation of the pure Hodge structure of weight 2 over $\mathscr{U}_1$. Given a $\mathbb{Q}$-Hodge structure A of weight 2 with $F^3 A_{\mathbb{C}} = 0$, let $T[A]$ denote the smallest $\mathbb{Q}$-Hodge substructure of A which contains $F^2 A_{\mathbb{C}}$. By the results of [17, Section 6], there is a Zariski open subset $\mathscr{U}_2 \subseteq \mathscr{U}_1$ such that

$$u \in \mathscr{U}_2 \Rightarrow T[\mathrm{Gr}_2^W(\mathscr{H}_u)] = T[H^2(\tilde{Z}_u)] \oplus T[H^2(\tilde{Y}_u)].$$

For generic $f \in V$, let $\varphi_f : \Delta^* \to \Gamma\backslash D$ denote the local period map of $\pi : \mathcal{S}(f) \to \mathbb{C}$ near $t = 0$. By the results of [17, Section 6], φ_f has finite local monodromy, and hence the limit mixed Hodge structure $H_{\lim}(f)$ of φ_f is pure. Moreover,

$$T[H_{\lim}(f)] = T[H^2(\tilde{Z}_f)] \oplus T[H^2(\tilde{Y}_f)]. \tag{7.4}$$

Thus, at the loss of the information contained in the finite monodromy of φ_f, we are justified in calling the transcendental part of $\mathscr{H}$ the limit variation of the Hodge structure of $\mathbf{M}$ along D_Σ.

Remark 7.2. The moduli count for the surfaces $\tilde{Z}_\Sigma$ in terms of the Milnor number of Σ is given by $29 - \mu_\Sigma$ whereas the moduli count for the surfaces $\tilde{Y}$ is $\mu_\Sigma - 2$, adding up to $27 = 28 - 1$ which suggests that we have a divisor. That indeed D_Σ is a divisor corresponds to the fact that these two components can be deformed independently (see [17] for details).

Since μ_Σ is the index of Σ in the list (7.2), the above formulae give the moduli for each of the components.

At this point, it is natural to ask the following:

(1) What is the birational type of surface $\tilde{Z}_\Sigma$?

(2) Does the period map of the limit variation of the Hodge structure constructed above have positive-dimensional fibers?

For the unimodal singularities of types Z_{11}, Z_{12}, Z_{13}, W_{12} and W_{13}, the answer to the first question is that they are birational to K3 surfaces. As explained in the last section of [17], one sees this by simply multiplying the defining equation of $\mathcal{S}_0(f)$ by y^2 and considering the birational transformation

$$(x : y : z : w) \in \mathbb{P}[1, 1, 2, 5] \dashrightarrow (xy : y^2 : z : yw) \in \mathbb{P}(2, 2, 2, 6) \cong P(1, 1, 1, 3)$$

which converts a limit surface of the type Z or W into an ADE K3 surface which is a double cover of $\mathbb{P}^2$ branched along a sextic which intersects a line L in a special configuration (for example, multiplicities 1–3 for the Z_{11} singularity).

All we know is that the answer to the second question is "no" for the E_{13} case, as we now explain. By (7.4), the period map is governed by the restriction to the transcendental part of the Z-surface and of the Y-surface. The Z-surface (the type E_{13} in $\mathbb{P}(1, 2, 3, 7)$) has injective differential by Table 6. The Y-surface is a K3 surface in the weighted projective space whose period map has an injective differential. Since the two surfaces Y and Z deform independently, this shows that the total period map has an injective differential.

7.2 Relation with the present paper

One of the observations which gave rise to this paper is that for the singularity types E_{13} and E_{14}, the resulting surfaces $\widetilde{Z}$ are birational to a singular hypersurface of degree 14 in $\mathbb{P}[1, 2, 3, 7]$ by simply multiplying the defining equation of $\mathcal{S}_0$ by z^2 and considering the birational transformation

$$(x : y : z : w) \in \mathbb{P}[1, 1, 2, 5] \dashrightarrow (x_0 : x_1 : x_2 : x_3) = (y : z : xz : zw) \in \mathbb{P}[1, 2, 3, 7].$$

Table 6 Examples, infinitesimal mixed Torelli

1	Defining polynomial	Generator of Kernel
1.	$x^{14}+y^7+yz^4+w^2+x^2y^3z^2$	$x^{12}y+(1/7)y^4z^2$
2.	$x^{12}+y^6+z^4+yw^2+x^2y^2z^2$	$x^{10}y+(1/6)y^3z^2$
$\mathscr{J}$	$x^2z^4+yz^4+x^5z^3+x^8z^2$	$x^8y^3+(4/5)x^6yz^2+(1/2)x^3yz^3$
	$+y^7+x^{10}y^2-w^2+x^2y^3z^2$	$+(1/5)y^4z^2+(1/5)yz^4$
E_{13}	$y^7+y^2x^{10}+yz^4+x^5z^3+z^2x^8-w^2$	$x^8y^3+(4/5)x^6yz^2+(1/2)x^3yz^3$
E_{14}	$y^7+y^2x^{10}+yz^4+x^3yz^3+z^2x^8-w^2$	$x^8y^3+(4/5)x^6yz^2+(3/10)xy^2z^3$

In fact, both the E_{13} and E_{14} singularities are subvarieties of the locus $\mathscr{J}$ of degree 14 surfaces in $\mathbb{P}[1,2,3,7]$ whose singular locus consists of the orbifold point $[0:0:1:0]$ of $\mathbb{P}[1,2,3,7]$ and exactly one A_1-singularity which occurs at a smooth point of $\mathbb{P}[1,2,3,7]$.

By a result of Burns and Wahl [9], $\mathscr{J}$ should have codimension 1 in the moduli of hypersurfaces of the type $(14,[1,2,3,7])$. For completeness, we give a direct algebraic proof here: The group of automorphisms of $\mathbb{P}[1,2,3,7]$ acts transitively on the smooth points of $\mathbb{P}[1,2,3,7]$ (consider the orbit of $\xi=[1:0:0:0]$). Therefore, we consider the hypersurfaces $V(f)\in\mathscr{J}$ which have an A_1-singularity at ξ and are given as double covers $x_3^2=g(x_0,x_1,x_2)$. The condition that $f(\xi)=0$ implies the vanishing of the coefficient of x_0^{14} in f. The condition that $(\nabla f)(\xi)=0$ forces the vanishing of the coefficients of $x_0^{12}x_1$ and $x_0^{11}x_2$ in f as well. Since we are considering double covers of $\mathbb{P}[1,2,3]$, the relevant automorphism group consists of invertible transformations of the form (see (B.3))

$$(x_0:x_1:x_2)\mapsto(a_0x_0:a_1x_1+a_2x_0^2:a_3x_2+a_4x_0x_1+a_5x_0^3).$$

The subgroup of elements which fix the point $[1:0:0]$ corresponds to transformations for which $a_2=a_5=0$, and hence (up to scaling) this subgroup has $5-2=3$ parameters. The number of monomials of degree 14 in $\mathbb{P}[1,2,3]$ is 24. So, the dimension count for $\mathscr{J}$ is $(24-3-1)-3=17$.

As noted above, the generic surface of the type $(14,[1,2,3,7])$ has elliptic fiber structure $2I_0+24\times I_1$. Let S be a generic point of $\mathscr{J}$, E_{13} or E_{14}. In Appendix A.4, it will be shown that in each case the singular locus of S consists of the orbifold point $[0:1:0:0]$ of $\mathbb{P}[1,2,3,7]$ and an A_1 singularity at a smooth point of S. In Appendix A.3, we compute the rank of the period map in the $\mathscr{J}$-case and the E_{13}-case. This is then shown to lead to the following theorem.

Theorem 7.3. *The generic $\mathscr{J}$-type surface as well as the generic E_{13} surface and the generic E_{14} surface is birational to a type $(14,[1,2,3,7])$ surface which has one singular point of the type A_1 at the point $(1:0:0:0)\in\mathbb{P}[1,2,3,7]$ and finite quotient singularities where it intersects the singular locus of $\mathbb{P}[1,2,3,7]$. These surfaces are properly elliptic with $p_g=1$. Furthermore,*

(1) *the elliptic fiber type of the elliptic fibration in the $\mathscr{J}$-case and the E_{13}-case is given by $2I_0+I_2+22\times I_1$ and in the E_{14}-case by $2I_0+I_3+21\times I_1$;*

(2) *let S_{triv} be the "trivial" lattice, spanned by the fibers and the "canonical" multisection; the invariants in the three cases are given in Table 7.*

If S_{triv} is the entire Picard lattice, then $S_{\mathrm{triv}}^{\perp}$ is the transcendental lattice and $D(S_{\mathrm{triv}}^{\perp})$ is the associated period domain.

Table 7 Data for generic models of the above 3 types of surfaces

	$\mathscr{J}$	E_{13}	E_{14}
$\dim S_{\mathrm{triv}}$	3	3	4
Hodge numbers of $S_{\mathrm{triv}}^{\perp}$	$(1,17,1)$	$(1,17,1)$	$(1,16,1)$
dim of $D(S_{\mathrm{triv}}^{\perp})$	17	17	16
Number of moduli	17	16	15
Rank of the period map	17	16	$\geqslant 14$

Remark 7.4. To prove our results on (mixed) period maps also to the family $\mathscr{J}$, it is required to extend the residue calculus for quasi-smooth hypersurfaces to the situation where supplementary ordinary nodes are allowed, e.g., by extending the results [12] of Dimca and Saito and so one expects the residue calculus to involve working with polynomials in the Jacobian ring which vanish at the nodes. Assuming this to be the case one argues as follows:

(a) Rigidity for the period map of the family $\mathscr{J}$ should follow from Proposition 5.1 upon applying [35, Theorem 3]. To show that the family of canonical curves is rigid, we can apply the same residue calculations of Subsection 6.1 since the canonical curve $x_0 = 0$ does not pass through the A_1-singularity at $(1 : 0 : 0 : 0)$.

(b) To prove also that rigidity and mixed Torelli hold in the $\mathscr{J}$-case, we need to calculate the derivative of the period map $\varphi : \mathscr{J} \to \Gamma\backslash D$ at a suitable surface $S \in \mathscr{J}$. If the residue calculus can indeed be applied, we find, as expected, that there are 17 deformation parameters. Furthermore, the code of Appendix A.3 shows that for the particular surface $S = V(f)$ defined by

$$g = x^2z^4 + yz^4 + x^5z^3 + x^8z^2 + y^7 + x^{10}y^2 - w^2 + x^2y^3z^2, \tag{7.5}$$

where $x = x_0$, $y = x_1$, $z = x_2$ and $w = x_3$, the kernel of the derivative of the period map has dimension 1. Indeed (see Table 6), the code shows that the kernel of the map

$$\ker((S/J)_d \xrightarrow{x} (S/J)_{d+1})$$

is given by

$$\eta = x^8y^3 + (4/5)x^6yz^2 + (1/2)x^3yz^3 + (1/5)y^4z^2 + (1/5)yz^4. \tag{7.6}$$

The canonical curve E_t of the surface $X_t = V(g + t\eta)$ is therefore defined by the equation

$$yz^4 + y^7 - w^2 + t(y^4z^2 + yz^4)/5.$$

A residue calculation shows that y generates $H^{1,0}(E_0)$ whereas y^5z^2 generates $H^{0,1}(E_0)$. Moreover, yz^4 is contained in the Jacobian ideal of E_0 whereas y^4z^2 does not reduce to zero modulo the Jacobian ideal. Therefore, the derivative of the period map of the family E_t at $t = 0$, which corresponds to multiplication by y^4z^2, is injective. Thus, as in Section 5, the derivative of the mixed Torelli map is injective at the generic point of $\mathscr{J}$.

Acknowledgements The authors thank Patricio Gallardo (University of California, Riverside) and Luca Schaffler (University Roma 3) for their comments on an earlier version of the manuscript, Miles Reid (Warwick University) and Matthias Schütt (Leibniz University, Hannover) for their help in understanding the geometry of the elliptic pencils, Wolfgang Ebeling (Leibniz University, Hannover) for answering questions about singularities and their monodromy groups, and János Kollár (Princeton University) for pointing out the reference [26]. In addition, the authors thank the referees for their remarks. The third author thanks Ronald van Luijk (Leiden University) for all his input and feedback on this work. The third author also thanks the second author for the opportunity to work on this problem and the interesting conversations he had on this topic.

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Appendix A Manual computations and SageMath code

Appendix A.1 Manual verifications

The Hodge number $h^{2,0} = p_g$ for any of the types 1–4 weighted quasi-smooth surfaces $F = 0$ equals $\dim R^1_F = 1$. The remaining Hodge number $\dim H^{1,1}_{\text{prim}} = \dim R^{d+1}_F$ (constant in families) can be checked by hand for the ***basic quasi-smooth surfaces*** from Proposition 2.5. This is also the case for the number of moduli

$$\mu_F = \dim(H^1(T_X)_{\text{proj}}) = \dim(R^d_F)$$

thereby giving a check for the results of Subsection 3.1. Finally, this can also be done for the dimension of the kernel of the derivative of the period map $\delta_F = \dim\ker(R^d_F \xrightarrow{\cdot x_0} R^{d+1}_F)$. The results are given in Tables A.1 and A.2.

Let us carry this out for the basic example in the case 1, where

$$J_F = \left(\frac{1}{16}x_0^{15} + x_2^3, \frac{1}{8}x_1^7 + x_3^2, \frac{1}{3}x_0x_2^2, x_1x_3\right).$$

Table A.1 Data for the basic examples

Basic example	$h^{1,1}_{\text{prim}}$	μ	δ
$x_0^{14} + x_1^7 + x_2^4x_1 + x_3^2$	18	18	1
$x_0^{12} + x_1^6 + x_2^4 + x_1x_3^2$	17	17	1
$x_0^{16} + x_1^8 + x_0x_2^3 + x_1x_3^2$	17	16	1
$x_0^{22} + x_1^{11} + x_0x_2^3 + x_3^2$	18	18	1

Table A.2 Basis for R^{16}_F and R^{17}_F

R^{16}_F	R^{17}_F	R^{16}_F	R^{17}_F
$(14,1,0,0)$	$(15,1,0) \sim (0,1,3,0)$	$(9,0,0,1)$	$(10,0,0,1)$
$(12,2,0,0)$	$(13,2,0,0))$	$(11,0,1,0)$	$(12,0,1,0)$
$(10,3,0,0)$	$(11,3,0,0)$	$(9,1,1,0)$	$(10,1,1,0)$
$(8,4,0,0)$	$(9,4,0,0)$	$(7,2,1,0)$	$(8,2,1,0)$
$(6,5,0,0)$	$(7,5,0,0)$	$(5,3,1,0)$	$(6,3,1,0)$
$(2,7,0,0)$	$(3,7,0,0)$	$(3,4,1,0)$	$(4,4,1,0)$
$(0,8,0,0)$	$(1,8,0,0)$	$(1,5,1,0)$	$(2,5,1,0)$
$(0,3,2,0)$	–	$(4,0,1,1)$	$(5,0,1,1)$
			$(0,6,1,0)$
			$(0,0,2,1)$

Table A.2 shows that $\delta_F = \ker(R_F^{16} \xrightarrow{\cdot x_0} R_F^{17}) = \mathbb{C} \cdot x_1^3 x_2^2$ and thus infinitesimal Torelli does not hold for this particular surface.

As noted earlier, for types 3 and 4, the basic hypersurfaces do not belong to the moduli spaces $\mathscr{M}_{5,7}$ (resp. $\mathscr{M}_{7,11}$).

Using the sage code listed below, we show that $\delta_F = 0$ at the generic point $V(F)$ of the moduli spaces in the cases 1–4.

Appendix A.2 Quasi-smoothness via Groebner basis

To quickly verify the quasi-smoothness of the hypersurface $V(g)$ for a given polynomial g using the SageMath code, we compute a Groebner basis of the Jacobian ideal of g. In each case, the basis contains some power of x_3. Setting $x_3 = 0$, one then finds that $x_2 = 0$ as well. This results in a system of equations which is easily solved by hand.

```
# a=3; b=7;  #Type 1
# a=3; b=5;  #Type 1
# a=5; b=7;  #Type 3
# a=7; b=11; #Type 4
d = a+b+4
R=PolynomialRing(QQ,"x,y,z,w",order=TermOrder("wdeglex",(1,2,a,b)))
x,y,z,w=R.gens() #x=x_0, y=x_1, z=x_2, w=x_3
# Examples from Prop. 3.3.1
# g = x^14 + y^7 + z^4*y + w^2 + x^11*z + x^5*z^3 # Type 1
# g = w^2*y + 2*w*x*(z^2 + x^6) + z^4 + y^3*z^2 + x*y^4*z + y^6 # Type 2
# A = y^7 -w^2;  B = x^8*w + x^3*z*w + (x^(15) + x^(10)*z + x^5*z^2 + z^3 );
# g = A*y + B*x; # Type 3
# g = x*z^3 + y^4*z^2 + (x*y^7+x^9*y^3+x^(15))*z + y^11 - w^2 # Type 4
J=g.jacobian_ideal()
B =J.groebner_basis()
[print(b) for b in B]
gx=g.derivative(x); gy=g.derivative(y);
gz=g.derivative(z); gw=g.derivative(w);
print(gx.subs(z=0,w=0),"|",gy.subs(z=0,w=0),"|",gz.subs(z=0,w=0),"|",gw.subs(z=0,w=0))
```

Note. If the equation is of the form $g = w^2 + wf(x,y,z) + h(x,y,z)$, one should first eliminate $wf(x,y,z)$ via a change of variable.

Appendix A.3 Rank of the period maps for modular families

In this appendix, we present code to compute the kernel of the differentials of the period maps we consider. The basis of the code is that SageMath has facilities to reduce polynomials relative to a given ideal and compute the coefficient matrix of a sequence of polynomials with respect to the monomials which occur in the sequence. In this way, the problem reduces to a straightforward linear algebra problem. This code has also been adapted to incorporate the singularities of types $\mathscr{J}$, E_{13} and E_{14}.

More precisely, let R be a graded ring and S and J be homogeneous ideals of R such that $J \subset S$. Then, the multiplicative structure of R induces a well-defined map

$$(R/J)_\alpha \times (S/J)_d \mapsto (S/J)_{d+\alpha}, \tag{A.1}$$

where $(-)_k$ denotes the degree k-component. As above, for types 1–4, the determination of the kernel of the derivative of the period map at the generic surface $V(g)$ amounts to the calculation of the kernel of (A.1) in the special case where R is the homogeneous coordinate ring of $P(1,2,a,b)$, $S = R$, J is the Jacobian ideal of g, $\alpha = 1$ and $d = \deg(g)$.

Let $\mathscr{M}$ denote a moduli space of surfaces of the type $\mathscr{J}$, E_{13}, E_{14} or $\mathscr{M}_{a,b}$. Let $V(g)$ be a generic element of $\mathscr{M}$, with Jacobian ideal $J \subseteq R$. To produce code which handles all of these moduli spaces at once, we observe that in each case, there exists a homogeneous ideal I of R such that every element of the tangent space $\mathscr{T}$ to $\mathscr{M}$ at $V(g)$ can be obtained by a first order deformation $t \mapsto V(g + t\zeta)$ for some $\zeta \in I$. We therefore set $S = I + J$, where $I = R$ for the moduli spaces $\mathscr{M}_{a,b}$.

Table A.3 Examples, infinitesimal pure Torelli

1	Defining polynomial
1.	$x^{14} + x^2z^4 + 2xy^2z^3 + y^7 + y^4z^2 + yz^4 + w^2$
2.	$x^{12} + y^6 + z^4 + yw^2 + x^2y^2z^2$
$\mathscr{J}$	$x^{10}y^2 + x^8z^2 + 2x^6yz^2 - 2x^5y^3z + x^2z^4$
	$-2xy^2z^3 + y^7 + y^4z^2 + yz^4 + w^2$
E_{13}	$x^{10}y^2 + x^8z^2 + 2x^5z^3 + 2x^4y^2z^2 + 2xy^2z^3$
	$+y^7 + y^4z^2 + yz^4 + w^2$
3.	$x^{16} + y^8 + xz^3 + yw^2 + y^3z^2$
4.	$w^2 + xz^3 + y^4z^2 + x^{20}y + y^{11}$

For the moduli spaces $\mathscr{M}$ under consideration, I_d always has a monomial basis B. Let $X = \{m \in B \mid m = m.\text{reduce}(J)\}$ using the reduce command of SageMath. Let $\tau(X)$ denote the subspace of $\mathscr{T}$ defined by the first order deformation through elements of span(X). Then, $\tau(X) = \mathscr{T}$ if and only if

(i) $|X| = \dim \mathscr{M}$;

(ii) $\dim(\text{span}(X) + J) = |X| + \dim J$.

This is easily checked by computer using the code found at the end of this appendix (Appendix A.3). In the code, *ex_*dim = dim $\mathscr{M}$ and the basis B is the complement of the monomials listed in the parameter *forbidden*. The results of these calculations are summarized in Tables 6 and A.3. For types 1–4, the code also verifies the previously stated dimensions of $(R/J)_d$ and $(R/J)_{d+1}$.

To check that the specific hypersurfaces used in these calculations are quasi-smooth, we use the Groebner basis method of Appendix A.2. In each case, we find that some power of z and w are contained in the Jacobian ideal of g (this can be checked directly using the reduce command in SageMath), and hence we must have $z = w = 0$ at the singular point. In the same way, we verify that the test surfaces of types $\mathscr{J}$, E_{13} and E_{14} do not have any extra singularities.

Since the only new feature arises in the case of types $\mathscr{J}$, E_{13} and E_{14}, we only treat these cases. For the examples for which the derivative of the period map has a non-trivial kernel, z^{10} and w belong to the Jacobian ideal. Moreover, the condition to have a singular point at p reduces to $g_x = 10x^9y^2 = 0$ and $g_y = 2x^{10}y + 7y^6 = 0$. The only singular point is therefore at $p = [1 : 0 : 0 : 0]$, as expected.

Code for infinitesimal period map calculations. The dimension of moduli space $\mathscr{M}$ is *ex_*dim. The basis B of I_d is the complement of the monomials listed in *forbidden*. For the moduli spaces of types $\mathscr{J}$, E_{13} and E_{14}, the code assumes $g = w^2 + h(x, y, z)$. The example for which mixed (infinitesimal) Torelli holds were found by perturbation of the basic examples. The examples for which the pure (infinitesimal) Torelli theorem holds were found by generating a random element of the moduli space.

```
#a=3; b=7;   ex_dim = 18; #Type 1. Different ex_dim for J, E13, E14 below.
#a=3; b=5;  ex_dim = 17; #Type 2
#a=5; b=7;  ex_dim = 16; #Type 3
#a=7; b=11; ex_dim = 18  #Type 4
d = a+b+4
R=PolynomialRing(QQ,"x,y,z,w",order=TermOrder("wdeglex",(1,2,a,b)))
x,y,z,w=R.gens()
# Code assumes g = w^2 + h(x,y,z) in the cases J, E_13 and E_14
# Examples with non-zero kernels and infinitesimal mixed Torelli
# g = x^(14) + y^7 + y*z^4 + w^2 + x^2*y^3*z^2 # Type 1
# g = x^(12) + y^6 + z^4 + y*w^2 + x^2*y^2*z^2 # Type  2
# Type J:
# g = x^2*z^4 + y*z^4 + x^5*z^3 +x^8*z^2 + y^7 +x^(10)*y^2 - w^2 + x^2*y^3*z^2
# g = y^7 +y^2*x^(10) + y*z^4 + x^5*z^3 + z^2*x^8 - w^2 #E13
# g = y^7 +y^2*x^(10) + y*z^4 + x^3*y*z^3 + z^2*x^8 - w^2 #E14
# Examples with trivial kernels (i.e.,  infinitesimal Torelli holds)
# g = x^(14) + x^2*z^4 + 2*x*y^2*z^3 + y^7 + y^4*z^2 + y*z^4 + w^2 # Type 1
# g = x^(12) + x*z^2*w + y^6 + y^2*z*w + y*w^2 + z^4 # Type 2
# g = x^(16) + y^8 + x*z^3 + y*w^2 + y^3*z^2 # Type 3
```

```
# g = w^2 + x*z^3 + y^4*z^2+x^(20)*y + y^(11) # Type 4
# Type J:
# A = x^(10)*y^2 + x^8*z^2 + 2*x^6*y*z^2 - 2*x^5*y^3*z + x^2*z^4;
# B = - 2*x*y^2*z^3 + y^7 + y^4*z^2 + y*z^4 + w^2; g = A+B;
# Type E13
# A = x^10*y^2 + x^8*z^2 + 2*x^5*z^3 + 2*x^4*y^2*z^2 + 2*x*y^2*z^3;
# B = y^7 + y^4*z^2 + y*z^4 + w^2; g = A+B;
# The parameters forbidden, ex_dim:
# forbidden=[]; #For types 1-4
# forbidden=[x^14,x^11*z,x^12*y]; ex_dim=17 #J surface
# forbidden=[x^14,x^11*z,x^12*y,x^2*z^4]; ex_dim=16; #E13 surface
# forbidden=[x^14,x^11*z,x^12*y,x^2*z^4,x^5*z^3]; ex_dim = 15; #E14 surface
Md=[R.monomial(*e) for e in WeightedIntegerVectors(d,(1,2,a,b))]
[Md.remove(m) for m in forbidden]
J=g.jacobian_ideal()
gx=g.derivative(x); gy=g.derivative(y);
gz=g.derivative(z); gw=g.derivative(w);
Z=Sequence([x*gx,x^2*gy,y*gy]);
Ma=[R.monomial(*e) for e in WeightedIntegerVectors(a,(1,2,a,b))]
[Z.append(m*gz) for m in Ma]
Mb=[R.monomial(*e) for e in WeightedIntegerVectors(b,(1,2,a,b))]
[Z.append(m*gw) for m in Mb]
# Z = Degree d component of J.
W,n=Z.coefficient_matrix(); jd = rank(W);
print("Dimension of J_d = ", jd)
X=Sequence([m for m in Md if m.reduce(J)==m]); L=Set(X)
rx = L.cardinality(); print("Cardinality of X = ",rx);
[Z.append(m) for m in Md if m.reduce(J)==m];
U,n=Z.coefficient_matrix(); ru=U.rank()
print("Dimension of (J_d + span(X)) = ",ru)
# X gives a basis for the tangent space to the deformation space if
# ru = rx + jd and rx = ex_dim
if ((ru==rx+jd) and (rx==ex_dim)): #This code must be indented.
     print("X is a basis of the tangent space, calculating the kernel dimension.")
     T=Sequence([x^2*gx, y*gx])
     M3 = [R.monomial(*e) for e in WeightedIntegerVectors(3,(1,2,a,b))]
     [T.append(m*gy) for m in M3]
     Ma = [R.monomial(*e) for e in WeightedIntegerVectors(a+1,(1,2,a,b))]
     [T.append(m*gz) for m in Ma]
     Mb = [R.monomial(*e) for e in WeightedIntegerVectors(b+1,(1,2,a,b))]
     [T.append(m*gw) for m in Mb]
     # Degree d+1 component of J.
     D, l = T.coefficient_matrix()
     r2 = D.rank(); print("Dimension of J_{d+1} = ",r2)
     [T.append((x*m)) for m in X]
     D, l = T.coefficient_matrix()
     r3 = D.rank(); print("Dimension of J_{d+1} + Im(span(X)) = ",r3)
     print("Kernel dimension = ",rx+r2-r3)
     if(rx+r2-r3>0):
     # Find the kernel.
         print("Calculating kernel.");
         c = D.ncols(); P = D.submatrix(0,0,r2,c); P1=P.row_space();
         Q = D.submatrix(r2,0,rx,c); Q1=Q.row_space();
         B = P1.intersection(Q1);
         B1 = B.basis_matrix();
         for j in range(B1.nrows()):
             s=[]
             [s.append(B1[j,k]*l[k]) for k in range(c)]
             m=(sum(s))[0]
             t, r = m.quo_rem(x)
             print("t=",t,"| x*t mod J=",(x*t).reduce(J),"| t mod J=|",t.reduce(J))
     if((Set(forbidden)).is_empty()):
         Md= [R.monomial(*e) for e in WeightedIntegerVectors(d,(1,2,a,b))]
         K = Set(Md)
```

```
        Md1=[R.monomial(*e) for e in WeightedIntegerVectors(d+1,(1,2,a,b))]
        L = Set(Md1)
        print("dim (R/J)_d = ",K.cardinality()-jd)
        print("dim (R/J)_{d+1} = ",L.cardinality()-r2)
else:
    print("X is not a basis of tangent space, exiting.")
```

Appendix A.4 Calculations involving the E_{12}, E_{13} and E_{14}-singularities

Appendix A.4.1 *Reduction to the type* $(14,[1,2,3,7])$

Recall that the singularity types determine data (p,q,d) from the exponents of the occurring monomials $x^ay^bz^b$ via the weight rule (7.3). For the the singularity types E_{12}, E_{13} and E_{14} in [17], these data are

$$E_{12}:(3,7,21),\quad E_{13}:(2,5,15),\quad E_{14}:(3,8,24).$$

As noted in [17, Section 3], the following 19 monomials $x^ay^bz^c\leftrightarrow(a,b,c)$ have non-negative weight for E_{12}, E_{13} and E_{14}:

$$\begin{aligned}&(0,0,5),\ (0,2,4),\ (0,4,3),\ (0,6,2),\ (0,8,1),\ (0,10,0),\\&(1,3,3),\ (1,1,4),\ (1,5,2),\ (1,7,1),\ (1,9,0),\ (2,0,4),\\&(2,2,3),\ (2,4,2),\ (2,6,1),\ (2,8,0),\ (3,1,3),\ (3,3,2),\ (4,0,3).\end{aligned}$$

The monomial $x^3y^5z\leftrightarrow(3,5,1)$ occurs in non-negative weight for both E_{12} and E_{13}. Finally, the monomial $x^3y^7\leftrightarrow(3,7,0)$ occurs in non-negative weight only for E_{12}. After multiplying each of the monomials in the previous list by z^2 and converting to the variables $x_0=y$, $x_1=z$ and $x_2=xz$, we see that the previous list becomes $x^ay^bz^c\mapsto x_0^bx_1^{c+2-a}x_2^a\leftrightarrow(b,c+2-a,a)$:

$$\begin{aligned}&(0,7,0),\ (2,6,0),\ (4,5,0),\ (6,4,0),\ (8,3,0),\ (10,2,0),\\&(3,4,1),\ (1,5,1),\ (5,3,1),\ (7,2,1),\ (9,1,1),\ (0,4,2),\\&(2,3,2),\ (4,2,2),\ (6,1,2),\ (8,0,2),\ (1,2,3),\ (3,1,3),\ (0,1,4).\end{aligned}$$

The remaining E_{13} monomial x^3y^5z maps to $x^3y^5z^3=x_0^5x_2^3$. The monomial x^3y^7, which occurs only in E_{12}, does not transform into a degree 14 monomial in x_0,x_1 and x_2 by this process.

Direct enumeration shows that there are 24 monomials of degree 14 in $\mathbb{P}[1,2,3]$. Thus, there are 4 monomials missing from the list for E_{13}: since this highest power of y which can appear in degree 10 in $\mathbb{P}[1,1,2]$ is 10, it follows that we miss the monomials x_0^{14}, $x_0^{12}x_1$ and $x_0^{11}x_2$. We also miss the monomial $x_0^2x_2^4=x^4y^2z^4$ since this comes by multiplying $x^4y^2z^2$ by z^2, and $x^4y^2z^2$ has weight $\omega=(2)(2)+(2)(5)-15=-1$.

In particular, since we do not have the monomial x_0^{14}, a curve $B=V(g)$ arising from the E_{13} or E_{14} singularity will always pass through the point $(x_0:x_1:x_2)=(1:0:0)$. Moreover, since we do not have the monomials $x_0^{12}x_1$ and $x_0^{11}x_2$ it follows that $\nabla g=0$ at $(1:0:0)$.

Appendix A.4.2 *The* $\mathscr{J}$*-locus*

Proposition A.1. *The singular locus of the degree* 14 *surface* $V(f)\subset\mathbb{P}[1,2,3,7]$ *defined by the equation*

$$f=x_0^2x_2^4+x_1x_2^4+x_0^5x_2^3+x_0^8x_2^2+x_1^7+x_0^{10}x_1^2-x_3^2 \tag{A.2}$$

consists of an A_1*-singularity at the point* $[1:0:0:0]$ *and the finite quotient singularity at the point* $[0:0:1:0]$ *which* $V(f)$ *inherits from* $\mathbb{P}[1,2,3,7]$*. Moreover, since the defining equation of this surface contains the term* $x_0^2x_2^4$*, it is not contained in the* E_{13} *or* E_{14} *locus. The associated smooth elliptic surface has the fiber structure* $2I_0+I_2+22\times I_1$.

Proof. Dividing by x_0^{14} and setting $\zeta=x_1/x_0^2$, $\nu=x_2/x_0^3$ and $\omega=x_3/x_0^7$ gives

$$\omega^2=(1+\zeta)\nu^4+\nu^3+\nu^2+\zeta^7+\zeta^2$$

and thus one has an I_2 fiber over $(1:0)$[15].

Taking the discriminant of the right-hand side of the previous equation with respect to ν gives

$$(\zeta^6-\zeta^5+\zeta^4-\zeta^3+\zeta^2)(256\zeta^{18}+1024\zeta^{17}+1536\zeta^{16}+1024\zeta^{15}+256\zeta^{14}\\+512\zeta^{13}+2048\zeta^{12}+3072\zeta^{11}+1920\zeta^{10}+272\zeta^9+133\zeta^8+1013\zeta^7\\+1536\zeta^6+896\zeta^5+16\zeta^4-123\zeta^3+5\zeta^2+28\zeta+12).$$

Factoring out ζ^2 and taking the resultant of the remaining two polynomials of degree 4 and 18 give 999,680. Thus, the discriminant has 22 simple roots and one double root at $\zeta=0$. The existence of $2I_0$ fiber is the same as the generic surface of the type $(14,[1,2,3,7])$. The fiber structure is, therefore, $2I_0+I_2+22\times I_1$ as claimed.

The analysis of the singular locus for this surface is exactly the same as the surface presented at the end of Appendix A. The only difference between the surface considered here and the surface presented there is the term $x^2y^3z^2$. The Jacobian ideal contains the monomials z^{10} and w, and the condition to have singular point at p is $g_x=10x^9y^2$ and $g_y=2x^{10}y+7y^6$. Therefore $(1:0:0:0)$ is the only singular point of the surface. □

Transferring equation (A.2) back to $\mathbb{P}(1,1,2,5)$ by dividing by z^2 after setting $x_0=y$, $x_1=z$, $x_2=xz$ and $x_3=zw$ yields the defining equation

$$w^2=f(x,y,z),\quad f(x,y,z)=x^4y^2z^2+x^4z^3+x^3y^5z+z^5+x^2y^8+y^{10}.$$

In this case, the branch curve $V(f)$ has a singularity of the type $J[2,2]$ at the point $(1:0:0)$. This can be confirmed by the following sage code, which also shows that this singularity has modality 1 and Milnor number $\mu=12$ (set $x=1$ to work in an affine chart).

```
r = singular.ring(0,'(y,z)', 'ds')
singular.lib('classify.lib')
h = singular.new('y^2*z^2 + z^3 + y^5z + z^5 + y^8 + y^(10)')
print(singular.eval('classify({})'.format(h.name())))
```

Remark A.2. As shown above, $\mathscr{J}$ has dimension 17, which matches the dimension formula $29-\mu$ of the other modality 1 singularities considered above.

Appendix A.4.3 *The case* E_{13}

A typical E_{13} branch curve is defined by the equation

$$f=z^5+y^{10}+x^4z^3+x^3y^5z+x^2y^8,$$

which becomes

$$g=x_1^7+x_1^2x_0^{10}+x_1x_2^4+x_2^3x_0^5+x_2^2x_0^8. \tag{A.3}$$

To analyze the singularity at the point $(x_0:x_1:x_2)=(1:0:0)$, we divide by x_0^{14} and introduce the new variables $\zeta=x_1/x_0^2$ and $\nu=x_2/x_0^3$ to obtain

$$\zeta^7+\zeta^2+\nu^4\zeta+\nu^3+\nu^2. \tag{A.4}$$

The lowest order term here is $\zeta^2+\nu^2$, which produces an A_1 surface singularity at $(1:0:0:0)$. As shown at the end of Appendix A, this surface has no other singularities.

To continue the analysis of the birational models of the E_{13} surfaces as degree 14 hypersurfaces in $\mathbb{P}[1,2,3,7]$, we consider the fibration to $\mathbb{P}^1$ given by $(x_0:x_1:x_2:x_3)\mapsto(x_0^2:x_1)$. By (A.4), the fiber over the point $(x_0^2:x_1)=(1:\zeta)$ is given by $\nu^4\zeta+\nu^3+\nu^2+\zeta^7+\zeta^2=\omega^2$, where $\zeta=x_1/x_0^2$, $\nu=x_2/x_0^3$

15) The only monomials $x_0^ax_1^bx_2^c$ which occur in $\mathscr{J}$ and survive evaluation at $(x_0^2:x_1)=(1:0)$ are $x_2^4x_0^2$, $x_2^2x_0^8$ and $x_2^3x_0^5$, so the analysis presented here is the generic case.

and $\omega = x_3/x_0^7$ in the affine chart $x_0 \neq 0$ of $\mathbb{P}[1,2,3,7]$. The fiber over $\zeta = 0$ is the nodal cubic[16)] $\omega^2 = \nu^2(\nu+1)$ whose singularity at the point $(0,0)$ in the (ν,ω)-plane coincides with the A_1-singularity at the point $(1:0:0:0)$ on the surface. Resolving this singularity, we obtain an I_2-fiber. Just like the generic degree 14 surface in $\mathbb{P}[1,2,3,7]$, the fiber over $(0:1)$ is of the type $2I_0$. To finish the analysis of the fibers of π, we calculate the discriminant D of the polynomial (A.4) with respect to ν:

$$D = \zeta^2(\zeta^5+1)(256\zeta^{17}+512\zeta^{12}-128\zeta^9+144\zeta^8+229\zeta^7-128\zeta^4+144\zeta^3-27\zeta^2+16\zeta-4).$$

The discriminant of the degree 17 factor of D with respect to the variable ζ is an 81 digit integer. Thus, π also has 22 I_1 fibers.

Appendix A.4.4 *The case E_{14}*

The analysis of this case is similar. A typical E_{14} branch curve is $f = z^5 + y^{10} + x^4z^3 + x^2y^8$ which becomes

$$g = x_1^7 + x_1^2x_0^{10} + x_1x_2^4 + x_2^2x_0^8. \tag{A.5}$$

Setting $x_0 = 1$ and letting $\zeta = x_1/x_0^2$ and $\nu = x_2/x_0^3$ as above, we have

$$\zeta^7 + \zeta^2 + \zeta\nu^4 + \nu^2 \tag{A.6}$$

so again we have an A_1 surface singularity at $(1:0:0:0)$.

To determine the elliptic fibration structure, we compute the discriminant D of

$$\zeta^7 + \zeta^2 + \zeta\nu^4 + \zeta\nu^3 + \nu^2, \tag{A.7}$$

which yields

$$\zeta^3(\zeta^5+1)(256\zeta^{16}+512\zeta^{11}-27\zeta^{10}+144\zeta^9-128\zeta^8+256\zeta^6-27\zeta^5+144\zeta^4-128\zeta^3-4\zeta+16).$$

The discriminant of the degree 16 factor of D is a 77 digit integer, and hence this factor has no multiple roots. The resultant of the degree 5 and 16 factors is 1,049,600. We also retain the $2I_0$ fiber over $(x_0^2 : x_1) = (0:1)$. Thus, the fibration structure of this surface is $2I_0 + I_3 + 21 \times I_1$. Observe that for the generic E_{14}-surface, the affine form of the fiber at $\zeta = 0$ is $\nu^2 - \omega^2 = 0$ and so gives 2 smooth rational curves; the A_1-singularity contributes another rational component, confirming the I_3-structure at $\zeta = 0$ for the generic E_{14}-surface. Thus, the generic E_{14}-surface also has the fiber type $2I_0 + I_3 + 21 \times I_1$.

Remark A.3. Let $S \subset \mathbb{P}[1,1,2,5]$ be the surface of the type E_{12} defined by the equation

$$x^4z^3 + x^3y^7 - xy^9 + y^{10} + z^5 - w^2 = 0$$

and $\pi : \tilde{S} \to \mathbb{P}^1$ be the elliptic surface obtained by resolving the indeterminacies of the map $(x:y:z:w) \mapsto (y^2:z)$. Then, $\pi^{-1}(1:\lambda)$ is $\lambda^3X^4 + X^3 - X + (1+\lambda^5) = W^2$, where $X = x/y$ and $W = w/y^5$. The discriminant of the left-hand side of this equation is a degree 24 polynomial without multiple roots, and the fiber over $(1:0)$ is an irreducible elliptic curve.

Appendix A.5 Code for Appendix C

The polynomial F of (C.1) is defined as follows:

```
WP.<x0,x1,x2,x3> = PolynomialRing(ZZ)
G  = x0*x2^2 + x0^4*x2 + 3*x1^2*x2
G0 = -1
G3 = x0^6 + 2*x0^4*x1 + x0^2*x1^2 + 2*x1^3
G4 = 4*x0^6*x1+2*x0^4*x1^2 + x0^2*x1^3 + 4*x1^4
G6 = x0^12 + 3*x0^10*x1 + 3*x0^8*x1^2 + x0^4*x1^4 + 3*x0^2*x1^5 + x1^6
F = x1*x3^2 + G*x3 + G0*x2^4 + G3*x2^2 + G4*x0*x2 + G6
```

For practical reasons, we introduced here instead $G = G_{1,2} \cdot x_1^2 + G_2 \cdot x_0$.

[16)] The only monomials $x_0^a x_1^b x_2^c$ which occur in E_{13} and survive evaluation at $(x_0^2 : x_1) = (1:0)$ are $x_2^2x_0^8$ and $x_2^3x_0^5$, so the analysis presented here is the generic case.

Appendix A.5.1 *Checking quasi-smoothness*

Next, we choose $p \in \{2, 3\}$ and check if the surface X_p over $\mathbb{F}_p$ is quasi-smooth. This is done by checking if the radical of the ideal generated by F and its derivatives is equal to the irrelevant ideal. The outcome of this check is 'true', which shows that the surface X_p is quasi-smooth.

```
p = 2    # (or p = 3)
Wp.<xp0,xp1,xp2,xp3> = PolynomialRing(GF(p))
f = F(xp0,xp1,xp2,xp3)
f0 = f.derivative(xp0)
f1 = f.derivative(xp1)
f2 = f.derivative(xp2)
f3 = f.derivative(xp3)
I = Ideal([f,f0,f1,f2,f3])
RI = I.radical()
RI == Ideal([xp0,xp1,xp2,xp3])
```

Appendix A.5.2 *Checks for the arithmetic surface*

Here, we give the code that checks if the arithmetic surface $\mathscr{C}$ is smooth. This is done on the two affine parts of the surface. In both cases, we check that the defining equation together with its derivatives generates the whole ring. The outcome of these checks is 'true', which shows that the surface is smooth.

```
CF = x1*F(x0,x1,x2,x3/x1)
R.<t,x,y> = PolynomialRing(GF(p))
CF1 = CF(1,t,x,y)
Ft1 = CF1.derivative(t)
Fx1 = CF1.derivative(x)
Fy1 = CF1.derivative(y)
I = Ideal([CF1,Ft1,Fx1,Fy1])
I == (1)

CF2 = x^4*CF(1,t,1/x,y/x^2)
Ft2 = CF2.derivative(t)
Fx2 = CF2.derivative(x)
Fy2 = CF2.derivative(y)
I = Ideal([CF2,Ft2,Fx2,Fy2])
I == (1)
```

Appendix A.5.3 *Calculating the discriminant*

In the next part of the code, we calculate the discriminant of the defining polynomial F' of the arithmetic surface $\mathscr{C}$. To calculate this discriminant, we first calculate it over $\mathbb{Z}[t]$, which is named `pDisc` in the code. The outcome of this part of the code gives the factorization of this polynomial modulo p, which is the polynomial Δ_p given in Lemma C.16.

```
Fd.<tt> = FunctionField(QQ)
Rd.<xd> = PolynomialRing(Fd)
fd = F(1,Fd.0,Rd.0,0)*Rd.0
hd = G(1,Fd.0,Rd.0,0)
pDisc = 4^(-4)*discriminant(hd^2-4*fd)
R.<t> = PolynomialRing(GF(p))
Disc = R(pDisc.numerator())
Disc.factor()
```

Appendix A.5.4 *Counting the points*

The following part of the code counts the points on the surface X'_p following the method described in the proof of Proposition C.21. For $1 \leqslant n \leqslant 9$, we count the $\mathbb{F}_{p^n}$-points of X'_p at once and save the number we find in the list with the name `Count`.

```
Count = []
for i in range(1, 10):
     q = p^i
    Fq = GF(q)
    A2 = AffineSpace(2, Fq)
    R.<t> = PolynomialRing(Fq)
   # count points above (0:1)
    f = -F(0,1,R.0,0)
    h = G(0,1,R.0,0)
    C = HyperellipticCurve(f,h)
    count = C.cardinality()
     # count points above (1:a)
    for a in Fq:
        if Disc(a) == 0:
            g = t^2+t-a
            r = Set(g.roots()).cardinality()
            f = CF(1,a,A2.0,A2.1)
            C = Curve(f,A2)
            count = count + C.count_points(1)[0] + r
        else:
            f = -F(1,a,R.0,0)*a
            h = G(1,a,R.0,0)
            C = HyperellipticCurve(f,h)
            count = count + C.cardinality()
   Count.append(count)
# print total number of Fq-points for q=p^i with 0<i<10:
Count
```

Remark A.4. The code for counting the points takes a lot of time when $p = 3$ (roughly 18 hours on Mac OS with Apple M1 processor and 8GB RAM). For finding the surfaces, we used other software, namely, MAGMA (see [7]). This software is faster and makes it easier to search through many surfaces over $\mathbb{F}_2$ and $\mathbb{F}_3$, respectively, to find those that satisfy the required conditions.

Appendix A.5.5 *Computing the characteristic polynomial*

The last part of the code is used for the proof of Proposition C.23. First, we compute the coefficients c_i.

```
# Calculate the values of the trace
Tr = []
for i in range(1, 10):
    Tr.append((Count[i-i] - 1 - p^(2*i)-3*p^i)/(p^i))
# Calculate the values of the coefficients
coef = [1,-Tr[0]]
for i in range(1, 9):
    sum = 0
    for j in range(1,i+1):
        sum = sum + Tr[i-j]*coef[j]
    coef.append(-(Tr[i]+sum)/(i+1))
# Print the coefficients; N.b. first value is c0, not c1:
coef
```

In the next part of the code, we define both possible polynomials using the functional equation.

```
R.<t>=PolynomialRing(QQ)
# Defining polynomial with positive sign of func eq
coefp = [0] * 20
for i in range (0,10):
    coefp[i] = coef[i]
    coefp[19-i] = coef[i]
wp = R(coefp)
# Defining polynomial with negative sign of func eq
coefn = [0] * 20
for i in range (0,10):
```

```
    coefn[i] = -coef[i]
    coefn[19-i] = coef[i]
wn = R(coefn)
```

To exclude the polynomial where the sign is positive, we can print the absolute values of the roots by using the following line of code:

```
for root, _ in wp.roots(CC): print(abs(root))
```

The outcome will give a list of absolute values of the roots, of which four are not equal to 1. As a sanity check, by using the same line of code with `wn` instead of `wp`, we can see that all the roots indeed have absolute value 1.

Lastly, we factored the polynomial by using the function `wn.factor()`. This gives us a factor $t-1$ and the other factor is the irreducible polynomials h_p of degree 18, which is given in Proposition C.23.

Appendix A.5.6 *A similar case for degree* 14

Here, we give the code from which one can deduce that for a general choice of a quasi-smooth surface X of degree 14 in $\mathbb{P}_k(1,2,3,7)$, and a minimal desingularization X' has Picard rank 2. In the below code, quasi-smoothness is omitted, but it can be checked with the exact same code as in Appendix A.5.1. Also, the verification that the model we use is correct, is omitted. We highlight the differences in the code with the degree 12 case.

```
# Defining polynomial
WP.<x0,x1,x2,x3> = PolynomialRing(ZZ)
G = x0*x2^2 + x0^4*x2 + x1^2*x2
G2 = x0^2*x1
G5 = x0^8*x1 + x0^2*x1^4 + x1^5
G7 = x0^14 + x0^12*x1 + x0^10*x1^2 + x0^6*x1^4 + x0^2*x1^6 + x1^7
F = x1*x2^4 + x3^2 + G*x3 + G2*x0*x2^3 + G5*x0*x2 + G7
# Calculating the discriminant
p = 2
Fd.<tt> = FunctionField(QQ)
Rd.<xd> = PolynomialRing(Fd)
fd = F(1,Fd.0,Rd.0,0)
hd = G(1,Fd.0,Rd.0,0)
pDisc = 4^(-4)*discriminant(hd^2-4*fd)
R.<t> = PolynomialRing(GF(p))
Disc = R(pDisc.numerator())
# Counting the points
Count = []
for i in range(1, 11):
# Note that we now count one more extension,
# also the code below is adjusted accordingly.
    q = p^i
    Fq = GF(q)
    A2 = AffineSpace(2, Fq)
    R.<t> = PolynomialRing(Fq)
    f = -F(0,1,R.0,0)
    h = G(0,1,R.0,0)
    C = HyperellipticCurve(f,h)
    count = C.cardinality()
    for a in Fq:
        if Disc(a) == 0:
            g = t^2+t+a #sign change, although not necessary
            r = Set(g.roots()).cardinality()
            f = F(1,a,A2.0,A2.1) #changed defining polynomial
            C = Curve(f,A2)
            count = count + C.count_points(1)[0] + r
        else:
            f = -F(1,a,R.0,0) #changed defining polynomial
            h = G(1,a,R.0,0)
```

```
            C = HyperellipticCurve(f,h)
            count = count + C.cardinality()
        Count.append(count)
    # Calculating the traces
    Tr = []
    for i in range(1, 11):
        Tr.append((Count[i-1] - 1 - p^(2*i)-2*p^i)/(p^i))
    # Note the slight change in the formula for the trace,
    # because we now know that there is a 2-dim subspace
    # on which Frobenius is acting trivial and not 3-dim.
     # Calculating the coefficients
    coef = [1,-Tr[0]]
    for i in range(1, 10):
        sum = 0
        for j in range(1,i+1):
            sum = sum + Tr[i-j]*coef[j]
        coef.append(-(Tr[i]+sum)/(i+1))
    coef[10]
    # Because the coefficient c10 is non-zero,
    # the functional equation gives us that the
    # other coefficients are positive as well.
    # Calculating the characteristic polynomial of Frobenius
    R.<t>=PolynomialRing(QQ)
    for i in range (0,10):
        coef.append(coef[9-i])
    wp = R(coef)
    wp.factor()
```

The outcome of the code gives us an irreducible polynomial h with

$$h := \tfrac{1}{2}(2t^{20} - 2t^{18} + t^{16} - t^{14} + t^{13} + t^{12} - t^{11} - t^{10} - t^9 + t^8 + t^7 - t^6 + t^4 - 2t^2 + 2),$$

which has no roots of unity as zeros. We deduce that the characteristic polynomial of Frobenius acting on $H^2_{\text{ét}}((X_2')_{\overline{\mathbb{F}}_2}, \mathbb{Q}_\ell(1))$ equals $(t-1)^2 \cdot h$. We conclude that for any minimal desingularization of a quasi-smooth surface X of degree 14 in $\mathbb{P}_\mathbb{Q}(1,2,3,7)$, for which the reduction at the prime 2 is isomorphic to X_2, we have $\rho(X') = \rho(X'_{\overline{\mathbb{Q}}}) = 2$.

Appendix B Normal forms: Proofs

We give indications of the proof of Proposition 3.1 concerning normal forms of quasi-smooth hypersurfaces $(F=0)$ in $\mathbb{P}(1,2,a,b)$ of degree $d = a+b+4$. Note that in the case where $(a,b)=(3,7)$ and $(a,b)=(7,11)$, one has $d=2c$ which means that the surface is a double cover of $\mathbb{P}(1,2,a)$ branched in a degree d quasi-smooth curve C. It then suffices to write a normal form for the polynomial F_C defining C and then $F = F_C - x_3^2$. This deals with 2 cases.

Lemma B.1. (1) *If $(a,b)=(3,7)$, then via the automorphism group of $\mathbb{P}(1,2,3)$, the polynomial F_C can be put in the form*

$$F_C = x_1x_2^4 + G_0x_0^5x_2^3 + G_4(x_0^2,x_1)x_2^2 + x_0G_5(x_0^2,x_1)x_2 + G_7(x_0^2,x_1), \tag{B.1}$$

where G_j is an ordinary *polynomial of degree j in two variables. The subgroup of* $\operatorname{Aut}\mathbb{P}(1,2,3)$ *preserving a normal form of the type* (B.1) *consists of transformations of the form $x_j \mapsto c_jx_j$ with $c_j \in \mathbb{C}^*$ and $c_2^4c_1 = 1$.*

(2) *If $(a,b)=(7,11)$, then provided that the coefficient of $x_1^4x_2^2$ is non-zero, via the automorphism group of $\mathbb{P}[1,2,7]$, the polynomial F_C can be put in the form*

$$F_C = x_0x_2^3 + G_0x_1^4x_2^2 + x_0G_7(x_0^2,x_1)x_2 + G_{11}(x_0^2,x_1), \quad G_0 \neq 0, \tag{B.2}$$

where G_j is an ordinary *polynomial of degree j in two variables, and the coefficient of x_0^{22} in $G_{11}(x_0^2,x_1)$ is zero. The subgroup of* $\operatorname{Aut}\mathbb{P}(1,2,7)$ *preserving a normal form of the type* (B.2) *consists of transformations of the form $x_j \mapsto c_jx_j$ with $c_j \in \mathbb{C}^*$ and $c_0c_3^2 = 1$.*

In both cases, the stabilizer of F_C is generically the identity.

Proof. (1) Since 3 is not a divisor of 14, every degree 14 curve in $\mathbb{P}(1,2,3)$ will pass through the singular point $[0,0,1]$ of $\mathbb{P}(1,2,3)$. Thus, to be quasi-smooth, the coefficient of $x_2^4x_1$ in F_C has to be non-zero, otherwise $\nabla F_C = 0$ at $[0,0,1]$. Accordingly, we can write $F_C = x_2^4P_2 + x_2^3P_5 + x_2^2P_8 + x_2P_{11} + P_{14}$, where $P_j = P_j(x_0,x_1)$ is homogeneous of weighted degree j and $P_2(x_0,x_1) = \alpha_1x_1 + \alpha_2x_0^2$ with $\alpha_1 \neq 0$.

The automorphism group of $\mathbb{P}(1,2,3)$ consists of invertible transformations of the form

$$[x_0,x_1,x_2] \mapsto [a_0x_0, a_1x_1 + a_2x_0^2, a_3x_2 + a_4x_0x_1 + a_5x_0^3]. \tag{B.3}$$

In particular, via the transformation $[x_0,x_1,x_2] \mapsto [x_0, P_2(x_0,x_1), x_2]$, we can reduce the defining equation of C to the form

$$x_2^4x_1 + x_2^3P_5 + x_2^2P_8 + x_2P_{11} + P_{14}. \tag{B.4}$$

Next, we observe that $P_5(x_0,x_1) = x_0(b_0x_1^2 + b_1x_1x_0^2 + b_2x_0^4) = x_1(b_0x_1x_0 + b_1x_0^3) + b_2x_0^5$. Therefore, setting $G_0 = b_2$ and using the transformation

$$[x_0,x_1,x_2] \mapsto \left[x_0, x_1, x_2 - \frac{1}{4}(b_0x_1x_0 - b_1x_0^3)\right],$$

we can reduce the definition of C to the form

$$x_2^4x_1 + G_0x_2^3x_0^5 + x_2^2P_8 + x_2P_{11} + P_{14}. \tag{B.5}$$

To obtain the normal form (B.1), we now observe that since x_0 has degree 1 while x_2 has degree 2, we can write $P_8 = G_4(x_0^2,x_1)$, $P_{11} = x_0G_5(x_0^2,x_1)$ and $P_{14} = G_7(x_0^2,x_1)$, where now G_j's are ordinary polynomials of degree j.

To finish the proof of (1), we note that the given set of automorphisms clearly acts on the normal form (B.1). On the other hand, to obtain the reduction (B.4), we must use a combination of transformations of the form $x_1 \mapsto a_1x_1 + a_2x_0^2$ and $x_2 \mapsto a_3x_2$. This fixes a_2 and the product a_1a_3. Likewise, the reduction (B.5) fixes the coefficients a_4 and a_5.

(2) This is a bit more involved. As in the case (1), we write $F_C = x_2^3x_0 + x_2^2P_8 + x_2P_{15} + P_{22}$, where $P_j = P_j(x_0,x_1)$ is weighted homogeneous of degree j and rewrite this equation as

$$F_C = x_2^3x_0 + x_2^2G_4(x_0^2,x_1) + x_2x_0G_7(x_0^2,x_1) + G_{11}(x_0^2,x_1)$$

in terms of ordinary degree j polynomials G_j in x_0^2 and x_1. Since the coefficient b_4 in $G_4(x_0^2,x_1) = b_0x_0^8 + b_1x_0^6x_1 + b_2x_0^4x_1^2 + b_3x_0^2x_1^3 + b_4x_1^4$ is non-zero, using an automorphism of $\mathbb{P}(1,2,7)$ of the form $x_1 \mapsto x_1 + \beta x_0^2$, we may assume that the coefficient of x_0^8 equals 3λ, where λ^3 is the coefficient of x_0^{22}. In this way, we obtain

$$F_C = x_2^3x_0 + x_2^2(3\lambda x_0^8 + x_0^2x_1q_2(x_0^2,x_1) + G_0x_1^4) + x_2x_0q_7(x_0^2,x_1) + G_{11}(x_0^2,x_1),$$

where $G_0 = b_4$. Next, we consider the automorphism

$$x_2 \mapsto x_2 - x_0x_1G_2(x_0^2,x_1)/3 - \lambda x_0^7.$$

Then, $x_0x_2^3$ transforms into $x_0x_2^3 - x_2^2(3\lambda x_0^8 + x_0^2x_1q_2(x_0^2,x_1)) + x_2(\cdots) - \lambda^3x_0^{22} + x_1(\cdots)$ and F_C becomes

$$F_C = x_0x_2^3 + G_0x_1^4x_2^2 + x_0x_2G_7(x_0^2,x_1) + G_{11}(x_0^2,x_1),$$

where now the coefficient of x_0^{22} is zero.

Finally, the given transformations preserve the normal form, and unipotent mixing of the variables destroys the given normal form.

The last assertion follows by considering the relations imposed on the coefficients of F_C if $(c_0,c_1,c_2) \in (\mathbb{C}^*)^3$ fixes each of them. $\square$

It is clear that the statement of Lemma B.1 implies the parts (1) and (4) in Proposition 3.1.

Now we consider the two cases which are not double covers. The next lemma implies the parts (2) and (3) in Proposition 3.1.

Lemma B.2. (1) *In the case* $(a,b)=(3,5)$ *via the automorphism group of* $\mathbb{P}(1,2,3,5)$*, the defining equation of* F *can be put in the form*

$$\begin{aligned} F = {} & x_1x_3^2 + x_0G_2(x_0^3,x_2)x_3 + G_0x_2^4 \\ & + G_3(x_0^2,x_1)x_2^2 + G_4(x_0^2,x_1)x_0x_2 + G_6(x_0^2,x_1), \quad G_0 \neq 0, \end{aligned} \tag{B.6}$$

where G_j *is an ordinary polynomial of degree* j *in two variables. The subgroup of* $\operatorname{Aut}\mathbb{P}(1,2,3,5)$ *preserving a normal form of the type* (B.6) *consists of transformations of the form* $x_j \mapsto c_jx_j$ *for* $c_j \in \mathbb{C}^*$ *with* $c_1c_3^2 = 1$.

(2) *In the case* $(a,b)=(5,7)$ *via the automorphism group of* $\mathbb{P}(1,2,5,7)$*, the defining equation of* F *can be put in the form*

$$F = x_1x_3^2 + x_0^2G_1(x_0^5,x_2)x_3 + r_0x_0x_2^3 + G_0x_1^3x_2^2 + x_0x_2G_5(x_0^2,x_1) + G_8(x_0^2,x_1), \tag{B.7}$$

where G_j *is an ordinary polynomial of degree* j *in two variables and* r_0 *is a non-zero constant. The subgroup of* $\operatorname{Aut}\mathbb{P}(1,2,5,7)$ *acting on the normal form* (B.7) *consists of transformations of the form* $x_j \mapsto c_jx_j$ *for* $c_j \in \mathbb{C}^*$ *with* $c_1c_3^2 = 1$.

In both cases, the stabilizer of F *is generically the identity.*

Proof. (1) The surface X will pass through the singular point $[0,0,0,1]$ and so the monomial $x_3^2x_1$ must therefore appear with a non-zero coefficient in f, otherwise $\nabla F = 0$ at $[0,0,0,1]$. We can therefore write $F = x_3^2p_2(x_1,x_0) + x_3P_7(x_0,x_1,x_2) + P_{12}(x_0,x_1,x_2)$, where $P_2(x_1,x_0) = \alpha_1x_1 + \alpha_2x_0^2$ with $\alpha_1 \neq 0$. Therefore, using the transformation $x_1 \mapsto \alpha_1x_1 + \alpha_2x_0^2$, we can reduce the defining equation of X to

$$F = x_1x_3^2 + x_3P_7(x_0,x_1,x_2) + P_{12}(x_0,x_1,x_2)$$

(of course, this changes P_7 and P_{12} as well).

We next simplify P_{12}. Note that if the coefficient of x_2^4 is zero, X passes through the singular point $[0,0,1,0]$ of $\mathbb{P}[1,2,3,5]$ and $\nabla F = 0$ at $[0,0,1,0]$ which violates the assumption that X be quasi-smooth. Thus we can write $P_{12} = q_0x_2^4 + b_3(x_0,x_1)x_2^3 + b_6(x_0,x_1)x_2^2 + b_9(x_0,x_1)x_2 + b_{12}(x_0,x_1)$, which can be rewritten as $P_{12} = G_0x_2^4 + G_1(x_0^2,x_1)x_0x_2^3 + G_3(x_0^2,x_1)x_2^2 + G_4(x_0^2,x_1)x_0x_2 + G_6(x_0^2,x_1)$, where each G_j is an ordinary polynomials of degree j. Finally, using the transformation $x_2 \mapsto x_2 - x_0G_1(x_0^2,x_1)/4G_0$, we can obtain the simplified form

$$P_{12} = G_0x_2^4 + G_3(x_0^2,x_1)x_2^2 + G_4(x_0^2,x_1)x_0x_2 + G_6(x_0^2,x_1),$$

possibly changing G_3, G_4 and G_6.

The previous transformation of x_2 will also have changed P_7 which we subsequently simplify as follows. Using a transformation of the form $x_3 \mapsto x_3 + P_5(x_0,x_1,x_2)$, we can remove all of the monomials from P_7 which are divisible by x_1, i.e., P_7 becomes

$$P_7 = x_0P_6(x_0,x_2), \quad \deg P_6 = 6$$

since x_2 has degree 3. This finally brings F in the desired form.

(2) Using that the surface X passes through the point $[0,0,0,1]$, we deduce that F must contain the monomial $x_3^2x_1$ so that the defining equation has the form

$$F = x_3^2P_2(x_0,x_1) + x_3P_9(x_0,x_1,x_2) + P_{16}(x_0,x_1,x_2),$$

where $P_2(x_0,x_1) = \alpha_0x_0^2 + \alpha_1x_1$ with $\alpha_1 \neq 0$. Therefore, using the transformation $x_1 \mapsto \alpha_0x_0^2 + \alpha_1x_1$, we can assume that

$$P_2(x_0,x_1) = x_1.$$

Next, we use a transformation of the form $x_3 \mapsto x_3 + P_7(x_0, x_1, x_2)$ to eliminate all of the terms of $P_9(x_0, x_1, x_2)$ which are divisible by x_1 so that

$$P_9 = P_9(x_0, x_2) = x_0^4 G_1(x_0^5, x_2)$$

with G_1 an ordinary polynomial of degree 1 in two variables. Note that we have potentially changed P_{12} which now will be written as

$$P_{12} = G_{12}(x_0, x_1, x_2).$$

Finally, we consider $P_{16}(x_0, x_1, x_2)$ which must contain $x_2^3 x_0$ to avoid creating a singularity at $[0, 0, 1, 0]$. Thus, we can write $P_{16}(x_0, x_1, x_2) = r_0 x_2^3 x_0 + x_2^2 P_6(x_0, x_1) + x_2 P_{11}(x_0, x_1) + R_{16}(x_0, x_1)$, which can be rewritten as $P_{16}(x_0, x_1, x_2) = r_0 x_0 x_2^3 + x_2^2 G_3(x_0^2, x_1) + x_0 x_2 G_5(x_0^2, x_1) + G_8(x_0^2, x_1)$, where G_j is an ordinary polynomial of degree j and r_0 is a constant. Using the transformation $x_2 \mapsto x_2 - x_0 q_2(x_0^2, x_1)/(3r_0)$, we can remove all of the terms of $x_2^2 G_3(x_0, x_2)$ which are divisible by x_0^2, i.e., all the terms except $x_1^3 x_2^2$. In other words,

$$P_{16}(x_0, x_1, x_2) = r_0 x_0 x_2^3 + G_0 x_1^3 x_2^2 + x_0 x_2 G_5(x_0^2, x_1) + G_8(x_0^2, x_1),$$

which brings F in the required shape. The group of substitutions which preserve this form is given by $x_j \mapsto a_j x_j$ for $a_j \in \mathbb{C}^*$, where $a_1 a_3^2 = 1$ to keep the coefficient of $x_3^2 x_1$ equal to 1.

The last assertion follows by considering the relations imposed on the coefficients of F_C if $(c_0, c_1, c_2, c_3) \in (\mathbb{C}^\times)^4$ fixes each of them. □

Appendix C The Picard number of the generic type 2 members (by Wim Nijgh)

Let k be an arbitrary field and let $\overline{k}$ be an algebraic closure of k. For any variety Y over k, we let $Y_{\overline{k}}$ denote its base change to $\overline{k}$. Furthermore, if Y is projective, we denote by $\mathrm{NS}(Y)$ the Neron-Severi group of Y and by $\mathrm{NS}(Y)_{\mathrm{tor}}$ its torsion subgroup. We denote by $\rho(Y)$ the Picard number of Y, which is the rank of $\mathrm{NS}(Y)$.

Appendix C.1 Overview

In the weighted projective space $\mathbb{P}_k(1, 2, 3, 5)$ with coordinates x_0, x_1, x_2 and x_3, we look at the family of quasi-smooth surfaces of degree 12. After some linear transformation, such a surface is given by an equation $F = 0$, where

$$\begin{aligned} F = {} & x_1 x_3^2 + G_0' x_0 x_1^3 x_3 + G_{1,1}(x_0^3, x_2) x_0^2 x_1 x_3 + G_{1,2}(x_0^3, x_2) x_1^2 x_3 + G_2(x_0^3, x_2) x_0 x_3 \\ & + G_0 x_2^4 + G_{1,3}(x_0^2, x_1) x_0 x_2^3 + G_3(x_0^2, x_1) x_2^2 + G_4(x_0^2, x_1) x_0 x_2 + G_6(x_0^2, x_1) \end{aligned} \tag{C.1}$$

such that $G_0, G_0' \in k$, each $G_{1,i}$ is homogeneous of degree 1 and each G_i is homogeneous of degree i. If $\mathrm{char}(k) \neq 2$, one can assume that

$$G_0' = G_{1,1} = G_{1,2} = G_{1,3} = 0$$

after some linear transformation and obtain the family described in Proposition 3.1(2).

Now let Y be a quasi-smooth surface of degree 12 in $\mathbb{P}_k(1, 2, 3, 5)$. Note that the only singular points in $\mathbb{P}_k(1, 2, 3, 5)$ are the points $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$. From (C.1), we observe that the point $(0 : 0 : 0 : 1)$ is always contained in the surface Y. If the coefficient of the monomial x_1^6 in G_6 is non-zero, then $(0 : 1 : 0 : 0)$ is not on the surface Y, and if $G_0 \neq 0$, then $(0 : 0 : 1 : 0)$ is not on Y.

From now on, we assume that we are in the general case where indeed the points $(0 : 1 : 0 : 0)$ and $(0 : 0 : 1 : 0)$ are not on Y. Let Y' be a minimal desingularization of Y. The following lemma shows that we can obtain Y' from a blowup in the point $(0 : 0 : 0 : 1)$.

Lemma C.1. *Suppose that* $\mathrm{char}(k) \neq 5$ *and let* Y *be as above. Then the blowup of* Y *in* $(0:0:0:1)$ *gives a minimal desingularization of* Y*. The exceptional locus contains two rational curves, i.e., each curve is isomorphic to* $\mathbb{P}^1$*, which are both defined over* k*. The self-intersection number of these curves equals* -2 *and* -3 *and they intersect each other transversally in one point.*

Proof. We can generalize the proof of Proposition 4.1(2) to deduce that the point $(0:0:0:1)$ is a quotient singularity of the type $\frac{1}{5}(1,3)$. The procedure of resolving this singularity generalizes to fields k with $\mathrm{char}(k) \neq 5$ (see [19, Proposition 2.5]), and the desired results all follow. □

From this observation, we deduce the following result.

Corollary C.2. *Let* Y' *be a minimal desingularization of a quasi-smooth surface of degree* 12 *in* $\mathbb{P}_k(1,2,3,5)$*. Then we have* $\rho(Y') \geqslant 3$*.*

Proof. The strict transform of the hyperplane section given by the equation $x_0 = 0$ and the two curves obtained from the blow-up are linearly independent of each other in $\mathrm{NS}(Y')$ and are all non-torsion (see Corollary 4.7(2)). □

These notes aim to prove that for a field of characteristic 0, and for a general enough choice, the geometric Picard number $\rho(Y'_{\overline{k}})$, and hence also the Picard number $\rho(Y')$, equals 3. We will do this by showing that it holds for the surface of Definition C.4.

Definition C.3. We define $F \in \mathbb{Z}[x_0, x_1, x_2, x_3]$ to be the polynomial given as in (C.1) with

$$\begin{aligned}
&G_0' = G_{1,1} = G_{1,3} = 0, \quad G_{1,2}(x_0^3, x_2) = 3x_2,\\
&G_2(x_0^3, x_2) = x_2^2 + x_0^3 x_2, \quad G_0 = -1,\\
&G_3(x_0^2, x_1) = x_0^6 + 2x_0^4 x_1 + x_0^2 x_1^2 + 2x_1^3,\\
&G_4(x_0^2, x_1) = 4x_0^6 x_1 + 2x_0^4 x_1^2 + x_0^2 x_1^3 + 4x_1^4,\\
&G_6(x_0^2, x_1) = x_0^{12} + 3x_0^{10} x_1 + 3x_0^8 x_1^2 + x_0^4 x_1^4 + 3x_0^2 x_1^5 + x_1^6.
\end{aligned}$$

Definition C.4. We define X to be the degree 12 surface in $\mathbb{P}_{\mathbb{Q}}(1,2,3,5)$ given by $F = 0$. We define X' over $\mathbb{Q}$ as the surface obtained by the blowup of X in the point $(0:0:0:1)$.

Theorem C.5. *The surface* X' *is smooth and* $\rho(X') = \rho(X'_{\overline{\mathbb{Q}}}) = 3$*.*

The proof of Theorem C.5 can be found in Appendix C.4. The proof uses a similar method as described in the proof of [50, Theorem 3.1] and [24, Section 4]. We look at good reductions of this surface over $\mathbb{F}_2$ and over $\mathbb{F}_3$, denoted by X_2' and X_3', respectively, and show that (i) $\rho((X_2')_{\overline{\mathbb{F}}_2}), \rho((X_3')_{\overline{\mathbb{F}}_3}) \leqslant 4$ and (ii) the discriminants of the geometric Neron-Severi lattices of X_2' and X_3' do not differ by a square factor. We see that this implies that $\rho(X'_{\overline{\mathbb{Q}}})$ is at most 3.

To calculate the discriminants (up to a square factor) of these Neron-Severi lattices, we use the Artin-Tate formula. This, together with a result about finding upper bounds for the Picard number, will be discussed in Appendix C.2.

Next, we define the surfaces of good reduction and determine the characteristic polynomial of Frobenius acting on some cohomology group. This characteristic polynomial will give the upper bound for the Picard number, and together with the Artin-Tate formula, it will give the necessary information we need in order to prove Theorem C.5. This work will be done in Appendix C.3.

Some of the proofs in Appendix C.3 are based on computations which are done in SageMath. The code which is used can be found in Appendix A.5.

Appendix C.2 The Neron-Severi group for varieties over finite fields

In this appendix, we recall some known results for the Neron-Severi group for varieties over finite fields. These results will be used in the proof of Theorem C.5 and some of the intermediate results in Appendix C.3.

Assume that k is a finite field. Set $p := \mathrm{char}(k)$ and $q := \#k$. Let Y denote any projective, smooth, and geometrically connected surface over k. Define

$$\alpha(Y) := \chi(Y, \mathscr{O}_Y) - 1 + \dim(\mathrm{Pic}_{Y/k}).$$

Let $\ell \neq p$ be any other prime. The absolute Galois group of k, which we will denote by $\mathrm{Gal}(\overline{k}/k)$ and which is generated by Frobenius, acts on the geometric Neron-Severi group $\mathrm{NS}(Y_{\overline{k}})$ as well as on the second cohomology group $\mathrm{H}^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell(1))$. We let Frob_q denote the linear map induced by Frobenius on $\mathrm{H}^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell(1))$ and let φ denote the characteristic polynomial of Frob_q.

Proposition C.6. *There is an inclusion*

$$\mathrm{NS}(Y_{\overline{k}}) \otimes \mathbb{Q}_\ell(1) \hookrightarrow \mathrm{H}^2_{\acute{e}t}(Y_{\overline{k}}, \mathbb{Q}_\ell(1))$$

of finite-dimensional vector spaces that respects the Galois action.

Proof. See [51, Proposition 6.2]. □

Corollary C.7. *Identify* $\mathrm{NS}(Y)$ *as a subset of* $\mathrm{NS}(Y_{\overline{k}})$. *Then the following holds*:

(i) *Under the embedding of Proposition* C.6*, we have the equality*

$$\mathrm{NS}(Y) \otimes \mathbb{Q}_\ell(1) = \mathrm{NS}(Y_{\overline{k}}) \otimes \mathbb{Q}_\ell(1) \cap \mathrm{H}^2_{\acute{e}t}(Y_{\overline{k}}, \mathbb{Q}_\ell(1))^{\mathrm{Gal}(\overline{k}/k)}.$$

(ii) *If* r *denotes the multiplicity of the eigenvalue 1 of* Frob_p*, then for the Picard number of* Y*, we have* $\rho(Y) \leqslant r$.

(iii) *The number of eigenvalues, counted with multiplicity, of* Frob_p *which are a root of unity, is an upper bound for* $\rho(Y_{\overline{k}})$.

(iv) *The Tate conjecture holds for* Y *if and only if the upper bounds in* (ii) *and* (iii) *are exactly the Picard numbers of the surfaces* Y *and* $Y_{\overline{k}}$*, respectively.* □

Remark C.8. For the surfaces we study, we have $\dim \mathrm{H}^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) = 22$ (see Proposition 4.1(2)). In particular, because 22 is even, we have as a corollary of the Weil conjectures, that in our case the upper bounds given in Corollary C.7 will be even.

Remark C.8 is the reason that we compare the reduction at two different primes in the proof of Theorem C.5. We use the following result to make this comparison.

Lemma C.9 (Artin-Tate formula). *Suppose that the Tate conjecture holds for* Y. *Then the group* $\mathrm{Br}(Y)$ *is finite, and*

$$\lim_{t\to 1} \frac{\varphi(t)}{(t-1)^{\rho(Y)}} = \frac{\#\,\mathrm{Br}(Y) \cdot \mathrm{disc}(\mathrm{NS}(Y)/\,\mathrm{NS}(Y)_{\mathrm{tor}})}{q^{\alpha(Y)}(\#\,\mathrm{NS}(Y)_{\mathrm{tor}})^2}.$$

Proof. See [43, Theorem 5.2]. □

Corollary C.10. *Suppose that the Tate conjecture holds for* Y. *Then the discriminant of the Neron-Severi lattice* $\mathrm{NS}(Y)/\,\mathrm{NS}(Y)_{\mathrm{tor}}$ *is up to a square factor equal to*

$$q^{\alpha(Y)} \cdot \lim_{t\to 1} \frac{\varphi(t)}{(t-1)^{\rho(Y)}}.$$

Proof. If the Brauer group is finite, its order $\#\,\mathrm{Br}(Y)$ is a square (see [28] and its corrigendum [29]). With this observation, the result follows directly from Lemma C.9. □

Appendix C.3 Good reductions at the primes 2 and 3

In this appendix, fix $p \in \{2, 3\}$. We define two surfaces over $\mathbb{F}_p$, which will be good reductions for the surfaces X and X' of Definition C.4, respectively.

Definition C.11. We define the surface X_p over $\mathbb{F}_p$ as the degree 12 surface in $\mathbb{P}_{\mathbb{F}_p}(1, 2, 3, 5)$ given by $F = 0$, where F from Definition C.3 is seen as a polynomial with coefficients in $\mathbb{F}_p$. We also define the surface X'_p to be the blowup of X_p in the point $(0 : 0 : 0 : 1)$.

Lemma C.12. *The surface X_p is quasi-smooth and the surface X'_p is smooth.*

Proof. A direct verification, done in SageMath (see Appendix A.5.1), shows that X_p is quasi-smooth. Because $(0:0:0:1)$ is the only singular point on X_p, it follows from Lemma C.1 that X'_p is smooth. □

Our next aim is to count the number of points on the surface X'_p, which will be used in the proof of Proposition C.23 to determine the characteristic polynomial of Frobenius. To do this, we use an elliptic fibration on the surface X'_p (see Subsection 4.2 for this notion) whose fibers do not contain a -1-curve.

The elliptic fibration we use is the morphism that is induced by the rational map $\tau\colon X_p \dashrightarrow \mathbb{P}^1$ defined by $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0^2 : x_1)$. The following lemma shows that the map τ extends to a minimal elliptic fibration $\tau'\colon X'_p \to \mathbb{P}^1$.

Lemma C.13. *The map $\tau\colon X_p \dashrightarrow \mathbb{P}^1$ extends to a minimal elliptic fibration $\tau'\colon X'_p \to \mathbb{P}^1$ and for the curves in the exceptional locus, we have that the -2-curve is in the fiber above the point $(1:0)$ and that the -3-curve is a double section for this fibration.*

Proof. We can apply the proof of Proposition 4.5 to the surfaces X_p and X'_p. □

Next, we define an arithmetic surface $\mathscr{C} \to \operatorname{Spec} \mathbb{F}_p[t]$. We refer the reader to [40, Section IV.4f] for the definition and standard results on arithmetic surfaces.

Definition C.14. The polynomial F from (C.1) defines the arithmetic surface

$$\mathscr{C} \subset \operatorname{Spec} \mathbb{F}_p[t] \times \mathbb{P}(1,2,1)$$

as the zero set of $F' = tz^4 \cdot F(1, t, x/z, y/tz^2)$, $F' \in \mathbb{F}_p[t][x,y,z]$.

Using the birational map $\mathscr{C} \dashrightarrow X_p$ given by $(t,(x:y:z)) \mapsto (1 : t : x/z : y/tz^2)$ (and with inverse given by $(x_0 : x_1 : x_2 : x_3) \mapsto (x_1/x_0^2, (x_0x_2 : x_0x_1x_3 : x_0^4)))$ induces an isomorphism

$$\mathscr{C} \setminus \{tz = 0\} \xrightarrow{\sim} X_p \setminus (\{x_0 = 0\} \cup \{x_1 = 0\}).$$

With $E \subset X'_p$, the exceptional locus of $X'_p \to X_p$ and $O := (0:0:0:1)$, we have $X'_p \setminus E \xrightarrow{\sim} X_p \setminus \{O\}$.

Next, setting $t = x_1/x_0^2$, we can identify $\operatorname{Spec} \mathbb{F}_p[t] \subset \mathbb{P}^1$ as a subscheme. This identification makes $U := X'_p \setminus {\tau'}^{-1}(0:1)$ an arithmetic surface over $\mathbb{F}_p[t]$. Combining the above observations, we get an embedding $\mathscr{C} \setminus \{tz = 0\} \hookrightarrow U$ of $\operatorname{Spec} \mathbb{F}_p[t]$-schemes. The next lemma shows that this extends to an isomorphism.

Lemma C.15. *The embedding $\mathscr{C} \setminus \{tz = 0\} \hookrightarrow U$ above, extends to an isomorphism $\mathscr{C} \xrightarrow{\sim} U$ as $\operatorname{Spec} \mathbb{F}_p[t]$-schemes.*

Proof. Note that because τ' is a minimal elliptic fibration, it follows that U is a minimal proper regular model for its generic fiber as defined in [40, Theorem IV.4.5b]. We show that $\mathscr{C}$ is a minimal proper regular model as well and then the result follows from [40, Theorem IV.4.5b].

To show this, we first note that this surface is projective over $\operatorname{Spec} \mathbb{F}_p[t]$, and hence proper over $\operatorname{Spec} \mathbb{F}_p[t]$. To check that it is smooth, we note that for each $a \in \overline{\mathbb{F}}_p$, the point $(a,(0:1:0))$ does not lie on $\mathscr{C}$. It follows that every point of this surface lies on the affine where x does not vanish or where z does not vanish. Now using SageMath (see Appendix A.5.2), we can check that $\mathscr{C}$ is smooth over $\mathbb{F}_p$ by checking both affines. It follows that $\mathscr{C}$ is regular. So it remains to show that this surface $\mathscr{C}$ is minimal.

Recall that there is an embedding $\mathscr{C} \setminus \{tz = 0\} \hookrightarrow U$. Because the fibration on U is minimal, we deduce that the only possible exceptional curves in the fibers of $\mathscr{C} \to \operatorname{Spec} \mathbb{F}_p[t]$ can be found at $z = 0$ or in the fiber $t = 0$. Note that for every fiber $t = a$, it is easy to see that $\mathscr{C} \cap \{t = a, z = 0\}$ is 0-dimensional (see the proof of Lemma C.20, from which we deduce that every fiber above $t = a \neq 0$ cannot contain an exceptional curve).

The fiber above $t = 0$ is given by the equation $y(y + x^2 + xz) = 0$ and so it consists of two rational curves E_1 and E_2 which intersect each other in two points, i.e., as a divisor, the fiber is given by $E_1 + E_2$. Because the intersection number of a fiber with every fibral divisor is zero (see [40, Proposition IV.7.3(b) and Remark IV.7.6]), it follows that $E_i \cdot (E_1 + E_2) = 0$, $i = 1, 2$. Hence $E_1^2 = E_2^2 = -E_1 \cdot E_2 = -2$.

This shows that there are no fibral exceptional curves on $\mathscr{C}$ and we conclude from [40, Remark IV.7.5.1] that $\mathscr{C}$ is minimal. □

One of the tools, we make use of is the discriminant related to this arithmetic surface $\mathscr{C}$. For a definition of the discriminant of a weighted homogeneous polynomial, we refer the reader to [45, 1.1].

Lemma C.16. *Define $\Delta_p := \operatorname{disc} F'$. Then we have*

$$\Delta_2(t) = t^2(t^{10}+t^9+t^8+t^7+t^2+t+1)(t^{12}+t^8+t^5+t^4+t^3+t+1),$$
$$\Delta_3(t) = 2t^2(t+2)(t^9+2t^8+2t^7+2t^6+t^5+2t^4+t^2+t+2)(t^{12}+2t^{10}+t^8+t^7+2t^6+t^5+t^2+2),$$

where the terms in between brackets are irreducible.

Proof. Recall that the equation of $\mathscr{C}$ is of the form $y^2+h_t(x,z)y+f_t(x,z)$, where

$$f_t(x,z) = F'(t,(x,0,z)),$$
$$h_t(x,z) = z^2(G_{1,2}(1,x/z)t^2+G_2(1,x/z)).$$

The formula for the discriminant of such a polynomial is given in [45, Example 3.5] combined with [44, Lemma 3.3]. From this formula, we deduce that the discriminant Δ of F' over $\mathbb{Z}[t]$ can be given by

$$\Delta = 4^{-4}\cdot \operatorname{disc}(h_t(x,z)^2-4f_t(x,z)) \in \mathbb{Z}[t],$$

where disc denotes the discriminant of a polynomial of degree 4 (see [18, Chapter 12.1.B (1.35)]).

Using SageMath (see Appendix A.5.3), we use the above formula to calculate this polynomial Δ. Then we factor its reduction mod p to obtain the above expressions. □

We use the discriminant Δ_p of Lemma C.16 to deduce the type of fibers of the fibration $\tau' : X_p' \to \mathbb{P}^1$.

Lemma C.17. *Let $\tau' : X_p' \to \mathbb{P}^1$ be the elliptic fibration as above. Then the fiber above $P\in\mathbb{P}^1(\overline{\mathbb{F}}_p)$ is singular if and only if $P=(1:t_0)$ with t_0 a zero of Δ_p. Moreover, if it is a singular fiber, then it is of the type I_2 if $P=(1:0)$ and it is of the type I_1 otherwise.*

Proof. Let t_0 be a zero of Δ_p. Recall from Lemma C.15 that the fiber above $(1:t_0)$ of τ' is isomorphic to the above fiber t_0 of the arithmetic surface $\mathscr{C}\to\operatorname{Spec}\mathbb{F}_p[t]$. We can use Tate's algorithm (see [40, Subsections IV.8 and IV.9]) to determine the type of fiber above t_0 on the arithmetic surface $\mathscr{C}$ as follows.

First, we choose some separable extension of $\mathbb{F}_p(t)$ which is unramified at t_0 and such that the base change of $\mathscr{C}$ to this field, gives an arithmetic surface $\mathscr{C}'$ which has a section. We can then put the defining equation of $\mathscr{C}'$ in the Weierstrass form to apply the algorithm. Now choose t_0' such that the fiber above t_0' on $\mathscr{C}'$ gets mapped to the fiber above t_0 on $\mathscr{C}$. Because the extension is unramified above t_0, the base change is regular and minimal around t_0' and so the fiber above t_0' on $\mathscr{C}'$ is isomorphic to the fiber above t_0 on $\mathscr{C}$ are isomorphic over some separable extension, and hence the fiber types are the same. By the defining property of the discriminant (see [45, Theorem 1.2]), the valuation of the discriminant of the Weierstrass form at t_0' will exactly equal the valuation of the polynomial Δ_p at t_0. From this, we deduce that we can use the valuation of Δ_p at t_0 to deduce the type of fiber above t_0.

Note that Δ_p has a factor t^2 and that all the other factors are separable. So for $t_0\neq 0$, we have multiplicity 1. Hence, above $(1:t_0)$, we deduce from [40, Subsection IV.9, Table 4.1], that this is a fiber of the type I_1.

For $t_0=0$, we have $v_0(\Delta_p)=2$. In characteristics 2 and 3, the order of vanishing of the discriminant will be bigger than 2 in the case where the fiber has multiplicative reduction due to wild ramification (see [38, Proposition 5.1]). We deduce from the above that the fiber above $(1:0)$ is of the type I_2.

Because Δ_p has degree 24, which equals the Euler characteristic of the surface (see Proposition 4.5), we deduce that these are all the singular fibers of this fibration $\tau' : X_p' \to \mathbb{P}^1$ and that all other fibers are smooth. □

Remark C.18. In the proof of Lemma C.17, we deduced that the fiber above $t=0$ is of the type I_2 by using the discriminant Δ_p, but we already encountered this fiber in the proof of Lemma C.15, from which we also could have concluded that it is of the type I_2.

We now define the following affine curves, which we will use in Proposition C.21 to count the points on the surface X'_p.

Definition C.19. Set $\tilde{F} = x_1 \cdot F$. For each $a \in \mathbb{F}_{p^n}$, we define the affine curve C_a over $\mathbb{F}_{p^n}$ in $\mathbb{A}^2_{\mathbb{F}_{p^n}}(x', y')$ by the equation $\tilde{F}(1, a, x', y'/a) = 0$. We define the curve C_∞ in $\mathbb{A}^2_{\mathbb{F}_p}(x', y')$ to be the curve given by the equation $F(0, 1, x', y') = 0$.

For the curves C_a, we have the following result.

Lemma C.20. *Let $a \in \mathbb{F}_{p^n}$, and $g \in \mathbb{F}_{p^n}[s]$ be given by $g = s^2 + s - a$. The number of $\mathbb{F}_{p^n}$-points on the fiber above the point $(1 : a)$ of $\tau' \colon X'_p \to \mathbb{P}^1$ is equal to the number of $\mathbb{F}_{p^n}$-points on C_a plus the number of roots in $\mathbb{F}_{p^n}$ of the polynomial g.*

Proof. By Lemma C.15, we have that the fiber above $(1 : a)$ of the morphism τ' is isomorphic to the fiber above $t = a$ of $\mathscr{C}$. The embedding $(x', y') \mapsto (a, (x' : y' : 1))$ embeds the curve C_a into the fiber above $t = a$ of the arithmetic surface $\mathscr{C}$ and is isomorphic to the affine part of this fiber where z does not vanish. So it follows that the number of points on the fiber above $(1 : a)$ equals the number of points on C_a plus the number of points on this fiber intersected with $\{z = 0\}$.

Recall that the defining polynomial of $\mathscr{C}$ is given by $F' := tz^4 \cdot F(1, t, x/z, y/tz^2) \in \mathbb{F}_p[t][x, y, z]$, and that $F'(a, (0, 1, 0)) = 1$. It follows that all the points on the intersection of $t = a$ with $z = 0$ and the arithmetic surface $\mathscr{C}$ are on the affine where x does not vanish. This means that these points are of the form $(a, (1 : s : 0))$ such that $F'(a, (1, s, 0)) = 0$. Following the steps defining F', we can deduce that

$$F'(1, s, 0) = s^2 + G_2(0, 1)s + a \cdot G_0 = s^2 + s - a,$$

from which the result follows. □

Now we combine the above results to count the points on the surface X'_p.

Proposition C.21. *The number of $\mathbb{F}_{p^i}$-points on X'_p is given by Table* C.1.

Proof. Let $1 \leqslant n \leqslant 9$ be given. As mentioned earlier, we count the $\mathbb{F}_{p^n}$-points of X'_p by counting for each point $P \in \mathbb{P}^1(\mathbb{F}_{p^n})$ the number of $\mathbb{F}_{p^n}$-points in the fiber of the map $\tau' \colon X'_p \to \mathbb{P}^1$ and sum their total. Using SageMath, we follow the steps described next and count for each fiber the number of points and add them together. The code can be found in Appendix A.5.4.

We start with the fiber above $(0 : 1)$. By Lemma C.17, we have that it is non-singular. Recall that $X'_p \setminus E \cong X_p \setminus \{(0 : 0 : 0 : 1)\}$, from which it follows that we can identify the affine curve C_∞ as an affine part of the fiber above $(0 : 1)$. This curve C_∞ is given by an equation of the form $y^2 + h(x)y = f(x)$, where $f = -F(0, 1, x, 0)$ and $h = G_{1,2}(0, 1, x, 0)$. A smooth projective closure can be defined by using the function `HyperellipticCurve` in SageMath. Because this defines a smooth projective curve, of which an affine is isomorphic to an affine part of the fiber above $(0 : 1)$, it is isomorphic to this fiber. In particular, the amount of $\mathbb{F}_{p^n}$-points will be the same and we can count the number of points on this hyperelliptic curve (see Remark C.22).

Table C.1 Count of the numbers of points

n	$\#X'_2(\mathbb{F}_{2^n})$	$\#X'_3(\mathbb{F}_{3^n})$
1	11	17
2	29	95
3	65	803
4	241	6,767
5	1,121	59,477
6	4,289	532,883
7	16,769	4,798,097
8	67,329	43,071,575
9	264,449	387,431,885

Above all the other fibers, i.e., the above $(1:a)$, we first check if the curve C_a is smooth by checking if the discriminant Δ_p vanishes (see Lemma C.17). If it is not smooth, we use Lemma C.20 to count the number of points. In the smooth case, we can count the points similarly as in the case for the fiber $(0:1)$ as follows. The defining polynomial of C_a is again of the form $y^2+h(x)y-f(x)$, where $f=-a\cdot F(1,a,x,0)$ and $h=G_{1,2}(1,x)a^2+G_2(1,x)$. Then, we can again define an isomorphic hyperelliptic curve using f and h and count the points on this curve. □

Remark C.22. The main reason to use the function `HyperellipticCurve` in SageMath in the proof of Proposition C.21 is that it has a built-in pointing count algorithm which is faster than naive point counting on curves.

Our next goal is to find the characteristic polynomial of Frobenius acting on the vector space $H_p := H^2_{\text{ét}}((X_p)_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell(1))$. We denote by Frob_p the linear map on H_p that is induced by the Frobenius morphism on $\overline{\mathbb{F}}_p$ and by f_p the characteristic polynomial of Frob_p.

Proposition C.23. *The characteristic polynomial of* Frob_p *equals* $f_p=(t-1)^4\cdot h_p$, *where* h_p *is irreducible and equals*

$$\begin{aligned} h_2(t) &= t^{18}+t^{17}+t^{16}+2t^{15}+3t^{14}+3t^{13}+\tfrac{7}{2}t^{12}+\tfrac{9}{2}t^{11}+\tfrac{9}{2}t^{10}\\ &\quad+\tfrac{9}{2}t^{9}+\tfrac{9}{2}t^{8}+\tfrac{9}{2}t^{7}+\tfrac{7}{2}t^{6}+3t^5+3t^4+2t^3+t^2+t+1,\\ h_3(t) &= t^{18}+\tfrac{5}{3}t^{17}+\tfrac{8}{3}t^{16}+\tfrac{10}{3}t^{15}+4t^{14}+\tfrac{14}{3}t^{13}+\tfrac{16}{3}t^{12}+\tfrac{16}{3}t^{11}+\tfrac{16}{3}t^{10}\\ &\quad+\tfrac{16}{3}t^{9}+\tfrac{16}{3}t^{8}+\tfrac{16}{3}t^{7}+\tfrac{16}{3}t^{6}+\tfrac{14}{3}t^{5}+4t^4+\tfrac{10}{3}t^3+\tfrac{8}{3}t^2+\tfrac{5}{3}t+1. \end{aligned}$$

Proof. Computations in this proof are done in SageMath (see Appendix A.5.5).

Recall from Proposition C.6 that there is an inclusion

$$\mathrm{NS}((X'_p)_{\overline{\mathbb{F}}_p})\otimes\mathbb{Q}_\ell(1)\hookrightarrow H_p$$

respecting the Galois action. From Corollary C.2, we deduce that there is a subspace U_p of H_p of dimension 3 on which Frobenius is acting trivially.

Let V_p denote the quotient space $V_p=H_p/U_p$. Because Frobenius leaves U_p invariant, we get an induced action on V_p, denoted by $\overline{\mathrm{Frob}}_p$, and we have the relation $f_p(t)=(t-1)^3\cdot g_p(t)$, where g_p denotes the characteristic polynomial of $\overline{\mathrm{Frob}}_p$.

From Proposition 4.1(2), we know that $\dim H_p=22$. It follows that

$$\dim V_p=\dim H_p-\dim U_p=22-3=19,$$

and so the polynomial g_p has degree 19. Moreover, it is equal to

$$g_p(t)=c_0t^{19}+c_1t^{18}+\cdots+c_{18}t+c_{19},$$

where $c_0=1$, $c_1=-\mathrm{Tr}(\overline{\mathrm{Frob}}_p)$, and the other c_i's are given recursively by Newton's identity (see [49, § 26, Exercise 3])

$$c_i=-\frac{\mathrm{Tr}(\overline{\mathrm{Frob}}_p^{\,i})+\sum_{j=1}^{i-1}c_j\,\mathrm{Tr}(\overline{\mathrm{Frob}}_p^{\,i-j})}{i}.$$

The functional equation gives us that $t^{19}g(1/t)=\pm g(t)$ and so we either have $c_i=c_{19-i}$ for all $0\leqslant i\leqslant 9$ or $c_i=-c_{19-i}$ for all $0\leqslant i\leqslant 9$.

From the Lefschetz Trace formula, the Weil conjectures, the relation

$$\mathrm{Tr}(\overline{\mathrm{Frob}}_p^{\,i})=\mathrm{Tr}(\mathrm{Frob}_p^i)-3$$

and the fact that the eigenvalues of Frob_p^i on H_p differ by a factor p^i from the eigenvalues of Frobenius acting on $\mathrm{H}^2_{\text{ét}}((X'_p)_{\overline{\mathbb{F}}_p},\mathbb{Q}_\ell)$, we deduce the equality

$$\mathrm{Tr}(\overline{\mathrm{Frob}}_p^{\,i})=\frac{\#X'_p(\mathbb{F}_{p^i})-1-p^{2i}-3p^i}{p^i}.$$

Table C.2 The numbers c_i, $i = 1, \ldots, 9$

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
$p = 2$	0	0	1	1	0	$\frac{1}{2}$	1	0	0
$p = 3$	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0	0	0

Using the above formula, Proposition C.21 and Newton's identity mentioned above, we get the following values of c_i for $1 \leqslant i \leqslant 9$ for both $p = 2, 3$.

Table C.2 gives us two options for the polynomials g_p, which depend on the sign of the functional equation. It follows from the Weil conjectures that g_p should have all the roots on the unit circle. By using SageMath, we can exclude, for both $p = 2$ and $p = 3$, the one for which the sign of the functional equation is positive because this polynomial has roots outside the unit circle. Moreover, using a factorization algorithm in SageMath, we can calculate that g_p contains a factor $t-1$ and an irreducible factor of degree 18 equal to h_p. From this, the statement follows. □

Corollary C.24. *For the Picard number of X'_p, we have $\rho(X'_p) = \rho((X'_p)_{\overline{\mathbb{F}}_p}) \leqslant 4$.*

Proof. Because the minimal polynomial of every root of unity has integral coefficients, it follows that the characteristic polynomial of Frobenius f_p, which we found in Proposition C.23, has no other roots that are a root of unity except for 1. This means that $\rho(X'_p) = \rho((X_p)_{\overline{\mathbb{F}}_p})$ and the upper bound follows from Corollary C.7. □

Appendix C.4 Proof of the main result, Theorem C.5

Let $\mathbb{Z}_{(6)}$ denote the localization of $\mathbb{Z}$ away from the ideal (6). Define the scheme $\mathfrak{X}$ over $\mathbb{Z}_{(6)}$ to be the blow-up of the scheme $\mathrm{Proj}(\mathbb{Z}_{(6)}[x_0, x_1, x_2, x_3]/(F))$ at the ideal $I = (x_0, x_1, x_2)$. Because blow-ups commute under flat morphisms (see [46, Lemma 0805]), the reduction of $\mathfrak{X}$ at the prime 2 is isomorphic to X'_2, the reduction at the prime 3 is isomorphic to X'_3, and the generic fiber is isomorphic to X'. From this observation, we conclude that the surface X' is smooth as well because it has a smooth reduction.

By the proof of [51, Proposition 6.2], $N := \mathrm{NS}(X'_{\overline{\mathbb{Q}}}) / \mathrm{NS}(X'_{\overline{\mathbb{Q}}})_{\mathrm{tor}}$ embeds for both $p = 2$ and $p = 3$ into the lattice

$$N_p := \mathrm{NS}((X'_p)_{\overline{\mathbb{F}}_p}) / \mathrm{NS}((X'_p)_{\overline{\mathbb{F}}_p})_{\mathrm{tor}}.$$

By Corollary C.24, we have $\rho(X'_{\overline{\mathbb{Q}}}) \leqslant \rho((X'_p)_{\overline{\mathbb{F}}_p}) \leqslant 4$. If the Tate conjecture does not hold for X'_2 or X'_3, then it follows from Corollary C.7 that $\rho(X'_{\overline{\mathbb{Q}}}) \leqslant 3$, and hence with Corollary C.2 that $\rho(X') = \rho(X'_{\overline{\mathbb{Q}}}) = 3$, and we would be done.

So assume for the remainder that the Tate conjecture holds for both surfaces X'_2 and X'_3. Combining Corollaries C.7 and C.24, we find for both $p = 2$ and $p = 3$ the equality $\rho(X'_p) = \rho((X'_p)_{\overline{\mathbb{F}}_p}) = 4$. Because $\mathrm{Br}(\mathbb{F}_p) = 0$, it follows that the Neron-Severi lattice of X'_p equals the lattice N_p for both $p = 2$ and $p = 3$.

Using Corollary C.10, we have

$$\begin{aligned} \operatorname{disc} N_2 &= s_2^2 \cdot 2^{\alpha(X'_2)} \cdot h_2(1) = s_2^2 \cdot 2^{\alpha(X'_2)} \cdot \frac{103}{2}, \\ \operatorname{disc} N_3 &= s_3^2 \cdot 3^{\alpha(X'_3)} \cdot h_3(1) = s_3^2 \cdot 3^{\alpha(X'_3)} \cdot 72 \end{aligned}$$

for some $s_2, s_3 \in \mathbb{Q}$. We deduce that the discriminants of N_2 and N_3 do not differ by a square factor.

From the theory of lattices, we know that the discriminant of a full-rank sublattice always differs by a square factor from the discriminant of the full lattice. It follows that N cannot be embedded in both the lattices N_p as a full-rank sublattice. We deduce $\rho(X'_{\overline{\mathbb{Q}}}) < \rho((X'_p)_{\overline{\mathbb{F}}_p}) = 4$. Combining this with Corollary C.2 gives the result of Theorem C.5. □

Remark C.25. (1) In the proof of Theorem C.5, one can use the valuation at the prime 103 to conclude that the discriminants do not differ by a square factor. So instead of using Corollary C.10, we could also have used Lemma C.9 with the original, slightly weaker, result of Tate [43, Theorem 5.1], which states that the Brauer group is a square or two times a square.

(2) One can apply the above methods to find that for a minimal desingularization of a general member of the family of quasi-smooth degree 14 surfaces in the weighted projective space $\mathbb{P}_k(1,2,3,7)$ with $\operatorname{char} k = 0$, the Picard number equals 2 (see Corollary 4.7(1)). We give a sketch of the proof here. The minimal desingularization X' of a quasi-smooth surface X of degree 14 in $\mathbb{P}_k(1,2,3,7)$ is given by blowing up in $(0:0:1:0)$ if $\operatorname{char} k \neq 3$. The exceptional locus of the blowup consists of only one curve, a -3-curve, which will be a double section for the elliptic fibration. From this, we deduce the lower bound $\rho(X') \geqslant 2$.

Now it suffices to find a surface over $\mathbb{F}_2$ with Picard rank at most 2. Then as in Propositions C.21 and C.23, we can count the $\mathbb{F}_{2^i}$ points and find the characteristic polynomial of Frobenius, with only minor adjustments to the proofs.

If we apply the method to the surface X_2 over $\mathbb{F}_2$ given by the equation $F = 0$, where

$$\begin{aligned}
&F = x_3^2 + Gx_3 + x_1x_2^4 + G_2x_0x_2^3 + G_5x_0x_2 + G_7,\\
&G = x_0x_2^2 + x_0^4x_2 + x_1^2x_2, \quad G_2 = x_0^2x_1,\\
&G_5 = x_0^8x_1 + x_0^2x_1^4 + x_1^5, \quad G_7 = x_0^{14} + x_0^{12}x_1 + x_0^{10}x_1^2 + x_0^6x_1^4 + x_0^2x_1^6 + x_1^7,
\end{aligned}$$

one can find that the surface X_2' has Picard rank 2 (see Appendix A.5.6). From this, one can deduce the desired result.