Entire curves and algebraic differential equations

Jean-Pierre Demailly

Institut Fourier, Université de Grenoble I, France

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- $X = \overline{\mathbb{C}} \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$ has no entire curves (Picard's theorem)
- A complex torus $X = \mathbb{C}^n/\Lambda$ (Λ lattice) has a lot of entire curves. As $\mathbb{C}$ simply connected, every $f : \mathbb{C} \to X = \mathbb{C}^n/\Lambda$ lifts as $\tilde{f} : \mathbb{C} \to \mathbb{C}^n$,

$$\tilde{f}(t) = (\tilde{f}_1(t), \ldots, \tilde{f}_n(t))$$

and $\tilde{f}_j : \mathbb{C} \to \mathbb{C}$ can be arbitrary entire functions.

Projective algebraic varieties

- Consider now the complex projective n-space

$$\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*, \qquad [z] = [z_0 : z_1 : \ldots : z_n].$$

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- An entire curve $f : \mathbb{C} \to \mathbb{P}^n$ is given by a map

$$t \longmapsto [f_0(t) : f_1(t) : \ldots : f_n(t)]$$

where $f_j : \mathbb{C} \to \mathbb{C}$ are holomorphic functions without common zeroes (so there are a lot of them).

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- More generally, look at a (complex) projective manifold, i.e.

$$X^n \subset \mathbb{P}^N, \qquad X = \{[z]\,;\ P_1(z) = ... = P_k(z) = 0\}$$

  where $P_j(z) = P_j(z_0, z_1, \ldots, z_N)$ are homogeneous polynomials (of some degree d_j), such that X is non singular.

Kobayashi metric / hyperbolic manifolds

- For a complex manifold, $n = \dim_{\mathbb{C}} X$, one defines the Kobayashi pseudo-metric : $x \in X$, $\xi \in T_X$

$$\kappa_x(\xi) = \inf\{\lambda > 0\,;\ \exists f : \mathbb{D} \to X,\ f(0) = x,\ \lambda f_*(0) = \xi\}$$

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- **Theorem.** (Brody) If X is compact then X is Kobayashi hyperbolic if and only if there are no entire holomorphic curves $f : \mathbb{C} \to X$ (Brody hyperbolicity).

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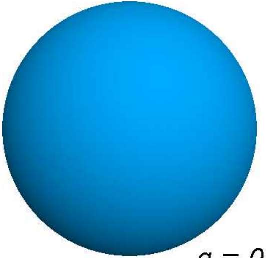
- **Theorem.** (Brody) If X is compact then X is Kobayashi hyperbolic if and only if there are no entire holomorphic curves $f : \mathbb{C} \to X$ (Brody hyperbolicity).
- Hyperbolic varieties are especially interesting for their expected diophantine properties :

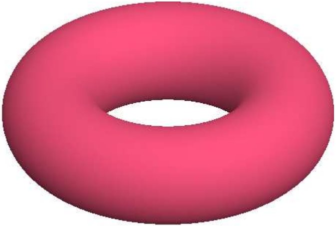
  Conjecture (S. Lang) *If a projective variety X defined over $\mathbb{Q}$ is hyperbolic, then $X(\mathbb{Q})$ is finite.*

Complex curves ($n = 1$) : genus and curvature



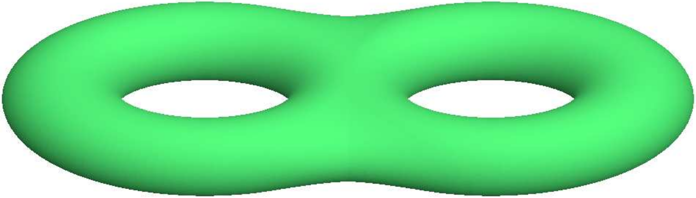
$g = 0\,,\ K_X < 0$

(positive curvature)



$g = 1\,,\ K_X = 0$

(zero curvature)



$g > 1\,,\ K_X > 0$

(negative curvature)

$K_X = \Lambda^n T^*_X, \quad \deg(K_X) = 2g - 2$

Curves : hyperbolicity and curvature

- Case $n = 1$ (compact Riemann surfaces):

$$\begin{array}{ll} X = \mathbb{P}^1 & (g = 0, \quad T_X > 0) \\ X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) & (g = 1, \quad T_X = 0) \end{array}$$

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- If $g \geq 2$, $X \simeq \mathbb{D}/\Gamma$ $(T_X < 0)$, then X hyperbolic.
- **The *n*-dimensional case** (Kobayashi)
  If T_X is negatively curved ($T_X^* > 0$, i.e. ample), then X is hyperbolic.
  Recall that a holomorphic vector bundle E is ample iff its symmetric powers $S^m E$ have global sections which generate 1-jets of (germs of) sections at any point $x \in X$.
- **Examples :** $X = \Omega/\Gamma$, Ω bounded symmetric domain.

Varieties of general type

- **Definition** *A non singular projective variety X is said to be of general type if the growth of pluricanonical sections*

$$\dim H^0(X, K_X^{\otimes m}) \sim cm^n, \qquad K_X = \Lambda^n T_X^*$$

is maximal.

(sections locally of the form $f(z)\,(dz_1 \wedge \ldots \wedge dz_n)^{\otimes m}$)

Example: A non singular hypersurface $X^n \subset \mathbb{P}^{n+1}$ of degree d satisfies $K_X = \mathcal{O}(d-n-2)$, it is of general type iff $d > n+2$.

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- **Conjecture GT.** *If a compact manifold X is hyperbolic, then it should be of general type, i.e. $K_X = \Lambda^n T_X^*$ should be of positive curvature ($Ricci < 0$, possibly with some degeneration).*

Conjectural characterizations of hyperbolicity

- **Theorem.** *Let X be projective algebraic. Consider the following properties :*
  (P1) *X is hyperbolic*
  (P2) *Every subvariety Y of X is of general type.*
  (P3) $\exists\varepsilon > 0$, $\forall C \subset X$ *algebraic curve*

  $$2g(\bar{C}) - 2 \geq \varepsilon \deg(C).$$

  (X *"algebraically hyperbolic"*)
  (P4) *X possesses a jet-metric with negative curvature on its k-jet bundle X_k [to be defined later], for $k \geq k_0 \gg 1$.*
  Then (P4) $\Rightarrow$ (P1), (P2), (P3),
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- It is expected that all 4 properties (P1), (P2), (P3), (P4) are equivalent for projective varieties.

Green-Griffiths-Lang conjecture

Conjecture (Green-Griffith-Lang = GGL) *Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.*

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- **Expected consequence** (of GT + GGL)
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  are equivalent.
- The main idea in order to attack GGL is to use differential equations. Let

$$\mathbb{C} \to X, \quad t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$$

be a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X.

Definition of algebraic differential operators

- Consider algebraic differential operators which can be written locally in multi-index notation

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\ f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k} \end{aligned}$$

where $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are holomorphic coefficients on X and $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet.

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$$\lambda \cdot f(t) = f(\lambda t), \quad (\lambda \cdot f)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$$

$\Rightarrow$ weighted degree $m = |\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k|$.

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- **Definition.** $E_{k,m}^{\mathrm{GG}}$ *is the sheaf (bundle) of algebraic differential operators of order k and weighted degree m.*

Vanishing theorem for differential operators

- **Fundamental vanishing theorem**
(Green-Griffiths '78, Demailly '95, Siu '96)
Let $P \in H^0(X, E^{\mathrm{GG}}_{k,m} \otimes \mathcal{O}(-A))$ *be a global algebraic differential operator whose coefficients vanish on some ample divisor* A*. Then for any* $f : \mathbb{C} \to X$, $P(f_{[k]}) \equiv 0$.

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- *Proof*. One can assume that A is very ample and intersects $f(\mathbb{C})$. Also assume f' bounded (this is not so restrictive by Brody !). Then all $f^{(k)}$ are bounded by Cauchy inequality. Hence

$$\mathbb{C} \ni t \mapsto P(f', f'', \ldots, f^{(k)})(t)$$

is a bounded holomorphic function on $\mathbb{C}$ which vanishes at some point. Apply Liouville's theorem ! □

Geometric interpretation of vanishing theorem

- Let $X_k^{\mathrm{GG}} = J_k(X)^*/\mathbb{C}^*$ be the projectivized k-jet bundle of X = quotient of non constant k-jets by $\mathbb{C}^*$-action. Fibers are weighted projective spaces.
  Observation. *If $\pi_k : X_k^{\mathrm{GG}} \to X$ is canonical projection and $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ is the tautological line bundle, then*

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- Saying that $f : \mathbb{C} \to X$ satisfies the differential equation $P(f_{[k]}) = 0$ means that

$$f_{[k]}(\mathbb{C}) \subset Z_P$$

  where Z_P is the zero divisor of the section

$$\sigma_P \in H^0(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}(-A))$$

  associated with P.

- **Consequence of fundamental vanishing theorem.**
If $P_j \in H^0(X, E_{k,m}^{\mathrm{GG}} \otimes \mathcal{O}(-A))$ *is a basis of sections then the image* $f(\mathbb{C})$ *lies in* $Y = \pi_k(\bigcap Z_{P_j})$*, hence property asserted by the GGL conjecture holds true if there are "enough independent differential equations" so that*

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- However, some differential equations are useless. On a surface with coordinates (z_1, z_2), a Wronskian equation $f_1' f_2'' - f_2' f_1'' = 0$ tells us that $f(\mathbb{C})$ sits on a line, but $f_2''(t) = 0$ says that the second component is linear affine in time, an essentially meaningless information which is lost by a change of parameter $t \mapsto \varphi(t)$.

Invariant differential operators

- The k-th order Wronskian operator

$$W_k(f) = f' \wedge f'' \wedge \ldots \wedge f^{(k)}$$

(locally defined in coordinates) has degree $m = \frac{k(k+1)}{2}$ and

$$W_k(f \circ \varphi) = \varphi'^m W_k(f) \circ \varphi.$$

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- **Definition.** *A differential operator P of order k and degree m is said to be invariant by reparametrization if*

$$P(f \circ \varphi) = \varphi'^m P(f) \circ \varphi$$

for any parameter change $t \mapsto \varphi(t)$. Consider their set

$$E_{k,m} \subset E^{\mathrm{GG}}_{k,m} \quad (\textit{a subbundle})$$

(Any polynomial $Q(W_1, W_2, \ldots W_k)$ is invariant, but for $k \geq 3$ there are other invariant operators.)

Category of directed manifolds

- **Definition.** *Category of directed manifolds :*
  - Objects are pairs (X, V) where X is a complex manifold and $V \subset T_X$ (subbundle or subsheaf)
  - Arrows $\psi : (X, V) \to (Y, W)$ are holomorphic maps with $\psi_* V \subset W$

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  - "Absolute case" (X, T_X)
  - "Relative case" $(X, T_{X/S})$ where $X \to S$
  - "Integrable case" when $[V, V] \subset V$ (foliations)

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- **Fonctor "1-jet" :** $(X, V) \mapsto (\tilde{X}, \tilde{V})$ where :

$$\tilde{X} = P(V) = \text{bundle of projective spaces of lines in } V$$

$$\pi : \tilde{X} = P(V) \to X, \quad (x, [v]) \mapsto x, \quad v \in V_x$$

$$\tilde{V}_{(x,[v])} = \{\xi \in T_{\tilde{X},(x,[v])} \,;\ \pi_* \xi \in \mathbb{C}v \subset T_{X,x}\}$$

Semple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$
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- **Basic exact sequences**

$$0 \to T_{\tilde{X}/X} \to \tilde{V} \xrightarrow{\pi_\star} \mathcal{O}_{\tilde{X}}(-1) \to 0 \qquad \Rightarrow \text{rk}\,\tilde{V} = r = \text{rk}\,V$$

$$0 \to \mathcal{O}_{\tilde{X}} \to \pi^\star V \otimes \mathcal{O}_{\tilde{X}}(1) \to T_{\tilde{X}/X} \to 0 \quad \text{(Euler)}$$

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$$0 \to T_{X_k/X_{k-1}} \to V_k \xrightarrow{(\pi_k)_\star} \mathcal{O}_{X_k}(-1) \to 0 \quad \Rightarrow \text{rk } V_k = r$$
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- For $n = \dim X$ and $r = \mathrm{rk}\, V$, get a tower of $\mathbb{P}^{r-1}$-bundles

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- **Direct image formula.** $(\pi_{k,0})_* \mathcal{O}_{X_k}(m) = E_{k,m}V^* =$ *invariant algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.*

Results obtained so far

- Using this technology and deep results of McQuillan for curve foliations on surfaces, D. – El Goul proved in 1998
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- **Higher dimensions.** Combining these ideas, E. Rousseau - J. Merker - S. Diverio (2006-2009) just proved the Green-Griffiths conjecture for hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree $d \geq d_n$ large.

Algebraic structure of differential rings

- Although very interesting, results are currently limited by lack of knowledge on jet bundles and differential operators
- **Unknown !** *Is the ring of germs of invariant differential operators on* $(\mathbb{C}^n, T_{\mathbb{C}^n})$ *at the origin*
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$$\mathcal{A}_{k,n} = \bigoplus_m E_{k,m} T^*_{\mathbb{C}^n} \quad \textit{finitely generated ?}$$
- At least this is OK for $\forall n,\ k \le 2$ and $n = 2,\ k \le 4$:

$$\mathcal{A}_{1,n} = \mathcal{O}[f_1', \ldots, f_n']$$
$$\mathcal{A}_{2,n} = \mathcal{O}[f_1', \ldots, f_n', W^{[ij]}], \quad W^{[ij]} = f_i' f_j'' - f_j' f_i''$$
$$\mathcal{A}_{3,2} = \mathcal{O}[f_1', f_2', W_1, W_2][W]^2, \quad W_i = f_i' DW - 3f_i'' W$$
$$\mathcal{A}_{4,2} = \mathcal{O}[f_1', f_2', W_{11}, W_{22}, S][W]^6, \quad W_{ii} = f_i' DW_i - 5f_i'' W_i$$

where $W = f_1' f_2'' - f_2' f_1''$ is 2-dim Wronskian and $S = (W_1 DW_2 - W_2 DW_1)/W$. Also known: $\mathcal{A}_{3,3}$ (E. Rousseau, 2004), $\mathcal{A}_{5,2}$ (J. Merker, 2007)

Strategy : evaluate growth of differential operators

- The strategy of the proofs is that the algebraic structure of $\mathcal{A}_{k,n}$ allows to compute the Euler characteristic $\chi(X, E_{k,m} \otimes \mathcal{O}(-A))$, e.g. on surfaces

$$\chi(X, E_{k,m} \otimes \mathcal{O}(-A)) = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3).$$

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- Therefore many global differential operators exist for surfaces with $13c_1^2 - 9c_2 > 0$, e.g. surfaces of degree large enough in $\mathbb{P}^3$, $d \geq 15$ (end of proof uses stability)

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- **Strategy.** OK by Ahlfors-Schwarz lemma if $r = \operatorname{rk} V = 1$. First try to get differential equations $f_{[k]}(\mathbb{C}) \subset Z \subsetneq X_k$. Take minimal such k. If $k = 0$, we are done! Otherwise $k \geq 1$ and $\pi_{k,k-1}(Z) = X_{k-1}$, thus $W = V_k \cap T_Z$ has rank $< \operatorname{rk} V_k = r$ and should have again $\det W^*$ big (unless some degeneration occurs ?). Use induction on r !

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- **Needed induction step.** *If (X, V) has* $\det V^*$ *big and $Z \subset X_k$ irreducible with $\pi_{k,k-1}(Z) = X_{k-1}$, then (Z, W), $W = V_k \cap T_Z$ has $\mathcal{O}_{Z_\ell}(1)$ big on (Z_ℓ, W_ℓ), $\ell \gg 0$.*

Use holomorphic Morse inequalities !

- **Simple case of Morse inequalities**
  (Demailly, Siu, Catanese, Trapani)
  If $L = \mathcal{O}(A - B)$ is a difference of big nef divisors A, B, then L is big as soon as

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- My PhD student S. Diverio has recently (2007-2008) worked out this strategy for hypersurfaces $X \subset \mathbb{P}^{n+1}$, with

$$L = \bigotimes_{1 \le j < k} \pi_{k,j}^* \mathcal{O}_{X_j}(2 \cdot 3^{k-j-1}) \otimes \mathcal{O}_{X_k}(1),$$

$$B = \pi_{k,0}^* \mathcal{O}_X(2 \cdot 3^{k-1}), \quad A = L + B \Rightarrow L = A - B.$$

  In this way, one obtains differential equations of order $k = n$, when $d \ge d_n$, e.g. for $d_n = n^{5n^4}$. One can check

$$d_2 = 15, \quad d_3 = 82, \quad d_4 = 329, \quad d_5 = 1222, \quad \ldots.$$

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- However (observation by Voisin & Siu), on the universal family $\mathcal{X}_k$, suitable vector fields exist (with B small).
- End of 2008, Diverio-Merker-Rousseau proved the Green-Griffiths-Lang conjecture for $X \subset \mathbb{P}^{n+1}$ of degree $d \geq d_n \geq n^{(n+1)^{n+5}}$.

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