ISBN 978-85-244-0273-9

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XV ESCOLA DE GEOMETRIA DIFERENCIAL

Em homenagem aos
80 anos de Manfredo do Carmo

Fortaleza, Ceará - 14 a 18 de julho de 2008

An Elementary Introduction to Eigenvalue Problems

with an application to catenoids in R^3

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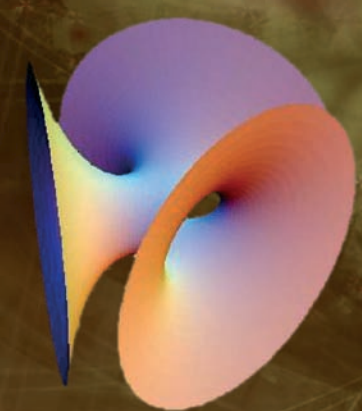

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Universidade Federal do Ceará
Departamento de Matemática

Au Professeur Manfredo do Carmo,
pour son quatre-vingtième anniversaire,
en amical et respecteux hommage.

An elementary introduction to eigenvalue problems

with an application to catenoids in $\mathbb{R}^3$

Pierre Bérard

Contents

Introduction

These notes were written for a mini-course given at the Differential Geometry School (Fortaleza - Brazil, July 2008) – *XV Escola de geometria diferencial, em homenagem aos 80 anos de Manfredo do Carmo).*

The notes are intended for geometry students and they aim at giving an elementary introduction to eigenvalue problems based on variational methods (min-max principle). They should in principle address undergraduate students with a fair understanding of advanced calculus (Ascoli's theorem and the Cauchy-Lipschitz theorem for linear ordinary differential equations). For this purpose, we have avoided using Hilbert space techniques. Chapter 1 is introductory and devoted to eigenvalues of symmetric matrices. Chapter 2 deals with the Dirichlet eigenvalue problem for a Sturm-Liouville operator with continuous potential on a closed interval. Chapter 3 gives an application of the techniques developed in Chapter 2 to the computation of the index of the catenoid in $\mathbb{R}^3$. This chapter should address more advanced geometry students. In Chapter 4 we give some glimpses at spectral geometry and eigenvalue problems in minimal surface theory. This chapter is meant as a motivation and encouragement to the students for further reading.

We thank the Organizing committee of the Differential Geometry School for giving us the opportunity to present this course and the Mathematics Department of PUC-Rio for their hospitality during the preparation of these notes.

Special thanks are due to Professor Ricardo Sá Earp (PUC-Rio) for discussions and support.

Chapter 1

Eigenvalue problems for real symmetric endomorphisms

Summary. In this chapter we show how to diagonalize real symmetric endomorphisms in a finite dimensional Euclidean space using the variational method. As a by-product, we derive some properties of the eigenvalues.

1.1 Notations

Let E be a *real* Euclidean space with finite dimension n, inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. We let $E^\bullet$ denote $E \setminus \{0\}$ and S_E denote the unit sphere $S_E := \{x \in E \mid \|x\| = 1\}$.

Let $A \in \mathcal{S}(E)$ be a symmetric endomorphism of E, *i.e.* $\langle A(x), y\rangle = \langle x, A(y)\rangle$ for all $x, y \in E$. We introduce the quadratic form Q_A,

$$Q_A(x) := \langle A(x), x\rangle$$

and the Rayleigh quotient R_A,

$$R_A(x) := \frac{\langle A(x), x\rangle}{\langle x, x\rangle}, \quad \text{for} \quad x \neq 0,$$

associated with A.

We recall the following result.

Proposition 1.1 *Let A be a symmetric endomorphism of a finite dimensional Euclidean space $(E, \langle\cdot,\cdot\rangle)$. Assume that $E = F \bigoplus G$ is an orthogonal decomposition of E (*i.e.* for all $x \in E$, there exists a unique pair $x_F \in F$, $x_G \in G$ such that $x = x_F + x_G$ and $\langle y, z\rangle = 0$ for all $y \in F$ and $z \in G$). Assume furthermore that the endomorphism A leaves the subspace F invariant (*i.e.* for all $y \in F, A(y) \in F$). Then A also leaves the subspace G invariant.*

Proof. Left to the reader. □

1.2 Existence of eigenvalues

Proposition 1.2 *Let $(E, \langle\cdot,\cdot\rangle)$ be a finite dimensional (real) Euclidean space and let A be a symmetric endomorphism of E. The Rayleigh quotient R_A of A is bounded on $E^\bullet$ and hence*

$$(1.1) \qquad \begin{cases} \lambda_{\min}(A) & := & \inf\left\{R_A(x) \;\middle|\; x \in E^\bullet\right\} \\ \textit{and} & & \\ \lambda_{\max}(A) & := & \sup\left\{R_A(x) \;\middle|\; x \in E^\bullet\right\} \end{cases}$$

exist. Furthermore, there exist unit vectors $e_{\min}$ and $e_{\max}$ in S_E (not necessarily unique) such that

$$(1.2) \qquad \begin{cases} \lambda_{\min}(A) & := & R_A(e_{\min}) = Q_A(e_{\min}), \\ \textit{and} & & \\ \lambda_{\max}(A) & := & R_A(e_{\max}) = Q_A(e_{\max}). \end{cases}$$

The vectors $e_{\min}$ *and* $e_{\max}$ *are eigenvectors of* A*, associated respectively to the eigenvalues* $\lambda_{\min}(A)$ *and* $\lambda_{\max}(A)$*,*

$$\begin{cases} A(e_{\min}) & = & \lambda_{\min}\, e_{\min}, \\ \textit{and} & & \\ A(e_{\max}) & = & \lambda_{\max}\, e_{\max}. \end{cases} \tag{1.3}$$

Proof. Observe that R_A is dilation invariant, *i.e.* $R_A(\alpha x) = R_A(x)$ for all $x \in E^\bullet$ and all $\alpha \in \mathbb{R}^\bullet$. It follows that it suffices to consider R_A, or equivalently Q_A, on the compact set S_E. Since these functions are continuous they are bounded from below and from above and they achieve their infimum and supremum on the compact set S_E.

Choose any $y \in E$ and take t small enough so that $e_{\min} + ty \neq 0$. For t small we have the expansion

$$R_A(e_{\min} + ty) = \lambda_{\min} + 2t\big\{\langle A(e_{\min}), y\rangle - \lambda_{\min}\,\langle e_{\min}, y\rangle\big\} + o(t)$$

where $o(t)$ denotes a function which tends to zero with t. Since $R_A(e_{\min} + ty) \geq \lambda_{\min}$ for all t sufficiently small, we conclude that $\langle A(e_{\min}), y\rangle = \lambda_{\min}\,\langle e_{\min}, y\rangle$ for all $y \in E$ and hence that $A(e_{\min}) = \lambda_{\min}\, e_{\min}$.

A similar proof applies for $\lambda_{\max}, e_{\max}$ as well. □

Exercise 1.1 *Give a proof of the preceding Proposition considering a* C^1 *curve* $c(t)$ *on* S_E *such that* $c(0) = e_{\min}$ *and* $\dot{c}(0) = y$*.*

Exercise 1.2 *Give a proof of the preceding Proposition using Lagrange multipliers.*

Theorem 1.3 *Let $(E, \langle\cdot,\cdot\rangle)$ be an n-dimensional (real) Euclidean space and let A be a symmetric endomorphism of E. Then there exists an orthonormal basis $\{e_1, \cdots, e_n\}$ of eigenvectors of A, associated with the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$.*
More precisely, the eigenvalues and eigenvectors can be constructed inductively as follows.

- *Let $\lambda_1 := \inf\left\{R_A(x) \;\middle|\; x \in E^{\bullet}\right\}$ and let e_1 be a unit vector at which R_A achieves the value λ_1.*
- *For $2 \leq k \leq n$, construct λ_k and e_k inductively in such a way that*
$$\lambda_k := \inf\left\{R_A(x) \;\middle|\; x \in F_{k-1}^{\perp}, x \neq 0\right\}$$
where $F_{k-1} := \mathrm{Vect}\{e_1, \cdots, e_{k-1}\}$ and e_k is a vector in $S_E \cap F_{k-1}^{\perp}$ at which R_A achieves its infimum. Here $F^{\perp}$ is the subspace orthogonal to F in E, with respect to the inner product $\langle\cdot,\cdot\rangle$.

In this construction, the eigenvalues appear in increasing order, $\lambda_1 \leq \cdots \leq \lambda_n$.

Proof. Apply Proposition 1.2 to the triple $(E, \langle\cdot,\cdot\rangle, A)$ and let (with obvious notations)

$$\text{(1.4)} \qquad \begin{cases} \lambda_1 & := & \lambda_{\min}(E, A), \\ e_1 & := & e_{\min}(E, A), \\ F_1 & := & \mathbb{R}\, e_1. \end{cases}$$

Then $A(e_1) = \lambda_1 e_1$ and hence A leaves F_1 invariant.

Let $E_2 := F_1^{\perp}$. By Proposition 1.1, A leaves E_2 invariant. Let A_2 be the restriction of A to E_2. Apply Proposition 1.2 to the triple $(E_2, \langle\cdot,\cdot\rangle, A_2)$ and let (with obvious notations)

$$\text{(1.5)} \qquad \begin{cases} \lambda_2 & := & \lambda_{\min}(E_2, A_2), \\ e_2 & := & e_{\min}(E_2, A_2), \\ F_2 & := & F_1 \bigoplus \mathbb{R}\, e_2. \end{cases}$$

Then $A(e_2) = A_2(e_2) = \lambda_2 \, e_2$ and A leaves F_2 invariant.

...

Let $E_{k+1} := F_k^{\perp}$. By Proposition 1.1, A leaves E_{k+1} invariant. Let A_{k+1} be the restriction of A to E_{k+1}. Apply Proposition 1.2 to the triple $(E_{k+1}, \langle \cdot, \cdot \rangle, A_{k+1})$ and let (with obvious notations)

$$\text{(1.6)} \qquad \begin{cases} \lambda_{k+1} & := & \lambda_{\min}(E_{k+1}, A_{k+1}), \\ e_{k+1} & := & e_{\min}(E_{k+1}, A_{k+1}), \\ F_{k+1} & := & F_k \bigoplus \mathbb{R} \, e_{k+1}. \end{cases}$$

Then $A(e_{k+1}) = A_{k+1}(e_{k+1}) = \lambda_{k+1} \, e_{k+1}$ and A leaves F_{k+1} invariant.

After $(n-1)$ such steps, we find an orthonormal basis $\{e_1, \cdots, e_n\}$. Note that this basis is not uniquely defined. □

Exercise 1.3 *Give another proof of the existence of an orthonormal basis of eigenvectors of A whose output is the eigenvalues arranged in decreasing order.*

1.3 Variational characterization of eigenvalues

One drawback of the method described in Section 1.2 is that in order to derive λ_2 (or more generally λ_k) we need to know e_1 (or more generally $e_1, \cdots, e_{k-1}$).

Given a vector space E, we denote by $\mathcal{G}_k(E)$ the set of all linear subspaces of E of dimension k, for $0 \le k \le \dim(E)$.

Theorem 1.4 *Let $(E, \langle \cdot, \cdot \rangle)$ be a (real) Euclidean space with dimension n and let A be a symmetric endomorphism of E. Write the eigenvalues of A in increasing order, $\lambda_1 \leq \cdots \leq \lambda_n$ (allowing multiplicities). Then we have the* min-max characterization*,*

$$\lambda_k = \inf_{F \in \mathcal{G}_k(E)} \big(\sup \big\{ R_A(x) \;\big|\; x \in F^\bullet \big\} \big). \tag{1.7}$$

Proof. Call μ_k the right-hand side of equality (1.7). Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of E associated with the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$, as given by Theorem 1.3. Let $F_k := \mathrm{Vect}\{e_1, \cdots, e_k\}$.

For $x \in F_k^\bullet$, write $x = x_1 e_1 + \cdots + x_k e_k$. It is clear that

$$R_A(x) = \sum_{j=1}^{k} \lambda_j x_j^2 \,/\, \sum_{j=1}^{k} x_j^2$$

and hence that $R_A(x) \leq \lambda_k$. It follows that

$$\sup \big\{ R_A(x) \;\big|\; x \in F_k^\bullet \big\} \leq \lambda_k$$

and we can conclude that $\mu_k \leq \lambda_k$.

Take any $F \in \mathcal{G}_k(E)$. Since $\dim F = k$ and $\dim F_{k-1}^\perp = n + 1 - k$, there exists some $u \in F \cap F_{k-1}^\perp \setminus \{0\}$. By construction,

$$R_A(u) \geq \inf \big\{ R_A(y) \;\big|\; y \in F_{k-1}^\perp, y \neq 0 \big\} =: \lambda_k$$

and hence

$$\sup \big\{ R_A(x) \;\big|\; x \in F^\bullet \big\} \geq \lambda_k.$$

Since this is true for any k-dimensional subspace F, it follows that $\mu_k \geq \lambda_k$. □

Theorem 1.5 *Let $(E, \langle \cdot, \cdot \rangle)$ be a (real) Euclidean space with dimension n and let A be a symmetric endomorphism of E. Write the eigenvalues of A in increasing order, $\lambda_1 \leq \cdots \leq \lambda_n$ (allowing multiplicities). Then we have the* max-min characterization*,*

$$\lambda_k = \sup_{F \in \mathcal{G}_{k-1}(E)} \Big(\inf \big\{ R_A(x) \; \big| \; x \in F^{\perp}, x \neq 0 \big\} \Big). \tag{1.8}$$

Proof. We use the same notations as in the proof of Theorem 1.4. Let ν_k denote the right hand side of equality (1.8). Take $F = F_{k-1}$.

Then we have

$$\lambda_k = \inf \big\{ R_A(x) \; \big| \; x \in F_{k-1}^{\perp}, x \neq 0 \big\}$$

and hence $\nu_k \geq \lambda_k$. Take any $F \in \mathcal{G}_{k-1}(E)$. Then $\dim F^{\perp} = n-k+1$ and hence there exists some $u \in F^{\perp} \cap F_k \setminus \{0\}$. For such a vector u, we have $R_A(u) \leq \lambda_k$ and hence $\inf \big\{ R_A(x) \; \big| \; x \in F^{\perp}, x \neq 0 \big\} \leq \lambda_k$. Since this is true for any $F \in \mathcal{G}_{k-1}(E)$, we conclude that $\nu_k \leq \lambda_k$. □

1.4 Applications

Let $(E, \langle \cdot, \cdot \rangle)$ be an n-dimensional real Euclidean space. Given a symmetric endomorphism A of E, we write the eigenvalues of A in increasing order (see Theorem 1.3),

$$\lambda_1(A) \leq \cdots \leq \lambda_n(A).$$

We leave the proofs of the following results to the reader.

1.4.1 Monotonicity of eigenvalues

Let A, B be two symmetric endomorphisms of E. Assume that $A \leq B$, *i.e.*

$$\langle A(x), x\rangle \leq \langle B(x), x\rangle \text{ for all } x \in E.$$

Then

$$\lambda_k(A) \leq \lambda_k(B), \quad \text{for all } k, \ 1 \leq k \leq n.$$

Exercise 1.4 *Let A be a symmetric endomorphism of E. Let F be a k-dimensional subspace of E and let B be the symmetric endomorphism of $(F, \langle\cdot,\cdot\rangle|F)$ associated with the restriction of the quadratic form Q_A to the subspace F.*

1. *Show that $Q_A|F$ is the quadratic form associated with the symmetric operator $P_F \circ A \circ P_F$ where P_F is the orthogonal projection onto the subspace F.*
2. *Prove that*

$$\lambda_j(A) \leq \lambda_j(B), \quad \textit{for any } j, \ 1 \leq j \leq k.$$

3. *Give a geometric interpretation of these inequalities for ellipsoids.*

1.4.2 Continuity of eigenvalues

Let A, B be two symmetric endomorphisms of E. Then

$$\big|\lambda_k(A) - \lambda_k(B)\big| \leq \|A - B\|_2 \text{ for any } k, 1 \leq k \leq n,$$

where $\|A\|_2$ denotes the norm of the endomorphism A associated with $\langle\cdot,\cdot\rangle$, namely

$$\|A\|_2 := \sup\big\{\|A(x)\| \;\big|\; x \in E, \|x\| \le 1\big\}.$$

In particular, if $A(t)$ is a continuous family of symmetric endomorphisms, depending on a parameter t, then the eigenvalues, $\lambda_1(A(t)) \le \cdots \le \lambda_n(A(t))$ written in increasing order, are continuous functions of t.

Note. When the symmetric endomorphism A depends smoothly or analytically on a parameter t the situation is more complicated. We refer the reader to [21, 25] for more details.

Exercise 1.5 *Let A, B be two symmetric endomorphisms of E. Give an estimate of the eigenvalue $\lambda_1\big(tA+(1-t)B\big)$ for $t \in [0,1]$ (convexity property).*

Chapter 2

Sturm-Liouville eigenvalue problems

2.1 Introduction

In this section we investigate the *Dirichlet eigenvalue problem for the Sturm-Liouville operator*.

For the sake of simplicity, we shall restrict ourselves to the particular *Sturm-Liouville operator*

$$L_V(y)(t) := -\ddot{y}(t) + V(t)y(t), \tag{2.1}$$

where $\dot{y}(t) := \frac{d}{dt}y(t)$, with *Dirichlet boundary conditions*, $y(a) = 0$ and $y(b) = 0$. We refer to [26] for the general case.

More precisely, given a closed interval $[a, b] \subset \mathbb{R}$ and a continuous real function $V : [a, b] \to \mathbb{R}$, we seek the pairs (λ, y) of a real number λ and a C^2 real valued function y ($y \neq 0$) such that

$$\begin{cases} -\ddot{y}(t) + V(t)y(t) = \lambda y(t) \quad \text{for} \quad t \in]a, b[, \\ \quad y(a) = y(b) = 0. \end{cases} \tag{2.2}$$

The eigenvalue problem (2.2) is closely related to the *Dirichlet boundary value problem for the Sturm-Liouville operator*, namely, given a continuous real valued function f and two real numbers α, β, find the functions y, if any, satisfying

$$\left\{ \begin{array}{lcl} L_V(y) & = & f, \\ y(a) & = & \alpha \text{ and,} \\ y(b) & = & \beta. \end{array} \right. \tag{2.3}$$

Historically, the motivations for studying the Dirichlet eigenvalue problem for the Sturm-Liouville operator came from mathematical physics (vibrating string ; reduction of the vibrating membrane equation to Sturm-Liouville problems after separation of variables). This was developped in the 19^{th} century and gave birth to the modern functional analysis (see [14] for an account).

The Sturm-Liouville equations are also very much related to the study of geodesics on Riemannian surfaces (or more generally on any Riemannian manifold, for vector-valued Sturm-Liouville problems). They also appear in the theory of minimal or constant mean curvature surfaces or hypersurfaces of revolution. More generally, they provide an interesting insight in the study of spectral geometry problems on Riemannian manifolds.

2.2 Initial value versus boundary value problems

Before proceeding with the study of the Dirichlet eigenvalue problem for Sturm-Liouville operators, we point out that the Dirichlet problem (2.3) is quite different from the initial value problem (or Cauchy problem) which is studied in the ordinary differential equations classes.

Recall the general existence and uniqueness theorem (or Cauchy-Lipschitz theorem, see [26]) for our Sturm-Liouville equation. Given $\alpha, \beta \in \mathbb{R}$ and a continuous real function f on $[a, b]$, there exists a unique function u satisfying the Cauchy problem

$$\begin{cases} L_V(y) & = & f \;\text{ in }\; [a,b], \\ y(a) & = & \alpha, \\ \dot{y}(a) & = & \beta. \end{cases} \tag{2.4}$$

In contrast with this result, we have the following examples for the Dirichlet problem.

- Take $V = 0, a = 0, b = 1$ and choose $f = 0, \alpha = 0, \beta = 0$. Then the corresponding Dirichlet problem (2.3) has only one solution 0.
- Take $V = 0, a = 0, b = \pi$ and choose $f = 0, \alpha = 0, \beta = 0$. Then the corresponding Dirichlet problem (2.3) has a one-dimensional set of solutions, $c \sin t$ $(c \in \mathbb{R})$.
- Take $V \equiv -1, a = 0, b = \pi$ and choose $\alpha = 0, \beta = 0$. A necessary condition for the corresponding Dirichlet problem (2.3) to have a solution is that $\int_0^\pi \sin(t) f(t)\, dt = 0$.

2.3 Setting-up the eigenvalue problem

• Spaces of functions for the Dirichlet eigenvalue problem

Considering the operator $L_V := -\ddot{y}(t) + V(t)y(t)$ (where $V : [a, b] \to \mathbb{R}$ is a continuous function), and the Dirichlet boundary conditions, a natural space of functions to work with is

$$E_2 := \{y \in C^2([a,b],\mathbb{R}) \mid y(a) = y(b) = 0\}. \tag{2.5}$$

the space of functions which are twice continuously differentiable in $]a,b[$, with first and second derivatives extending continuously to $[a,b]$, and which vanish at a and b (to take into account the Dirichlet boundary conditions).

In comparison with the eigenvalue problem for symmetric endomorphisms in finite dimensional Euclidean spaces, one difficulty here is that the operator L_V does not leave the space E_2 invariant ($L_V(E_2) \not\subset E_2$).

For later use, we introduce the space

$$E_1 := \{y \in C^1_{pw}([a,b],\mathbb{R}) \mid y(a) = y(b) = 0\}. \tag{2.6}$$

of functions which are piece-wise C^1 and continuous on $[a,b]$, and which vanish at a and b and the space

$$E_0 := \{y \in C^0([a,b],\mathbb{R}) \mid y(a) = y(b) = 0\}. \tag{2.7}$$

of continuous functions on $[a,b]$, vanishing at a and b.

• Adapted norms for the Dirichlet eigenvalue problem

We consider the inner product $\langle\cdot,\cdot\rangle_0$ given by

$$\langle u, v\rangle_0 := \int_a^b u(t)v(t)\,dt \tag{2.8}$$

on any of the spaces $E_i, i \in \{0,1,2\}$ and the associated norm $\|\cdot\|_0$.

We consider the inner product $\langle\cdot,\cdot\rangle_1$ given by

(2.9) $$\langle u, v\rangle_1 := \int_a^b \big(\dot u(t)\dot v(t) + u(t)v(t)\big)\, dt$$

on the spaces $E_i, i \in \{1, 2\}$, and the associated norm $\|\cdot\|_1$.

Observation. The above notations are classical in functional analysis and correspond to natural norms in the scale of Sobolev spaces H^k (where H^0 is the usual L^2 space of functions). The indices $0, 1$ refer to the number of derivatives we consider.

We finally introduce the uniform norm

(2.10) $$\|u\|_\infty := sup\big\{|u(t)| \;\big|\; t \in [a, b]\big\}.$$

Exercise 2.1 *Consider the space $E_1([a, b])$.*

1. *Show that the following inequalities hold for any $y \in E_1$ and any $t \in [a, b]$.*

   (2.11) $$\begin{cases} |y(t)| & \leq \sqrt{t-a}\;\|\dot y\|_0 \leq \sqrt{b-a}\;\|y\|_1, \\ \|y\|_\infty & \leq \sqrt{b-a}\;\|\dot y\|_0 \leq \sqrt{b-a}\;\|y\|_1. \end{cases}$$

2. *Let $\{y_n\}$ be a $\|\cdot\|_1$-Cauchy sequence in E_1. Show that $\{y_n\}$ is a $\|\cdot\|_\infty$-Cauchy sequence in E_0 and hence that there exists some $y \in E_0$ such that $\{y_n\}$ converges uniformly to y in E_0.*

3. *Let H_0^1 be the completion of the space $(E_1, \|\cdot\|_1)$. Show that H_0^1 can be viewed as a subspace of E_0.*

Indication. Use the formula $y(t) - y(a) = \int_a^t \dot y(s)\, ds$ for any $y \in E_1$.

Exercise 2.2 *1. Let $z \in E_2$. Let $\{y_n\}$ be a $\|\cdot\|_1$-Cauchy sequence in E_1 and let y be its limit in E_0 (see Exercise 2.1). Show that*

$$\lim_{n\infty} \langle y_n, z\rangle_1 = \int_a^b y(t)\big(z(t) - \ddot{z}(t)\big)\, dt = \langle y, z - \ddot{z}\rangle_0.$$

2. Show that one can extend the $\langle\cdot,\cdot\rangle_1$-inner product on E_1 to an inner product on H_0^1. For such an extension, show that

$$\langle y, z\rangle_1 = \int_a^b y(t)\big(z(t) - \ddot{z}(t)\big)\, dt = \langle y, z - \ddot{z}\rangle_0.$$

For more details on the space H_0^1, see [9].

2.4 Existence of eigenvalues, variational method

Let us first make the following observation.

Lemma 2.1 *The operator L_V is symmetric on $(E_2, \langle\cdot,\cdot\rangle_0)$,* i.e.

$$\langle L_V(u), v\rangle_0 = \langle u, L_V(v)\rangle_0, \quad \textit{for all} \;\; u, v \in E_2.$$

Proof. Use integration by parts and the fact that u and v vanish at a and b. □

Observation. Yet another difference with the finite dimensional case: the boundary conditions appear in the fact that L_V is symmetric.

2.4.1 First eigenvalue and eigenfunction

Mimicking Chapter 1, we introduce the quadratic form Q_{L_V} and the Rayleigh quotient R_{L_V} associated with the Sturm-Liouville operator L_V with Dirichlet boundary conditions, namely

$$(2.12)\quad \begin{cases} Q_{L_V}(y) & := \int_a^b \big(\dot{y}^2(t) + V(t)y^2(t)\big)\,dt \quad \text{for} \quad y \in E_1, \\ R_{L_V}(y) & := Q_{L_V}(y) \;/\; \int_a^b y^2(t)\,dt \quad \text{for} \quad y \in E_1, y \neq 0. \end{cases}$$

We also view Q_{L_V} as a bilinear form.

Note. The natural space to work with would actually be $L^2([a,b], dt)$.

Proposition 2.2 *The quadratic form Q_{L_V} is continuous and bounded from below, more precisely*

$$\big(\inf_{[a,b]} V - 1\big)\,\|y\|_0^2 \leq Q_{L_V}(y), \quad \textit{for all} \quad y \in E_1.$$

Proof. Note that $Q_{L_V}(y) = \|y\|_1^2 + \int_a^b \big(V(t)-1\big)y^2(t)\,dt$. The second term is bounded from below by $\big(\inf_{[a,b]} V - 1\big) \int_a^b y^2(t)\,dt$. Furthermore

$$\Big|\int_a^b \big(V(t)-1\big)\big(y^2(t)-z^2(t)\big)\,dt\Big| \leq \big(\sup|V|+1\big)\int_a^b |y^2 - z^2|\,dt$$

and the right-hand side can be bounded by $\big(\sup|V|+1\big)\,\|y-z\|_0\,\big(\|y\|_0 + \|z\|_0\big)$. □

Theorem 2.3 *Let*

$$\begin{aligned}\lambda_1 &:= \inf\left\{R_{L_V}(y) \;\middle|\; y \in E_1, y \neq 0\right\}\\ &= \inf\left\{Q_{L_V}(y) \;\middle|\; y \in E_1, \|y\|_0 = 1\right\}.\end{aligned}$$

Then, there exists a function $u_1 \in E_2$ such that $\|u_1\|_0 = 1$ and $\lambda_1 = Q_{L_V}(u_1)$. Furthermore u_1 satisfies the Dirichlet eigenvalue problem for the Sturm-Liouville operator L_V, with eigenvalue λ_1, i.e.

$$L_V(u_1) = \lambda_1\, u_1 \quad \textit{and} \quad u(a) = u(b) = 0.$$

Proof. The existence of λ_1 follows from Proposition 2.2. The proof of the other assertions is divided into three steps.

Step 1. By the definition of the infimum, there exists a sequence $\{x_n\}$ in E_1 such that $\|x_n\|_0 = 1$ and $\lambda_1 \leq Q_{L_V}(x_n) \leq \lambda_1 + \frac{1}{n}$. Then

$$0 \leq \|x_n\|_1^2 = \int_a^b \left(\dot{x}_n^2(t) + x_n^2(t)\right) dt \leq 1 + \lambda_1 + \frac{1}{n} + \int_a^b |V| x_n^2\, dt$$

and hence

$$\|x_n\|_1^2 \leq C := |\lambda_1| + 2 + \sup |V|.$$

Using the fact that $y(t) - y(s) = \int_s^t \dot{y}(\tau)\, d\tau$ we have the inequality

$$|y(t) - y(s)| \leq \sqrt{|t-s|}\; \|y\|_1.$$

The preceding inequality applied to $\{x_n\}$ and the fact that $\|x_n\|_1^2 \leq C$ tell us that $\{x_n(t)\}$ is uniformly bounded for all t and that $\{x_n\}$ is an equicontinuous sequence. It follows from Ascoli's theorem ([13, 23]) that $\{x_n\}$ is relatively compact in $(E_0, \|\cdot\|_\infty)$.

By redefining the sequence $\{x_n\}$ if necessary, we may assume that $\{x_n\}$ converges uniformly to some $u_1 \in E_0$.

Step 2. By the definition of λ_1 and for any $s \in \mathbb{R}$ and any $y \in E_2$, we have $Q_{L_V}(x_n + sy) \geq \lambda_1 \|x_n + sy\|_0^2$. Developing this inequality, we find

$$Q_{L_V}(x_n)+2sQ_{L_V}(x_n, y)+s^2Q_{L_V}(y) \geq \lambda_1\big(\|x_n\|_0^2+2s\langle x_n, y\rangle_0+s^2\|y\|_0^2\big)$$

The second term in the left-hand side can be written as

$$Q_{L_V}(x_n, y) = \int_a^b \big(\dot{x}_n\dot{y} + Vx_ny\big)\,dt = \int_a^b x_n\big(-\ddot{y} + Vy\big)\,dt$$

and hence tends to $\int_a^b u_1\big(-\ddot{y} + Vy\big)\,dt$ when n tends to infinity. Letting n tend to infinity in the above inequality, it follows that

$$2s\int_a^b u_1\big(-\ddot{y} + Vy - \lambda_1 y\big)\,dt + s^2\big(Q_{L_V}(y) - \lambda_1\|y\|_0^2\big) \geq 0$$

for all $s \in \mathbb{R}$ and all $y \in E_2$. Using again the definition of λ_1, the coefficient of s^2 is non-negative and we may conclude that

$$\int_a^b u_1\big(-\ddot{y} + Vy - \lambda_1 y\big)\,dt = 0 \quad \text{for all} \quad y \in E_2. \tag{2.13}$$

In view of equality (2.13), we say that the function u_1 is a *weak solution* of the Dirichlet eigenvalue problem for the Sturm-Liouville equation associated with the value λ_1, or that it satisfies the equation in the *weak sense.*

Step 3. Because V and u_1 are continuous, there is a uniquely defined C^2 function w such that $\ddot{w} = (V - \lambda_1)u_1$ and $w(a) = w(b) = 0$. For such a function w, the equality

$$\int_a^b u_1\big(\ddot y - Vy + \lambda_1 y\big)\,dt = 0 \tag{2.14}$$

can be rewritten as

$$\int_a^b \big(u_1\ddot y - y\ddot w\big)\,dt = 0$$

and hence, after integration by parts taking into account the fact that the functions vanish at a and b,

$$\int_a^b (u - w)\ddot y = 0.$$

We can now choose $y \in E_2$ such that $\ddot y = u_1 - w$ and $y(a) = y(b) = 0$. We then get $\int_a^b (u-w)^2\,dt = 0$ and hence $u_1 \equiv w$ which shows that $u_1 \in E_2$.

Once we know that $u_1 \in E_2$, we can integrate equation (2.14) by parts twice (using the fact that the functions vanish at a and b) and conclude that $L_V(u_1) - \lambda_1 u_1$ is $\langle\cdot,\cdot\rangle_0$-orthogonal to all functions y in E_2. It follows, from a density argument, that $L_V(u_1) - \lambda_1 u_1 \equiv 0$. Multiplying this last identity by u_1 and integrating by parts once, we conclude that $Q_{L_V}(u_1) = \lambda_1$. □

Remarks.

- Step 1 is a compacity argument. This argument is always present in similar proofs (using compact Sobolev embeddings).
- Step 2 is a trick which shows that the continuous function u_1 given by Step 1 is a weak solution. In general this can be achieved more simply using Hilbert space methods (namely weak convergence).
- Step 3 is a regularity result. It is very simple here because we deal with ordinary differential equations. The regularity argument in higher dimensions is much more involved.

2.4.2 Higher eigenvalues

Let us first of all mention the following result which is very specific to the Dirichlet eigenvalue problem for the Sturm-Liouville operator (it is not true in higher dimensions ; it is not true for the Sturm-Liouville operator with periodic boundary conditions).

Proposition 2.4 *The eigenvalues of the Dirichlet problem for the Sturm-Liouville operator are simple,* i.e. *if* (λ, u) *is a non-trivial solution of the Dirichlet eigenvalue problem (2.2), then* u *is unique up to a multiplicative constant.*

Proof. This is an immediate consequence of the uniqueness of the Cauchy problem for the Sturm-Liouville equation. □

By Section 2.4.1, there exists an eigenvalue λ_1 and an associated eigenfunction $u_1 \in E_2$ of the Dirichlet eigenvalue problem (2.2) for the Sturm-Liouville operator. We now introduce

$$\lambda_2 := \inf \left\{ Q_{L_V}(y) \;\middle|\; y \in E_1,\ \langle y, u_1 \rangle_0 = 0\ \|y\|_0 = 1 \right\}. \tag{2.15}$$

The infimum exists by Proposition 2.2. We have the following result.

Lemma 2.5 *There exists a function* $u_2 \in E_2$ *such that*

$$\langle u_2, u_1 \rangle_0 = 0 \quad \textit{and} \quad \|u_2\|_0 = 1,$$

$$Q_{L_V}(u_2) = \lambda_2 \quad \textit{and} \quad L_V(u_2) = \lambda_2 u_2.$$

The couple (λ_2, u_2) *is a solution of the Dirichlet eigenvalue problem (2.2) for the Sturm-Liouville operator.*

Proof. Follow the steps of the proof of Theorem 2.3.

Step 1. As in the proof of Theorem 2.3, there exists a sequence $\{x_n\}$ in E_1 and a function $u_2 \in E_0$ such that $Q_{L_V}(x_n)$ tends to λ_2, $\|x_n\|_0 = 1$, $\langle x_n, u_1\rangle_0 = 0$ for all n and $\{x_n\}$ tends to u_2 in the $\|\cdot\|_\infty$-norm. Furthermore, $\langle u_1, u_2\rangle_0 = 0$.

Step 2. As in the proof of Theorem 2.3, one shows that $\langle u_2, \ddot{y} - Vy + \lambda_2 y\rangle_0 = 0$ for any function $y \in E_2$ such that $\langle u_1, y\rangle_0 = 0$. Note that $\langle u_2, \ddot{u}_1 - Vu_1 + \lambda_2 u_1\rangle_0 = 0$ is true also because u_1 is an eigenfunction of L_V and $\langle u_1, u_2\rangle_0 = 0$. It follows that $\langle u_2, \ddot{y} - Vy + \lambda_2 y\rangle_0 = 0$ for any function $y \in E_2$ and that u_2 is a weak solution of the Dirichlet eigenvalue problem for the Sturm-Liouville equation, associated with the eigenvalue λ_2.

Step 3. The proof of Theorem 2.3 applies without modification. □

Note that by Proposition 2.4, we have $\lambda_1 < \lambda_2$.

We can now repeat this construction to obtain higher eigenvalues. Assume that we have constructed couples (λ_1, u_1), ..., (λ_k, u_k) for $k \geq 2$, with

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k \text{ and } u_1, \ldots, u_k \ \langle\cdot,\cdot\rangle_0\text{-orthogonal functions } \in E_2.$$

We obtain the couple (λ_{k+1}, u_{k+1}) by minimizing $Q_{L_V}(y)$ over the functions $y \in E_1$ such that y is $\langle\cdot,\cdot\rangle_0$-orthogonal to $u_1, \ldots, u_k$ and $\|y\|_0 = 1$.

Because E_1 is infinite dimensional, we obtain an infinite sequence $\{\lambda_k, u_k)\}_{k\geq 1}$ with $\{\lambda_k\}_{k\geq 1}$ strictly increasing and $\{u_k\}_{k\geq 1}$ an $\langle\cdot,\cdot\rangle_0$-orthonormal family in E_2.

Lemma 2.6 *The sequence $\{\lambda_k\}_{k\geq 1}$ tends to infinity.*

Proof. Since the sequence $\{\lambda_k\}_{k\geq 1}$ is increasing it either tends to infinity or it is bounded. If it were bounded, the sequence $\{u_k\}_{k\geq 1}$ would be uniformly bounded in the $\|\cdot\|_1$-norm and hence by Ascoli's theorem (compare with Step 1 in the proof of Theorem 2.3) it would have a converging subsequence in the $\|\cdot\|_\infty$-norm and hence in the $\|\cdot\|_0$-norm which is not possible because $\{u_k\}_{k\geq 1}$ is an orthonormal sequence. □

Lemma 2.7 *The vector space* $\mathrm{Vect}\{u_1, u_2, \ldots\}$ *(finite linear combinations in the functions u_k's) is dense in E_1 with respect to the $\|\cdot\|_0$-norm.*

Proof. Assume $\mathrm{Vect}\{u_1, u_2, \ldots\}$ is not dense in E_1 with respect to the $\|\cdot\|_0$-norm. Then there would exist a function $u \in E_1$ such that $\langle u, u_k\rangle_0 = 0$ for all $k \geq 1$ and $\|u\|_0 = 1$. By the construction of the sequence $\{(\lambda_k, u_k)\}_{k\geq 1}$, we would have $Q_{L_V}(u) \geq \lambda_k$ for all $k \geq 1$ and hence the sequence $\{\lambda_k\}_{k\geq 1}$ would be bounded from above, a contradiction. □

One can summarize the preceding results in the following theorem.

Theorem 2.8 *There exists a sequence $\lambda_1 < \lambda_2 < \cdots$ of real numbers and an $\langle\cdot,\cdot\rangle_0$-orthonormal sequence $u_k, k \geq 1$, of C^2 functions vanishing at a and b, which satisfy the following properties.*

1. *For $k = 1$, $\lambda_1 = \inf\left\{Q_{L_V}(y) \;\middle|\; y \in E_1, \|y\|_0 = 1\right\}$ and the infimum is achieved at u_1.*

2. *For $k \geq 2$, $\lambda_k = \inf\left\{Q_{L_V}(y) \;\middle|\; y \in E_1, \|y\|_0 = 1, \langle y, F_{k-1}\rangle_0 = 0\right\}$, where $F_{k-1} := \mathrm{Vect}\{e_1, \ldots, u_{k-1}\}$ and the infimum is achieved at u_k.*

3. *The sequence of eigenvalues* $\{\lambda_k\}_{k\geq 1}$ *tends to infinity and the sequence of eigenfunctions* $\{u_k\}_{k\geq 1}$ *is dense in* E_1 *in the* $\|\cdot\|_0$*-norm.*

4. *The pairs* (λ_k, u_k) *are solutions of the Dirichlet eigenvalue problem (2.2) for the Sturm-Liouville operator* L_V, i.e.

$$L_V(u_k) := -\ddot{u}_k(t) + V(t)u_k(t) = \lambda_k u_k(t),\ u_k(a) = u_k(b) = 0.$$

5. *Any non-trivial solution* (λ, u) *of the Dirichlet eigenvalue problem (2.2) for the Sturm-Liouville operator* L_V *is simple (*i.e. *u is unique up to multiplication by a scalar) and is one of the* (λ_k, u_k).

As in the finite dimensional case, we have the min-max and max-min principles.

Theorem 2.9 *Let* $\mathcal{G}_k$ *denote the set of k-dimensional linear subspaces of the space* E_1 *of piece-wise* C^1 *functions in* $[a, b]$*, vanishing at a and b. Write the eigenvalues of the Dirichlet problem (2.2) for the Sturm-Liouville operator in increasing order* $\lambda_1 < \lambda_2 < \cdots$. *Then,*

1. *max-min characterization,*

$$\lambda_1 = \inf\left\{R_{L_V}(y) \;\middle|\; y \in E_1^\bullet\right\}$$

and,

$$\lambda_k = \sup_{N\in\mathcal{G}_{k-1}} \inf\left\{R_{L_V}(y) \;\middle|\; y \in E_1^\bullet, \langle y, N\rangle_0 = 0\right\} \quad \textit{for} \quad k \geq 2$$

2. *min-max characterization,*

$$\lambda_k = \inf_{N\in\mathcal{G}_k} \sup\left\{R_{L_V}(y) \;\middle|\; y \in N^\bullet\right\}$$

.

Proof. The proof is left to the reader (mimick the proofs of Theorems 1.4 and 1.5). □

2.5 Nodal sets and nodal domains of eigenfunctions

Proposition 2.10 *Let $\{\lambda_k\}_{k\geq 1}$ be the eigenvalues of the Dirichlet problem (2.2) for the Sturm-Liouville operator written in increasing order. Let $\{u_k\}_{k\geq 1}$ be a corresponding orthonormal sequence of associated eigenfunctions. The function u_k is characterized by the fact that it has exactly $(k-1)$ zeroes in $]a,b[$.*

Proof.

Step 1. We first prove that an eigenfunction u associated with the k^{th}-eigenvalue λ_k has at most $(k-1)$ zeroes in $]a,b[$.

Assume u is not identically 0. Let $z \in]a,b[$ be a zero of u. By the uniqueness of the Cauchy problem for second order differential equations, $\dot{u}(z) \neq 0$. It follows that u can only have finitely many zeroes in $[a,b]$ and that u changes sign whenever it vanishes in $]a,b[$. Let $z_0 := a < z_1 < z_2 < \cdots < z_m < z_{m+1} := b$ be the zeroes of u and assume that $k \leq m$. For $1 \leq j \leq k$, define the functions v_j by the relations

$$v_j(t) = u(t), \quad \text{for} \quad t \in [z_{j-1}, z_j] \quad \text{and} \quad v_j(t) = 0, \quad \text{for} \quad t \notin [z_{j-1}, z_j].$$

The function $v := \sum_{j=1}^{k} a_j v_j$ belongs to E_1 and vanishes identically on the open set $z_k < t < b$. One can choose the coefficients $a_j, 1 \leq j \leq k$, in such a way that v is orthogonal to the eigenfunctions $u_1, \ldots, u_{k-1}$ and v not identically zero. Furthermore, it is easy to check that $R_{L_V}(v) = \lambda_k$. It follows from the proof of Theorem

2.3 (see steps 2 and 3), that v is an eigenfunction associated with the eigenvalue λ_k. It follows that v satisfies the differential equation $-\ddot{v}(t) + V(t)\,v(t) - \lambda_k\,v(t) = 0$ and vanishes on an open set. By the uniqueness of the Cauchy problem this implies that v is identically zero, a contradiction.

Step 2. We prove that an eigenfunction u associated with the k^{th}-eigenvalue λ_k has at least $(k-1)$ zeroes in $]a,b[$.

We first prove the following assertion. Let α, β be two consecutive zeroes of an eigenfunction v associated with λ_{k-1}, then u must vanish in the open interval $]\alpha,\beta[$.

Indeed, assume u does not vanish in the interval $]\alpha,\beta[$. We may assume that u and v are positive on this interval. Consider the function $W(t) := W(u,v)(t) := u(t)\dot{v}(t) - v(t)\dot{u}(t)$. Then $\dot{W}(t) = (\lambda_k - \lambda_{k-1})u(t)v(t) > 0$. On the other-hand, $W(\alpha) = u(\alpha)\dot{v}(\alpha) > 0$ and $W(\beta) = u(\beta)\dot{v}(\beta) < 0$, a contradiction.

To conclude the proof we reason by induction. According to Step 1, u_1 cannot vanish in $]a,b[$. The function u_2 is orthogonal to u_1 and hence it must vanish at least once in $]a,b[$. According to Step 1, it must vanish exactly once in $]a,b[$. Assume (induction assumption) that u_k vanishes exactly $(k-1)$ times in $]a,b[$. According to the previous argument, the function u_{k+1} must vanish at least once between two consecutive zeroes of u_k (including a and b) and hence it must vanish at least k times in $]a,b[$. According to Step 1, the function u_{k+1} vanishes exactly k times in $]a,b[$. □

Remark. A nodal domain of an eigenfunction u, is a connected component of $[a,b] \setminus u^{-1}(0)$ *i.e.* a maximal interval on which u does not vanish. The preceding Theorem can be restated as follows

1. An eigenfunction corresponding to the k^{th} eigenvalue has at most k nodal domains.

2. An eigenfunction corresponding to the k^{th} eigenvalue has at least k nodal domains.

The first assertion is true in higher dimensions. This is the so-called *Courant nodal domain theorem* (see [12]). The second assertion is very specific to the Dirichlet boundary value problem for the Sturm-Liouville operator. It holds also for the Neumann problem (vanishing derivatives at a and b instead of vanishing of functions), but not for the periodic boundary value problem (periodic functions) for the Sturm-Liouville operator.

2.6 Further properties of eigenvalues

We leave the following properties as Exercises for the reader.

2.6.1 Monotonicity of eigenvalues

Proposition 2.11 *Let V and W be two continuous real valued functions. Let $\{\lambda_k(L_V)\}_{k\geq 1}$ and $\{\lambda_k(L_W)\}_{k\geq 1}$ be the eigenvalues of the Dirichlet problem for the corresponding Sturm-Liouville operators, listed in increasing order. Assume that $V(t) \leq W(t)$ for all $t \in [a,b]$. Then $\lambda_k(L_V) \leq \lambda_k(L_W)$ for all $k \geq 1$.*

Proposition 2.12 *Let L_V be a Sturm-Liouville operator on $[a,b]$ and let*

$$\{\lambda_k(L_V, [a,b])\}_{k\geq 1}$$

be the eigenvalues of L_V for the Dirichlet problem, listed in increasing order. If $[\alpha,\beta] \subset [a,b]$, then

$$\lambda_k(L_V, [a,b]) \leq \lambda_k(L_V, [\alpha,\beta])$$

for all $k \geq 1$.

2.6.2 Continuity of eigenvalues

Proposition 2.13 *Let V and W be two continuous real valued functions. Let $\{\lambda_k(L_V)\}_{k\geq 1}$ and $\{\lambda_k(L_W)\}_{k\geq 1}$ be the eigenvalues of the Dirichlet problem for the corresponding Sturm-Liouville operators, listed in increasing order. Then*

$$\big|\lambda_k(L_V) - \lambda_k(L_W)\big| \leq \|V - W\|_\infty, \quad \textit{for all} \;\; k \geq 1.$$

Exercise 2.3 *Let V and W be two continuous real valued functions. Give and estimate of $\lambda_1\big(tV + (1-t)W\big)$ for $t \in [0,1]$ (convexity property).*

2.6.3 Asymptotic behaviour of eigenvalues

Proposition 2.14 *Let L_V be a Sturm-Liouville operator on $[a,b]$ with a continuous potential V. Let $\{\lambda_k(L_V,[a,b]), k \geq 1\}$ be the eigenvalues of L_V for the Dirichlet problem, listed in increasing order. Then*

$$\lambda_k(L_V,[a,b]) \sim k^2\pi^2/(b-a)^2 \quad \textit{when} \;\; k \to \infty$$

Chapter 3

Application to catenoids in $\mathbb{R}^3$

We will show how the preceding chapter can be used to understand the stability properties of catenoids in $\mathbb{R}^3$.

Given $a > 0$, let

$$X_a(t,\varphi) := \big(a\cosh(t)\cos(\varphi), a\cosh(t)\sin(\varphi), at\big) \tag{3.1}$$

for $t \in \mathbb{R}$ and $\varphi \in [0, 2\pi]$.

This is a parametrization of a family of catenoids in $\mathbb{R}^3$ (minimal surfaces indexed by a). The associated *Jacobi operator* (or stability operator, see [22]) is the operator $J_a = -\Delta_a + 2K_a$ where Δ_a is the (non-positive) Laplacian and K_a the intrinsic curvature for the induced metric on the catenoids. This operator is associated with the second variation of the area functional when the catenoids are deformed in the normal direction. In the parameters (t, φ), the Jacobi operator is given by

$$J_a = -\frac{1}{a^2\cosh^2(t)}\Big(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \varphi^2}\Big) - \frac{2}{a^2\cosh^4(t)}. \tag{3.2}$$

We now consider the domains $C_{a,T}$ defined, for $T > 0$, by

$$C_{a,T} := X_a\big([-T,T]\times[0,2\pi]\big) \tag{3.3}$$

and the eigenvalue problem for J_a, with Dirichlet boundary conditions in $C_{a,T}$,

$$J_a(f) = \lambda f \quad \text{in} \quad C_{a,T} \quad \text{and} \quad f|\partial C_{a,T} = 0. \tag{3.4}$$

We will prove the following result.

Theorem 3.1 *Let $J_a = -\Delta_a + 2K_a$ be the Jacobi operator with Dirichlet boundary conditions in $C_{a,T}$. Let T_0 be the positive zero of the equation $t\tanh(t) = 1$. Then,*

1. *For $0 < T < T_0$ the operator J_a has only positive eigenvalues (we say that the domain $C_{a,T}$ is* stable*).*
2. *For $T = T_0$ the operator J_a is non-negative and has 0 as simple eigenvalue (we say that the domain C_{a,T_0} is* weakly stable*).*
3. *For $T > T_0$ the operator J_a has exactly one negative simple eigenvalue and all other eigenvalues are positive (we say that the domain $C_{a,T}$ is* unstable *and has* index 1*).*

Proof. The eigenvalues of problem (3.4) are decreasing in T and positive when T is small. We investigate for which value of T an eigenvalue can pass from a positive value to a negative value.

In the parameters (t, φ), problem (3.4) for the domains $C_{a,T}$ boils down to

$$(3.5)\quad \Big(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \varphi^2}\Big) f(t,\varphi) + \frac{2}{\cosh^2(t)} f(t,\varphi) + \lambda a^2 \cosh^2(t) f(t,\varphi) = 0$$

with $f(-T, \varphi) = f(T, \varphi) = 0$.

Assume that, for some T, problem (3.4) has a negative eigenvalue λ, with associated eigenfunction $f(t, \varphi)$. Fix t and expand $f(t, \varphi)$ in Fourier series in the φ variable,

$$f(t,\varphi) = \sum_{p \in \mathbb{Z}} f_p(t) e^{ip\varphi}.$$

Using (3.5), we see that the coefficient f_p satisfies the equation

$$(3.6)\quad \ddot{u}(t) + \Big(\frac{2}{\cosh^2(t)} - p^2\Big) u(t) + \lambda a^2 \cosh^2(t) u(t) = 0, \quad \text{in }]-T, T[$$

with the boundary conditions $u(-T) = u(T) = 0$.

We introduce the operators

$$(3.7)\qquad L_p(u) := -\ddot{u}(t) + \big(p^2 - \frac{2}{\cosh^2(t)}\big) u(t)$$

with Dirichlet boundary conditions in $[-T, T]$.

It follows that equation (3.4) has a negative eigenvalue if and only if equation (3.6) has a (non-trivial) solution f_p for some value p.

We know investigate the eigenvalue problem (3.6). The quadratic form associated with L_p is given by

$$\mathcal{L}_p(u) := \int_{-T}^{T} \big(\dot{u}^2(t) + (p^2 - \frac{2}{\cosh^2(t)})u(t)\big)\, dt. \tag{3.8}$$

It is clear that the quadratic form $\mathcal{L}_p$ is positive for $|p| \geq 2$, so that we only need look at the cases $p = 0$ and $|p| = 1$.

We leave the following lemma as an exercise for the reader.

Lemma 3.2 *Define the functions* $k_v(t) := \tanh(t)$, $k_h(t) := \dfrac{1}{\cosh(t)}$ *and* $k_a(t) := 1 - t\,\tanh(t)$. *Then*

1. *The function* k_v *is positive on* $]0, \infty[$ *and satisfies* $L_0(k_v) = 0$.
2. *The function* k_h *is positive on* $\mathbb{R}$ *and satisfies* $L_{\pm 1}(k_h) = 0$.
3. *The function* k_a *has exactly one zero* T_0 *on* $[0, \infty[$. *It satisfies* $L_0(k_a) = 0$, $k_a(0) = 1$ *and* $\dot{k}_a(0) = 0$.

Remark. The function k_v (*resp.* k_h) has a geometric interpretation. It comes from the Killing field associated with translations parallel to the rotation axis of the catenoids in $\mathbb{R}^3$ (*resp.* the Killing field associated with the x-translations in $\mathbb{R}^3$). There is also a geometric interpretation for the function k_a : it is associated with the variation of the family of catenoids with respect to the parameter a. It describes the point where a catenoid touches the envelope $(T_0|t|, t)$ of the family.

We need another lemma.

Lemma 3.3 *Let L_V be the Sturm-Liouville operator*

$$L_V(y) := -\ddot{y} + Vy \quad in \quad [a,b],$$

with $V : [a,b] \to \mathbb{R}$ a continuous function. Assume that there exists some function $w \geq 0, w \not\equiv 0$ such that $L_V(w) = 0$ on $[a,b]$. Then, the eigenvalues of L_V for the Dirichlet problem in $[a,b]$ are non-negative. Furthermore, if $w(a) > 0$ or $w(b) > 0$, then the eigenvalues are positive.

Proof. Assume there exists a negative eigenvalue of the Dirichlet problem for L_V on $[a,b]$. Then the least eigenvalue μ is negative and we may select an associated eigenfunction u such that $u(a) = u(b) = 0$ and $u > 0$ in $]a,b[$. It follows that $\dot{u}(a) > 0$ and $\dot{u}(b) < 0$. Consider the Wronskian of u and w, namely the function

$$W(t) := w(t)\dot{u}(t) - u(t)\dot{w}(t).$$

We have $\dot{W}(t) = -\mu\, u(t)w(t) \geq 0$. It follows from the previous inequality and our assumption that, $W(b) > W(a)$. On the other-hand, $W(b) \leq 0$ and $W(a) \geq 0$, a contradiction. The proof of the second assertion is similar. □

Proof of Theorem 3.1.

Step 1. Let us first analyze L_0. Recall that a first eigenfunction may be chosen to be positive and that it is characterized by this property. It follows that k_a is a first eigenfunction of the operator L_0 in $[-T_0, T_0]$ (associated with the eigenvalue 0). By the (strict) monotonicity of eigenvalues, we deduce that L_0 is positive on $[-T,T]$ for $T < T_0$. Because $\mathcal{L}_p \geq \mathcal{L}_0$, we have proved the first assertion of the Theorem.

Step 2. We already know that L_p is positive for $|p| \geq 2$. Using the function k_h, it follows from Lemma 3.3 that $L_{\pm 1}$ is positive as well. This means that an eigenfunction associated with a negative eigenvalue must be a radial function (all Fourier coefficients f_p of f are zero except f_0). Using the function k_a and the preceding obervation, Assertion 2 follows.

Step 2. Assume that, for some $[-T, T]$, the operator L_0 has at least two negative eigenvalues. An eigenfunction associated with the second negative eigenvalue must vanish exactly once in $]-T, T[$. One can then get a contradiction by using the function k_v and Lemma 3.3. □

Remark. One can use the same method to study the index of complete minimal rotation hypersurfaces in $\mathbb{R}^n$ and in other spaces (see for example [7] for a recent result).

Chapter 4

Eigenvalues in geometry

In this Chapter we describe some extensions of the results described in the preceding sections as a motivation for further study. We do not aim at giving an exhaustive account. We rather point out some striking results.

The higher dimensional analogues of the Sturm-Liouville operator are operators of the form $L_V := -\Delta + V$ (the so-called *Schrödinger operators* of mathematical physics). Here Δ is the Laplace operator $\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ acting on functions on some open domain $\Omega \subset \mathbb{R}^n$ and V is a real function on Ω.

More generally, one considers Schrödinger operators on an n-dimensional Riemannian manifold (M, g). The corresponding operator Δ_g, the so-called *Laplace-Beltrami operator* is given in local coordinates $\{x_j\}_{j=1}^n$ on M by the formula

$$\Delta_g = v^{-1} \sum_{j,k} \frac{\partial}{\partial x_j} \big(v g^{jk} \frac{\partial}{\partial x_k} \big)$$

where (g^{jk}) denotes the inverse of the matrix (g_{ab}) representing the metric g in the local coordinates and $v = \sqrt{\mathrm{Det}(g_{ab})}$ is the density of the Riemannian measure in the local coordinates.

As for Sturm-Liouville operators, and under suitable assumptions (*e.g.* M is compact without boundary or M is compact with boundary and Dirichlet boundary conditions are imposed on the boundary; V is continuous) one can prove the existence of an infinite sequence of eigenvalues

$$\lambda_1(M, g) < \lambda_2(M, g) \leq \cdots \lambda_k(M, g) \leq \cdots$$

and of an orthonormal sequence of corresponding eigenfunctions $\{u_k\}_{k \geq 1}$.

The motivations for studying eigenvalue problems for Schrödinger operators on domains in $\mathbb{R}^n$ come from mathematical physics. We refer the reader to the classical books by R. Courant and D. Hilbert [12] and by M. Reed and B. Simon [24]. An important historical reference is the paper by H. Weyl [28].

4.1 Spectral geometry for itself

The book [8] by M. Berger, P. Gauduchon and E. Mazet, has had a seminal influence on a domain of research known as *spectral geometry* which comprises studying eigenvalue problems on Riemannian manifolds. We refer to [4, 5, 10] for more recent introductions.

The main issue is the relationship between the eigenvalues of the Laplace-Beltrami operator and the geometry of the manifold (M, g). More precisely, one can ask two types of questions: (i) Given a Riemannian manifold (M, g), what kind of information on the eigenvalues (lower bounds, upper bounds, ...) or eigenfunctions can one derive from information on geometric data (*direct problem*) ? (ii) Assuming one knows the eigenvalues of Δ_g on some (unknown) manifold (M, g), what kind of geometric information can one derive on the geometry of (M, g) (*inverse problem*) ?

Let us give some classical results.

Courant's nodal domain theorem

Let (M, g) we a compact manifold with or without boundary (in that case, use for example the Dirichlet boundary conditions). Let $\{\lambda_k(M, g)\}_{k \leq 1}$ be the eigenvalues of (M, g) listed in increasing order. Given an non-zero eigenfunction u associated with λ_k, denote by $D(u)$ the number of connected components of $M \setminus u^{-1}(0)$, the so-called *nodal domains* of u. Courant's nodal domain theorem (compare with Section 2.5) states that

$$D(u) \leq k.$$

It turns out that, in contrast with the one-dimensional case, there are examples of domains and eigenfunctions (with large eigenvalue order) with only 2 or 3 nodal domains. For a proof of Courant's theorem and examples see [12]. This theorem has been used to obtained bounds on eigenvalues multiplicities in dimension 2. An interesting and difficult related question is to give bounds (upper and lower) on the $(n-1)$-volume of the set $u^{-1}(0)$, the so-called *nodal set*, see for example [16] for a recent result.

Faber-Krahn inequality

Let Ω be a compact domain in $\mathbb{R}^n$ and let Ω^* be the Euclidean ball with volume $\mathrm{Vol}(\Omega)$. Let $\lambda_1(\Omega)$ be the least eigenvalue of the Laplace operator $-\Delta$ with Dirichlet boundary conditions on $\partial\Omega$, and let $\lambda_1(\Omega^*)$ be the corresponding eigenvalue for Ω^*. Then

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$$

and equality holds if and only if Ω is a Euclidean ball.

This inequality is both a direct result (lower bound on the least eigenvalue in terms of the volume) and an inverse result (the Euclidean ball is characterized up to isometry by the least eigenvalue of the Laplace operator). The main underlying argument is the classical isoperimetric inequality in $\mathbb{R}^n$. Isoperimetric inequalities play an important rôle in eigenvalue estimates (see [4, 5]).

Hermann Weyl's asymptotic law

Let (M, g) be an n-dimensional compact Riemannian manifold without boundary. Let $N(\lambda)$ denote the *counting function*

$$N(\lambda) = \sharp\{j \mid \lambda_k(M, g) \leq \sqrt{\lambda}\}$$

where the eigenvalues $\lambda_k(M, g)$ are listed in increasing order. Weyl's asymptotic law states (compare with Section 2.6.3) that

$$N(\lambda) = (2\pi)^{-n}\mathrm{Vol}(B^n)\mathrm{Vol}(M, g)\lambda^n + O(\lambda^{n-1})$$

where B^n is the unit ball in $\mathbb{R}^n$ and where $O(\lambda^{n-1})$ denotes a function such that $\lambda^{-(n-1)}O(\lambda^{n-1})$ is bounded when λ tends to infinity. The principal part $(2\pi)^{-n}\mathrm{Vol}(B^n)\mathrm{Vol}(M, g)\lambda^n$ is due to H. Weyl [27] for Euclidean domains. Understanding the remainder term $N(\lambda) - (2\pi)^{-n}\mathrm{Vol}(B^n)\mathrm{Vol}(M, g)\lambda^n$ turned out to be quite difficult and this term is related to properties of the geodesic flow on (M, g). The best result was obtained by L. Hörmander in 1968, see [20] for a review.

As in the previous example, Weyl's asymptotics can be interpreted both as a direct result (asymptotic information on the eigenvalues in terms of dimension and volume) and as an inverse result (knowing the spectrum determines the dimension and the volume).

Isospectral manifolds

Of particular interest is the question – known as Mark Kac's question, *Can one here the shape of a drum ?* – whether the set of eigenvalues (*the spectrum*) of the Laplace-Beltrami operator on a compact manifolds actually determines the manifold (M, g) up to isometry. Some manifolds are indeed characterized by their spectrum. This is the case for Euclidean balls (see the paragraph on Faber-Krahn inequality) or for spheres. Sporadic pairs of non-isometric compact manifolds with the same spectrum (*isospectral manifolds*) were already known in 1964; many examples of such pairs (or even continuous families) have been constructed since 1982. We refer the reader to the surveys [6, 19].

Spectral geometry is still an active area of research.

4.2 Eigenvalues and minimal submanifolds

Let Σ an oriented surface immersed into a Riemannian 3-manifold M, with unit normal field N_Σ. The first variation of the area functional for normal deformations is given by

$$A'(f) = -2 \int_\Sigma f\, H\, d\mu_\Sigma$$

where the variation $f N_\Sigma$ has compact support in Σ, where H is the normalized mean curvature in the direction N_Σ and $d\mu_\Sigma$ the Riemannian measure.

Critical points of the area functional are minimal surfaces. The second variation of the area functional is given by

$$A''(f) = \int_\Sigma f\big(-\Delta_\Sigma f + 2K_\Sigma f\big)\, d\mu_\Sigma$$

where Δ_Σ is the Laplace-Beltrami operator for the induced metric on Σ and K_Σ the Gauss curvature. See [11] for more details.

We call *Jacobi (or stability) operator* the operator $J_\Sigma := -\Delta_\Sigma + 2K_\Sigma$. We say that a compact domain $\Omega \subset \Sigma$ is *stable* if the Jacobi operator J_Σ (with Dirichlet boundary conditions on $\partial\Omega$) has only positive eigenvalues. This means that the domain Ω minimizes area up to second order. In any case, since Ω is compact, the operator J_Ω has at most finitely many negative eigenvalues. We call the number of negative eigenvalues the *index* of Ω and we denote this number by $\mathrm{Ind}(\Omega)$. This measures the number of ways in which one can decrease the area of Ω.

We say that a minimal surface Σ is *stable* if any compact domain $\Omega \subset \Sigma$ is stable. We also define the index of a minimal surface Σ as the supremum of the indices of all the compact domains,

$$\mathrm{Ind}(\Sigma) := \sup \big\{ \mathrm{Ind}(\Omega) \;\big|\; \Omega \Subset \Sigma \big\}.$$

It is an interesting question in the theory of minimal surfaces to determine stable domains and stable surfaces (see Chapter _}refS-sl-appli) and to relate the index with total curvature. Let us mention some classical important results (which have been generalized to different frameworks).

On the size of a minimal surface in $\mathbb{R}^3$

Recall that the Gauss map of an oriented minimal surface $\Sigma \looparrowright \mathbb{R}^3$ is the map G which sends a point $x \in \Sigma$ to the point $G(x) \in S^2 \subset \mathbb{R}^3$ which represents the normal $N_\Sigma(x)$ of the surface at the point x. L. Barbosa and M. do Carmo [1] proved the following very nice result which has had a seminal influence on the subject (see also [2]).

$$\mathrm{Area}_{S^2}\big(G(\Omega)\big) < 2\pi \Rightarrow \Omega \;\text{ is stable.}$$

The idea is to relate the Jacobi operator on Σ to an operator on the sphere and to estimate the least eigenvalue of this latter operator using an argument à la Faber-Krahn.

Stable complete orientable minimal surfaces in $\mathbb{R}^3$

It has been shown by various authours, independently (M. do Carmo – C.K. Peng ; D. Fischer-Colbrie – R. Schoen ; A.V. Pogorelov) that *stable complete orientable minimal surface in* $\mathbb{R}^3$ *are planes.* We refer to [18] for the proof which involves a careful analysis of the Jacobi operator.

Index and total curvature

The following result is due to D. Fischer-Colbrie [17].
Let Σ *be a complete orientable minimal surface in* $\mathbb{R}^3$*. Then* Σ *has finite index if and only if* Σ *has finite total curvature,* i.e. *the integral* $\int_\Sigma |K_\Sigma| d\mu_\Sigma$ *is finite.*

See [15] for a generalization of this result to the case of surfaces with constant mean curvature 1 in the hyperbolic space with curvature -1 (see [3] for an introduction to stability questions for constant mean curvature hypersurfaces).

See [7] for a recent result on the index of minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$.

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