J. Math. Pures Appl.,
78, 1999, *p.* 667-700

GENERAL CURVATURE ESTIMATES FOR STABLE H-SURFACES IMMERSED INTO A SPACE FORM

Pierre BÉRARD [a,1], **Laurent HAUSWIRTH** [b,2]

[a] *Institut Fourier, UMR 5582 UJF-CNRS, Université Joseph Fourier, B.P. 74, 38402 St Martin d'Hères Cedex, France*
[b] *Departamento de Matemática, Universidade Federal do Ceará, Campus do Pici, 60455-760 Fortaleza, Ceará, Brazil*

Manuscript received 6 November 1998

ABSTRACT. – In this paper, we give general curvature estimates for constant mean curvature surfaces immersed into a simply-connected 3-dimensional space form. We obtain bounds on the norm of the traceless second fundamental form and on the Gaussian curvature at the center of a relatively compact stable geodesic ball (and, more generally, of a relatively compact geodesic ball with stability operator bounded from below). As a by-product, we show that the notions of weak and strong Morse indices coincide for complete non-compact constant mean curvature surfaces. We also derive a geometric proof of the fact that a complete stable surface with constant mean curvature 1 in the usual hyperbolic space must be a horosphere.

Keywords: Constant mean curvature, Curvature estimates, Stability, Morse index

RÉSUMÉ. – Dans cet article, on établit une estimée de la courbure pour des surfaces de courbure moyenne constante immergées dans un espace de dimension 3, simplement connexe et de courbure constante. On obtient des bornes pour la courbure de Gauss et pour la norme de la seconde forme fondamentale à trace nulle au centre d'une boule géodésique stable relativement compacte (et plus généralement d'une boule géodésique d'indice de Morse fini). Comme conséquence, on montre que les notions d'indices de Morse faible et fort coincident pour les surfaces de courbure moyenne constante. On utilise ces estimées pour avoir une preuve géométrique du fait qu'une surface de courbure moyenne 1 complète et stable dans l'espace hyperbolique doit être une horosphère.

1. Introduction

In 1983, R. Schoen [16] proved a curvature estimate for stable minimal surfaces in $\mathbb{R}^3$. The Gauss curvature K of a stable minimal surface M, with boundary ∂M, immersed in $\mathbb{R}^3$, satisfies the estimate

$$\left|K(x_0)\right| \leqslant C d(x_0, \partial M)^{-2}, \tag{1}$$

where C is a universal constant and $d(x_0, \partial M)$ the distance of the point x_0 to the boundary. This estimate is very useful to study minimal surfaces. For instance, when M is a complete stable minimal surface immersed in $\mathbb{R}^3$, letting R tend to infinity, estimate (1) implies that M is a plane (a result proved independently by do Carmo and Peng, and Fischer-Colbrie and Schoen).

[1] E-mail: Pierre.Berard@ujf-grenoble.fr.
[2] E-mail: laurent@mat.ufc.br.

Previously, Heinz [10], Osserman [13] had proved similar estimates in some particular cases and Schoen, Simon and Yau [18] other curvature type estimates in higher dimensions.

The purpose of the present paper is to prove similar estimates for stable surfaces M with constant mean curvature H immersed in a 3-manifold $\overline{M}(c)$ with constant curvature c. The methods are very much inspired by those of [16].

Denoting by A^0 the traceless second fundamental form of the immersion, we shall prove estimates of the form

$$\left|A^0\right|^2(x_0) \leqslant C(\Lambda)R^{-2} \quad \text{and} \quad \left|K_g(x_0)\right| \leqslant C(\Lambda)R^{-2}$$

provided that the ball $B(x_0, R) \subset M$ is relatively compact and that R satisfies one of the following conditions:

$$\text{(A)} \qquad c + H^2 \leqslant 0 \quad \text{and} \quad 4R^2\left(c + H^2\right)_- \leqslant \Lambda,$$

or

$$\text{(B)} \qquad c + H^2 > 0 \quad \text{and} \quad 4\left(c + H^2\right)R^2 \leqslant \pi^2,$$

where Λ is a free parameter.

Note that F. Sauvigny [15] obtained an estimate of the form $|K(x_0)| \leqslant CR^{-2}$, with a constant which depends on the product HR for surfaces immersed in $\mathbb{R}^3$. Let us also point out that the estimate of Heinz and Osserman has been generalized to the constant mean curvature case by Spruck [21] and that Ecker and Huisken [6] obtained similar curvature estimates for graphs with prescribed mean curvature in the Euclidean n-space.

When $c + H^2 = 0$, there are no restrictions on the size of R in our estimate. This is not very surprising in view of Schoen's result [16] and of the Lawson correspondance between minimal surfaces in $\mathbb{R}^3$ and surfaces with constant mean curvature 1 in $\mathbb{H}^3$ (see [2,12]). We are then able to give a different proof of Silveira's result [19] which states that a complete stable surface with constant mean curvature 1 in $\mathbb{H}^3$ is a horosphere. We refer to Section 4 for more details. When $c + H^2 > 0$, the results of R. Freire de Lima [9,14] show that the limitation on the radius R is necessary.

We shall in fact give stronger results and consider the case in which the immersion is only assumed to have finite index (see Theorem 4.2 for a precise statement).

As is well-known, there are two different notions of stability for complete constant mean curvature surfaces. Both involve the stability operator L of the immersion. For *strong stability*, one considers the operator L acting on all smooth functions with compact support in M, while for *weak stability*, one considers the operator L acting on smooth functions with compact support having mean-value equal to zero on M. Using our curvature estimates and [1], one can show that these notions coincide for complete non-compact surfaces.

Notations. – Let $i : (M, g) \to (\overline{M}^3(c), \overline{g})$ be an isometric immersion of an oriented Riemann surface into a simply-connected 3-manifold with constant curvature c. We choose a unit normal field ν along the immersion. Let $A : T_pM \to T_pM$ be the shape operator associated to the second fundamental form and let k_1, k_2 be the eigenvalues of A. The mean curvature H of the immersion is given by $2H = k_1 + k_2$. We assume $H = Ct$ and we note $A^0 = A - H\,\mathrm{Id}$ the operator associated with the traceless second fundamental form. Both tensors A, A^0 satisfy the Codazzi equation. The stability operator L_g is given by:

$$L_g = \Delta_g + \left\{\left|A^0\right|^2 + 2\left(c + H^2\right)\right\},$$

where Δ_g is the non-positive Laplacian.

We assume furthermore that the immersion i is (strongly) stable, i.e., that the second variation of the area is non-negative for all deformations with compact support:

$$-\int_M \phi L_g \phi \, dv_g \geqslant 0$$

for all smooth functions ϕ with compact support in M, with ϕ vanishing on ∂M if M has a boundary. Here dv_g is the Riemannian measure associated with the metric g.

The stability assumption implies that the inequality

$$\int_M \zeta^2 \phi L_g \phi \, dv_g \leqslant \int_M \phi^2 |d\zeta|_g^2 \, dv_g \tag{2}$$

holds for any C^∞ function ϕ and for any Lipschitz function with compact support ζ on M. We have denoted by $|d\zeta|_g$ the norm of the differential of the function ζ in the metric g.

As in [16], the proof of our curvature estimates consists in applying (2) to different well chosen functions. The paper is organized as follows.

In Section 2, we recall the well-known iteration method of de Giorgi, Moser and Nash; it will be used repeatedly in the paper.

Section 3 is devoted to studying conformal isometric immersions of the unit disk. Similar results, in the stable case, were obtained in [16] (Theorem 1) and in [5] (in a more general setting). Our result (Theorem 3.2) is more precise and applies in the finite index case as well.

In Section 4, we state our curvature estimates and we give some applications (in particular to the equivalence between weak and strong stability in the complete case).

Section 5 is devoted to the proof of Theorem 4.2.

The authors are grateful to Manfredo do Carmo, Pascal Collin, Geraldo de Oliveira Filho, Harold Rosenberg and Walcy Santos for stimulating discussions during the preparation of this paper.

2. The de Giorgi–Moser–Nash iteration method

In this paper, we will apply the de Giorgi–Moser–Nash iteration method repeatedly, with slight variations, in order to obtain our curvature estimates. The purpose of this section is to recall the main lines of this method for the convenience of the reader. The iteration method is based on Sobolev inequalities.

2.1. Sobolev inequalities

Let (M,g) be a Riemannian surface. The main assumption we need is that (M,g) satisfies a Sobolev inequality of the form

$$\left(\int_M f^2 \, dv_g\right)^{1/2} \leqslant A_M\left(\int_M |df|_g \, dv_g + \int_M B_M |f| \, dv_g\right), \tag{3}$$

for all real valued, C^1-functions with compact support in M, $f \in C_0^1(M,\mathbb{R})$. Here $|df|_g$ denotes the pointwise norm of the differential of f (or equivalently of its gradient) with respect to the Riemannian metric g and dv_g denotes the Riemannian measure. This Sobolev inequality involves a constant A_M and a non-negative function B_M which a priori depend on the geometry of (M,g).

Such an inequality, with $A_M = A(n)$, a constant which only depends on the dimension n, and with $B_M = 0$, holds when (M, g) is the Euclidean space $(\mathbb{R}^2, e)$ or when M is a minimal surface immersed in Euclidean 3-space.

A similar inequality, with $A_M = A(n)$ and $B_M = |H|$, holds when (M, g) is isometrically immersed with mean curvature H into a simply-connected Riemannian manifold $(\overline{M}, \overline{g})$ with non-positive sectional curvatures (see [11], where a more general situation is described and [4], where it is shown that one can choose $A_M = A(n, h)$ and $B_M = 0$ when $\overline{M}$ has constant sectional curvatures equal to -1 and $|H| \leqslant h < 1$). Note that no completeness assumption is made on (M, g).

Given $p \geqslant 1$ and $u \in C_0^1(M, \mathbb{R})$ we apply inequality (3) to $f = |u|^p$ and we obtain

$$\left(\int_M |u|^{2p}\, dv_g\right)^{1/2} \leqslant pA_M\left(\int_M |u|^{p-1}|du|_g\, dv_g + \int_M B_M|u|^p\, dv_g\right).$$

Using Hölder's inequality repeatedly, we obtain

$$\left(\int_M |u|^{2p}\, dv_g\right)^{1/p} \leqslant 2p^2 A_M^2\left(\int_{\mathrm{Supp}(u)} dv_g\right)^{1/p}\left\{\int_M |du|_g^2\, dv_g + \int_M B_M^2 u^2\, dv_g\right\}, \tag{4}$$

for any $u \in C_0^1(M, \mathbb{R})$ and any $p \geqslant 1$.

2.2. The de Giorgi–Moser–Nash lemma

LEMMA 2.1. – *Assume that the Riemannian manifold (M, g) satisfies the Sobolev inequality* (3)*. Let $B(R)$ be some relatively compact geodesic ball in (M, g), centered at some point x_0 and assume that it satisfies the volume estimate*:

(a) *there exists some constant C_1 such that $\int_{B(R)} dv_g \leqslant C_1 R^2$.*

Let f, h be real valued C^2 functions on $B(R)$ such that $h \geqslant 0$ and $\Delta_g h + fh \geqslant 0$ pointwise in $B(R)$, where Δ_g is the non-positive Laplacian on (M, g). Assume furthermore that:

(b) *there exist some number $q \geqslant 6$ and some constant C_2 such that*

$$\left(\int_{B(3R/4)} h^{2q}\, dv_g\right)^{1/q} \leqslant C_2 R^{-2+2/q},$$

(c) *there exists some constant C_3 such that for all $\alpha \in [0, 1/2]$,*

$$\int_{B(3R/4)} \left(f + B_M^2\right)_+^{1+\alpha} dv_g \leqslant C_3 R^{-2\alpha}.$$

Then there exists a constant $C := C(A_M, C_1, C_2, C_3)$ such that:

$$\sup_{B(R/2)} h^2 \leqslant q^2 C R^{-2}.$$

Proof. – The proof of this lemma uses Sobolev inequality (4) and the de Giorgi–Nash–Moser iteration method. In the proof, we will denote by c_i constants which only depend on A_M, C_1, C_2, C_3.

Step 1: Integration by parts. Let ζ be a non-negative Lipschitz function with compact support in $B(R)$. Let $k \in \mathbb{R}$, with $k \geqslant 1$. Then

$$\left|d\left(\zeta h^k\right)\right|_g^2 = h^{2k}|d\zeta|_g^2 + k^2\zeta^2 h^{2k-2}|dh|_g^2 + 2k\zeta h^{2k-1}\langle dh, d\zeta\rangle_g$$

and

$$\left\langle d\left(\zeta^2 h^{2k-1}\right), dh\right\rangle_g = 2\zeta h^{2k-1}\langle dh, d\zeta\rangle_g + (2k-1)\zeta^2 h^{2k-2}|dh|_g^2.$$

Since $k \geqslant 1$, we obtain

$$\left|d\left(\zeta h^k\right)\right|_g^2 \leqslant h^{2k}|d\zeta|_g^2 + k\left\langle d\left(\zeta^2 h^{2k-1}\right), dh\right\rangle_g.$$

Multiplying the inequality $(\Delta_g + f)h \geqslant 0$ by $\zeta^2 h^{2k-1}$ and integrating by parts, we find

$$-\int_{B(R)} \zeta^2 h^{2k-1}\Delta_g h\, dv_g = \int_{B(R)} \left\langle d\left(\zeta^2 h^{2k-1}\right), dh\right\rangle_g dv_g \leqslant \int_{B(R)} f\zeta^2 h^{2k}\, dv_g$$

and finally

$$\int_{B(R)} \left|d\left(\zeta h^k\right)\right|_g^2 dv_g \leqslant \int_{B(R)} h^{2k}|d\zeta|_g^2\, dv_g + k\int_{B(R)} f\zeta^2 h^{2k}\, dv_g. \tag{5}$$

Now given $a \in [1/2, 3/4[$ and $r \in [0, 3/4 - a]$, we define:

$$B_a := B(aR) \subset B_{a+r} := B\left((a+r)R\right) \subset B(3R/4)$$

and we choose a family of Lipschitz functions $\zeta = \theta \circ \rho$ depending on a, r, R, where ρ is the geodesic distance to the given point x_0 in (M, g) and where θ is a smooth function such that $0 \leqslant \theta \leqslant 1$, $\theta = 1$ on $[0, aR]$, $\theta = 0$ on $[(a+r)R, \infty[$ and $|\theta'| \leqslant 2/(rR)$.

Step 2: Using the Sobolev inequality. Plugging $u := \zeta h^k$ into the Sobolev inequality (4), with ζ as above and $p = q$, we obtain

$$\left(\int_{B(R)} \left(\zeta h^k\right)^{2q} dv_g\right)^{1/q} \leqslant q^2 c_1 \left(\int_{B(R)} dv_g\right)^{1/q} \times \left\{\int_{B(R)} \left|d\left(\zeta h^k\right)\right|_g^2 dv_g + \int_{B(R)} B_M^2\left(\zeta h^k\right)^2 dv_g\right\}$$

which gives, using formula (5)

$$\left(\int_{B_a} h^{2kq}\, dv_g\right)^{1/q} \leqslant kq^2 c_1 \left(\int_{B(R)} dv_g\right)^{1/q} \times \left\{\int_{B_{a+r}} h^{2k}|d\zeta|_g^2\, dv_g + \int_{B_{a+r}} \left(f + B_M^2\right)_+ h^{2k}\, dv_g\right\}. \tag{6}$$

Step 3: Applying Hölder's inequality. We now apply Hölder's inequality with $\frac{2}{q}+\frac{q-2}{q}=1$. Since $q\geqslant 6$, we have $q/(q-2)\leqslant 3/2$ and

$$\left(\int_{B_a} h^{2kq}\,dv_g\right)^{\frac{1}{q}} \leqslant kq^2c_1\left(\int_{B(R)} dv_g\right)^{\frac{1}{q}}\left\{\left(\int_{B_{a+r}} |d\zeta|_g^{\frac{2q}{q-2}}\,dv_g\right)^{\frac{q-2}{q}}\right.$$

$$\left.+\left(\int_{B_{a+r}} (f+B_M^2)_+^{\frac{q}{q-2}}\,dv_g\right)^{\frac{q-2}{q}}\right\}\left(\int_{B_{a+r}} h^{qk}\,dv_g\right)^{\frac{2}{q}}. \tag{7}$$

Applying assumption (a) and the fact that $|\theta'|\leqslant 2/(rR)$ we get

$$\left(\int_{B_{a+r}} |d\zeta|_g^{\frac{2q}{q-2}}\,dv_g\right)^{\frac{q-2}{q}} \leqslant c_2r^{-2}R^{-2}\left(\int_{B(R)} dv_g\right)^{\frac{q-2}{q}} \leqslant c_3r^{-2}R^{-\frac{4}{q}}. \tag{8}$$

Using assumption (c) we obtain

$$\left(\int_{B_{a+r}} \left(f+B_M^2\right)_+^{\frac{q}{q-2}}\,dv_g\right)^{\frac{q-2}{q}} \leqslant c_4R^{-\frac{4}{q}}. \tag{9}$$

We can now plug inequalities (8) and (9) into (7) to obtain

$$\left(\int_{B_a} h^{2qk}\,dv_g\right)^{1/q} \leqslant c_5q^2kR^{-2/q}\left(r^{-2}+1\right)\left(\int_{B_{a+r}} h^{qk}\,dv_g\right)^{2/q} \tag{10}$$

for all $k\geqslant 1$.

Step 4: The iteration. We now define $k_i=2^i,\ r_i=2^{-i-3},\ a_0=3/4,\ a_{i+1}=a_i-r_i$, for $i\geqslant 0$, i.e., $a_i=\frac{1}{2}+\frac{1}{2^{i+2}}$, and

$$I(i)=\left(\int_{B_{a_i}} h^{2q2^i}\,dv_g\right)^{1/q2^i}.$$

Rewriting the formula (10) with $k_{i+1},\ a_{i+1}$ and r_i, we obtain

$$I(i+1)^{2^{i+1}} \leqslant q^2c_62^{i+1}R^{-2/q}\left(2^{2i+6}+1\right)I(i)^{2^{i+1}}.$$

Then:

$$I(i+1)\leqslant\left(c_62^6q^2\right)^{1/2^{i+1}}\left(2^{3(i+1)}\right)^{1/2^{i+1}}R^{-2/q2^{i+1}}I(i). \tag{11}$$

Iterating (11), we obtain

$$I(i+1)\leqslant C(i+1)R^{-2d_{i+1}/q}I(0)$$

where

$$d_{i+1}=\sum_{j=1}^{i+1}\frac{1}{2^j}\quad\text{and}\quad C(i+1)=\left(q^2c_62^6\right)^{d_{i+1}}\prod_{j=1}^{i+1}\left(8^j\right)^{1/2^j}.$$

Assumption (b) gives the initial estimate for $I(0)$:

$$I(0) = \left(\int\limits_{B(3R/4)} h^{2q}\, dv_g \right)^{1/q} \leqslant C_2 R^{-2+2/q}.$$

Thus we get:

$$I(i+1) \leqslant C(i+1) C_2 R^{-2+2/q-2d_{i+1}/q}. \tag{12}$$

Letting i tend to infinity in (12), we obtain

$$\lim_{i\to\infty} I(i+1) = \sup_{B(R/2)} h^2 \leqslant q^2 C R^{-2}, \tag{13}$$

where the constant C only depends on A_M, C_1, C_2, C_3. □

3. On conformal disks

Let $i : (D_r, g) \to (\overline{M}^3(c), \overline{g})$ be a conformal isometric immersion of the disk of radius r in $\mathbb{R}^2$ into a 3-manifold with constant sectional curvatures c (we do not need $\overline{M}$ to be simply-connected in this section). Assume furthermore that the immersion has constant mean curvature H. Write the metric g as

$$g = i^*\overline{g} = \lambda^2 e = h^{-2} e$$

where $e = |dz|^2$ is the Euclidean metric in D_r and $2\lambda^2 = |di|_g^2$ (where $|di|_g^2 := |di(\frac{\partial}{\partial x})|_g^2 + |di(\frac{\partial}{\partial y})|_g^2$). The purpose of this section is to give a lower bound on the function λ (or equivalently an upper bound on the function h) under a stability assumption on the immersion i. Theorem 3.1 below generalizes Theorem 1 in [16]. The method of proof is similar.

THEOREM 3.1. – *Let $i : (D_r, g) \to (\overline{M}^3(c), \overline{g})$ be a conformal isometric immersion of the disk of radius r in $\mathbb{R}^2$ into $\overline{M}^3(c)$. Assume that i has constant mean curvature H. Let $B(R)$ denote the geodesic g-ball of radius R with center at 0 and assume that $B(R)$ is relatively compact in D_r. Assume finally that the immersion i is stable on $B(R)$, i.e., that the stability operator $-L_g$ is non-negative on the space $C_0^\infty(B(R))$.*

Then there exists a universal constant $C_0 > 0$ such that

$$\inf_{B(R/2)} |di|_g^2 \geqslant C_0 r^{-2} R^2 \bigl(1 + R^2 (c + H^2)_-\bigr)^{-1}.$$

We shall in fact prove the following stronger result:

THEOREM 3.2. – *Let $i : (D_r, g) \to (\overline{M}^3(c), \overline{g})$ be a conformal isometric immersion of the disk of radius r in $\mathbb{R}^2$ into $\overline{M}^3(c)$. Assume that i has constant mean curvature H. Let $B(R)$ denote the geodesic g-ball of radius R with center at 0 and assume that $B(R)$ is relatively compact in D_r. Assume finally that the the stability operator L_g of the immersion i is bounded from above by some non-negative number 2ℓ on the space $C_0^\infty(B(R))$.*

Then there exists a universal constant $C_0 > 0$ such that:

$$\inf_{B(R/2)} |di|_g^2 \geqslant C_0 r^{-2} R^2 \bigl\{1 + R^2 \bigl((c + H^2)_- + \ell\bigr)\bigr\}^{-1}.$$

Remarks. – The assumption that L_g is bounded from above on $C_0^\infty(B(R))$ is equivalent to saying that the least eigenvalue of L_g on $B(R)$, with Dirichlet boundary condition, is bounded from below by -2ℓ. Such an assumption is verified for a complete immersion with finite Morse index in the sense of [7].

Proof of Theorem 3.2. – In the following, $c_i > 0,\ i = 1, 2, \dots,$ will denote universal constants.

By rescaling, we may assume that $r = 1$ and we will denote by D the unit disk D_1.

Recall the general formula which relates the curvatures K_g, K_{g_0} of two conformal metrics $g = e^{2u} g_0$ in dimension 2:

$$K_g = e^{-2u}\{K_{g_0} - \Delta_{g_0} u\} = e^{-2u} K_{g_0} - \Delta_g u$$

(with non-positive Laplacians).

Since the metric g is conformal to the Euclidean metric $e = |dz|^2$, the (intrinsic) Gauss curvature of the metric g is given by $K_g = -\Delta_g \ln\lambda = \Delta_g \ln h$, where

$$\Delta_g = \lambda^{-2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

is the Laplacian for the metric g. Since $\Delta_g \ln h = h^{-1}\Delta_g h - h^{-2}|dh|_g^2$, it follows that

$$\Delta_g h = K_g h + h^{-1}|dh|_g^2 \geqslant K_g h. \tag{14}$$

The Gauss equation of the immersion can be written as

$$2K_g = -\left|A^0\right|^2 + 2\left(c + H^2\right), \tag{15}$$

where A^0 is the traceless second fundamental form of the immersion so that inequality (14) gives

$$\Delta_g h + \left\{\frac{1}{2}\left|A^0\right|^2 - \left(c + H^2\right)\right\} h \geqslant 0. \tag{16}$$

Let $a := (c + H^2)_- = \max\{0, -(c + H^2)\}$ denote the negative part of $(c + H^2)$ and let f denote the function $f := \frac{1}{2}|A^0|^2 + a$. With these notations, inequality (16) implies that

$$\Delta_g h + fh \geqslant 0. \tag{17}$$

We will apply a variant of the de Giorgi–Moser–Nash lemma to this inequality in order to bound h from above. For this purpose, we need some initial estimates (compare with Lemma 2.1). As in [16], they will be given by the stability assumption (more precisely, by the following lemma applied to suitable functions ζ and ϕ). We will repeatedly use the fact that the metric g is conformal to the Euclidean metric e.

In the sequel, we denote by $|d\varphi|_g$ and $|d\varphi|_e$ the norm of the differential of a function φ respectively in the metrics g and e. Recall that $g = h^{-2} e$ and observe that $|d\varphi|_g^2 = h^2 |d\varphi|_e^2$ and that the Riemannian measures are related by $dv_g = h^{-2}\, dv_e$ (notice that $|d\varphi|_e^2 = \varphi_x^2 + \varphi_y^2$ on D).

LEMMA 3.3. – *Under the assumptions of Theorem* 3.2*, the stability inequality*

$$\int_D \zeta^2 \phi L_g \phi\, dv_g \leqslant \int_D \phi^2 |d\zeta|_g^2\, dv_g + 2\ell \int_D \phi^2 \zeta^2\, dv_g \tag{18}$$

holds for all $\phi \in C^\infty(D)$ *and all* $\zeta \in C_0^\infty(B(R))$.

We leave the proof to the reader.

Step 1: Initial estimates. As in Lemma 2.1, we need estimates of $\int_{B(3R/4)} h^{2p}\, dv_e$, for some $p \in [6, +\infty[$, and of $\int_{B(3R/4)} f^\alpha\, dv_e$, for all $1 \leqslant \alpha \leqslant 3/2$. Using the expression of the stability operator $L_g = \Delta_g + \{|A^0|^2 + 2(c + H^2)\}$, Eq. (14) and the stability condition (18) with $\phi = h$, we obtain

$$\int_D \zeta^2 \big\{|dh|_g^2 + \big(K_g + 2\big(c + H^2\big) + \big|A^0\big|^2\big)h^2\big\}\, dv_g \leqslant \int_D h^2 |d\zeta|_g^2\, dv_g + 2\ell \int_D h^2 \zeta^2\, dv_g$$

for any function $\zeta \in C_0^\infty(B(R), \mathbb{R})$ and, more generally, for any Lipschitz function with compact support in $B(R)$. Using (15), taking into account the relations between the metrics g and e and using the conformal invariance of the Dirichlet integral, we obtain

$$\int_D \zeta^2 \left\{|dh|_e^2 + \left(\frac{1}{2}\big|A^0\big|^2 + 3\big(c + H^2\big) - 2\ell\right)\right\} dv_e \leqslant \int_D h^2 |d\zeta|_e^2\, dv_e. \tag{19}$$

Estimate of $\int_{B(3R/4)} h^{2p}\, dv_e$**.** Let us denote by a' the number $a' := (c + H^2)_- + \ell = a + \ell$. Using Eq. (19) and the relationship between the metrics g and e, we obtain the inequalities

$$\begin{aligned}
\int_D \big|d(\zeta h)\big|_e^2\, dv_e &\leqslant 2 \int_D \zeta^2 |dh|_e^2\, dv_e + 2 \int_D h^2 |d\zeta|_e^2\, dv_e \\
&\leqslant 4 \int_D h^2 |d\zeta|_e^2\, dv_e + 6a' \int_D \zeta^2\, dv_e \\
&= 4 \int_D |d\zeta|_g^2\, dv_e + 6a' \int_D \zeta^2\, dv_e. \qquad (20)
\end{aligned}$$

Let ζ be a cut-off function of the geodesic distance in the g-metric, with $\zeta = 1$ on $B(3R/4)$, $\zeta = 0$ outside $B(7R/8)$, and $|d\zeta|_g^2 \leqslant c_1 R^{-2}$ for some universal constant $c_1 > 0$ independent of R. The Euclidean Sobolev inequality, i.e., inequality (4) with $B_M = 0$, applied to the function $u = \zeta h$ gives

$$\left(\int_D (\zeta h)^{2p}\, dv_e\right)^{1/p} \leqslant p^2 c_2 \left(\int_D dv_e\right)^{1/p} \int_D \big|d(\zeta h)\big|_e^2\, dv_e.$$

Taking (20) into account, with the above choice of ζ, we obtain our first initial estimate (to be compared with assumption (b) in Lemma 2.1)

$$\left(\int_{B(3R/4)} h^{2p}\, dv_e\right)^{1/p} \leqslant 4p^2 c_2 \left(4 \int_D |d\zeta|_g^2\, dv_e + 6a' \int_D \zeta^2\, dv_e\right) \leqslant p^2 c_3 \big(R^{-2} + a'\big), \tag{21}$$

for any $p \in [1, +\infty[$, where c_3 is a universal constant.

Estimate of $\int_{B(3R/4)} f^\alpha \, dv_e$. From the definitions of a and f, we have $2f = |A^0|^2 + 2a$. Choosing a suitable cut-off function ζ, with compact support in $B(R)$ and such that $\zeta = 1$ on $B(7R/8)$, we obtain from (19)

$$\int_{B(7R/8)} \left(\left|A^0\right|^2 + 2a\right) dv_e \leqslant 2 \int_D |d\zeta|_g^2 \, dv_e + 8a' \int_D \zeta^2 \, dv_e \leqslant c_4\left(R^{-2} + a'\right). \tag{22}$$

LEMMA 3.4. – *Let $\phi := h(|A^0|^2 + 2a)^\alpha$, for $0 < \alpha \leqslant 1$. Then*

$$\phi \Delta_g \phi \geqslant \phi^2 \Delta_g \ln \phi \geqslant -\left(\frac{1}{2} + 2\alpha\right) h^2 \left(\left|A^0\right|^2 + 2a\right)^{1+2\alpha}.$$

Proof of the lemma. – Using moving frame techniques as in [22] or J. Simons' equation, we have the equality $\Delta_g \ln |A^0|^2 = 4K_g$, which implies that

$$\Delta_g \ln\left(\left|A^0\right|^2 + 2a\right) \geqslant \frac{4K_g |A^0|^2}{|A^0|^2 + 2a}.$$

Since $K_g \leqslant 0$ when $a \neq 0$, if follows that $\Delta_g \ln(|A^0|^2 + 2a) \geqslant 4K_g$. One can now write

$$\Delta_g \ln \phi \geqslant (1 + 4\alpha) K_g \geqslant -\left(\frac{1}{2} + 2\alpha\right)\left(\left|A^0\right|^2 + 2a\right)$$

and the lemma follows. □

Using the stability condition (18), with $\phi = h(|A^0|^2 + 2a)^\alpha$ and with a suitable cut-off function ζ, we obtain

$$\begin{aligned} &\left(\frac{1}{2} - 2\alpha\right) \int_{B(3R/4)} h^2 \left(\left|A^0\right|^2 + 2a\right)^{1+2\alpha} dv_g \\ &\quad \leqslant \int_D h^2 \left(\left|A^0\right|^2 + 2a\right)^{2\alpha} |d\zeta|_g^2 \, dv_g + (4a + 2\ell) \int_D h^2 \zeta^2 \left(\left|A^0\right|^2 + 2a\right)^{2\alpha} dv_g. \end{aligned} \tag{23}$$

Choosing α small enough, for example $\alpha = 1/8$, and taking the relationship between the metrics g and e into account, we obtain

$$\begin{aligned} &\frac{1}{4} \int_{B(3R/4)} \left(\left|A^0\right|^2 + 2a\right)^{5/4} dv_e \\ &\quad \leqslant \int_D \left(\left|A^0\right|^2 + 2a\right)^{1/4} |d\zeta|_g^2 \, dv_e + 4a' \int_D \zeta^2 \left(\left|A^0\right|^2 + 2a\right)^{1/4} dv_e \end{aligned} \tag{24}$$

since $a' = a + \ell$. Using (22), Hölder's inequality and the fact that the Euclidean volume is controled we have

$$\int_{B(7R/8)} \left(\left|A^0\right|^2 + 2a\right)^{1/4} dv_e \leqslant c_5 \left(\int_{B(7R/8)} \left(\left|A^0\right|^2 + 2a\right) dv_e \right)^{1/4} \leqslant c_6 \left(R^{-2} + a'\right)^{1/4}, \tag{25}$$

for some universal constants c_5, c_6. With a suitable choice of ζ in (24), we get

$$\int_{B(3R/4)} \left(|A^0|^2+2a\right)^{5/4} dv_e \leqslant c_7\left(R^{-2}+a'\right) \int_{B(7R/8)} \left(|A^0|^2+2a\right)^{1/4} dv_e \leqslant c_8\left(R^{-2}+a'\right)^{5/4}.$$

Finally, we obtain our second initial estimate

$$\left(\int_{B(3R/4)} \left(|A^0|^2+2a\right)^{5/4} dv_e\right)^{4/5} \leqslant c_9\left(R^{-2}+a'\right). \tag{26}$$

Step 2: The iteration. With estimates (21) and (26) at hand, we can apply the de Giorgi–Moser–Nash iteration method to (17), $\Delta_g h + fh \geqslant 0$. Formula (5) is still valid

$$\int_{B(R)} \left|d\left(\zeta h^k\right)\right|_g^2 dv_g \leqslant \int_{B(R)} h^{2k}|d\zeta|_g^2\, dv_g + k\int_{B(R)} f\zeta^2 h^{2k}\, dv_g$$

and gives, by conformal invariance and using the relation $h^2 dv_g = dv_e$,

$$\int_{B(R)} \left|d\left(\zeta h^k\right)\right|_e^2 dv_e \leqslant \int_{B(R)} h^{2k-2}|d\zeta|_g^2\, dv_e + k\int_{B(R)} f\zeta^2 h^{2k-2}\, dv_e. \tag{27}$$

Now given $t \in [1/2, 3/4[$ and $r \in [0, 3/4-t]$, we define:

$$B_t := B(tR) \subset B_{t+r} := B\big((t+r)R\big) \subset B(3R/4)$$

and we choose a family of Lipschitz functions $\zeta = \theta \circ \rho$ depending on t, r, R, where ρ is the geodesic distance to the point 0 in (D, g) and where θ is a smooth function such that $0 \leqslant \theta \leqslant 1$, $\theta = 1$ on $[0, tR]$, $\theta = 0$ on $[(t+r)R, \infty[$ and $|\theta'| \leqslant 2/(rR)$.

We apply the Euclidean Sobolev inequality to the function ζh^k and, using (27), we get

$$\left(\int_{B_t} h^{2kp}\, dv_e\right)^{1/p} \leqslant p^2 k c_{10} \int_{B_{t+r}} \left\{h^{2k-2}|d\zeta|_g^2 + f\zeta^2 h^{2k-2}\right\} dv_e. \tag{28}$$

Let

$$p_1 := \frac{kp}{2(k-1)}, \qquad p_2 := \frac{kp}{k(p-2)+2}$$

and choose $p \geqslant 10$ so that $p_2 \leqslant 5/4$. We can apply Hölder's inequality, with $\frac{1}{p_1}+\frac{1}{p_2}=1$, to inequality (28) and we obtain the analog of inequality (6):

$$\begin{aligned}
&\left(\int_{B_t} h^{2kp}\, dv_e\right)^{1/p} \\
&\quad\leqslant p^2 k c_{11}\left\{\left(\int_{B_{t+r}} |d\zeta|_g^{2p_2}\, dv_e\right)^{1/p_2} + \left(\int_{B_{t+r}} f^{p_2}\, dv_e\right)^{1/p_2}\right\}\left(\int_{B_{t+r}} h^{pk}\, dv_e\right)^{2(k-1)/pk}.
\end{aligned} \tag{29}$$

Since $B_{t+r} \subset D$, we have

$$\left(\int\limits_{B_{t+r}} |d\zeta|_g^{2p_2}\, dv_e\right)^{1/p_2} \leqslant c_{12} r^{-2} R^{-2}$$

and, applying Hölder's inequality and using (26),

$$\left(\int\limits_{B_{t+r}} f^{p_2}\, dv_e\right)^{1/p_2} \leqslant c_{13}\left(\int\limits_{B_{t+r}} f^{5/4}\, dv_e\right)^{4/5} \leqslant c_{14}\big(R^{-2} + a'\big)$$

since $p_2 \leqslant 5/4$. Plugging these inequalities into (29) we obtain

$$\left(\int\limits_{B_t} h^{2pk}\, dv_e\right)^{1/p} \leqslant p^2 k c_{15}\big(R^{-2} + a'\big)\big(r^{-2} + 1\big)\left(\int\limits_{B_{t+r}} h^{pk}\, dv_e\right)^{2(k-1)/pk}. \tag{30}$$

We now perform the iteration (as in the proof of Lemma 2.1, Step 4). We choose $q \geqslant 1$ and we define $k_i = q2^i,\ r_i = 2^{-i-3},\ t_0 = 3/4$ and $t_{i+1} = t_i - r_i$, for $i \geqslant 0$, and

$$I(i,q) := \left(\int\limits_{B_{t_i}} h^{2pk_i}\, dv_e\right)^{1/pk_i}.$$

We can rewrite (30) with $k_{i+1},\ t_{i+1}$ and r_i as

$$I(i+1,q)^{k_{i+1}} \leqslant p^2 c_{15} k_{i+1}\big(R^{-2} + a'\big)\big(r_i^{-2} + 1\big) I(i,q)^{k_{i+1}-1}.$$

From which we obtain

$$I(i+1,q) \leqslant \big(p^2 c_{15} k_{i+1} 2^{2i+8}\big)^{1/k_{i+1}} \big(R^{-2} + a'\big)^{1/k_{i+1}} I(i,q)^{1-1/k_{i+1}}. \tag{31}$$

Define the sequence

$$s_{i,j} = \begin{cases} \big(1 - \frac{1}{k_i}\big)\big(1 - \frac{1}{k_{i-1}}\big)\cdots\big(1 - \frac{1}{k_{j+1}}\big)\frac{1}{k_j} & \text{if } j < i, \\ \frac{1}{k_i} & \text{if } j = i. \end{cases}$$

Iterating inequality (31), we obtain

$$I(i+1,q) \leqslant C(i+1,q)\big(R^{-2} + a'\big)^{\alpha(i+1,q)} I(0,q)^{\beta(i+1,q)} \tag{32}$$

with

$$\alpha(i,q)=\sum_{j=1}^{i}s_{i,j},$$
$$\beta(i,q)=\left(1-\frac{1}{k_i}\right)\left(1-\frac{1}{k_{i-1}}\right)\cdots\left(1-\frac{1}{k_1}\right),$$
$$C(i,q)=\left(p^2c_{15}2^6\right)^{\alpha(i,q)}\prod_{j=1}^{i}\left(4^jk_j\right)^{s_{i,j}}.$$

Applying inequality (21), we have

$$I(0,q)=\left(\int\limits_{B(3R/4)}h^{2pq}\,dv_e\right)^{1/pq}\leqslant p^2q^2c_3\left(R^{-2}+a'\right).$$

Since $\beta(i+1,q)=1-\alpha(i+1,q)$, we have

$$I(i+1,q)\leqslant C(i+1,q)\left(p^2q^2c_3\right)^{\beta(i+1,q)}\left(R^{-2}+a'\right).$$

Let us define $\beta(q)=\lim_{i\to\infty}\beta(i+1,q)$ and $\alpha(q)=\lim_{i\to\infty}\alpha(i+1,q)=1-\beta(q)$. A straightforward computation gives

$$\frac{1}{q}\leqslant-\ln\beta(q)\leqslant\frac{1}{q}+\frac{4}{3q^2}$$

which implies that

$$\beta(q)=1-\frac{1}{q}+\mathrm{O}\left(\frac{1}{q^2}\right)\quad\text{and}\quad\alpha(q)=\frac{1}{q}+\mathrm{O}\left(\frac{1}{q^2}\right)$$

when q goes to infinity. We also have, $\lim_{i\to\infty}(p^2c_{15}2^6)^{\alpha(i+1,q)}=(p^2c_{15}2^6)^{\alpha(q)}$.

Moreover

$$\lim_{i\to\infty}\prod_{j=1}^{i+1}\left(4^jk_j\right)^{s_{i+1,j}}=\lim_{i\to\infty}\prod_{j=1}^{i+1}\left(q8^j\right)^{s_{i+1,j}}=\lim_{i\to\infty}e^{\alpha(i+1,q)\ln q}e^{\gamma(i+1,q)\ln 8}$$

where

$$\gamma(i+1,q)=\sum_{j=1}^{i+1}js_{i+1,j}.$$

Then $js_{i+1,j}\leqslant j/q2^j$ implies that $\gamma(i+1,q)\leqslant\delta/q$ where

$$\delta=\sum_{j=1}^{\infty}\frac{j}{2^j}.$$

Thus

$$1 \leqslant \lim_{i\to\infty} \prod_{j=1}^{i+1} \left(4^j k_j\right)^{s_{i+1,j}} \leqslant q^{\alpha(q)} 8^{\delta/q}.$$

This gives

$$\sup_{B(R/2)} h^2 \leqslant c_{17} p^2 q^{1+\beta(q)} \left(R^{-2} + a'\right)$$

and

$$\inf_{B(R/2)} \lambda^2 \geqslant C_0 \left(R^{-2} + a'\right)^{-1} = C_0 R^2 \left(1 + R^2 a'\right)^{-1},$$

once we have fixed some $p \geqslant 10$ and some $q \geqslant 1$. This is the estimate we wanted to prove. □

4. Curvature estimates

The purpose of this section is to prove the following theorem which generalizes theorem 3 in [16].

THEOREM 4.1. – *Let (M, g) be an oriented Riemannian surface. Let $i : (M, g) \to (\overline{M}^3(c), \overline{g})$ be an isometric immersion with constant mean curvature H of (M, g) into a simply-connected 3-manifold with constant curvature c. Assume there are positive numbers $R' > R$ such that the geodesic g-ball $B(R')$ centered at some x_0 in M, with radius R', is relatively compact in (M, g) and that the stability operator L_g, with Dirichlet boundary conditions, is non-positive on $B(R')$. Let A^0 and K_g denote respectively the traceless second fundamental form and the Gaussian curvature of the immersion i.*

Given $\Lambda > 0$, there exists a constant $C(\Lambda)$, which only depends on Λ, such that

$$\left|A^0\right|^2(x_0) \leqslant C(\Lambda) R^{-2} \quad \textit{and} \quad \left|K_g(x_0)\right| \leqslant C(\Lambda) R^{-2}$$

under one of the following conditions:

(A) $$c + H^2 \leqslant 0 \quad \textit{and} \quad 4R^2\left(c + H^2\right)_- \leqslant \Lambda,$$

or

(B) $$c + H^2 > 0 \quad \textit{and} \quad 4\left(c + H^2\right) R^2 \leqslant \pi^2.$$

We shall in fact prove the following, more general theorem:

THEOREM 4.2. – *Let (M, g) be an oriented Riemannian surface. Let $i : (M, g) \to (\overline{M}^3(c), \overline{g})$ be an isometric immersion with constant mean curvature H of (M, g) into a simply-connected 3-manifold with constant curvature c. Assume there are positive numbers $R' > R$ such that the geodesic g-ball $B(R')$ centered at some x_0 in M, with radius R', is relatively compact in (M, g) and that the stability operator L_g, with Dirichlet boundary conditions, is bounded from above by some number $2\ell \geqslant 0$ on $B(R')$. Let A^0 and K_g denote respectively the traceless second fundamental form and the Gaussian curvature of the immersion i.*

Given $\Lambda > 0$, there exist positive constants $C(\Lambda)$, $c(\Lambda)$, which only depends on Λ, such that

$$\left|A^0\right|^2(x) \leqslant C(\Lambda) R^{-2} \quad \textit{and} \quad \left|K_g(x)\right| \leqslant C(\Lambda) R^{-2},$$

for all $x \in B(c(\Lambda)R)$, provided one of the following conditions holds,

$$\text{(A)} \qquad c + H^2 \leqslant 0 \quad \textit{and} \quad 4R^2\big((c+H^2)_- + \ell\big) \leqslant \Lambda,$$

or

$$\text{(B)} \qquad \begin{cases} c + H^2 > 0,\ 4\big(c+H^2\big)R^2 \leqslant \pi^2, \\ \textit{and}\ 4\ell R^2 \leqslant \Lambda. \end{cases}$$

Remarks. – Notice that when $c + H^2 = 0$, the condition $4R^2(c+H^2)_- \leqslant \Lambda$ in Theorem 4.1 is empty. Theorem 4.2 implies Theorem 4.1 by taking $\ell = 0$. The assumption that L_g is bounded from above by 2ℓ in Theorem 4.2 means that the least eigenvalue of the operator L_g in $B(R')$, with Dirichlet boundary condition, is bounded from below by -2ℓ. Such an assumption is verified when i is an immersion of a complete surface (M, g), with finite Morse index in the sense of [7].

We postpone the proof of Theorem 4.2 to Section 5 and we now give two applications.

Let $(M, g) \to (\overline{M}^3(c), \overline{g})$ be a complete immersion with constant mean curvature H. Once c, H are fixed, choosing some $\Lambda > 0$, we obtain from Theorem 4.2 uniform estimates for $|A^0|$ and for K_g over M. Recall that one can introduce two indices for the stability operator L_g, namely the *strong index* $\mathrm{Ind}_s(L_g)$, resp. the *weak index* $\mathrm{Ind}_w(L_g)$, defined as the maximal dimension of a subspace E of $C_0^\infty(M)$, resp. as the maximal dimension of a subspace E of $C_0^\infty(M) \cap \{f \mid \int_M f\, dv_g = 0\}$, such that $\int_M \varphi L_g \varphi\, dv_g \geqslant 0$ for all $\varphi \in E$ (see [7]). These indices satisfy the inequality

$$\mathrm{Ind}_w(L_g) \leqslant \mathrm{Ind}_s(L_g) \leqslant \mathrm{Ind}_w(L_g) + 1. \tag{33}$$

APPLICATION 1. – It follows from inequality (33) that weak stability implies that the strong index is at most 1. On the other-hand, Theorem 4.1 implies a uniform estimate for the Gauss curvature of a weakly stable surface M and hence, according to [1], that the surface is strongly stable provided it is complete and non-compact.

APPLICATION 2. – Let M be a weakly stable complete immersion with constant mean curvature 1 in $\mathbb{H}^3$. It follows from the preceding application that the immersion is in fact strongly stable. We can then apply Theorem 4.1 with no restriction on R because $c + H^2 = 0$. This implies that $A^0 \equiv 0$ and hence that the immersion is totally umbilic. We have therefore obtained a new proof of Silveira's theorem [19].

5. Proof of Theorem 4.2

Proof of Theorem 4.2. – The proof will take the remainder of this section. For the sake of clarity, and although some arguments will be repeated, we will give two separate proofs, one for each condition (A) and (B). Notice that we may slightly restrict R in the arguments if necessary.

In the course of the proof, we will use the following notations:

$$\begin{cases} S := c + H^2 \text{ and } a := S_- = (c+H^2)_-, \\ S' := S - \ell \text{ and } a' := a + \ell. \end{cases} \tag{34}$$

Recall that the Gauss equation of the immersion i can be written as

$$K_g = -\frac{1}{2}\left|A^0\right|^2 + S. \tag{35}$$

5.1. Proof of Theorem 4.2 under Condition (A)

Step 0. Since $S \leqslant 0$, we have $K_g \leqslant 0$ and the exponential map $\exp_{x_0}$ is a local diffeomorphism. Let us consider the ball $B_0(R') \subset (T_{x_0}M, \exp_{x_0}^* g)$.

We need the following lemma which appears in [8]:

LEMMA 5.1. – *Let (M, g) be a Riemannian manifold. Let $q : M \to \mathbb{R}$ be a continuous function and let L be the operator $L := \Delta_g + q$.*

(i) *If $\Omega \Subset M$ is a smooth relatively compact domain in M, then $L \leqslant 0$ on $C_0^\infty(\Omega)$ if and only if there exists a function $u : \Omega \to \mathbb{R}$, with $u \geqslant 0$ and $u \not\equiv 0$ on Ω, such that $Lu \leqslant 0$ in Ω;*

(ii) *If M is complete non-compact, $L \leqslant 0$ on $C_0^\infty(M)$ if and only if there exists a function $u : M \to \mathbb{R}$, with $u \geqslant 0$ and $u \not\equiv 0$ in M, such that $Lu = 0$ in M.*

Proof. – We shall in fact only use Assertion (i) which follows by applying Green's formula and the fact that the first eigenfunction of L does not vanish in the interior of Ω. We refer to [8] for Assertion (ii). □

Applying Assertion (i) of the above lemma to $\Omega = B(R')$, gives us a non-negative function u on Ω, such that $(L_g - 2\ell)u \leqslant 0$. We now consider the ball $\widetilde{\Omega} := B_0(R') \subset (T_{x_0}M, \tilde{g})$, where $\tilde{g} = \exp_{x_0}^* g$. The function $\tilde{u} = u \circ \exp_{x_0}$ is non-negative in $\widetilde{\Omega}$. Since $\exp_{x_0}$ is a local isometry, we have $(L_{\tilde{g}} - 2\ell)\tilde{u} \leqslant 0$. Assertion (i) of the lemma implies that the operator $L_{\tilde{g}} - 2\ell$ is non-positive on $B_0(R')$.

Step 1. Since the immersion $i \circ \exp_{x_0} : (B_0(R'), \exp_{x_0}^* g) \to (\overline{M}^3(c), \overline{g})$ is also an isometric immersion with constant mean curvature H, it follows from the preceding argument that we can from now on assume that M is diffeomorphic to a disk.

Since the ball $B_0(R')$ is simply-connected, there exists some R'', with $R' \geqslant R'' > R$, such that $(B_0(R''), \exp_{x_0}^* g)$ is conformally equivalent to the unit disk (D, e), i.e., there exists a diffeomorphism $\varphi : D \to B_0(R'')$ such that $\varphi(0) = 0_{T_{x_0}M}$ and $\varphi^*(\exp_{x_0}^* g) = \lambda^2 e$ for some function λ. We may also assume that $R'' \leqslant 2R$.

Since $\exp_{x_0}$ is a local isometry, curvature estimates in $B_0(R')$ imply curvature estimates in $B(R)$. We are then reduced to proving the following result.

PROPOSITION 5.2. – *Let D be the unit disk in $\mathbb{C}$, equiped with a Riemannian metric g. Let $i : (D, g) \to (\overline{M}^3(c), \overline{g})$ be a conformal isometric immersion (i.e., $g = i^*\overline{g} = \lambda^2 e$, where e is the Euclidean metric), with constant mean curvature H, such that the geodesic g-ball $B(R) := B^g(0, R)$ is relatively compact in D. Assume furthermore that $c + H^2 \leqslant 0$ and $4R^2\big((c + H^2)_- + \ell\big) \leqslant \Lambda$ for some $\Lambda > 0$ and that the stability operator L_g of the immersion, with Dirichlet boundary condition, is bounded from above by 2ℓ on $B(R)$. Then, there exists a constant $C(\Lambda)$ such that*:

$$\left|A^0\right|^2(x_0) \leqslant C(\Lambda) R^{-2} \quad \textit{and} \quad \left|K_g(x_0)\right| \leqslant C(\Lambda) R^{-2}.$$

Step 2. We shall now continue with the proof of Proposition 5.2.

Let ρ denote the Riemannian distance to the point $0 \in D$ with respect to the metric g. Since $K_g \leqslant 0$ by (35), Bishop's theorem gives

$$\Delta_g \rho^2 \geqslant 2, \tag{36}$$

where Δ_g is the non-positive Laplacian for the metric g. It follows from Lemma 3.3 that the following stability inequality holds

$$\int_D \zeta^2 \phi L_g \phi \, dv_g \leqslant \int_D \phi^2 |d\zeta|_g^2 \, dv_g + 2\ell \int_D \phi^2 \zeta^2 \, dv_g \tag{37}$$

for any C^∞ function ϕ and any Lipschitz function ζ with compact support in $B(R)$. Recall that the stability operator L_g is given by

$$L_g = \Delta_g + \left|A^0\right|^2 + 2S. \tag{38}$$

A recurrent idea in the proof is to get estimates by plugging well chosen functions ϕ and ζ into (37). In the sequel, we shall denote by c_i universal constants, by $c_i(\Lambda)$ constants which only depend on Λ, etc. We shall also denote by $D(1/2)$ the Euclidean disk of radius $1/2$.

We begin by choosing $\phi = e^{\Lambda\rho^2/R^2}$, where Λ is the positive number given in the assumptions of Theorem 4.2. Since we may assume $R'' \leqslant 2R$ in the above construction, it follows that

$$1 \leqslant \phi \leqslant e^{4\Lambda} \quad \text{in } D.$$

Using (36) a direct computation gives

$$\begin{aligned}
\phi L_g \phi &= \left(\Lambda R^{-2} \Delta_g \rho^2 + \Lambda^2 R^{-4} \left|d\rho^2\right|_g^2 + \left|A^0\right|^2 + 2S\right)\phi^2 \\
&\geqslant \left(2\Lambda R^{-2} + 2S + \left|A^0\right|^2\right)\phi^2.
\end{aligned}$$

Using this inequality, the stability condition (37) and the conformal invariance of the Dirichlet integral, we obtain

$$\begin{aligned}
\int_D \left(2\Lambda R^{-2} - 2a + \left|A^0\right|^2\right)\phi^2\zeta^2 \, dv_g &\leqslant \int_D \phi^2 |d\zeta|_g^2 \, dv_g + 2\ell \int_D \phi^2\zeta^2 \, dv_g \\
&= \int_D \phi^2 |d\zeta|_e^2 \, dv_e + 2\ell \int_D \phi^2\zeta^2 \, dv_g,
\end{aligned}$$

where $a := (c + H^2)_-$, see (34). Using a suitable function ζ of the Euclidean distance to $0 \in D$, we obtain the following important estimate

$$\int_{D(1/2)} \left(2\Lambda R^{-2} - 2a' + \left|A^0\right|^2\right) dv_g \leqslant c_1(\Lambda), \tag{39}$$

where $a' = a + \ell$, see (34). This estimate is meaningfull only when the integrand in the left-hand side is positive. This is why we have to assume that $4a'R^2 \leqslant \Lambda$ (unless $a' = 0$, i.e., $c + H^2 = 0$ and $\ell = 0$), see Condition (A) in the statement of Theorem 4.2.

Step 3. The above estimate involves the Euclidean disk $D(1/2)$; in order to be able to take the metric $g=\lambda^2 e$ into account, we make use of Theorem 3.2 which gives the estimate

$$\lambda^2 \geqslant C_0 R^2\big(1+a'R^2\big)^{-1} \quad \text{on } B(R/2). \tag{40}$$

We then have the following lemma:

LEMMA 5.3. – *Under the assumptions of Proposition* 5.2, *define*

$$b(\Lambda) := \min\left\{\frac{1}{2}, (1+\Lambda)^{-1/2}\frac{\sqrt{C_0}}{2}\right\} \quad \textit{and} \quad R_\Lambda := b(\Lambda)R, \tag{41}$$

where C_0 is the universal constant given by Theorem 3.2. *We then have:*

$$B(R_\Lambda) := B^g(R_\Lambda) \subset D(1/2) \tag{42}$$

and

$$\int\limits_{B(R_\Lambda)} \big(\Lambda R^{-2} + \big|A^0\big|^2\big)\, dv_g \leqslant c_1(\Lambda). \tag{43}$$

In particular

$$\begin{cases} \int_{B(R_\Lambda)} |A^0|^2\, dv_g \leqslant c_1(\Lambda),\\ \operatorname{Vol}\big(B(R_\Lambda)\big) \leqslant c_2(\Lambda) R_\Lambda^2. \end{cases} \tag{44}$$

Proof. – We have $g=\lambda^2 e$, with $\lambda^2 \geqslant C_0 R^2(1+a'R^2)^{-1}$ on the ball $B(R/2)$. Let c be a geodesic issued from 0 and parametrized by arc-length in the metric g. The inequality $\int_0^{R_\Lambda} |\dot c(t)|_e\, dt \leqslant 1/2$ implies that $c(R_\Lambda) \in D(1/2)$. To achieve the inequality, it suffices to have $\int_0^{R_\Lambda} \lambda^{-1}|\dot c(t)|_g\, dt \leqslant 1/2$, i.e., if $R_\Lambda \leqslant R/2$, $R_\Lambda C_0^{-1} R^{-1}(1+a'R^2)^{1/2} \leqslant 1/2$ and $4a'R^2 \leqslant \Lambda$. These conditions are satisfied if $2R_\Lambda/R \leqslant C_0^{1/2}(1+\Lambda)^{-1/2}$. $\square$

Step 4. For any $m>0$, the equality $\Delta_g \ln u = f$ implies the inequality $\Delta_g(u+m)^\beta \geqslant \beta f u(u+m)^{\beta-1}$. Since $\Delta_g \ln|A^0|^2 = 4K_g$ (see [22]), the function $f := (|A^0|^2+m)^{1/2}$, with $m \geqslant 0$, satisfies the inequality

$$f^{1/2}\Delta_g f^{1/2} \geqslant f K_g \big|A^0\big|^2 f^{-2},$$

where K_g is the Gaussian curvature of the metric g. It follows that

$$f^{1/2}(L_g - 2\ell) f^{1/2} = f^{1/2}\Delta_g f^{1/2} + \big|A^0\big|^2 f + 2S' f \geqslant F f, \tag{45}$$

where $F := |A^0|^2 + 2S' + K_g|A^0|^2 f^{-2}$ (recall formula (38) and the notation $S' := c + H^2 - \ell$).

We now define the function

$$h := \big(\Lambda R^{-2} + 2S' + \big|A^0\big|^2\big)^{1/2} = \big(\Lambda R^{-2} - 2a' + \big|A^0\big|^2\big)^{1/2}, \tag{46}$$

where $S' := S - \ell,\ a' := a + \ell$, see (34). Using the assumption $4a'R^2 \leqslant \Lambda$, we see that h actually exists and that

$$\frac{1}{2}\big(\Lambda R^{-2} + \big|A^0\big|^2\big)^{1/2} \leqslant h \leqslant \big(\Lambda R^{-2} + \big|A^0\big|^2\big)^{1/2}. \tag{47}$$

We can deduce from Lemma 5.3 the estimate

$$\int_{B(R_\Lambda)} h^2\,dv_g \leqslant c_3(\Lambda). \tag{48}$$

Choose $0 < t < t + r < 1$ and let θ be a C^∞ function with compact support on $\mathbb{R}$ such that $0 \leqslant \theta \leqslant 1$, $\theta|[0, tR_\Lambda] = 1$, $\theta|[(t+r)R_\Lambda, \infty[= 0$ and $|\theta'| \leqslant c_4(rR_\Lambda)^{-1}$. Applying the stability inequality (37) with $\phi = h^{1/2}$ and $\zeta = \theta \circ \rho$ and inequality (45) with $f = h$, we obtain

$$\begin{aligned}\int_{B(tR_\Lambda)} Fh\,dv_g &\leqslant \int_{B(R_\Lambda)} \zeta^2 h^{1/2}(L_g - 2\ell)h^{1/2}\,dv_g\\ &\leqslant \int_{B(R_\Lambda)} h|d\zeta|_g^2\,dv_g \leqslant c_6(rR_\Lambda)^{-2}\int_{B(R_\Lambda)} h\,dv_g\\ &\leqslant c_6(rR_\Lambda)^{-2}\Bigl(\int_{B(R_\Lambda)} dv_g\Bigr)^{1/2}\Bigl(\int_{B(R_\Lambda)} h^2\,dv_g\Bigr)^{1/2},\end{aligned}$$

where $F := K_g|A^0|^2h^{-2} + |A^0|^2 + 2S'$, and it follows from Lemma 5.3 and (48) that

$$\int_{B(tR_\Lambda)} Fh\,dv_g \leqslant c_7(\Lambda)r^{-2}R_\Lambda^{-1}. \tag{49}$$

Since $K_g \leqslant 0$, we have

$$F := K_g\bigl|A^0\bigr|^2h^{-2} + \bigl|A^0\bigr|^2 + 2S - 2\ell > K_g + \bigl|A^0\bigr|^2 + 2S - 2\ell = \frac{1}{2}\bigl|A^0\bigr|^2 + 3S - 2\ell$$

and finally (recall that $S' := S - \ell,\ \ell \geqslant 0$)

$$F \geqslant \frac{1}{2}\bigl|A^0\bigr|^2 + 3S'. \tag{50}$$

One can then write

$$\begin{aligned}\int_{B(tR_\Lambda)} h^3\,dv_g &= \int_{B(tR_\Lambda)} h\bigl(\Lambda R^{-2} + 2S' + \bigl|A^0\bigr|^2\bigr)\,dv_g\\ &= \int_{B(tR_\Lambda)} \bigl(6S' + \bigl|A^0\bigr|^2\bigr)h\,dv_g + \bigl(\Lambda R^{-2} - 4S'\bigr)\int_{B(tR_\Lambda)} h\,dv_g\\ &\leqslant 2\int_{B(tR_\Lambda)} Fh\,dv_g + 2\Lambda R^{-2}\int_{B(tR_\Lambda)} h\,dv_g,\end{aligned}$$

where we have used the inequality (50) and the assumption $4a'R^2 \leqslant \Lambda$. Finally, using Cauchy–Schwarz, Lemma 5.3 and inequalities (48) and (49), we obtain

$$\int_{B(tR_\Lambda)} h^3\,dv_g \leqslant 2c_7(\Lambda)r^{-2}R_\Lambda^{-1} + c_8(\Lambda)R_\Lambda^{-1}. \tag{51}$$

Choosing $t = 7/8$ and $r = 1/16$, we obtain:

LEMMA 5.4. – *Recall the notations* (34). *Define* $h := (\Lambda R^{-2} + 2S' + |A^0|^2)^{1/2}$ *and* $f := -2K_g|A^0|^2 h^{-2}$. *Under the assumptions* $S := c + H^2 \leqslant 0$ *and* $4a'R^2 \leqslant \Lambda$, *we have*

(i) $$\Delta_g h + fh \geqslant 0.$$

There exists a constant $c_{10}(\Lambda)$, *which only depend on* Λ, *such that*:

(ii) $$\int_{B(7R_\Lambda/8)} h^2\, dv_g \leqslant c_{10}(\Lambda),$$

(iii) $$\int_{B(7R_\Lambda/8)} h^3\, dv_g \leqslant c_{10}(\Lambda) R_\Lambda^{-1},$$

and, for all $\alpha \in [0, 1/2]$,

(iv) $$\int_{B(7R_\Lambda/8)} h^{2(\alpha+1)}\, dv_g \leqslant c_{10}(\Lambda) R_\Lambda^{-2\alpha}.$$

Furthermore,

(v) $$f \leqslant h^2 \quad \text{and hence} \quad \int_{B(7R_\Lambda/8)} f_+^{1+\alpha}\, dv_g \leqslant c_{10}(\Lambda) R_\Lambda^{-2\alpha}.$$

Proof. – Assertion (i) follows from the equality $\Delta_g \ln|A^0|^2 = 4K_g$ and from previous computations. We have already proved Assertions (ii) and (iii) and Assertion (iv) follows by interpolation. Furthermore, since $K_g \leqslant 0$,

$$f = f_+ \leqslant -2K_g = \left|A^0\right|^2 - 2S = h^2 - 4S + 2\ell - \Lambda R^{-2} \leqslant h^2 + 4a' - \Lambda R^{-2}.$$

The assumption $4a'R^2 \leqslant \Lambda$ implies that $f \leqslant h^2$ which proves Assertion (v). □

This lemma says that Assumptions (a) and (c) of Lemma 2.1 are satisfied (indeed, (a) follows from Lemma 5.3, control of the volume of the ball, and (c) follows from Lemma 5.4 if we can prove that $B_M = 0$). In order to be able to apply Lemma 2.1 to the present situation, it therefore remains to show that Assumption (b) in that lemma is satisfies as well.

Step 5. Let us prove

LEMMA 5.5. – *Given* $q \geqslant 1$, *there exists a constant* $c_{11}(q, \Lambda)$ *such that*

$$\left(\int_{B(3R_\Lambda/4)} h^{2q}\, dv_g \right)^{1/q} \leqslant c_{11}(q, \Lambda) R^{-2+2/q}$$

provided that $4a'R^2 \leqslant \Lambda$, *see notation* (34).

In order to prove this result, we need another lemma.

LEMMA 5.6. – *Under the assumption of Proposition* 5.2, *the surface* (D,g) *satisfies a Sobolev inequality of Euclidean type* (*i.e., an inequality of the form* (3), *with* $B_M = 0$)

$$\forall \phi \in C_0^\infty(D), \quad \Bigl(\int_D \phi^2\, dv_g\Bigr)^{1/2} \leqslant A_D \int_D |d\phi|_g\, dv_g.$$

Proof. – To prove this lemma, we use the Lawson correspondence. Recall that $i:(D,g) \to (\overline{M}^3(c), \overline{g})$ is an isometric immersion with constant mean curvature H and that $c + H^2 \leqslant 0$. Let W be the shape operator associated with i. This operator satisfies

$$\begin{cases} \operatorname{Det} W + c = K_g \quad \text{(Gauss equation)}, \\ \bigl(D_X^g W\bigr)(Y) = \bigl(D_Y^g W\bigr)(X) \quad \text{(Codazzi equation)}, \end{cases}$$

for all vector fields X, Y. The operator $W^0 := W - H\,\mathrm{Id}$ satisfies the equations

$$\begin{cases} \operatorname{Det} W^0 + c + H^2 = K_g, \\ \bigl(D_X^g W^0\bigr)(Y) = \bigl(D_Y^g W^0\bigr)(X), \end{cases}$$

for all vector fields X, Y, because H is constant. Since D is simply-connected, it follows from [17] (Volume IV, Theorem 19, p. 71 ff) that there exists an isometric immersion $j:(D,g) \to (\overline{M}^3(c+H^2), \overline{g})$ whose shape operator is precisely W^0. Since Trace $W^0 = 0$, this immersion is a minimal immersion of (D,g) into a simply-connected space form with non-positive curvature $c+H^2$. We may therefore apply the Sobolev inequality given by Hoffman and Spruck [11]: there exists a universal constant A_D such that

$$\forall \phi \in C_0^\infty(D), \quad \Bigl(\int_D \phi^2\, dv_g\Bigr)^{1/2} \leqslant A_D \int_D |d\phi|_g\, dv_g. \qquad \square \tag{52}$$

From inequality (52), we deduce that

$$\Bigl(\int_D |\phi|^{2p}\, dv_g\Bigr)^{1/p} \leqslant p^2 A_D^2 \Bigl(\int_{\operatorname{Supp}\phi} dv_g\Bigr)^{1/p} \int_D |d\phi|_g^2\, dv_g, \tag{53}$$

for all $\phi \in C_0^\infty(D)$ and for all $p \geqslant 1$.

In order to prove Lemma 5.5, we apply inequality (53) to the function $\phi = \zeta h$, with h as in Lemma 5.4 and $\zeta \in C_0^\infty(B(R_\Lambda))$ (recall that $B(R)$ is relatively compact in D). Assuming that $\zeta | B(3R_\Lambda/4) = 1$, we get

$$\Bigl(\int_{B(3R_\Lambda/4)} h^{2p}\, dv_g\Bigr)^{1/p} \leqslant c_{15}(p) \Bigl(\int_{B(R_\Lambda)} dv_g\Bigr)^{1/p} \int_{B(R_\Lambda)} \bigl|d(\zeta h)\bigr|_g^2\, dv_g. \tag{54}$$

Using

$$\bigl|d(\zeta h)\bigr|_g^2 = h^2 |d\zeta|_g^2 + \zeta^2 |dh|_g^2 + 2\zeta h \langle d\zeta, dh\rangle_g$$

and

$$\langle d(\zeta^2 h), dh\rangle_g = 2\zeta h\langle d\zeta, dh\rangle_g + \zeta^2 |dh|_g^2,$$

integration by parts and the inequality $\Delta_g h + fh \geqslant 0$, we obtain

$$\begin{aligned}\int\limits_{B(R_\Lambda)} |d(\zeta h)|_g^2\, dv_g &= \int\limits_{B(R_\Lambda)} h^2 |d\zeta|_g^2\, dv_g + \int\limits_{B(R_\Lambda)} \langle d(\zeta^2 h), dh\rangle_g\, dv_g \\ &\leqslant \int\limits_{B(R_\Lambda)} h^2 |d\zeta|_g^2\, dv_g + \int\limits_{B(R_\Lambda)} f\zeta^2 h^2\, dv_g.\end{aligned}$$

Using Lemma 5.3, one can rewrite inequality (54) as

$$\Bigg(\int\limits_{B(3R_\Lambda/4)} h^{2p}\, dv_g\Bigg)^{1/p} \leqslant c_{16}(p,\Lambda) R_\Lambda^{2/p} \Bigg\{ \int\limits_{B(7R_\Lambda/8)} h^2 |d\zeta|_g^2\, dv_g + \int\limits_{B(7R_\Lambda/8)} fh^2\, dv_g \Bigg\},$$

provided that $\operatorname{Supp}\zeta \subset B(7R_\Lambda/8)$. Using the inequality $f \leqslant h^2$ of Lemma 5.4, one finally deduces that

$$\Bigg(\int\limits_{B(3R_\Lambda/4)} h^{2p}\, dv_g\Bigg)^{1/p} \leqslant c_{16}(p,\Lambda) R_\Lambda^{2/p} \Bigg\{ \int\limits_{B(7R_\Lambda/8)} h^2 |d\zeta|_g^2\, dv_g + \int\limits_{B(7R_\Lambda/8)} h^4\, dv_g \Bigg\}. \tag{55}$$

Choosing ζ such that $\zeta | B(3R_\Lambda/4) = 1$, $\operatorname{Supp}\zeta \subset B(7R_\Lambda/8)$ and $|d\zeta|_g^2 \leqslant c_{17} R_\Lambda^{-2}$ and applying Lemma 5.3, we obtain

$$\int\limits_{B(7R_\Lambda/8)} h^2 |d\zeta|_g^2\, dv_g \leqslant c_{18}(p,\Lambda) R_\Lambda^{-2} \int\limits_{B(7R_\Lambda/8)} h^2\, dv_g \leqslant c_{19}(p,\Lambda) R_\Lambda^{-2}.$$

We now need an analogous control of $\int_{B(7R_\Lambda/8)} h^4\, dv_g$. For this purpose, we apply the Sobolev inequality (52) to the function ζh^2 and we choose ζ such that $\zeta | B(7R_\Lambda/8) = 1$, $\operatorname{Supp}\zeta \subset B(15R_\Lambda/16)$ and we obtain

$$\Bigg(\int\limits_{B(7R_\Lambda/8)} h^4\, dv_g\Bigg)^{1/2} \leqslant c_{20} \int\limits_{B(R_\Lambda)} |d(\zeta h^2)|_g\, dv_g.$$

We also have

$$\int\limits_{B(R_\Lambda)} |d(\zeta h^2)|_g\, dv_g \leqslant \int\limits_{B(R_\Lambda)} h^2 |d\zeta|_g\, dv_g + \int\limits_{B(R_\Lambda)} \zeta |dh^2|_g\, dv_g,$$

$$\int\limits_{B(R_\Lambda)} |d(\zeta h^2)|_g\, dv_g \leqslant c_{21}(\Lambda) R_\Lambda^{-1} + \int\limits_{B(15R_\Lambda/16)} |dh^2|_g\, dv_g,$$

and

$$\int\limits_{B(tR_\Lambda)} |dh^2|_g\, dv_g = \int\limits_{B(tR_\Lambda)} |d(h^{1/2})^4|_g\, dv_g = 4 \int\limits_{B(tR_\Lambda)} h^{3/2} |dh^{1/2}|_g\, dv_g$$

$$\leqslant 4\Bigl(\int_{B(tR_\Lambda)} h^3\,dv_g\Bigr)^{1/2}\Bigl(\int_{B(tR_\Lambda)} \bigl|dh^{1/2}\bigr|_g^2\,dv_g\Bigr)^{1/2}$$
$$\leqslant c_{22}(\Lambda)R_\Lambda^{-1/2}\Bigl(\int_{B(tR_\Lambda)} \bigl|dh^{1/2}\bigr|_g^2\,dv_g\Bigr)^{1/2},$$

where we have used Lemma 5.4 in the last inequality.

From the definition of the function $h := (\Lambda R^{-2} + 2S' + |A^0|^2)^{1/2}$ and from the equation $\Delta_g \ln|A^0|^2 = 4K_g$ (see [22]), we deduce the equality

$$\Delta_g \ln h^{1/2} = K_g\bigl|A^0\bigr|^2h^{-2} + \frac{1}{4}\bigl|d\bigl|A^0\bigr|^2\bigr|^2\bigl(\Lambda R^{-2}+2S'\bigr)\bigl|A^0\bigr|^{-2}h^{-4}$$

and, since $\Lambda R^{-2} + 2S' \geqslant 0$,

$$\Delta_g \ln h^{1/2} \geqslant K_g\bigl|A^0\bigr|^2h^{-2}$$

and

$$h^{1/2}\Delta_g h^{1/2} \geqslant \bigl|dh^{1/2}\bigr|_g^2 + hK_g\bigl|A^0\bigr|^2h^{-2}.$$

According to (45) with $f = h$, one also has

$$h^{1/2}(L_g - 2\ell)h^{1/2} \geqslant \bigl|dh^{1/2}\bigr|_g^2 + h\bigl(K_g\bigl|A^0\bigr|^2h^{-2} + 2S' + \bigl|A^0\bigr|^2\bigr).$$

With an appropriate choice of ζ, with $\operatorname{Supp}\zeta \subset B((t+r)R_\Lambda)$, $\zeta|B(tR_\Lambda) = 1$, we can write

$$\int_{B(R_\Lambda)} \zeta^2h^{1/2}(L_g-2\ell)h^{1/2}\,dv_g \geqslant \int_{B(tR_\Lambda)} h^{1/2}(L_g-2\ell)h^{1/2}\,dv_g$$
$$\geqslant \int_{B(tR_\Lambda)} \bigl\{\bigl|dh^{1/2}\bigr|_g^2 + h\bigl(K_g\bigl|A^0\bigr|^2h^{-2} + 2S' + \bigl|A^0\bigr|^2\bigr)\bigr\}\,dv_g.$$

On the other hand, using the stability inequality (37), we have

$$\int_{B(R_\Lambda)} \zeta^2h^{1/2}(L_g-2\ell)h^{1/2}\,dv_g = \int_{B((t+r)R_\Lambda)} \zeta^2h^{1/2}(L_g-2\ell)h^{1/2}\,dv_g$$
$$\leqslant \int_{B((t+r)R_\Lambda)} h|d\zeta|_g^2\,dv_g \leqslant c_{24}(\Lambda,r)R_\Lambda^{-2}\int_{B((t+r)R_\Lambda)} h\,dv_g$$
$$\leqslant c_{25}(\Lambda,r)R_\Lambda^{-1}$$

using Cauchy–Schwarz, Lemma 5.3 and Lemma 5.4. On the other hand, since $K_g \leqslant 0$, we have

$$K_g\bigl|A^0\bigr|^2h^{-2} + 2S' + \bigl|A^0\bigr|^2 \geqslant K_g + 2S' + \bigl|A^0\bigr|^2 \geqslant \frac{1}{2}\bigl|A^0\bigr|^2 + 3S',$$

which implies that

$$\int_{B(tR_\Lambda)} h\bigl(K_g\bigl|A^0\bigr|^2h^{-2} + 2S' + \bigl|A^0\bigr|^2\bigr)\,dv_g \geqslant -3a'\int_{B(tR_\Lambda)} h\,dv_g$$

and

$$-\int_{B(tR_\Lambda)} h\big(K_g\big|A^0\big|^2h^{-2}+2S'+\big|A^0\big|^2\big)\,dv_g \leqslant 3a'\int_{B(tR_\Lambda)} h\,dv_g$$

and hence, using Lemma 5.3, (47) and the assumption $4a'R^2\leqslant\Lambda$,

$$-\int_{B(tR_\Lambda)} h\big(K_g\big|A^0\big|^2h^{-2}+2S'+\big|A^0\big|^2\big)\,dv_g \leqslant 3a'R_\Lambda\leqslant c_{26}(\Lambda)R_\Lambda^{-1}.$$

Finally,

$$\int_{B(tR_\Lambda)} \big|dh^{1/2}\big|_g^2\,dv_g\leqslant c_{27}(\Lambda)R_\Lambda^{-1}$$

which implies

$$\int_{B(tR_\Lambda)} \big|dh^2\big|_g\,dv_g\leqslant c_{28}(\Lambda)R_\Lambda^{-1}$$

and, choosing t, r appropriately,

$$\int_{B(7R_\Lambda/8)} h^4\,dv_g\leqslant c_{29}(\Lambda)R_\Lambda^{-2}.$$

Plugging this inequality into (55), we conclude that

$$\forall p\geqslant 1,\quad \Big(\int_{B(3R_\Lambda/4)} h^{2p}\,dv_g\Big)^{1/p}\leqslant c_{30}(p,\Lambda)R_\Lambda^{-2+2/p}.$$

We can now apply Lemma 2.1 to obtain the estimate

$$\sup_{B(R_\Lambda/2)} h^2\leqslant c_{31}(\Lambda)R_\Lambda^{-2}. \tag{56}$$

Recalling that $h:=\big(\Lambda R^{-2}+2S'+|A^0|^2\big)^{1/2}$, that $|K_g|\leqslant\frac12|A^0|^2+a$ and the assumption $4a'R^2\leqslant\Lambda$, we obtain the estimates

$$\big|A^0\big|^2\leqslant c_{32}(\Lambda)R_\Lambda^{-2}\quad\text{on } B(R_\Lambda/2),$$
$$|K_g|\leqslant c_{33}(\Lambda)R_\Lambda^{-2}\quad\text{on } B(R_\Lambda/2).$$

This finishes the proof of Theorem 4.2 under Condition (A).

5.2. Proof of Theorem 4.2 under Condition (B)

We assume that there exist two positive numbers R_1', R_1, with $2R_1\geqslant R_1'>R_1$, such that

$$B(R_1'):=B(x_0,R_1')\Subset(M,g),$$

i.e., $B(R_1')$ is relatively compact in M. Recall that $S:=c+H^2>0$.

We also assume that the stability operator $L_g := \Delta_g + |A^0|^2 + 2S$ is bounded from above by some number $2\ell \geqslant 0$ on $C_0^\infty(B(R_1))$. Then, according to Lemma 3.3, we have the *stability condition*

$$\int_M \zeta^2 \phi L_g \phi \, dv_g \leqslant \int_M \phi^2 |d\zeta|_g^2 \, dv_g + 2\ell \int_M \phi^2 \zeta^2 \, dv_g, \tag{57}$$

for all $\phi \in C^\infty(B(R_1))$ and for all $\zeta \in C_0^\infty(B(R_1))$ (or more generally for any Lipschitz function ζ with compact support in $B(R_1)$).

Note. – We shall in fact work in smaller balls, namely in balls $B(R)$ with

$$R \leqslant \min\bigl\{R_1, \pi/2\sqrt{c+H^2}, \sqrt{\Lambda/4\ell}\bigr\}.$$

Since we reduce the size of the domain, (57) is still valid in such balls.

According to (35), $K_g \leqslant S$ and hence the exponential map $\exp_{x_0}$ is a local diffeomorphism on $B(0, R_2') \subset T_{x_0}M$ where $R_2' := \min\{R_1', \pi/\sqrt{S}\}$.

First reduction. Applying Assertion (i) of Lemma 5.1 to $\Omega = B(R_2')$, gives us a non-negative function u on Ω, such that $(L_g - 2\ell)u \leqslant 0$. We now consider the ball

$$\widetilde{\Omega} := B\bigl(0, R_2'\bigr) \subset \bigl(T_{x_0}M, \tilde{g}\bigr),$$

where $\tilde{g} = \exp_{x_0}^* g$. The function $\tilde{u} = u \circ \exp_{x_0}$ is non-negative in $\widetilde{\Omega}$. Since $\exp_{x_0}$ is a local isometry, we have $(L_{\tilde{g}} - 2\ell)\tilde{u} \leqslant 0$. Assertion (i) of the same lemma implies that the operator $L_{\tilde{g}} - 2\ell$ is non-positive on $B(0, R_2')$.

By reducing R_2' if necessary, we may assume that

$$2R_2 \geqslant R_2' > R_2, \quad \text{where } R_2 := \min\bigl\{R_1, \pi/\sqrt{S}\bigr\}$$

and that $B(0, R_2')$ is conformally equivalent to the unit disk $D \subset \mathbb{C}$, i.e., that there exists a diffeomorphism $\Phi : D \to B_0(R_2')$ such that $\Phi(0) = 0_{T_{x_0}M}$ and $\Phi^*(\exp_{x_0}^* g) = \lambda^2 e$, where e is the Euclidean metric in D.

This is our first reduction: we shall now work in a ball $B(R)$ such that $2R \geqslant R' > R$, with $B(R')$ conformally equivalent to the unit disk D. We shall also assume that:

$$\begin{cases} \text{(i)} & 4SR^2 \leqslant \pi^2, \\ \text{(ii)} & 4\ell R^2 \leqslant \Lambda. \end{cases} \tag{58}$$

Condition (i) in (58) is a strong restriction. Condition (ii) depends on the free parameter Λ, a positive constant which can be chosen appropriately. Note that condition (ii) is empty if $\ell = 0$.

Second reduction. In order to make the second reduction, we make use of the Lawson correspondence back and forth.

• Assume we are given an isometric immersion $i_1 : (D, g) \to (\overline{M}(c_1), \overline{g})$ with constant mean curvature H_1 (with $H_1 = H, c_1 = c$) such that $c_1 < 0$ and $0 < c_1 + H_1^2 < -c_1$. Let W_1 denote its shape operator.

Consider the operator

$$W_2 := W_1 - \Bigl(H_1 - \sqrt{c_1 + H_1^2}\Bigr) \,\mathrm{Id}.$$

The Lawson correspondence gives us an isometric immersion $i_2:(D,g)\to(\mathbb{R}^3,e)$, with constant mean curvature $H_2=\sqrt{c_1+H_1^2}$ and with the same metric g on D, hence with the same Gauss curvature K_g. It follows that the corresponding stability operators L_1 and L_2 coincide.

• Define $\mu:=H_2/\sqrt{-c_1}\in\,]0,1[$ and make a dilation in $\mathbb{R}^3$ to obtain an isometric immersion $i_3:(D,g_3)\to(\mathbb{R}^3,e)$ with constant mean curvature $H_3=\mu^{-1}H_2=\sqrt{-c_1}$, metric $g_3=\mu g$ and shape operator W_3. It follows that the Gauss curvature is $K_3=\mu^{-2}K_g$.

Define $W_4:=W_3+(-H_3+\sqrt{-2c_1})\,\mathrm{Id}$. The Lawson correspondence gives us an immersion $i_4:(D,g_4)\to(\overline{M}(c_4),\overline{g})$ with metric $g_4=g_3$, constant mean curvature $H_4=\sqrt{-2c_1}$, $c_4=c_1$, $c_4+H_4^2=-c_4$.

Since $g_4=\mu^2 g$, we have

$$K_4=\mu^{-2}K_g,\qquad \left|A_4^0\right|^2=\mu^{-2}\left|A^0\right|^2,\qquad B^{g_4}(\mu R)=B^g(R)\Subset(D,g_4).$$

The corresponding stability operator satisfies $L_4=\mu^{-2}L_g$ which implies that $L_4\leqslant 2\ell\mu^{-2}$ on $C_0^\infty(B^{g_4}(\mu R))$ and we have:

$$\begin{aligned}&S_4:=c_4+H_4^2=-c_4,\\&\left(2\ell\mu^{-2}\right)(\mu R)^2\leqslant\Lambda,\\&4S_4(\mu R)^2=-4c_1H_2^2/(-c_1)R^2=4S_1R^2\leqslant\pi^2.\end{aligned}$$

We are therefore reduced to studying an immersion satisfying

$$\begin{cases}\text{either } c\geqslant 0,\\ \text{or } c<0\quad\text{and}\quad S=c+H^2\geqslant -c>0.\end{cases}\tag{59}$$

Assuming the estimates are proved under the assumptions (59), we obtain for the immersion i_4 the estimates $|K_4|,|A_4^0|^2\leqslant C(\Lambda)(\mu R)^{-2}$ in the ball $B^{g_4}(c(\Lambda)\mu R)$. Using the relations between the invariants in the g_4 and in the g metric, we obtain the desired estimates for the immersion i_1

$$|K_1|,\left|A_1^0\right|^2\leqslant C(\Lambda)R^{-2}\quad\text{in } B^{g_4}\big(c(\Lambda)R\big).$$

In order to prove Assertion (B) of Theorem 4.1, we are now reduced to proving the following proposition:

PROPOSITION 5.7. – *Let $D\subset\mathbb{C}$ be the unit disk and let g be a Riemannian metric on D. Fix some positive constant Λ. Make the following assumptions*:

(1) *There exists a conformal isometric immersion $i:(D,g)\to(\overline{M}^3(c),\overline{g})$, $i^*\overline{g}=g=\lambda^2 e$, with constant mean curvature H, with $S:=c+H^2>0$ if $c\geqslant 0$ and $S:=c+H^2\geqslant -c$ if $c<0$;*

(2) *There exists $R>0$ such that the ball $B(R)$ is relatively compact in (D,g), where $B(R):=B^g(0,R)\subset D$, and such that $D\subset B(2R)$;*

(3) *The stability operator L_g of the immersion, $L_g:=\Delta_g+|A^0|^2+2S$, is bounded from above by 2ℓ on $C_0^\infty(B(R))$, for some $\ell\geqslant 0$;*

(4) *The number R satisfies* (58):

$$\begin{cases}\text{(i)}\quad 4SR^2\leqslant\pi^2,\\ \text{(ii)}\quad 4\ell R^2\leqslant\Lambda.\end{cases}$$

Then, there exist positive constants $C(\Lambda)$, $c(\Lambda)$, which only depend on Λ, such that

$$\begin{cases} |A^0|^2 \leqslant C(\Lambda)R^{-2}, \\ |K_g| \leqslant C(\Lambda)R^{-2}, \end{cases} \quad \textit{on } B\big(c(\Lambda)R\big).$$

Proof. – The remainder of this section will be devoted to the proof of Proposition 5.7. In the course of the proof, we will denote by $c_i, c_i(\Lambda), \dots$ constants which depend on the indicated arguments.

Step 1. Let ρ denote the Riemannian distance to the point $0 \in D$ with respect to the metric g. Since $K_g \leqslant S$ by (35), using (58)(i), Bishop's theorem gives

$$\Delta_g \rho^2 \geqslant 2, \tag{60}$$

where Δ_g is the non-positive Laplacian for the metric g.

We define $\phi = e^{\Lambda\rho^2/R^2} \in C^\infty(D)$, where Λ is the positive number given in the assumptions of Theorem 4.2. Then, we have $1 \leqslant \phi \leqslant e^{4\Lambda}$ because $D \subset B(2R)$ and

$$\begin{aligned} \phi L_g \phi &= \big(\Lambda R^{-2}\Delta_g \rho^2 + \Lambda^2 R^{-4}\big|d\rho^2\big|_g^2 + \big|A^0\big|^2 + 2S\big)\phi^2 \\ &\geqslant \big(2\Lambda R^{-2} + 2S + \big|A^0\big|^2\big)\phi^2. \end{aligned}$$

It follows from (57) that

$$\int_D \big(2\Lambda R^{-2} + 2S + \big|A^0\big|^2\big)\phi^2\zeta^2\,dv_g \leqslant \int_D \phi^2|d\zeta|_g^2\,dv_g + 2\ell\int_D \phi^2\zeta^2\,dv_g.$$

Using the conformal invariance of the Dirichlet integral and (58)(ii), we obtain

$$\int_D \big(\Lambda R^{-2} + 2S + \big|A^0\big|^2\big)\phi^2\zeta^2\,dv_g \leqslant \int_D \phi^2|d\zeta|_e^2\,dv_e.$$

Using a suitable function ζ of the Euclidean distance to $0 \in D$ and the inequality $1 \leqslant \phi \leqslant e^{4\Lambda}$, we obtain the following important estimate

$$\int_{D(1/2)} \big(\Lambda R^{-2} + 2S + \big|A^0\big|^2\big)\,dv_g \leqslant c_1(\Lambda). \tag{61}$$

Step 2. The above estimate involves the Euclidean disk $D(1/2)$; in order to be able to take the metric g into account, we make use of Theorem 3.2 which gives the estimate

$$\lambda^2 \geqslant C_0 R^2\big(1 + \ell R^2\big)^{-1} \quad \text{on } B(R/2). \tag{62}$$

We obtain the analogue of Lemma 5.3.

LEMMA 5.8. – *Under the assumptions of Proposition* 5.7, *define*

$$A = A(\Lambda) := \min\big\{1, \sqrt{C_0(1+\Lambda)^{-1}}\big\}/2 \quad \textit{and} \quad R_A := AR, \tag{63}$$

where C_0 *is given by Theorem* 3.2. *Then*

$$B(R_A) \subset D(1/2), \tag{64}$$

$$\int_{B(R_A)} \left(\Lambda R^{-2} + 2S + \left|A^0\right|^2\right) dv_g \leqslant c_1(\Lambda). \tag{65}$$

In particular,

$$\operatorname{Vol}\big(B(R_A)\big) \leqslant c_2(\Lambda) R_A^2. \tag{66}$$

Remarks. – Note that $A(\Lambda)$ does not depend on Λ when $\ell = 0$. By Gauss equation (35), and since $S > 0$, we have

$$|K_g| \leqslant \max\left\{S, \frac{1}{2}\left|A^0\right|^2\right\}.$$

In order to control K_g it therefore suffices to control $|A^0|^2$ or any function of the form $(m + |A^0|^2)$, for some $m > 0$. From now on, let

$$h := \left(\Lambda R^{-2} + 2S + \left|A^0\right|^2\right)^{1/2}. \tag{67}$$

As in Step 4 of the proof of Theorem 4.2, under Condition (A), we have

$$h^{1/2} L_g h^{1/2} \geqslant h\left(\left|A^0\right|^2 + 2S + K_g\left|A^0\right|^2 h^{-2}\right)$$

and it follows easily, looking at the cases $K_g \leqslant 0$ and $K_g \geqslant 0$, that

$$h^{1/2} L_g h^{1/2} \geqslant h\left(\frac{1}{2}\left|A^0\right|^2 + 2S\right). \tag{68}$$

We then have the following lemma:

LEMMA 5.9. – *Define* $h := \left(\Lambda R^{-2} + 2S + |A^0|^2\right)^{1/2}$ *and* $f := -2K_g|A^0|^2 h^{-2}$ *and let*

$$f_1 := \begin{cases} (f + H^2)_+ & \text{if } c \leqslant 0, \\ (f + c + H^2)_+ & \text{if } c \geqslant 0. \end{cases}$$

Under the assumptions of Proposition 5.7, *we have*:

(i) $\Delta_g h + f h \geqslant 0$.

There exists a constant $c_{10}(\Lambda)$, *which only depends on* Λ, *such that*:

(ii) $\int_{B(7R_A/8)} h^2\, dv_g \leqslant c_{10}(\Lambda)$,

(iii) $\int_{B(7R_A/8)} h^3\, dv_g \leqslant c_{10}(\Lambda) R_A^{-1}$,

(iv) $\forall \alpha \in [0, 1/2]$, $\int_{B(7R_A/8)} h^{2(1+\alpha)}\, dv_g \leqslant c_{10}(\Lambda) R_A^{-2\alpha}$,

(v) $f_1 \leqslant h^2$ *and hence*, $\forall \alpha \in [0, 1/2]$, $\int_{B(7R_A/8)} f_1^{1+\alpha}\, dv_g \leqslant c_{10}(\Lambda) R_A^{-2\alpha}$.

Proof. – Assertion (i) follows from the equality $\Delta_g \ln|A^0|^2 = 4K_g$, Assertion (ii) directly from the definition of h and from inequality (65). Take $0 < a < a + r < 1$ (to be chosen later) and choose a smooth function θ such that $\theta = 1$ on $[0, aR_A]$, $\theta = 0$ on $[(a + r)R_A, R_A]$ and $|\theta'| \leqslant$

$c_4(rR_A)^{-1}$. Plugging $\phi = h^{1/2}$ and $\zeta = \theta \circ \rho$ into (57), using the inequality $h^{1/2}L_g h^{1/2} \geqslant hF$, with

$$F := \left(\left|A^0\right|^2 + 2S + K_g\left|A^0\right|^2 h^{-2}\right),$$

we obtain

$$\int\limits_{B(aR_A)} Fh\,dv_g \leqslant \int\limits_{B(R_A)} \zeta^2 h^{1/2} L_g h^{1/2}\,dv_g \leqslant \int\limits_{B(R_A)} h|d\zeta|_g^2\,dv_g + 2\ell \int\limits_{B(R_A)} \zeta^2 h\,dv_g.$$

Using (65) and (66), we have

$$\int\limits_{B(R_A)} h\,dv_g \leqslant \Bigg(\int\limits_{B(R_A)} h^2\,dv_g\Bigg)^{1/2}\Bigg(\int\limits_{B(R_A)} dv_g\Bigg)^{1/2} \leqslant c_7(\Lambda)R_A \tag{69}$$

and hence, using (58)(ii),

$$\int\limits_{B(aR_A)} hF\,dv_g \leqslant c_8(\Lambda)r^{-2}R_A^{-1}. \tag{70}$$

Since

$$F \geqslant \frac{1}{2}\left|A^0\right|^2 + 2S \tag{71}$$

by (68), we can finally write

$$\int\limits_{B(aR_A)} h^3\,dv_g = \int\limits_{B(aR_A)} h\left(\Lambda R^{-2} + 2S + \left|A^0\right|^2\right)dv_g \leqslant \Lambda R^{-2} \int\limits_{B(aR_A)} h\,dv_g + 2\int\limits_{B(aR_A)} hF\,dv_g.$$

Using (69) and (70) this leads to

$$\int\limits_{B(aR_A)} h^3\,dv_g \leqslant c_9(\Lambda)r^{-2}R_A^{-1}. \tag{72}$$

Finally, using interpolation, the values $a := 7/8$ and $r := 1/16$, (ii) and (72) give Assertion (iv).

In order to prove Assertion (v), it suffices to show that $f_1 \leqslant h^2$.

CLAIM. – $f_1 \leqslant h^2$. Indeed, using the definition of f, h, f_1 and (58), we have:

- If $c \geqslant 0$ and $K_g \geqslant 0$, then $f \leqslant 0$ and $f + c + H^2 \leqslant c + H^2 = S \leqslant h^2$.
- If $c \geqslant 0$ and $K_g \leqslant 0$, then $f \leqslant -2K_g$ and $f + c + H^2 \leqslant -2K_g + c + H^2 \leqslant |A^0|^2 - S \leqslant h^2$.
- If $c < 0$ and $K_g \leqslant 0$, then $f \leqslant -2K_g$, since $H^2 \leqslant 2(H^2 + c)$ under the assumptions of Proposition 5.7. We obtain $0 \leqslant f + H^2 \leqslant |A^0|^2 \leqslant h^2$.
- If $c < 0$ and $K_g \geqslant 0$, then $f + H^2 \leqslant H^2 \leqslant 2(c + H^2) \leqslant h^2$.

This finishes the proof of Lemma 5.9. □

Note. – With the above notations (in particular under the additionnal condition $-cR^2 \leqslant \Lambda$), we claim that $f_1 \leqslant h^2$ provided that $4(c + H^2)R^2 \leqslant \pi^2$ and $-cR^2 \leqslant \Lambda$, under the sole assumption $c + H^2 > 0$ (this avoids using the Lawson correspondence to reduce to the case $c + H^2 > 0$ if $c \geqslant 0$, or $c + H^2 \geqslant -c$ if $c < 0$). Indeed,

- If $c \geqslant 0$ and $K_g \geqslant 0$, as above.
- If $c \geqslant 0$ and $K_g \leqslant 0$, as above.
- If $c < 0$ and $K_g \leqslant 0$, then $f \leqslant -2K_g = |A^0|^2 - 2S$. Furthermore,

$$0 \leqslant f + H^2 \leqslant \left|A^0\right|^2 - \left(c + H^2\right) - c \leqslant \left|A^0\right|^2 - c \leqslant \left|A^0\right|^2 + \Lambda R^{-2} \leqslant h^2.$$

- If $c < 0$ and $K_g \geqslant 0$, then $f \leqslant 0$ and hence

$$f + H^2 \leqslant H^2 = c + H^2 - c \leqslant S + \Lambda R^{-2} \leqslant h^2.$$

Step 3. Let us prove

LEMMA 5.10. – *Under the assumption of Proposition* 5.7, *given* $q \geqslant 1$, *there exists a constant* $c_{11}(q, \Lambda)$ *such that*

$$\left(\int_{B(3R_A/4)} h^{2q}\, dv_g \right)^{1/q} \leqslant c_{11}(q, \Lambda) R_{R_A}^{-2+2/q}.$$

In order to prove this result, we need another lemma.

LEMMA 5.11. – *Under the assumptions of Proposition* 5.7 (*in particular* $c + H^2 > 0$), *the surface* (D, g) *satisfies the Sobolev inequality*

$$\left(\int_D \phi^2\, dv_g \right)^{1/2} \leqslant A_D \left\{ \int_D |d\phi|_g\, dv_g + B_D \int_D |\phi|\, dv_g \right\}$$

for all $\phi \in C_0^\infty(D)$, *where* A_D *is a universal constant and where* B_D *is defined by*

$$B_D = \begin{cases} H & \text{if } c \leqslant 0, \\ \sqrt{c + H^2} & \text{if } c \geqslant 0. \end{cases}$$

Proof. – To prove this lemma, we use the fact that the immersion

$$i : (D, g) \to \left(\overline{M}^3(c, \overline{g})\right)$$

is an isometric immersion with constant mean curvature H. When $c \leqslant 0$ we can directly apply the Sobolev inequality in [11]. When $c > 0$, we compose i with the isometric immersion of the 3-sphere of curvature c into $\mathbb{R}^4$. This gives an isometric immersion j whose mean curvature vector has norm $\sqrt{c + H^2}$ and we apply [11] again. □

From this inequality, we deduce that

$$\left(\int_D |\phi|^{2p}\, dv_g \right)^{1/p} \leqslant c_{15}(p) \left(\int_{\operatorname{Supp}\phi} dv_g \right)^{1/p} \left\{ \int_D |d\phi|_g^2\, dv_g + B_D^2 \int_D \phi^2\, dv_g \right\}, \tag{73}$$

for all $\phi \in C_0^\infty(D)$ and for all $p \geqslant 1$.

In order to prove Lemma 5.10, we apply inequality (73) to the function $\phi = \zeta h$, with h as in Lemma 5.9 and $\zeta \in C_0^\infty(B(R_A))$ with $\zeta \big| B(3R_A/4) = 1$. We get

$$\left(\int_{B(3R_A/4)} h^{2p}\,dv_g\right)^{1/p} \leqslant c_{15}(p)\left(\int_{B(R_A)} dv_g\right)^{1/p}\left\{\int_{B(R_A)} \left|d(\zeta h)\right|_g^2\,dv_g + B_D^2\int_{B(R_A)} h^2\zeta^2\,dv_g\right\}. \tag{74}$$

Using

$$\left|d(\zeta h)\right|_g^2 = h^2|d\zeta|_g^2 + \zeta^2|dh|_g^2 + 2\zeta h\langle d\zeta, dh\rangle_g$$

and

$$\left\langle d\left(\zeta^2 h\right), dh\right\rangle_g = 2\zeta h\langle d\zeta, dh\rangle_g + \zeta^2|dh|_g^2$$

integration by parts and the inequality $\Delta_g h + fh \geqslant 0$, we obtain:

$$\begin{aligned}\int_{B(R_A)} \left|d\left(\zeta h\right)\right|_g^2\,dv_g &= \int_{B(R_A)} h^2|d\zeta|_g^2\,dv_g + \int_{B(R_A)} \left\langle d\left(\zeta^2 h\right), dh\right\rangle_g\,dv_g\\ &\leqslant \int_{B(R_A)} h^2|d\zeta|_g^2\,dv_g + \int_{B(R_A)} f\zeta^2h^2\,dv_g.\end{aligned}$$

Using Lemma 5.8, one can rewrite inequality (74) as

$$\left(\int_{B(3R_A/4)} h^{2p}\,dv_g\right)^{1/p} \leqslant c_{16}(p,\Lambda)R_A^{2/p}\left\{\int_{B(7R_A/8)} h^2|d\zeta|_g^2\,dv_g + \int_{B(7R_A/8)} \left(f + B_D^2\right)_+ h^2\,dv_g\right\}$$

provided that $\operatorname{Supp}\zeta \subset B(7R_A/8)$ and $0 \leqslant \zeta \leqslant 1$. Using the inequality $f_1 \leqslant h^2$ of Lemma 5.9 and a suitable function ζ, we obtain

$$\left(\int_{B(3R_A/4)} h^{2p}\,dv_g\right)^{1/p} \leqslant c_{16}(p,\Lambda)R_A^{2/p}\left\{R_A^{-2}\int_{B(7R_A/8)} h^2\,dv_g + \int_{B(7R_A/8)} h^4\,dv_g\right\}. \tag{75}$$

We now need to control $\int_{B(7R_A/8)} h^4\,dv_g$. For this purpose, we apply Lemma 5.11 to the function ζh^2 and we choose a suitable function ζ such that

$$\zeta|B(7R_A/8) = 1, \qquad \operatorname{Supp}\zeta \subset B(15R_A/16).$$

We obtain

$$\left(\int_{B(7R_A/8)} h^4\,dv_g\right)^{1/2} \leqslant c_{20}\left\{\int_{B(R_A)} \left|d\left(\zeta h^2\right)\right|_g\,dv_g + B_D\int_{B(R_A)} \zeta h^2\,dv_g\right\}.$$

We also have

$$\int_{B(R_A)} \left|d\left(\zeta h^2\right)\right|_g\,dv_g \leqslant \int_{B(R_A)} h^2|d\zeta|_g\,dv_g + \int_{B(R_A)} \zeta\left|dh^2\right|_g\,dv_g,$$

$$\int_{B(R_A)} \left|d\left(\zeta h^2\right)\right|_g dv_g \leqslant c_{21}(\Lambda)R_A^{-1} + \int_{B(15R_A/16)} \left|dh^2\right|_g dv_g,$$

and

$$\begin{aligned}\int_{B(aR_A)} \left|dh^2\right|_g dv_g &= \int_{B(aR_A)} \left|d\left(h^{1/2}\right)^4\right|_g dv_g = 4\int_{B(aR_A)} h^{3/2}\left|dh^{1/2}\right|_g dv_g \\ &\leqslant 4\left(\int_{B(aR_A)} h^3\, dv_g\right)^{1/2}\left(\int_{B(aR_A)} \left|dh^{1/2}\right|_g^2 dv_g\right)^{1/2} \\ &\leqslant c_{22}(\Lambda)R_A^{-1/2}\left(\int_{B(aR_A)} \left|dh^{1/2}\right|_g^2 dv_g\right)^{1/2},\end{aligned}$$

where we have used Lemma 5.9 in the last inequality.

From the definition of the function $h := (\Lambda R^{-2} + 2S + |A^0|^2)^{1/2}$ and from the equation $\Delta_g \ln|A^0|^2 = 4K_g$ [22], we deduce the equality

$$\Delta_g \ln h^{1/2} = K_g\left|A^0\right|^2 h^{-2} + \frac{1}{4}\left|d\left|A^0\right|^2\right|^2\left(\Lambda R^{-2} + 2S\right)\left|A^0\right|^{-2}h^{-4}$$

and, since $\Lambda R^{-2} + 2S \geqslant 0$,

$$\Delta_g \ln h^{1/2} \geqslant K_g\left|A^0\right|^2 h^{-2}$$

and

$$h^{1/2}\Delta_g h^{1/2} \geqslant \left|dh^{1/2}\right|_g^2 + hK_g\left|A^0\right|^2 h^{-2}.$$

According to (45) with $f = h$, one also has

$$h^{1/2}L_g h^{1/2} \geqslant \left|dh^{1/2}\right|_g^2 + h\left(K_g\left|A^0\right|^2 h^{-2} + 2S + \left|A^0\right|^2\right).$$

If $K_g \geqslant 0$, we have

$$\left|dh^{1/2}\right|_g^2 \leqslant h^{1/2}L_g h^{1/2}.$$

If $K_g \leqslant 0$ we have

$$K_g\left|A^0\right|^2 h^{-2} + 2S + \left|A^0\right|^2 \geqslant K_g + 2S + \left|A^0\right|^2 \geqslant \frac{1}{2}\left|A^0\right|^2 + 3S > 0$$

and again

$$\left|dh^{1/2}\right|_g^2 \leqslant h^{1/2}L_g h^{1/2}.$$

With an appropriate choice of ζ, this implies

$$\begin{aligned}\int_{B(aR_A)} \left|dh^{1/2}\right|_g^2 &\leqslant \int_{B(aR_A)} h^{1/2}L_g h^{1/2}\, dv_g \leqslant \int_{B(aR_A)} \zeta^2 h^{1/2}L_g h^{1/2}\, dv_g \\ &\leqslant \int_{B(R_A)} h|d\zeta|_g^2\, dv_g + 2\ell\int_{B(R_A)} h\zeta^2\, dv_g \leqslant c_{24}(\Lambda)R_A^{-2}\int_{B(R_A)} h\, dv_g,\end{aligned}$$

where we have used (58)(ii). Using (69), Lemma 5.8, Lemma 5.9 and (58)(ii) we get:

$$\int_{B(aR_A)} \left|dh^{1/2}\right|_g^2 \,dv_g \leqslant c_{25}(\Lambda)R_A^{-1},$$
$$\int_{B(aR_A)} \left|dh^2\right|_g \,dv_g \leqslant c_{28}(\Lambda)R_A^{-1},$$
$$\int_{B(aR_A)} \left|d\left(\zeta h^2\right)\right|_g \,dv_g \leqslant c_{29}(\Lambda)R_A^{-1},$$
$$\left(\int_{B(7R_A/8)} h^4 \,dv_g\right)^{1/2} \leqslant c_{30}(\Lambda)\left\{R_A^{-1} + B_D \int_{B(R_A)} h^2 \,dv_g\right\}.$$

Using the definition of B_D,

$$B_D h^2 = \begin{cases} Hh^2 & \text{if } c \leqslant 0,\\ \sqrt{c+H^2}h^2 & \text{if } c > 0.\end{cases}$$

In the case $c \leqslant 0$ we have, using $H^2 \leqslant 2(H^2+c)$,

$$h = \left(\left|A^0\right|^2 + 2\left(c+H^2\right) + \Lambda R^2\right)^{1/2} \geqslant H.$$

In the case $c > 0$, we have $h \geqslant \sqrt{c+H^2}$. These inequalities imply that $B_D h^2 \leqslant h^3$. Using Lemma 5.9, Assertion (iii), we obtain

$$\int_{B(7R_A/8)} h^4 \,dv_g \leqslant c_{31}(\Lambda)R_A^{-2}.$$

Finally, we have obtained the estimate

$$\left(\int_{B(3R_A/4)} h^{2p} \,dv_g\right)^{1/p} \leqslant c_{32}(p,\Lambda)R_A^{-2+2/p} \tag{76}$$

and we can now apply Lemma 2.1 to obtain the estimate

$$\sup_{B(R_A/2)} h^2 \leqslant c_{33} R_A^{-2}$$

from which Assertion (B) follows in view of the Gauss equation (35). □

REFERENCES

[1] J.M.L. BARBOSA and P. BÉRARD, On a twisted eigenvalue problem, Preprint, 1998.
[2] R. BRYANT, Surfaces of mean curvature one in hyperbolic space, *Astérisque* 154–155 (1987) 321–347.
[3] P. BÉRARD, Simons' equation revisited, *An. Acad. bras. Ci.* 66 (1994) 397–403.
[4] P. CASTILLON, Sur l'opérateur de stabilité des sous-variétés à courbure moyenne constante dans l'espace hyperbolique, *Manuscr. Math.* 94 (1997) 385–400.

[5] L.-F. CHEUNG, Geometric properties of stable noncompact constant mean curvature surfaces, Dissertation, Rheinische Friedrich-Wilhems-Universität Bonn, 1990. *Bonner Math. Schriften*, 220. Universität Bonn, Math. Institut, Bonn (1991) iv+65 pp.

[6] K. ECKER and G. HUISKEN, Interior curvature estimates for hypersurfaces of prescrided mean curvature, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 6 (1989) 251–260.

[7] D. FISCHER-COLBRIE, On complete minimal surfaces with finite Morse index in three manifolds, *Inventiones Math.* 82 (1987) 121–132.

[8] D. FISCHER-COLBRIE and R. SCHOEN, The structure of complete minimal surfaces in 3-manifolds of non-negative scalar curvature, *Comm. Pure Appl. Math.* 33 (1980) 199–211.

[9] R. FREIRE DE LIMA, Thesis in progress, IMPA, 1998.

[10] E. HEINZ, Über die Lösungen der Minimalflächengleichung, *Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. II* (1952) 51–56.

[11] D. HOFFMAN and J. SPRUCK, Sobolev and isoperimetric inequalities for Riemannian submanifolds, *Comm. Pure Appl. Math.* 27 (1974) 715–727.

[12] H.B. LAWSON, Complete minimal surfaces in S^3, *Annals of Math.* 92 (1970) 335–374.

[13] R. OSSERMAN, An extension of certain results in function theory to a class of surfaces, *Proc. Nat. Acad. Sci.* 45 (1959) 1031–1035.

[14] H. ROSENBERG, Private communication, 1998.

[15] F. SAUVIGNY, A priori estimates of the principal curvatures for immersions of prescribed mean curvature and a theorem of Bernstein type, *Math. Z.* 205 (1990) 567–582.

[16] R. SCHOEN, Estimates for stable minimal surfaces in three dimensional manifolds, in: E. Bombieri (Ed.), *Seminar on Minimal Submanifolds*, Annals of Math. Studies, Princeton University Press, 1983.

[17] M. SPIVAK, *A Comprehensive Introduction to Differential Geometry*, Vol. IV (2nd ed.), Publish or Perish Inc., 1975.

[18] R. SCHOEN, L. SIMON and S.-T. YAU, Curvature estimates for minimal hypersurfaces, *Acta Math.* 134 (1975) 275–288.

[19] A. DA SILVEIRA, Stability of complete noncompact surfaces with constant mean curvature, *Math. Ann.* 277 (1987) 629–638.

[20] J. SIMONS, Minimal varieties in Riemannian manifolds, *Annals of Math*. 88 (1968) 62–105.

[21] J. SPRUCK, Gauss curvature estimates for surfaces of constant mean curvature, *Comm. Pure Appl. Math*. 27 (1974) 547–557.

[22] M. TRAIZET, On stable surfaces of constant Gauss curvature in spaces forms, *Annals Global Anal. Geom.* 13 (1995) 141–148.