Spectral problems on Riemannian manifolds

Pierre Bérard

Université Joseph Fourier - Grenoble

GEOMETRIAS GÉOMÉTRIES

IMPA, April 13-15,2009

Introduction to the spectrum

Let (M, g) be a compact Riemannian manifold (possibly with boundary). We consider the Laplacian on M, acting on functions,

$$\Delta_g(f) = \delta_g(df),$$

where δ_g is the divergence operator on 1-forms.

The divergence of a 1-form ω is given by

$$\delta_g(\omega) = \sum_{j=1}^{n} (D^g_{E_j}\omega)(E_j) = \sum_{j=1}^{n} \left[E_j \cdot \omega(E_j) - \omega(D^g_{E_j} E_j) \right],$$

where $\{E_j\}_{j=1}^n$ a local orthonormal frame.

In a local coordinate system $\{x_j\}_{j=1}^n$, the Laplacian is given by

$$\Delta f = -\frac{1}{v_g(x)} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(v_g(x) g^{ij}(x) \frac{\partial f}{\partial x_j}),$$

where $(g^{ij}(x))$ is the inverse matrix $(g_{ij}(x))^{-1}$, the $g_{ij}(x) = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are the coefficients of the Riemannian metric in the local coordinates, and $v_g(x) = (\mathrm{Det}(g_{ij}(x)))^{1/2}$.

In local coordinates, the Riemannian measure dv_g on (M, g) is given by

$$dv_g = v_g(x)\, dx_1 \ldots dx_n.$$

We are interested in the eigenvalue problem for the Laplacian on (M, g), *i.e.* in finding the pairs (λ, u), where λ is a (real) number and u a non-zero function, such that

$$\Delta u = \lambda u$$

and, when M has a boundary ∂M, $u|\partial M = 0$ (Dirichlet eigenvalue problem).

We have the following theorem.

Theorem

Let (M, g) be a compact Riemannian manifold. Then there exist a sequence $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ of non-negative real numbers with finite multiplicities, and an $L^2(M, dv_g)$-orthonormal basis $\{\varphi_1, \varphi_2, \ldots, \varphi_k, \ldots\}$ of real C^∞ functions such that $\Delta\varphi_j = \lambda_j \varphi_j$, and $\varphi_j|\partial M = 0$ if M has a boundary.

The set $\sigma(M, g) = \{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\}$ is called the spectrum of the Riemannian manifold (M, g) (Dirichlet spectrum, if M has a boundary). This is a Riemannian invariant (*i.e.* two isometric Riemannian manifolds have the same spectrum).

The main questions addressed by spectral geometry are the following.

- Given a compact Riemannian manifold (M, g), can one describe $\sigma(M, g)$?
- What information on $\sigma(M, g)$ can one draw from geometric information on (M, g) ?
- What geometric information on (M, g) can one draw from $\sigma(M, g)$?

By *information on* $\sigma(M, g)$, we mean bounds on the eigenvalues, their asymptotic behaviour, *etc*.

By *information on* (M, g), we mean bounds on curvature, on the volume, on the diameter, *etc*.

Given (M, g), describe $\sigma(M, g)$. Two examples.

- Flat tori. Let Γ be a lattice in $\mathbb{R}^n$, $\Gamma^\star$ the dual lattice and let $T_\Gamma = \mathbb{R}^n/\Gamma$ be the corresponding flat torus. Then,

$$\sigma(T_\Gamma) = \{4\pi^2 \|\gamma^\star\|^2 \mid \gamma^\star \in \Gamma^\star\},$$

  with associated eigenfunctions $(\mathrm{Vol}(T_\Gamma))^{-1/2} e^{2i\pi\langle\gamma^\star, x\rangle}$.

- Round spheres. Let S^2 be the unit sphere in $\mathbb{R}^3$, with induced metric. Then,

$$\sigma(S^2) = \{k(k+1), \ \text{ with multiplicity } \ 2k+1 \mid k \in \mathbb{N}\}.$$

  The associated eigenfunctions are the restrictions to the sphere of harmonic homogeneous polynomials in $\mathbb{R}^3$.

Heat kernel

As an introduction to Gérard Besson's lectures, I will first discuss some results obtained using the heat equation.

Let (M, g) be a closed (*i.e.* compact without boundary) n-dimensional Riemannian manifold.

We are interested in solving the Cauchy problem for the heat equation,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \Delta_x u(t,x) & = & 0 \\ u(0,x) & = & f(x) \end{cases}$$

where f is a given continuous function on M.

One can prove that the solution $u(t,x)$ is given by the formula

$$u(t,x) = \int_M k_M(t,x,y)\, dv_g(y),$$

where $k_M(t,x,y) \in C^\infty(\mathbb{R}_+^\bullet \times M \times M)$ is the so-called fundamental solution of the heat equation (or heat kernel) of M, given by the formula

$$k_M(t,x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y)$$

where the series converges for $t > 0, x, y \in M$.

Heat kernel, proofs

Let (M, g) be a closed Riemannian manifold.

Definition

A fundamental solution of the heat equation is a function $k : \mathbb{R}_+^\bullet \times M \times M \to \mathbb{R}$ *with the following properties.*

1. *The function* k *is in* $C^0(\mathbb{R}_+^\bullet \times M \times M)$*. It admits one derivative with respect to the first variable and first and second derivatives in the third variable and these derivatives are in* $C^0(\mathbb{R}_+^\bullet \times M \times M)$*.*
2. *The function* k *satisfies the equation*

$$\frac{\partial k}{\partial t} + \Delta_y k = 0, \text{ in } \mathbb{R}_+^\bullet \times M \times M.$$

3. *For any* $f \in C^0(M), \lim_{t\to 0_+} \int_M k(t,x,y) f(y)\, dv_g(y) = f(x)$*.*

Uniqueness of the heat kernel

Let (M, g) be a closed Riemannian manifold, and let $\{\varphi_j, j \geq 1\}$ be an orthonormal basis of eigenfunctions of the Laplacian, with associated eigenvalues $\lambda_j, j \geq 1$.

Proposition (Gaffney)

Assume that (M, g) admits a heat kernel k. Then, the series

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} \varphi_j(x) \varphi_j(y)$$

converges for all $(t, x, y) \in \mathbb{R}_+^\bullet \times M \times M$ and its sum is $k(t, x, y)$. As a consequence, for all $x, y \in M$, one has $k(t, x, y) = k(t, y, x)$.

Corollary

The series $\sum_{j=1}^{\infty} e^{-\lambda_j t}$ *converges for all* $t > 0$ *and its sum is equal to* $\int_M k(t,x,x)\, dv_g(x)$.

Parametrix

Let (M, g) be a closed Riemannian manifold.

Definition

A parametrix for the heat equation is a function $p : \mathbb{R}_+^\bullet \times M \times M \to \mathbb{R}$ with the following properties.

1. *The function p is in $C^\infty(\mathbb{R}_+^\bullet \times M \times M)$.*
2. *The function $\dfrac{\partial p}{\partial t} + \Delta_y p$ extends to a function in $C^0(\mathbb{R}_+ \times M \times M)$.*
3. *For any $f \in C^0(M), \lim_{t \to 0_+} \int_M p(t,x,y) f(y)\, dv_g(y) = f(x)$.*

Duhamel's principle

Let $k(t,x,y)$ be the fundamental solution (assuming it exists). Given $u \in C^0(M)$, the function

$$u(t,x) = \int_M k(t,x,y) u_0(y)\, dv_g(y)$$

solves the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Delta u & = 0, \\ u_0(0,\cdot) & = u_0. \end{cases}$$

The function

$$v(t,x) = \int_0^t u(\tau,x)\,d\tau = \int_0^t \Big(\int_M k(\tau,x,y)u_0(y)\,dv_g(y)\Big)\,d\tau$$

solves the inhomogeneous Cauchy problem

$$\begin{cases} \dfrac{\partial v}{\partial t} + \Delta u &= u_0, \\ u_0(0,\cdot) &= 0. \end{cases}$$

Assume that $p(t,x,y)$ is a parametrix of the heat equation. Then,

$$\begin{cases} \dfrac{\partial p}{\partial t} + \Delta_y p & = q, \\ \lim_{t\to 0_+} p(t,x,\cdot) & = \delta_x, \end{cases}$$

where $q \in C^\infty(\mathbb{R}_+^\bullet \times M \times M) \cap C^0(\mathbb{R}_+ \times M \times M)$.

Consider $\tilde{p}(t,x,y) = k(t,x,y) - p(t,x,y)$. This function satisfies

$$\frac{\partial \tilde{p}}{\partial t} + \Delta_y \tilde{p} = -q, \;\; \tilde{p}(0,\cdot,\cdot) = 0.$$

Using Duhamel's principle, we have that

$$k(t,x,y) + \int_0^t \left(\int_M q(\tau,x,z)k(t-\tau,z,y)\, dv_g(z) \right) d\tau = p(t,x,y),$$

or $(I + T)(k) = p$, where the operator T is defined by

$$\begin{aligned} T(a)(t,x,y) &=: q \star a(t,x,y) \\ &:= \textstyle\int_0^t \left(\int_M q(\tau,x,z)a(t-\tau,z,y)\, dv_g(z) \right) d\tau. \end{aligned}$$

It follows that the heat kernel can be expressed in terms of a parametrix,

$$k = \sum_{j=0}^{\infty} (-1)^j q^{\star j} \star p.$$

The next step is to look for a parametrix in the form

$$\begin{aligned} p_k(t, x, y) = \quad & (4\pi t)^{-n/2} e^{-d^2(x,y)/4t} \eta(x, y) \left(u_0(x, y) + t\, u_1(x, y)\right. \\ & \left. + \cdots + t^k\, u_k(x, y)\right), \end{aligned}$$

where the functions $u_j(x, y)$ are defined inductively in such a way that $\dfrac{\partial p_k}{\partial t} + \Delta_y p$ has lowest possible order in t as t goes to 0_+.

We introduce the so-called partition function

$$Z_M(t) = \int_M k_M(t,x,x)\, dv_g(x) = \sum_{i=1}^{\infty} e^{-\lambda_i t}.$$

Giving the function $Z_{(M,g)}(t)$ is equivalent to giving $\sigma(M,g)$.

That this function (and hence the spectrum) carries interesting geometric information is already apparent in Poisson's formula.

Asymtotic formulas

Poisson formula

Let Γ be a lattice in $\mathbb{R}^n$ and let $\Gamma^\star$ be the dual lattice. Then,

$$Z_{T_\Gamma}(t) = \sum_{\gamma^\star \in \Gamma^\star} e^{-4\pi^2 \|\gamma^\star\|^2 t} = (4\pi t)^{-n/2} \mathrm{Vol}(T_\Gamma) \sum_{\gamma \in \Gamma} e^{-\|\gamma\|^2/4t}.$$

Minakshisundaram-Pleijel formula

There exist coefficients $a_j(M)$ such that, when t tends to 0_+,

$$Z_M(t) \sim (4\pi t)^{-n/2} \{\mathrm{Vol}(M) + a_1(M)t + \cdots + a_k(M)t^k + O(t^{k+1})\}.$$

Weyl's formula

When j tends to infinity, the eigenvalues of a compact n-dimensional manifold have the following asymptotic behaviour

$$\lambda_j \sim 4\pi^2 \left(\frac{\mathrm{Vol}(B_n)}{\mathrm{Vol}(M)}\right)^{2/n} j^{2/n}.$$

Embedding Riemannian manifolds by their heat kernel

Let ℓ^2 be the Hilbert space of real sequences $\{a_i\}_{i\geq 1}$ such that $\sum a_i^2 < \infty$.

Definition

Given a closed n-dimensional Riemannian manifold M and an orthonormal basis $\mathcal{A}$ of eigenfunctions of the Laplacian of M, one defines a family of maps

$$\Psi_t^a : M \to \ell^2 \quad for \quad t > 0,$$

by

$$x \to \sqrt{2}(4\pi)^{n/4} t^{(n+2)/4} \{e^{-\lambda_j t/2} \varphi_j^{\mathcal{A}}(x)\}_{j\geq 2}$$

(notice that we have suppressed the constant eigenfunction for convenience).

Theorem (P. B. - G. Besson - S. Gallot)

Fix a closed n-dimensional Riemannian manifold (M, g) and an orthonormal basis $\mathcal{A}$ of eigenfunctions of its Laplacian. Let can *denote the Euclidean scalar product on ℓ^2 .*

- *For all positive t, the map $\Psi_t^{\mathcal{A}}$ is an embedding of M into ℓ^2.*
- *The pulled–back metric $(\Psi_t^{\mathcal{A}})^* \mathrm{can}$ is asymptotic to the metric g of M when t goes to zero. More precisely,*

$$(\Psi_t^{\mathcal{A}})^* \mathrm{can} = g + \frac{t}{3}(\frac{1}{2}\mathrm{Scal}_g \cdot g - \mathrm{Ric}_g) + O(t^2)$$

when $t \to 0_+$ (Scal_g is the scalar curvature and Ric_g the Ricci curvature tensor of the metric g).

An application

Theorem (PB)

Let (M, g) be an n-dimensional closed Riemannian manifold and let $\{\varphi_j, j \geq 1\}$ be an orthonormal basis of eigenfunctions of the Laplacian. Let $N(\lambda) = \mathrm{Card}\{j \geq 1 \mid \lambda_j \leq \lambda\}$. Let $a = (a_1, \dots, a_{N(\lambda)}) \in \mathbb{R}P^{N(\lambda)-1}$. Consider the function $\Phi_a = \sum_{j=1}^{N(\lambda)} a_j\varphi_j$ and let $\mathcal{Z}_a = \Phi_a^{-1}(0)$ be the nodal set of Φ_a. Then

$$\frac{1}{\mathrm{Vol}(\mathbb{R}P^{N(\lambda)-1})} \int_{\mathbb{R}P^{N(\lambda)-1}} \mathrm{Vol}_{n-1}(\mathcal{Z}_a)\, da \sim \frac{\mathrm{Vol}(S^{n-1})\mathrm{Vol}(M, g)}{\sqrt{n+2}\mathrm{Vol}(S^n)}\sqrt{\lambda}$$

when λ tends to infinity.

Lower bounds on the eigenvalues

Theorem (P.B. - G. Besson - S. Gallot)

Let (M, g) be a closed n-dimensional Riemannian manifold. Define $r_{\min}(M) = \inf\{\mathrm{Ric}(u, u) \mid u \in UM\}$ and let $d(M)$ be the diameter of M. Assume that (M, g) satisfies $r_{\min}(M)d(M)^2 \geq (n-1)\varepsilon\alpha^2$ for some $\varepsilon \in \{-1, 0, 1\}$ and some positive number α. Then, there exists a real number $R = a(n, \varepsilon, \alpha)d(M)$, such that

$$\mathrm{Vol}(M)k_M(t, x, x) \leq Z_{S^n(1)}(t/R^2).$$

As a consequence of the previous theorem, we have the following estimates for the eigenvalues and eigenfunctions of a closed n-dimensional Riemannian manifold (M, g) such that $\mathrm{Ric}_g \geq (n-1)kg$ and $d(M) \leq D$.

There exist explicit constants $A(n, k, D)$ and $B(n, k, D)$ such that

$$\begin{cases} \lambda_j(M, g) & \geq \quad A(n, k, D)\, j^{2/n} \\ \mathrm{Vol}(M, g) \sum_{\lambda_j \leq \lambda} \varphi_j^2(x) & \leq \quad B(n, k, D)\lambda^{n/2}. \end{cases}$$

Generalized Faber-Krahn inequality

Theorem (P. B. - G. Besson - S. Gallot)

Let (M, g) be a closed n-dimensional Riemannian manifold such that $r_{\min}(M)d(M)^2 \geq (n-1)\varepsilon\alpha^2$ for some $\varepsilon \in \{-1, 0, 1\}$ and some positive number α. Then, there exists a real number $b(n, \varepsilon, \alpha)$, such that

$$\lambda_2(M, g) \geq b(n, \varepsilon, \alpha)d(M)^{-2}.$$

Another embedding

We define another embedding of a closed n-dimensional Riemannian manifold (M, g). Given some $t > 0$ and an orthonormal basis $\mathcal{A}$ of eigenfunctions of the Laplacian, let

$$I_t^{\mathcal{A}}(x) = \{\sqrt{\mathrm{Vol}(M)}\, e^{-\lambda_j t/2}\, \varphi_j^{\mathcal{A}}(x)\}_{j\geq 2}.$$

We also introduce the set

$$\mathcal{M}_{n,k,D} = \Big\{(M,g) \big|\, \dim M = n, \mathrm{Ric}_g \geq (n-1)kg, \mathrm{Diam}(M) \leq D\Big\}$$

of closed Riemannian manifolds.

Theorem (P. B. - G. Besson - S. Gallot)

Define

$$d_t(M,M') = \max\{ \sup_{\mathcal{A}\in\mathcal{B}(M)} \inf_{\mathcal{A}'\in\mathcal{B}(M')} \mathrm{HD}(I_t^{\mathcal{A}}(M), I_t^{\mathcal{A}'}(M')),$$
$$\sup_{\mathcal{A}'\in\mathcal{B}(M')} \inf_{\mathcal{A}\in\mathcal{B}(M)} \mathrm{HD}(I_t^{\mathcal{A}}(M), I_t^{\mathcal{A}'}(M'))\},$$

where HD *is the Hausdorff distance between subsets of* ℓ^2. *Then,*

- *For all* $t > 0$, d_t *is a distance between isometry classes of Riemannian manifolds.*
- *For any* $t > 0$, *the space* $\mathcal{M}_{n,k,D}$ *is* d_t*-precompact.*

Variational method and applications

Let (M, g) be a compact, connected Riemannian manifold (M, g).

Notations.

- We denote by L the real vector space $L^2(M, dv_g)$ with the scalar product $\langle u, w\rangle_0 = \int_M u\, w\, dv_g(x)$ and associated norm $\|u\|_0$.
- We denote by H the completion of $C_0^\infty(M, \mathbb{R})$ for the norm $\|u\|_1$ associated with the scalar product $\langle u, w\rangle_1 = \int_M \left(g^\star(du, dw) + u\, w\right) dv_g$.

When $\partial M = \emptyset$, the Hilbert space H is the space $H^1(M, g)$. When $\partial M \neq \emptyset$, this is the space $H^1_0(M, g)$ of H^1-functions whose trace on the boundary is zero.

- For $u \in H$, we denote by $D(u)$ the Dirichlet integral $D(u) = \int_M g^\star(du, dw)\, dv_g$.
- Finally, for $u \in H \setminus \{0\}$, we define the Rayleigh quotient of u by $\dfrac{D(u)}{\|u\|_0^2}$.

One can prove the existence of eigenvalues and eigenfunctions inductively by minimizing the Rayleigh quotient on a chain of Hilbert spaces starting from H.

Variational method, proofs

Recall our first theorem.

Theorem

Let (M, g) be a compact Riemannian manifold. Then there exist a sequence $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ of non-negative real numbers with finite multiplicities, and an $L^2(M, dv_g)$-orthonormal basis $\{\varphi_1, \varphi_2, \ldots \varphi_k, \ldots\}$ of real C^∞ functions such that $\Delta\varphi_j = \lambda_j \varphi_j$ and $\varphi_j | \partial M = 0$ if M has a boundary.

We now sketch a proof of this theorem using the variational method.

Lemma

Let A be a closed subspace of H. Then, the infimum

$$\mu_A = \inf\{D(u) \mid u \in A, \|u\|_0 = 1\} = \inf\{R(u) \mid u \in A \setminus \{0\}\}$$

exists and is achieved on a finite dimensional subspace $E_A \subset H$ which is characterized by

$$u \in E_A \Leftrightarrow \forall v \in H,\ D(u, v) = \mu_A \langle u, v\rangle_0.$$

- The first eigenvalue is obtained by choosing $A = H$ in the previous lemma.
- Higher eigenvalues.
- Other statements.

Min-max and Max-min

Theorem

Let $\mathcal{G}_k$ be the set of k-dimensional subspaces in H.

The eigenvalues satisfy the min-max principle.

$$\lambda_k = \inf_{F \in \mathcal{G}_k} \sup\{D(u) \mid u \in F, \|u\|_0 = 1\}.$$

The eigenvalues satisfy the max-min principle.

$$\lambda_1 = \inf\{D(u) \mid u \in H, \|u\|_0 = 1\} \text{ and, for } k \geq 2$$

$$\lambda_k = \sup_{F \in \mathcal{G}_{k-1}} \inf\{D(u) \mid u \in F, \|u\|_0 = 1, \langle u, F \rangle_0\}.$$

The least eigenvalue has a remarkable property. Any eigenfunction associated with λ_1 does not vanish in the interior of M, the corresponding eigenspace E_1 has dimension 1, any eigenfuction which does not vanish in the interior of M must be associated with the least eigenvalue λ_1.

Monotonicity principle

Proposition

Let (M, g) be a Riemannian manifold and let $\Omega_1 \subset \Omega_2 \subset M$ be two relatively compact domains. Then

$$\lambda_1^D(\Omega_1) \geq \lambda_1^D(\Omega_2)$$

and strict inequality holds if the interior of $\Omega_2 \setminus \Omega_1$ is not empty.

Courant's nodal domain theorem

Theorem

Let u be an eigenfunction associated with the k-th eigenvalue. Then the number of nodal domains of u (i.e. of connected components of $M \setminus u^{-1}(0)$) is at most k.

Eigenvalue comparison theorems

Theorem (S.Y. Cheng)

Let (M, g) be any complete n-dimensional Riemannian manifold such that $\mathrm{Ric}_g \geq (n-1)kg$. Then, for any $x \in M$ and any $R > 0$, one has

$$\lambda_1^D(B(x, R)) \leq \lambda_1^D(B_k(R)),$$

where $B_k(R)$ denotes a ball with radius R in the simply-connected n-dimensional model manifold with constant sectional curvature k. Furthermore, equality holds if and only if $B(x, R)$ is isometric to $B_k(R)$.

Theorem (S.Y. Cheng, M. Gromov)

Let (M, g) be a closed n-dimensional Riemannian manifold such that $\mathrm{Ric}_g \geq (n-1)kg$. Let $\lambda(k, \epsilon) = \lambda_1^D(B_k(\epsilon))$. Then, for all $\epsilon > 0$ and all $j \leq \mathrm{Vol}(M)/\mathrm{Vol}(B_k(2\epsilon)$, $\lambda_j \leq \lambda(k, \epsilon)$. In particular, there exists a constant $C(n, k)$ such that

$$\lambda_j \prec C(n,k) \, (\frac{j}{\mathrm{Vol}(M)})^{2/n}$$

when j tends to infinity.

Note that this estimate is coherent with Weyl's asymptotic estimate.

Geometric operators

We have so far only considered the Laplacian on a closed Riemannian manifold. One can consider other interesting geometric operators. Of particular interest is the Jacobi operator associated with the second variation of the area of minimal or constant mean curvature hypersurfaces.

Let $M^n \looparrowright \widehat{M}^{n+1}$ be a complete minimal orientable hypersurface immersed into some Riemannian manifold $\widehat{M}$. Let N_M be a unit normal field along the immersion and let A_M be the associated second fundamental form. The Jacobi operator of the immersion is the operator

$$J_M = \Delta_M - (\widehat{\mathrm{Ric}}(N_M, N_M) + \|A_M\|^2).$$

Let $\lambda_1^D(J_M, \Omega)$ denote the least eigenvalue of the operator J_M with Dirichlet boundary conditions in Ω.

We say that the domain Ω is stable if $\lambda_1^D(J_M, \Omega) > 0$ and weakly stable if $\lambda_1^D(J_M, \Omega) \geq 0$.

The index of the domain Ω is the number of negative eigenvalues of the operator J_M in Ω with Dirichlet boundary condition.

Positivity and applications

Theorem (I. Glazman / D. Fischer-Colbrie - R. Schoen)

Let (M, g) be a complete Riemannian manifold and let $q : M \to \mathbb{R}$ be a smooth function. For a relatively compact domain $\Omega \subset M$, let $\lambda_1(\Omega)$ be the least eigenvalue of the operator $\Delta + q$ in Ω, with Dirichlet boundary condition. The following assertions are equivalent.

1. *For all $\Omega \Subset M$, $\lambda_1(\Omega) \geq 0$.*
2. *For all $\Omega \Subset M$, $\lambda_1(\Omega) > 0$.*
3. *There exists a positive function u on M such that $\Delta u + qu = 0$.*

Complete metrics on the unit disk

Theorem (D. Fischer-Colbrie – R. Schoen)

Let $(\mathbb{D}, g)$ *be the unit disk equiped with a metric* $g = \mu g_e$ *conformal to the Euclidean metric. Let* K *and* Δ *denote respectively the Gauss curvature and the Laplacian for the metric* g *(then* $2K = \Delta \ln \mu$*).*

If g *is complete, then for any* $a \geq 1$*, there is no positive solution of the equation* $(\Delta + aK)f = 0$ *on* $\mathbb{D}$*.*

As a matter of fact, one has the following result (which has been improved by Ph. Castillon).

Proposition

Let $g = \mu g_e$ a complete conformal metric on the unit disk $\mathbb{D}$. Then, there exists some numer $0 \leq a_0(g) < 1$ such that

- *for $a \leq a_0$, there exist no positive solution to $(\Delta + aK)f = 0$ on $\mathbb{D}$,*
- *for $a > a_0$, there exists a positive solution to $(\Delta + aK)f = 0$ on $\mathbb{D}$.*

Corollary

Let g be a complete conformal metric on the unit disk $\mathbb{D}$. For $a \geq 1$ (a constant) and for $p \geq 0$ (a function on $\mathbb{D}$), there exist no positive solution to $(\Delta + aK)f = pf$.

Stable minimal surfaces in $\mathbb{R}^3$

Theorem (M. do Carmo - C. Peng / D. Fischer-Colbrie - R. Schoen)

The only complete oriented stable minimal surface in $\mathbb{R}^3$ is the plane.

Jacobi fields

Let J_M be the Jacobi operator of a minimal hypersurface $M \looparrowright \widehat{M}$. A Jacobi field is a function u on M such that $J_M(u) = 0$.

The geometry provides natural Jacobi fields. Indeed, we have the following properties.

1. Let $\mathcal{K}$ be a Killing field in $\widehat{M}$. Then the function $u_{\mathcal{K}} = \widehat{g}(\mathcal{K}, N_M)$ is a Jacobi field on M.
2. Let $\psi_a : M \looparrowright \widehat{M}$ be a family of minimal immersions. Then the function $v_a = \widehat{g}(\frac{d\psi_a}{da}, N_M)$ is a Jacobi field on M.

Catenoïds in $\mathbb{R}^3$

Part of the proof of the positivity theorem can be restated as the following corollary.

Corollary

Let Ω be any bounded open domain. Assume that there exists a positive function u on Ω such that $(\Delta + q)u = 0$. Then, $\lambda_1(\Omega) \geq 0$.

As an application, we can prove Lindelöf's theorem for catenoïds in $\mathbb{R}^3$.

Theorem

One can characterize the maximal rotation-invariant stable domains of the catenoids as generated by the arcs of the catenary whose end-points have tangents meeting on the axis of the catenary. In particular, the half vertical catenoïd $x^2 + y^2 = \cosh^2(z), z \geq 0$ *is a maximal weakly stable rotation invariant domain.*

Observe that the catenoïd $x^2 + y^2 = \cosh^2(z)$ has index 1.

Generalizations

The same analysis can be applied to catenoids in $\mathbb{H}^2 \times \mathbb{R}$ or in $\mathbb{H}^3$ and to their higher dimensional analogues, with the occurence of interesting phenomena. (work in progress PB - R. Sá Earp).

I would like to take this opportunity to thank IMPA for their support and for hosting the main part of this event,

and to thank the sponsors: Programme ARCUS Brésil (Région Rhône-Alpes – MAE), CNRS, IMPA, Université Joseph Fourier.

Thank you for your attention.

Pierre.Berard@ujf-grenoble.fr