INVARIANT SUBSPACES FOR SOME SURFACE GROUPS ACTING ON A_2-EUCLIDEAN BUILDINGS.

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ABSTRACT. This paper deals with non-Archimedean representations of punctured surface groups in PGL_3, associated actions on Euclidean buildings (of type A_2), and degenerations of real convex projective structures on surfaces. The main result is that, under good conditions on Fock-Goncharov generalized shear parameters, non-Archimedean representations acting on the Euclidean building preserve a cocompact weakly convex subspace, which is part flat surface and part tree. In particular the eigenvalue and length(s) spectra are given by an explicit finite A_2-*complex*. We use this result to describe degenerations of real convex projective structures on surfaces for an open cone of parameters. The main tool is a geometric interpretation of Fock-Goncharov parametrization in A_2-buildings.

INTRODUCTION

One motivation for the study of actions of surface groups on nondiscrete Euclidean A_2-buildings is that, in the same way that degenerations of hyperbolic structures on surfaces give rise to actions of the surface group on real trees (see Bestvina [Bes88], Paulin [Pau88]), degenerations of representations in $\mathrm{SL}_3(\mathbb{R})$ give rise to actions on nondiscrete Euclidean A_2-buildings (see for example [Pau97], [KlLe97], [Par11]). More specifically, in [Par11] we constructed a compactification of the space $\mathcal{P}(\Sigma)$ of convex real projective structures on a closed surface Σ, whose boundary points are marked length spectra of actions of $\Gamma = \pi_1(\Sigma)$ on nondiscrete Euclidean A_2-buildings. These actions come from representations of Γ in $\mathrm{SL}_3(\mathbb{K})$ for some ultrametric valued fields $\mathbb{K}$. Degenerations of convex projective structures, or more generally of Hitchin representations, have been studied by numerous people, including J. Loftin [Lof07], D. Cooper, K. Delp, D. Long and M. Thistlethwaite (forthcoming work), D. Alessandrini [Al08], I. Le [Le12], T. Zhang [Zha13], B. Collier, Q. Li [CoLi14].

Given an action of a group Γ on a Euclidean building X, a natural question, in the spirit of minimal invariant subtrees for actions on trees and convex cores for actions on hyperbolic space, is whether it is possible to find a nice invariant convex subset $Y \subset X$, for example cocompact or a minimal subbuilding... One of the motivations is that the length spectrum for instance would then be recoverable from Y alone. But convexity is a quite rigid property in higher rank (see for instance [Quint05], [KlLe06]). We introduce here a more flexible notion of *weak convexity* for subsets Y of Euclidean buildings X, that we call $\mathfrak{C}$-*convexity*, such that the length spectrum is still recoverable - in a more indirect way - from Y.

In the case where Σ is a compact oriented surface with nonempty boundary, and $\mathbb{K}$ any ultrametric valuated field, for a large family of representations $\rho : \Gamma \to \mathrm{PGL}_3(\mathbb{K})$, we construct explicitly a simple, weakly convex, invariant subcomplex Y in the associated Euclidean building X, on which Γ acts freely properly cocompactly. The subcomplex Y which is piecewise a flat surface or a tree.

We introduce also the notion of A_2-*surface*, and more generally of $(\mathbb{A}, W)$-*complexes*, that is simplicial complexes modelled on a finite reflection group $(\mathbb{A}, W)$. Natural examples are subcomplexes of Euclidean buildings with model flat $(\mathbb{A}, W)$. A A_2-*structure* on a surface Σ is a $(\mathbb{A}, W)$-structure with singularities, for the finite reflection group $(\mathring{\mathbb{A}}, W)$ of type A_2, (corresponding to PGL_3) (see section 3.1.1). Such structures are analogous to translation and half-translation surfaces (and will be called $\frac{1}{3}$-*translation surfaces*), and are closely related to cubic holomorphic differentials on the surface (for which we refer to Labourie [Lab07], Loftin [Lof01], Benoist-Hulin [BeHu14], Dumas-Wolf [DuWo14]). As a consequence of the previous result, we construct a family of explicit finite A_2-complexes K, homotopy equivalent to Σ, parametrized by a $8\,|\chi(\Sigma)|$-dimensional real parameter (z, s), which encodes the absolute values of eigenvalues of the representations ρ above $(K \simeq Y/\rho(\Gamma))$. The main tool is the Fock-Goncharov parametrization of representations $\rho : \Gamma \to \mathrm{PGL}_3(\mathbb{K})$ (generalized shear coordinates).

We now state the definitions and results in more details.

The model flat (of type A_2) is the 2-dimensional real vector space

$$\mathbb{A} = \{\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 / \sum_i \alpha_i = 0\}$$
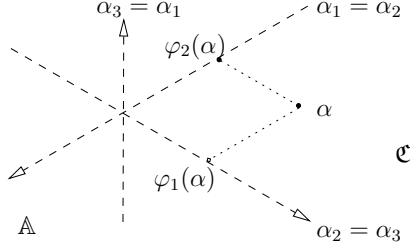
endowed with the action of the Weyl group $W = \mathfrak{S}_3$ acting on $\mathbb{A}$ by permutation of coordinates (finite reflection group). The model Weyl chamber is the cone

$$\mathfrak{C} = \{\alpha \in \mathbb{A}/ \; \alpha_1 > \alpha_2 > \alpha_3\}$$

in $\mathbb{A}$. Its closure $\overline{\mathfrak{C}}$ is a strict fundamental domain for the action of W on $\mathbb{A}$. A vector $\alpha \in \mathbb{A}$ is *singular* if it belongs to one of the three *singular lines* $\alpha_i = \alpha_j$. The two distinct types of singular directions (rays) in $\mathbb{A}$, corresponding to the orbits under W of two rays $\alpha_1 > \alpha_2 = \alpha_3$ and $\alpha_1 = \alpha_2 > \alpha_3$ bounding $\mathfrak{C}$, which will respectively be called *type* 1 and *type* 2. In the figures (Figure 1 and the sequel), the type of singular directions will be represented by an arrow $\triangleright$ indicating the induced orientation on singular lines (towards the type 1 extremity). We will use as canonical coordinates on $\mathbb{A}$ the simple roots, i.e. the linear forms $\varphi_1(\alpha) = \alpha_1 - \alpha_2$ and $\varphi_2(\alpha) = \alpha_2 - \alpha_3$, hence we will identify $\alpha \in \mathbb{A}$ with $(\varphi_1(\alpha), \varphi_2(\alpha)) \in \mathbb{R} \times \mathbb{R}$ (see Figure 1). The W-invariant Euclidean norm $\|\;\|$ on $\mathbb{A}$ (unique up to rescaling) is normalized so that the simple roots φ_i measure the distance to the corresponding singular line $\varphi_i = 0$.

When X is a (real) Euclidean building or a symmetric space of type A_2, i.e. with maximal flats isomorphic to $(\mathbb{A}, W)$, the usual metric $d : X \times X \to \mathbb{R}_{\geq 0}$ (induced by the Euclidean norm $\|\;\|$ on $\mathbb{A}$) has a natural vector-valued refinement,

$$d^{\mathfrak{C}} : X \times X \to \overline{\mathfrak{C}}$$

FIGURE 1. Simple roots coordinates in the model flat $\mathbb{A}$.

that we will call the $\mathfrak{C}$-*distance*: it is the canonical projection induced by the natural markings $f : \mathbb{A} \to X$ of flats, whose transition maps are in W up to translation. The corresponding refinement of the usual (translation) length (*Euclidean length*)

$$\ell_{euc}(g) = \{d(x, gx), \; x \in X\}$$

of an automorphism g of X is the $\mathfrak{C}$-*length* $\ell^{\mathfrak{C}}(g)$ of g. It may be defined as the unique vector of minimal length in (the closure in $\overline{\mathfrak{C}}$ of) $\{d^{\mathfrak{C}}(x, gx), \; x \in X\}$, and we have

$$\ell_{euc}(g) = \left\| \ell^{\mathfrak{C}}(g) \right\|.$$

For g in $\mathrm{SL}_3(\mathbb{K})$ acting on its associated Euclidean building (for ultrametric $\mathbb{K}$) or symmetric space (for $\mathbb{K} = \mathbb{R}$) it corresponds to

$$\ell^{\mathfrak{C}}(g) = (\log |a_i|)_i$$

where the a_i are the eigenvalues of g (in nonincreasing order). The $\mathfrak{C}$-length refines another notion of length of particular interest, the *Hilbert length*, which is the length of g for the Hilbert metric in the context of convex projective structures. It may be defined by

$$\ell_H(g) = N_H(\ell^{\mathfrak{C}}(g))$$

where N_H is the *hex-norm* on $\mathbb{A}$ i.e. the W-invariant norm defined by $N_H(\alpha) = \alpha_1 - \alpha_3$ for α in $\mathfrak{C}$ (whose unit ball is the singular regular hexagon).

We will here introduce the naturally associated notion of $\mathfrak{C}$-*geodesics*, which are paths on which the $\mathbb{C}$-distance is additive. Note that, unlike for the usual distance, $\mathfrak{C}$-geodesics between two given points are not unique, and that usual geodesics are $\mathfrak{C}$-geodesics, but the converse is not true. The notion of weak convexity is then defined, by analogy with the usual setting, as follows: we say that a subset $Y \subset X$ is $\mathfrak{C}$-*convex* if for any two points x, y in Y, there exists a $\mathfrak{C}$-geodesic from x to y that is contained in Y.

We now turn to the Fock-Goncharov parametrization of representations of the fundamental group Γ of a compact oriented surface Σ with nonempty boundary. More precisely, following [FoGo07], we explain how to associate, to an ideal triangulation $\mathcal{T}$ and $8\chi(\Sigma)$ parameters in $\mathbb{K}$ (one per triangle and two per edge), a representation $\rho : \Gamma \to \mathrm{PGL}_3(\mathbb{K})$. This construction is in fact valid any field $\mathbb{K}$. It is based on projective geometry, through the action of $\mathrm{PGL}_3(\mathbb{K})$ on the projective plane $\mathbb{P}(\mathbb{K}^3)$. Denote by $\mathbf{b}(a_1, a_2, a_3, a_4)$ the cross ratio on $\mathbb{P}(\mathbb{K}^2)$, with the convention $\mathbf{b}(\infty, -1, 0, a) = a$. Let $\mathrm{Flags}(\mathbb{P})$ be the space of flags in the projective plane $\mathbb{P} = \mathbb{P}(\mathbb{K}^3)$, that is the space

of pairs (p, D), where p is a point and D a line of $\mathbb{P}$, with $p \in D$. Denote by $\mathcal{F}_\infty(\Sigma)$ the *Farey set* of the surface, which may be defined as the set of boundary components of the universal cover $\widetilde{\Sigma}$ of Σ (see section 2.1), with the induced cyclic order. Let $\mathcal{T}$ be an ideal triangulation of Σ. Denote by $\widetilde{\mathcal{T}}$ the lift of $\mathcal{T}$ to the universal cover $\widetilde{\Sigma}$ of Σ. Shrinking boundary components of $\widetilde{\Sigma}$ to points, we may see $\widetilde{\mathcal{T}}$ as a triangulation of $\widetilde{\Sigma}$ with vertex set the Farey set $\mathcal{F}_\infty(\Sigma)$. Denote by T the set of triangles of $\mathcal{T}$, by $\overrightarrow{E}$ the set of oriented edges of $\mathcal{T}$, which are finite sets of respective cardinality $2\,|\chi(S)|$ and $6\,|\chi(S)|$. Fix a *FG-parameter* $(Z, S) = ((Z_\tau)_\tau, (S_e)_e)$ in $(\mathbb{K}_{\neq 0,-1})^T \times (\mathbb{K}_{\neq 0})^{\overrightarrow{E}}$. There exists then a unique (up to $\mathrm{PGL}(\mathbb{K}^3)$ action) associated *flag map* $F_{Z,S} : \mathcal{F}_\infty(\Sigma) \to \mathrm{Flags}(\mathbb{P})$, $i \mapsto (p_i, D_i)$, equivariant with respect to a unique representation $\rho_{Z,S} : \Gamma \to \mathrm{PGL}(\mathbb{K}^3)$, such that the flag map $F_{Z,S}$ sends each triangle $\tilde{\tau} = (i, j, k)$ of $\widetilde{\mathcal{T}}$ to a generic triple of flags of triple ratio

$$\mathbf{b}(D_i, p_i p_j, p_i(D_j \cap D_k), p_i p_k) = Z_\tau$$

where τ is the triangle of $\mathcal{T}$ with lift $\tilde{\tau}$, and for any two ajdacent triangles (i, j, k) and (k, ℓ, i) of $\widetilde{\mathcal{T}}$ with common edge $\tilde{e} = (k, i)$ we have

$$\mathbf{b}(D_i, p_i p_j, p_i p_k, p_i(D_k \cap D_\ell)) = S_e$$

where e is the oriented edge of $\mathcal{T}$ with lift $\tilde{e}$, and i, j, k, ℓ in $\mathcal{F}_\infty(\Sigma)$ are positively ordered. When $\mathbb{K} = \mathbb{R}$, the representations $\rho_{Z,S}$ with positive FG-parameters ($Z_\tau, S_e \in \mathbb{R}_{>0}$ for all τ, e) correspond to the holonomies of convex projective structures on Σ.

We now define the *A_2-complex K associated with* a *left-shifting geometric FG-parameter* (z, s) in $\mathbb{R}^T \times \mathbb{R}^{\overrightarrow{E}}$. Consider a *geometric FG-parameter* $(z, s) = ((z_\tau)_\tau, (s_e)_e)$ in $\mathbb{R}^T \times \mathbb{R}^{\overrightarrow{E}}$. We suppose that (z, s) is *left-shifting* i.e. satisfies the following condition:

(L) For each $e \in \overrightarrow{E}$, with left and right triangles τ and τ', we have
$$s_e > \max\{-z_\tau^-, -z_{\tau'}^+\}$$

where $t^+ = \max(t, 0)$ and $t^- = \max(-t, 0)$ for $t \in \mathbb{R}$. For each triangle τ of the triangulation $\mathcal{T}$, pick a singular equilateral triangle K^τ in the model plane $\mathbb{A}$, with vertices $\alpha_1, \alpha_2, \alpha_3$, and sides of $\mathfrak{C}$-length $d^\mathfrak{C}(\alpha_1, \alpha_2) = (z_\tau^+, z_\tau^-)$ in simple roots coordinates (well-defined up to translations and action of W), see figure 2.
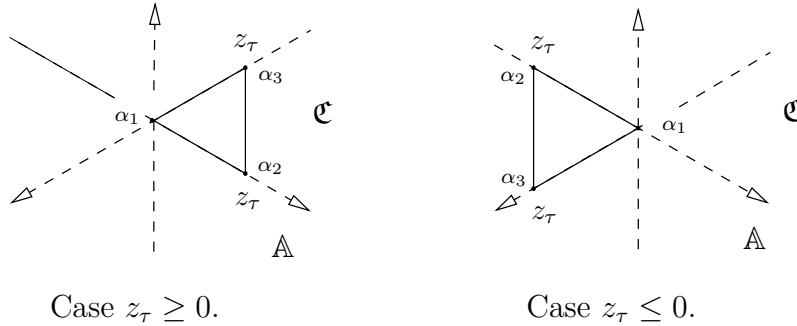

Case $z_\tau \geq 0$.   Case $z_\tau \leq 0$.

FIGURE 2. The singular triangle K^τ in $\mathbb{A}$.

When τ, τ' are adjacent along an edge e (oriented according to τ), we connect the end of the edge corresponding to e of the triangle K^τ to the beginning of the edge corresponding to e of the triangle $K^{\tau'}$, by gluing either a segment K^e in $\mathbb{A}$ of $\mathfrak{C}$-length $(s_{\bar{e}}, s_e)$, when $s_e, s_{\bar{e}} \geq 0$, or a flat strip $K^e \subset \mathbb{A}$ such that $K^e = [0, s_{\bar{e}}] \times [0, s_e]$ (in simple roots coordinates), when $s_e < 0$ or $s_{\bar{e}} < 0$, as in figure 3 (note that under hypothesis (L) $s_e < 0$ implies that $s_{\bar{e}} \geq 0$).
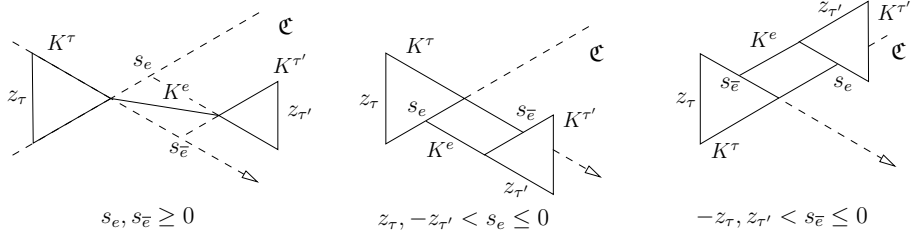

FIGURE 3. Gluings (local development in $\mathbb{A}$).

The resulting finite 2-dimensional complex K (see figure 4) is a deformation retract of Σ, and its fundamental group has canonical identification with $\Gamma = \pi_1(\Sigma)$. The length metric on K induced by the Euclidean W-invariant metric on $\mathbb{A}$ will be denoted by d. Furthermore, the complex K is endowed with a A_2-*structure* (charts in $\mathbb{A}$ with transition maps in W). Hence we may define the $\mathfrak{C}$-length of piecewise affine paths in K. The $\mathfrak{C}$-length $\ell^{\mathfrak{C}}(\gamma, K)$ of $\gamma \in \Gamma$ is then defined as the $\mathfrak{C}$-length of one (any) closed geodesic representing γ. We define the $\mathfrak{C}$-distance $d^{\mathfrak{C}}$ on the universal cover $\widetilde{K}$ of K as the $\mathfrak{C}$-length of the unique geodesic between two points. Note that, unlike in Euclidean buildings, in A_2-complexes the $\mathfrak{C}$-distance does not refine the usual metric d, in the sense that the inequality $\left\| d^{\mathfrak{C}}(x,y) \right\| \leq d(x,y)$ may be strict.

There are several particular cases of special interest, providing a continuous transition from graphs to surfaces. The geometric FG-parameters (z, s) satisfying the condition

$$(T) \begin{cases} z_\tau = 0 \text{ for all triangles } \tau \text{ of } \mathcal{T} \\ s_e > 0 \text{ for all oriented edges } e \text{ of } \mathcal{T} \end{cases}$$

(which imply (L)), correspond to the case where K is a graph (the 3-valent ribbon graph dual to the ideal triangulation), endowed with a $\mathfrak{C}$-*metric*. Relaxing the hypotheses, the condition

$$(TT) \; s_e \geq 0 \text{ for all oriented edge } e \text{ of } \mathcal{T}$$

means that all the K^e are segments so K is obtained from the previous graph by replacing vertices by triangles (*graph of triangles*). At the opposite of the spectrum, when

$$(Sf) \; s_{\bar{e}} < 0 \text{ or } s_e < 0 \text{ for all oriented edge } e \text{ of } \mathcal{T},$$

then K is a $\frac{1}{3}$-translation surface homeomorphic to Σ.

We now state the main result (see Theorem 4.2). We will need the following hypothesis: A geometric FG-parameter (z, s) will be called *edge-separating* if it satisfies the following condition.
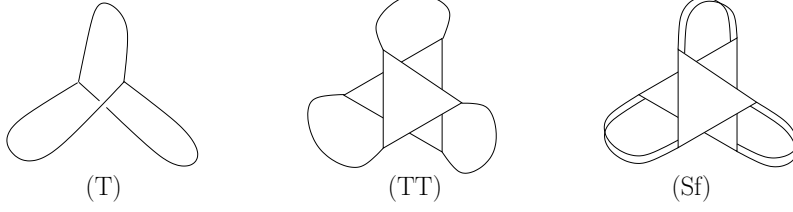
(T)   (TT)   (Sf)

FIGURE 4. Examples of A_2-complex K on a pair of pants, corresponding to the conditions (T), (TT), and (Sf) on the parameter (z, s).

(L) For each τ in $\mathcal{T}$ and every pair of edges e_1, e_2 of τ, we have
$$\begin{cases} -s_{e_1} - s_{e_2} & < z_\tau^- \\ -s_{\overline{e_1}} - s_{\overline{e_2}} & < z_\tau^+ \end{cases} \, .$$

Theorem 1. *Let $(Z, S) = ((Z_\tau)_\tau, (S_e)_e)$ in $(\mathbb{K}_{\neq 0, -1})^T \times (\mathbb{K}_{\neq 0})^{\overrightarrow{E}}$, and denote by ρ the representation $\rho_{Z,S} : \Gamma \to \mathrm{PGL}_3(\mathbb{K})$ of FG-parameter (Z, S). Let $z_m = \log|Z_m|$, $s_m = \log|S_m|$ and $z = (z_m)_m$, $s = (s_m)_m$. Suppose that*

(FT) For each triangle τ in $\mathcal{T}$, we have $|Z_\tau + 1| \geq 1$;
(FE) For each oriented edge e in $\mathcal{T}$, we have $|S_e + 1| \geq 1$;
(L) (z, s) is left-shifting ;
(S) (z, s) is edge-separating ;

Let K be the A_2-complex of geometric FG-parameter (z, s). Then there exists a ρ-equivariant map
$$\Psi : \widetilde{K} \to X$$
preserving the $\mathfrak{C}$-distance $d^{\mathfrak{C}}$.

Corollary 2. *Under the hypotheses of Theorem 1, the following assertions holds.*

(i) The $\mathfrak{C}$-length spectra coincide, i.e. for all $\gamma \in \Gamma$
$$\ell^{\mathfrak{C}}(\rho(\gamma)) = \ell^{\mathfrak{C}}(\gamma, K) \, .$$

In particular, the usual Euclidean and Hilbert length are given by
$$\ell_{euc}(\rho(\gamma)) = \left\| \ell^{\mathfrak{C}}(\gamma, K) \right\|,$$
$$\text{and } \ell_H(\rho(\gamma)) = N_H(\ell^{\mathfrak{C}}(\gamma, K)) \, .$$

(ii) The map Ψ is bilipschitz. In particular the representation ρ is undistorted, i.e. for any fixed point x in X, we have $d(x, \rho(\gamma)x) \simeq \|\gamma\|$ where $\|\gamma\|$ is the word length of γ in Γ.
(iii) The representation ρ is faithfull and proper (hence discrete).

Remarks.   (i) The image Y of Ψ is a closed $\mathfrak{C}$-convex subset of X preserved by ρ, and Γ acts freely discontisnuously cocompactly on Y.
       (ii) The $\mathfrak{C}$-length spectrum of $\rho_{Z,S}$ depends only on $z = \log|Z|$, $s = \log|S|$ (in particular it does not determine the representation up to conjugacy).
       (iii) Note that, for positive representations (that is, with positive FG-parameters $z_\tau, s_e > 0$) in ordered fields $\mathbb{K}$, the hypothesis (FT) and (FE) are always satisfied.

(iv) Note that (L) and (S) are a finite system of strict linear inequations in $z_\tau^-, z_{\tau'}^+$, in particular the subset O_{LS} of left-shifting and edge-separating (z, s) is a finite union of open convex polyhedral cones in $\mathbb{R}^T \times \mathbb{R}^{\overrightarrow{E}}$. It contains the non empty cone $\{0\}^T \times \mathbb{R}^{\overrightarrow{E}}_{>0}$ of (z, s) satisfying (T). For arbitrary fixed triangle parameters z_τ, conditions (L) and (S) are always satisfied for big enough edge parameters s_e. In particular O_{LS} is a nonempty open cone.

(v) The result holds in fact in a more general setting including exotic buildings, see Theorem 4.1.

A special case with much simpler hypotheses (and proof) is when (Z, S) satisfies simply

$$(T')\begin{cases} |Z_\tau| = |Z_\tau + 1| = 1 \text{ for all } \tau \\ |S_e| > 1 \text{ for all } e \ . \end{cases}$$

Then all hypotheses of Theorem 1 are satisfied, (z, s) satisfies (T) and K is a graph, and the image Y of Ψ is an invariant cocompact $\mathfrak{C}$-convex (in particular bilipschitz) tree in the building. The hypotheses of Theorem 1 are also satisfied in the other particular case corresponding to the following open simple condition

$$(TT')\begin{cases} |Z_\tau| \neq 1 \text{ for all } \tau \\ |S_e| > 1 \text{ for all } e, \end{cases}$$

and (z, s) satisfies (TT), providing an invariant $\mathfrak{C}$-convex "tree of triangles" Y. On the other end of the spectrum, Theorem 1 provides (for (z, s) satisfies (Sf)) examples of representations whose image preserves a $\mathfrak{C}$-geodesic (in particular, bilipschitz) surface Y in the building.

In the last part of the paper, we use Theorem 1 to describe limit of length functions (in the associated symmetric space) for a large family of degenerations of representations $\Gamma \to \mathrm{PGL}(\mathbb{R}^3)$ corresponding to convex $\mathbb{RP}^2$-structures on Σ.

Theorem 3. *Let $((z^n, s^n))_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^T \times \mathbb{R}^{\overrightarrow{E}}$. Let $Z_\tau^n = \exp(z_\tau^n)$ and $S_e^n = \exp(s_e^n)$. Let $\rho_n : \Gamma \to \mathrm{PGL}_3(\mathbb{R})$ be the representation of FG-parameter $(Z^n, S^n) = ((Z_\tau^n)_\tau, (Z_e^n)_e)$. Let $(\lambda_n)_n$ be a sequence of real numbers going to $+\infty$, such that the sequence $\frac{1}{\lambda_n}(z^n, s^n)$ converges to a nonzero (z, s) in $\mathbb{R}^T \times \mathbb{R}^{\overrightarrow{E}}$. Suppose that (z, s) is left-shifting and edge-separating ((L) and (S)). Let K be the A_2-complex of FG-parameter (z, s). Then the renormalized $\mathfrak{C}$-length spectrum of ρ_n converges to the $\mathfrak{C}$-length spectrum of K, that is: for all $\gamma \in \Gamma$ we have*

$$\frac{1}{\lambda_n}\ell^{\mathfrak{C}}(\rho_n(\gamma)) \to \ell^{\mathfrak{C}}(\gamma, K)$$

in $\overline{\mathfrak{C}}$. In particular for Euclidean and Hilbert lengths, we have then:

$$\frac{1}{\lambda_n}\ell_{euc}(\rho_n(\gamma)) \to \left\|\ell^{\mathfrak{C}}(\gamma, K)\right\|$$

$$\frac{1}{\lambda_n}\ell_H(\rho_n(\gamma)) \to N_H(\ell^{\mathfrak{C}}(\gamma, K))$$

for all $\gamma \in \Gamma$.

A similar result holds in more general valued field $\mathbb{K}$ (see Theorem 5.8). Note that, for a given sequence $((z^n, s^n))_{n \in \mathbb{N}}$ going to infinity, there always exists a convenient sequence λ_n, taking $\lambda_n = \max_{\tau,e} |z^n(\tau)|, |s^n(e)|$. This describes a part (corresponding to the open cone O_{LS} of FG-parameters) of the boundary (constructed in [Par11]) of the space $\mathcal{P}(\Sigma)$ of convex real projective structures on Σ (see Coro. 5.9). Note that D. Cooper, K. Delp, D. Long and M. Thistlethwaite announced results similar to Theorem 3.

Our proofs involve a geometric interpretation of FG-parameters in Euclidean buildings of type A_2, relying on results from [Par15a] describing the geometry of triples of ideal chambers in relation with their triple ratio as triples of flags. It allows to associate with each triangle τ of the triangulation $\widetilde{\mathcal{T}}$ a singular flat triangle Δ_τ in the building in a canonical way. The map Ψ is then defined by sending $\widetilde{K}^\tau$ to Δ^τ. The main technical difficulty is to prove that the map Ψ is globally $\mathfrak{C}$-geodesic. Note that in the case (T') of trees the proofs are *much* simpler. Application to degenerations of representations uses asymptotic cones, and basically reduces to prove that the Fock-Goncharov parametrization behaves well under ultralimits (Proposition 5.5).

The structure of the paper is the following: in Section 1, we recall some basic facts about Euclidean buildings of type A_2 that will be used throughout the article, and we establish a criterion for a local $\mathfrak{C}$-geodesic to be a global $\mathfrak{C}$-geodesic (Proposition 1.7) that will be used to prove global $\mathfrak{C}$-geodesicity for Ψ. In Section 2, we explain Fock-Goncharov parametrization for representations in any field $\mathbb{K}$. In Section 3, we introduce the notion of A_2-complexes, and we construct the A_2-complex K associated with a left-shifting geometric FG-parameter (z, s) and discuss the special cases (from trees to surfaces). In Section 4, we study actions on Euclidean buildings (possibly exotic), introducing a purely geometric version of FG-invariants, and we prove the main result (Theorem 1) in this wider setting. Finally, in Section 5, we study degenerations of representations and prove Theorem 3, introducing asymptotic cones of projective spaces and studying the asymptotic behaviour of Fock-Goncharov parametrizations.

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1. Geometric preliminaries

1.1. **Projective geometry.** We here collect notations for projective geometry which will be used throughout this article.

Nondegenerated quadruples on a projective line. Cross ratios on projective lines will be defined on quadruples $(\xi_1, \xi_2, \xi_3, \xi_4)$ of points satisfying the following nondegeneracy condition: (*no triple point*, i.e. any three of the points are not equal, or, equivalently,

$$(1.1) \qquad (\xi_1 \neq \xi_4 \text{ and } \xi_2 \neq \xi_3) \text{ or } (\xi_1 \neq \xi_2 \text{ and } \xi_3 \neq \xi_4).$$

The quadruple $(\xi_1, \xi_2, \xi_3, \xi_4)$ is then called *nondegenerated*.

Projective planes. Let $\mathbb{P}$ be a projective plane. We denote by $\mathbb{P}^*$ the dual projective plane, i.e. the set of lines in $\mathbb{P}$. We will denote $p \oplus q$ or pq the line joining two distinct point p, q in $\mathbb{P}$.

We denote by Flags($\mathbb{P}$) the set of (complete) flags $F = (p, D) \in \mathbb{P} \times \mathbb{P}^*$, $p \in D$, in the projective plane $\mathbb{P}$. Two flags are called *opposite* if they are in generic position.

Triples of flags. Let $T = (F_1, F_2, F_3)$ be a triple of flags $F_i = (p_i, D_i)$ in $\mathbb{P}$. We will denote by p_{ij} the point $D_i \cap D_j$ (resp. D_{ij} the line $p_i p_j$), when defined.

The natural nondegeneracy condition on the triple (F_1, F_2, F_3) for the triple ratios to be well defined is the following:

(ND) either for all i, $p_i \notin D_{i+1}$ or for all i, $p_i \notin D_{i-1}$.

This condition is clearly equivalent to: the points are pairwise distinct, the lines are pairwise distinct, none of the points is on the three lines (i.e. $D_i \cap D_j \neq p_k$ for all $\{i, j, k\} = \{1, 2, 3\}$) and none of the lines contains the three points (i.e. $p_i p_j \neq D_k$ for all i, j, k). We will then say that the triple (F_1, F_2, F_3) is *nondegenerated.*

It is easy to check that the triple T defines then a nondegenerated quadruple of well-defined lines D_i, $p_i p_j$, $p_i p_{jk}$, $p_i p_k$ through each point p_i, and a nondegenerated quadruple of well-defined points p_i, $D_i \cap D_j$, $D_i \cap D_{jk}$, $D_i \cap D_k$ on each line D_i.

The triple of flags $T = (F_1, F_2, F_3)$ is *generic* if the flags $F_i = (p_i, D_i)$ are pairwise opposite, the points $(p_i)_i$ are not collinear and the lines $(D_i)_i$ are not concurrent. In particular, T is then nondegenerated, and the induced quadruples of points on each line (resp. of lines through each point) are generic (pairwise distinct).

1.2. **The model finite reflection group** $(\mathbb{A}, W)$ **of type** A_2**.**

The *model flat* (of type A_2) is the vector space $\mathbb{A} = \mathbb{R}^3 / \mathbb{R}(1, 1, 1)$, endowed with the action of the *Weyl group* $W = \mathfrak{S}_3$ acting on $\mathbb{A}$ by permutation of coordinates, which is a finite reflection group. We denote by W_{aff} the subgroup of affine isomorphisms of $\mathbb{A}$ with linear part in W. We denote by $[\alpha]$ the projection in $\mathbb{A}$ of a vector α in $\mathbb{R}^3$. The vector space $\mathbb{A}$ will be identified with the hyperplane $\{\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 / \sum_i \alpha_i = 0\}$ of $\mathbb{R}^3$.

Recall that a vector in $\mathbb{A}$ is called *singular* if it belongs to one the three lines $\alpha_i = \alpha_j$, and *regular* otherwise. A *(open) (vectorial) Weyl chamber* of $\mathbb{A}$ is a connected component of regular vectors. The *model Weyl chamber* is $\mathfrak{C} = \{\alpha \in \mathbb{A} / \alpha_1 > \alpha_2 > \alpha_3\}$. Its closure $\overline{\mathfrak{C}}$ is a strict fundamental domain for the action of W on $\mathbb{A}$, and we denote by $p^{\mathfrak{C}} : \mathbb{A} \to \overline{\mathfrak{C}}$ the canonical projection, which maps a vector $\alpha \in \mathbb{A}$ to its *type* in $\overline{\mathfrak{C}}$. We denote by $\partial \mathbb{A}$ the subset of unitary vectors in $\mathbb{A}$, identified with the set $\mathbb{P}^+(\mathbb{A}) = (\mathbb{A} - \{0\})/\mathbb{R}_{>0}$ of rays issued from 0, and $\partial : \mathbb{A} \to \partial \mathbb{A}$ the corresponding projection. The *type (of direction)* of a nonzero vector $\alpha \in \mathbb{A}$ is its canonical projection $\partial(p^{\mathfrak{C}}(\alpha))$ in $\partial \overline{\mathfrak{C}}$.

The *simple roots* (associated with $\mathfrak{C}$) are the linear forms

$$\varphi_1 : \alpha \mapsto \alpha_1 - \alpha_2$$

$$\varphi_2 : \alpha \mapsto \alpha_2 - \alpha_3$$

and we denote by $\varphi_3 : \alpha \mapsto \alpha_3 - \alpha_1$ the root satisfying $\varphi_1 + \varphi_2 + \varphi_3 = 0$.

A singular vector α is said to be *of type* 1 if its type in $\overline{\mathfrak{C}}$ satisfies $\alpha_1 > \alpha_2 = \alpha_3$, and *of type* 2 if its type satisfies $\alpha_1 = \alpha_2 > \alpha_3$.

Recall that two nonzero vectors α and α' of $\mathbb{A}$ are called *opposite* if $\alpha' = -\alpha$. Similarly, two Weyl chambers C and C^+ of $\mathbb{A}$ are *opposite* if $C^+ = -C$. We denote by w^{opp} the unique element of W sending $\mathfrak{C}$ to $-\mathfrak{C}$, and by $\alpha^{\mathrm{opp}} = w^{\mathrm{opp}}(-\alpha) = (-\alpha_3, -\alpha_2, -\alpha_1)$ the image of α by the *opposition involution* opp of $\mathbb{A}$.

We will normalize the W-invariant Euclidean norm $\|\cdot\|$ on $\mathbb{A}$ by requiring that the simple roots have unit norm. The associated Euclidean metric on $\mathbb{A}$ is denoted by d.

The $\mathfrak{C}$-*distance* on $\mathbb{A}$ (or $\mathfrak{C}$-*length* of segments) is the canonical projection $d^{\mathfrak{C}} : \mathbb{A} \times \mathbb{A} \to \overline{\mathfrak{C}}$ which is defined by $d^{\mathfrak{C}}(\alpha, \beta) = p^{\mathfrak{C}}(\beta - \alpha)$.

We will denote by N_H the *hex*-norm, that is the W-invariant norm on $\mathbb{A}$ defined by

$$N_H(\alpha) = \alpha_1 - \alpha_3 = -\varphi_3(\alpha)$$

for α in $\mathfrak{C}$, whose unit ball is a regular hexagon with singular sides.

1.3. **Euclidean buildings.** The Euclidean buildings considered in this article are $\mathbb{R}$-buildings, in particular they are not necessarily discrete (have no simplicial complex structure) nor locally compact. We refer to [Par99] for their definition and basic properties (see also [Tits86], [KlLe97], [Rou09]). Let X be a Euclidean building of type A_2. Recall that X is a CAT(0) metric space endowed with a (maximal) collection $\mathcal{A}$ of isometric embeddings $f : \mathbb{A} \to X$ called *marked apartments*, or *marked flats* by analogy with Riemannian symmetric spaces, satisfying the following properties

 (A1) $\mathcal{A}$ is invariant by precomposition by W_{aff} ;
 (A2) If f and f' are two marked flats, then the transition map $f^{-1} \circ f'$ is in W_{aff} ;
 (A3') Any two rays of X are initially contained in a common marked flat.

The *flats* (resp. the *Weyl chambers*) of X are the images of $\mathbb{A}$ (resp. of $\mathfrak{C}$) by the marked flats.

We say that we are in the *algebraic case* when X is the Euclidean building $X(V)$ associated with some 3-dimensional vector space V on an ultrametric field $\mathbb{K}$. We then denote by $|\cdot|$ the absolute value of $\mathbb{K}$.

Recall that, in Euclidean buildings, two (unit speed) geodesic segments issued from a common point x have zero angle if and only if they have same germ at x (i.e. coincide in a neighborhood of x). A *direction* at $x \in X$ is a germ of (unit speed) geodesic segment from x. A direction, geodesic segment, ray or line has a well-defined *type (of direction)* in $\partial\overline{\mathfrak{C}}$, which is its canonical projection (through a marked flat) in $\partial\overline{\mathfrak{C}}$. It is called *singular* or *regular* accordingly.

The *space of directions* (or *unit tangent cone*) at x is denoted by $\Sigma_x X$. It is endowed with the angular metric. We denote by $\Sigma_x : X - \{x\} \to \Sigma_x X$ the associated projection.

The space of directions $\Sigma_x X$ is a spherical building of type A_2, whose apartment are the germs $\Sigma_x A$ at x of the flats A of X passing through x,

and whose chambers (i.e. 1-dimensional simplices) are the germs $\Sigma_x C$ at x of the Weyl chambers C of X with vertex x (see for example [Par99]).

The *local projective plane at x* $\mathbb{P}_x = \mathbb{P}_x(X)$ is the projective plane associated to the spherical A_2-building $\Sigma_x X$, i.e. the projective plane whose incidence graph is $\Sigma_x X$: Its points are the singular directions of type 1 and its lines are the singular directions of type 2 at x.

Recall that, in a spherical building, any two points (resp. chambers) are contained in a common apartment, and that they are *opposite* if they are opposite in that apartment.

Two Weyl chambers C, C^+ of X with common vertex x are *opposite* (at x) if their union contains a regular geodesic line passing by x, or, equivalently, if they define opposite chambers $\Sigma_x C$, $\Sigma_x C^+$ in the spherical building $\Sigma_x X$ of directions at x. Then there exists a unique flat of X containing both C and C^+.

1.4. The boundary of a A_2-Euclidean building and its projective geometry.

1.4.1. *The projective plane at infinity.* We denote by $\partial_\infty X$ the CAT(0) boundary of X. The *type* of an ideal point $\xi \in \partial_\infty X$ is the type in $\partial\overline{\mathfrak{C}}$ of any ray to ξ. The boundary $\partial_\infty X$ of X is the incidence graph of a projective plane $\mathbb{P} = \mathbb{P}_\infty(X)$ whose points are the singular points of type 1 of $\partial_\infty X$ and lines are the singular points of type 2 of $\partial_\infty X$. The set $\partial_F X$ of chambers at infinity of X (Furstenberg boundary) identifies then with the set Flags($\mathbb{P}$) of (complete) flags $F = (p, D) \in \mathbb{P} \times \mathbb{P}^*$, $p \in D$, in the projective plane $\mathbb{P}$.

In the algebraic case, the projective plane $\mathbb{P}$ at infinity of $X = X(V)$ is the classical projective plane $\mathbb{P}(V)$.

For $x \in X$, we denote by $\Sigma_x : y \to \Sigma_x y$ the canonical projection from $\partial_\infty X$ to the unit tangent cone $\Sigma_x X$ at x. The canonical projection $\Sigma_x : \partial_\infty X \to \Sigma_x X$ preserves the simplicial structure and the type (in $\partial\overline{\mathfrak{C}}$) of points, and in particular it induces the *canonical projection* $\Sigma_x : \mathbb{P} \to \mathbb{P}_x$, which is a surjective morphism of projective planes (i.e. if $p \in \mathbb{P}$ and $D \in \mathbb{P}^*$, then $\Sigma_x p \in \mathbb{P}_x$ and $\Sigma_x D \in \mathbb{P}_x^*$, and $p \in D$ implies $\Sigma_x p \in \Sigma_x D$).

If c_+ and c_- are opposite flags in $\mathbb{P}$ (i.e. chambers at infinity of X), then we denote by $A(c_-, c_+)$ the unique flat joining c_- to c_+ in X. A basic fact is that given a generic (i.e. non collinear) triple of points p_1, p_2, p_3 in $\mathbb{P}$ there exists a unique flat $A(p_1, p_2, p_3)$ of X containing them in its boundary (and the analog holds for lines).

1.4.2. *Transverse trees at infinity.* (See for example [Tits86, §8], [Leeb00, 1.2.3], [MSVM14, §4].) We denote by X_ξ the transverse tree at a singular ideal point ξ in $\partial_\infty X$ which may be defined, from the metric viewpoint, as the space of classes of strongly asymptotic rays to ξ the quotient space of the space of all rays to ξ by the pseudodistance d_ξ given by

$$d_\xi(r_1, r_2) = \inf_{t_1, t_2} d(r_1(t_1), r_2(t_2)) .$$

We denote by $\pi_\xi : X \to X_\xi$ the canonical projection. Recall that X_ξ is a $\mathbb{R}$-tree, and that its boundary $\partial_\infty X_\xi$ identifies with the set of singular points of $\partial_\infty X$ adjacent to ξ. In particular, if p is a point in $\mathbb{P}$, then the boundary

of the associated tree X_p is identified with the set p^* of lines D through p in the projective plane $\mathbb{P}$. Similarly, the boundary of the tree X_D associated with a line D of $\mathbb{P}$ is identified with the set D^* of points p of $\mathbb{P}$ that belong to D.

1.4.3. *The $\mathbb{A}$-valued Busemann cocycle.* We denote by $B_c : X \times X \to \mathbb{A}$ the $\mathbb{A}$-valued Busemann cocycle associated with an ideal chamber c of X, which is defined by

$$B_c(f(\alpha), f'(\alpha')) = \alpha' - \alpha$$

for all marked flats $f, f' : \mathbb{A} \to X$ sending $\partial\mathfrak{C}$ to c and *very strongly asymptotic* that is such that $d(f(r(t)), f'(r(t)))$ goes to zero when $t \to +\infty$ for one (all) regular ray r in $\mathfrak{C}$ (which in Euclidean buildings is equivalent to: $f = f'$ on some subchamber $\alpha" + \mathfrak{C}$). Note that in rank one (when $\dim \mathbb{A} = 1$) this is the usual Busemann cocycle, which is defined by

$$B_\xi(x, y) = \lim_{z \to \xi} d(x, z) - d(y, z)$$

We will use the following basic property, that describes the behaviour of Busemann cocycle associated with ideal chamber $c = (p, D)$ upon projections to transverse trees at infinity X_p and X_D.

$$(1.2) \qquad \begin{aligned} \varphi_1(B_{(p,D)}(x,y)) &= B_p(\pi_D(x), \pi_D(y)) \\ \varphi_2(B_{(p,D)}(x,y)) &= B_D(\pi_p(x), \pi_p(y)) \end{aligned}$$

If c_+ and c_- are opposite chambers at infinity, then

$$(1.3) \qquad B_{c_+}(x, y) = -(B_{c_-}(x, y))^{\mathrm{opp}} \text{ for } x, y \text{ in the flat } A(c_-, c_+)$$

1.4.4. *Cross ratio on the boundary of a tree.* (See [Tits86, §7], and for a more general setting [Otal92], [Bou96]). In this section, we suppose that X is a $\mathbb{R}$-tree, and we denote by $\partial_\infty X$ its boundary at infinity. Given three distinct ideal points ξ_1, ξ_2, ξ_3 in $\partial_\infty X$, we denote by $c(\xi_1, \xi_2, \xi_3)$ the *center* of the ideal triple ξ_1, ξ_2, ξ_3, that is the unique intersection point of the three geodesics joining two of the three points.

The *cross ratio* of four pairwise distinct points ξ_1, ξ_2, ξ_3, ξ_4 in $\partial_\infty X$ is defined as the oriented distance on the geodesic from ξ_3 to ξ_1, from the center x of the ideal triple ξ_3, ξ_1, ξ_2 to the center y of the ideal triple ξ_3, ξ_1, ξ_4
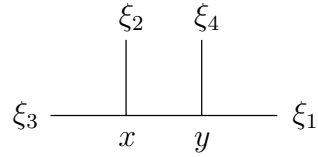
$$(1.4)$$
$$\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) = \overrightarrow{xy} = B_{\xi_1}(x, y) .$$

In the case where some of the points coincide, the cross ratio is still defined if the quadruple $(\xi_1, \xi_2, \xi_3, \xi_4)$ is nondegenerated (see section 1.1). It is then set to 0 when $\xi_1 = \xi_3$ or $\xi_2 = \xi_4$, $-\infty$ when $\xi_1 = \xi_2$ or $\xi_3 = \xi_4$, and $+\infty$ when $\xi_1 = \xi_4$ or $\xi_2 = \xi_3$.

We recall that the cross ratio is invariant under double transpositions and satisfies the following properties.

Proposition 1.1. *We have*

    (i) $\boldsymbol{\beta}(\xi_3, \xi_2, \xi_1, \xi_4) = \boldsymbol{\beta}(\xi_1, \xi_4, \xi_3, \xi_2) = -\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4)$ *;*
    (ii) $\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) + \boldsymbol{\beta}(\xi_1, \xi_4, \xi_2, \xi_3) + \boldsymbol{\beta}(\xi_1, \xi_3, \xi_4, \xi_2) = 0$ *;*

(iii) *if* $\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) > 0$,
   then $\boldsymbol{\beta}(\xi_1, \xi_3, \xi_4, \xi_2) = 0$ *and* $\boldsymbol{\beta}(\xi_1, \xi_4, \xi_2, \xi_3) = -\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4)$;
(iv) $\boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_4) + \boldsymbol{\beta}(\xi_1, \xi_4, \xi_3, \xi_5) = \boldsymbol{\beta}(\xi_1, \xi_2, \xi_3, \xi_5)$.

1.4.5. *Cross ratio on the boundary of a A_2-Euclidean building.* See [Tits86].
Let X be a Euclidean building of type A_2 and $\mathbb{P}$ the associated projective
plane at infinity. We denote by $\boldsymbol{\beta}(p_1, p_2, p_3, p_4)$ the *(geometric) cross ratio*
(*projective valuation* in [Tits86]) of a nondegenerated quadruple (p_1, p_2, p_3, p_4)
of points lying on a common line D of $\mathbb{P}$. We recall that it is defined as their
cross ratio as points in the boundary of the transverse tree X_D at ideal point
D of X. We similarly denote by $\boldsymbol{\beta}(D_1, D_2, D_3, D_4)$ the geometric cross ra-
tio of four lines D_1, D_2, D_3, D_4 through a common point p of $\mathbb{P}$, which is
defined as their cross ratio as points in the boundary of the transverse tree
X_p at ideal point p of X. Recall that perspectivities preserve cross ratios,
that is

$$\boldsymbol{\beta}(p_1, p_2, p_3, p_4) = \boldsymbol{\beta}(qp_1, qp_2, qp_3, qp_4)$$

$$\boldsymbol{\beta}(D_1, D_2, D_3, D_4) = \boldsymbol{\beta}(L \cap D_1, L \cap D_2, L \cap D_3, L \cap D_4)$$

(when defined).
   In the algebraic case, $\mathbb{P} = \mathbb{P}(\mathbb{K}^3)$ and the geometric cross ratio $\boldsymbol{\beta}$ is then
obtained from the usual (algebraic) cross ratio $\mathbf{b}$ (see section 2.2 for the
definition) by

$$(1.5) \qquad \begin{array}{rcl} \boldsymbol{\beta}(p_1, p_2, p_3, p_4) & = & \log|\mathbf{b}(p_1, p_2, p_3, p_4)| \\ \boldsymbol{\beta}(D_1, D_2, D_3, D_4) & = & \log|\mathbf{b}(D_1, D_2, D_3, D_4)| \end{array}$$

(see for example §1.10 in [Par15a]).

1.5. $\mathfrak{C}$-distance, translation lengths, and $\mathfrak{C}$-geodesics.

The $\mathfrak{C}$-distance. The $\mathfrak{C}$-*distance* on X is the map $d^{\mathfrak{C}} : X \times X \to \overline{\mathfrak{C}}$ defined
by $d^{\mathfrak{C}}(f(\alpha), f(\beta)) = d^{\mathfrak{C}}(\alpha, \beta)$ for any marked flat $f : \mathbb{A} \to X$ and $\alpha, \beta \in \mathbb{A}$.
Note that we have $d^{\mathfrak{C}}(y, x) = d^{\mathfrak{C}}(x, y)^{\mathrm{opp}}$. The $\mathfrak{C}$-distance may be seen as a
refinement of the usual distance d, since

$$d(x, y) = \left\| d^{\mathfrak{C}}(x, y) \right\| .$$

The $\mathfrak{C}$-length of an autorphism. Let g be an automorphism of X. The usual
(translation) length of g is $\ell_{euc}(g) = \inf_{x \in X} d(x, gx)$, and will be called the
Euclidean (translation) length of g.
   We will denote by $\ell^{\mathfrak{C}}(g)$ the $\mathfrak{C}$-*(translation) length* of g (called *vecteur de
translation* in [Par11]), which is the unique vector of minimal length in (the
closure in $\overline{\mathfrak{C}}$ of) $\{d^{\mathfrak{C}}(x, gx),\ x \in X\}$. We recall that in the algebraic case, for
$g \in \mathrm{PGL}_3(\mathbb{K})$, we have

$$\ell^{\mathfrak{C}}(g) = [(\log|a_i|)_i]$$

where the a_i are the eigenvalues of g. The $\mathfrak{C}$-length refines the Euclidean
length as $\ell_{euc}(g) = \left\| \ell^{\mathfrak{C}}(g) \right\|$. We will also consider the *Hilbert length*

$$\ell_H(g) = N_H(\ell^{\mathfrak{C}}(g)$$

of g, which correspond to the translation length for the Hilbert metric in
the case of holonomies of convex projective structures.

The $\mathfrak{C}$-geodesics. The $\mathfrak{C}$-*length* of a piecewise affine path σ with vertices x_0, $x_1,\ldots,x_N$ in X is the vector

$$\ell^{\mathfrak{C}}(\sigma) = \sum_n d^{\mathfrak{C}}(x_n, x_{n+1})$$

in the closed Weyl chamber $\overline{\mathfrak{C}}$.

Definition 1.2. A piecewise affine path $\sigma : [0,s] \to X$ will be called a $\mathfrak{C}$-*geodesic* if there is a marked flat $f : \mathbb{A} \to X$ such that σ is the image by f of a (piecewise affine) path $\eta : [0,s] \to \mathbb{A}$ such that $\dot\eta(t) \in \overline{\mathfrak{C}}$ for almost all $t \in [0,s]$.

Note that a piecewise affine path in $\mathbb{A}$ is a $\mathfrak{C}$-geodesic if and only if it is a geodesic for the hex-metric (that is the metric induced by the hex-norm N_H). The following proposition collects some obvious properties of $\mathfrak{C}$-geodesics that are needed in this article (they actually satisfy stronger properties, see [Par15b]).

Proposition 1.3. *Let $\sigma : [0,s] \to X$ be a $\mathfrak{C}$-geodesic from x to y in X. Then*

(i) the $\mathfrak{C}$-length $\ell^{\mathfrak{C}}(\sigma)$ of σ is equal to the $\mathfrak{C}$-distance $d^{\mathfrak{C}}(x,y)$,
(ii) any flat containing x and y contains σ. $\square$

A local criterion. We say that two directions in $\Sigma_x X$ are $\mathfrak{C}$-*opposite* if they are contained in opposite closed chambers of $\Sigma_x X$. For $y \neq x$ in X, we denote by $Fac_x(y)$ the minimal closed simplex of $\Sigma_x X$ containing $\Sigma_x y$.

Proposition 1.4. *Let $x,y,z \in X$, with $y \neq x,z$. The following are equivalent:*

(i) The path (x, y, z) is $\mathfrak{C}$-geodesic ;
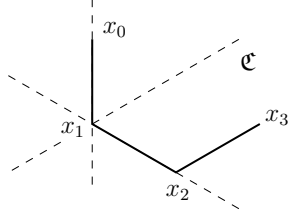*(ii) The directions $\Sigma_y x$ and $\Sigma_y z$ are $\mathfrak{C}$-opposite in $\Sigma_y X$.*

Then $x \neq z$ and $\Sigma_x(y)$ belongs to $Fac_x(z)$.

Proof. This follows from the fact that two opposite Weyl chambers at y are contained in a flat. $\square$

Remark 1.5. A key difficulty is that, unlike in the usual cases, a path may be locally $\mathfrak{C}$-geodesic but not globally $\mathfrak{C}$-geodesic, even for arbitrary close deformations. Easy examples can be found in products of two trees, taking in any flat identified with $\mathbb{R} \times \mathbb{R}$ a "U"-path: for instance the piecewise affine path with successive vertices $x_0 = (0,1)$, $x_1 = (0,0)$, $x_2 = (1,0)$, $x_3 = (1,1)$. In Euclidean buildings of type A_2, an example is the piecewise affine path σ in $\mathbb{A}$ with vertices $x_0 = [(-1,2,-1)]$, $x_1 = 0$, $x_2 = [(2,-1,-1)]$ and $x_3 = [(3,0,-3)]$, which is a local $\mathfrak{C}$-geodesic but not globally $\mathfrak{C}$-geodesic (see Figure 5). This phenomenon makes it hard to prove global preservation of the $\mathfrak{C}$-distance for maps between subset of Euclidean buildings, since it is not enough to check it locally.

A local to global criterion. For piecewise regular $\mathfrak{C}$-geodesic paths, we have the following fundamental local-to-global property:

Corollary 1.6. *Let $(x_n)_n$ be a (finite or not) sequence in X. Suppose that for all n the segment $[x_n, x_{n+1}]$ is regular, and the path (x_{n-1}, x_n, x_{n+1}) is $\mathfrak{C}$-geodesic. Then the whole path $(x_n)_n$ is $\mathfrak{C}$-geodesic.*

FIGURE 5. A local, but not global, $\mathfrak{C}$-geodesic in $\mathbb{A}$.

We now state a criterion for a general locally $\mathfrak{C}$-geodesic piecewise affine path to be $\mathfrak{C}$-geodesic, which will be used in the proof of the main theorem (Section 4.4).

Proposition 1.7. *Suppose that* $\dim \mathbb{A} = 2$. *Let* $(x_n)_n$ *be a (finite or not) sequence in* X, *such that for all* n *the point* x_n *is not in the segment* $[x_{n-1}, x_{n+1}]$. *Suppose that:*

  (i) *(local $\mathfrak{C}$-geodesic) For all n the directions $\Sigma_{x_n} x_{n-1}$ and $\Sigma_{x_n} x_{n+1}$ are $\mathfrak{C}$-opposite in $\Sigma_{x_n} X$.*
  (ii) *For all n such that $[x_{n-1}, x_n]$ is singular, $\Sigma_{x_n} x_{n-2}$ and $\Sigma_{x_n} x_{n+1}$ are $\mathfrak{C}$-opposite in $\Sigma_{x_n} X$.*

Then $(x_n)_n$ is $\mathfrak{C}$-geodesic.

Note that all involved directions are well defined, since we have $x_n \neq x_{n-1}, x_{n+1}$ for all n, and hypothesis (i) implies that $x_{n-1} \neq x_{n+1}$ for all n.

Proof. Suppose that $(x_0, x_1, \ldots, x_n)$ is $\mathfrak{C}$-geodesic for some $n \geq 2$. In the spherical building $\Sigma_{x_n} X$ of directions at x_n, Proposition 1.4 implies the following inclusions of simplices: $Fac_{x_n}(x_{n-1}) \subset Fac_{x_n}(x_{n-2}) \subset Fac_{x_n}(x_0)$. Note that, since x_{n-1} is not in $[x_n, x_{n-2}]$, the segment $[x_n, x_{n-2}]$ is necessarily regular, hence $Fac_{x_n}(x_{n-2})$ is a closed chamber (i.e. a maximal simplex), and then $Fac_{x_n}(x_{n-2}) = Fac_{x_n}(x_0)$.

If the segment $[x_{n-1}, x_n]$ is regular, then $Fac_{x_n}(x_{n-1}) = Fac_{x_n}(x_{n-2}) = Fac_{x_n}(x_0)$. By hypothesis $\Sigma_{x_n} x_{n+1}$ is in a closed chamber opposite to the closed chamber $Fac_{x_n}(x_{n-1}) = Fac_{x_n}(x_0)$, hence $\Sigma_{x_n} x_0$ is $\mathfrak{C}$-opposite to $\Sigma_{x_n} x_{n+1}$.

If the segment $[x_{n-1}, x_n]$ is singular, then by hypothesis $\Sigma_{x_n} x_{n+1}$ is in a closed chamber opposite to the closed chamber $Fac_{x_n}(x_{n-2}) = Fac_{x_n}(x_0)$, hence $\Sigma_{x_n} x_0$ is also $\mathfrak{C}$-opposite to $\Sigma_{x_n} x_{n+1}$.

Then in all cases x_0, x_n, x_{n+1} is $\mathfrak{C}$-geodesic (Proposition 1.4), and it follows that $(x_0, x_1, \ldots, x_{n+1})$ is $\mathfrak{C}$-geodesic. $\qquad\square$

2. FOCK-GONCHAROV PARAMETERS FOR SURFACE GROUP REPRESENTATIONS

In this section, following Fock and Goncharov [FoGo07], we explain in detail how to build representations of a punctured surface group in $\mathrm{PGL}_3(\mathbb{K})$ for any field $\mathbb{K}$ using ideal triangulations and projective geometry. The goal is to define the representation $\rho_{(Z,S)}$ associated with a *FG-parameter* $(Z, S) = ((Z_\tau)_\tau, (S_e)_e)$. Note that our edge parameters S_e are in fact a slight

modification of those in [FoGo07], and are more symmetric with respect to natural point-line duality (see §2.6 for the precise relationship).

In this section, $\mathbb{K}$ is any field and $\mathbb{P} = \mathbb{P}(\mathbb{K}^3)$.

2.1. **Surfaces and ideal triangulations.** Consider a compact oriented connected surface Σ with non empty boundary and negative Euler characteristic $\chi(\Sigma) < 0$. Boundary components of Σ are oriented in such a way that the surface lies to their right. They will also be seen as punctures. Let $\Gamma = \pi_1(\Sigma)$ be the fundamental group of Σ.

We denote by $\mathcal{F}_\infty(\Sigma)$ the *Farey set* of Σ, which may be defined as the set of boundary components of the universal cover $\widetilde{\Sigma}$ of Σ (see [FoGo06, §1.3]). This set inherits a cyclic order from the orientation of the surface. For each $i \in \mathcal{F}_\infty(\Sigma)$, we denote by γ_i the corresponding element of Γ, i.e the primitive element translating the boundary component i in the positive direction. Then for the induced order on $\mathcal{F}_\infty(\Sigma) - \{i\}$, we have $\gamma_i(j) > j$ for all $j \neq i$. The fundamental group $\Gamma = \pi_1(\Sigma)$ acts on the Farey set $\mathcal{F}_\infty(\Sigma)$, and γ_i fixes i for each $i \in \mathcal{F}_\infty(\Sigma)$.

Let $\mathcal{T}$ be an ideal triangulation of Σ, i.e a triangulation with vertices the boundary components, considered as punctures. We denote by $T(\mathcal{T})$ the set of triangles of $\mathcal{T}$ and by $\overrightarrow{E}(\mathcal{T})$ the set of oriented edges of $\mathcal{T}$. Lift $\mathcal{T}$ to an ideal triangulation $\widetilde{\mathcal{T}}$ of the universal cover $\widetilde{\Sigma}$ of Σ. The set of vertices of $\widetilde{\mathcal{T}}$ then identifies with the Farey set $\mathcal{F}_\infty(\Sigma)$ of Σ. We will identify the oriented edges e of $\widetilde{\mathcal{T}}$ with the corresponding pairs (i, j) of points in $\mathcal{F}_\infty(\Sigma)$ (vertices of e). A *marked* triangle of $\widetilde{\mathcal{T}}$ is a triple (i, j, k) of points in $\mathcal{F}_\infty(\Sigma)$ that are the common vertices of a triangle of $\widetilde{\mathcal{T}}$.

2.2. **Cross ratio.** We use the following convention for cross ratios (following Fock-Goncharov [FoGo07]). When V is a two dimensional vector space over a field $\mathbb{K}$, the cross ratio of a four points a_1, a_2, a_3, a_4 in the projective line $\mathbb{P}(V)$ is defined by

$$(2.1) \qquad \mathbf{b}(a_1, a_2, a_3, a_4) = \frac{(a_1 - a_2)(a_3 - a_4)}{(a_1 - a_4)(a_2 - a_3)}$$

in any affine chart $\mathbb{P}(V) \xrightarrow{\sim} \mathbb{K} \cup \{\infty\}$, that is in order that $\mathbf{b}(\infty, -1, 0, a) = a$. It is well-defined (in $\mathbb{K} \cup \{\infty\}$) when the quadruple is nondegenerated, i.e. when either the numerator or the denominator is nonzero (see section 1.1).

We now recall the natural symmetries. For a permutation σ in $\mathfrak{S}_4$, we denote

$$(\sigma \cdot \mathbf{b})(a_1, a_2, a_3, a_4) = \mathbf{b}(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) \ .$$

Recall that $\sigma \cdot \mathbf{b} = \mathbf{b}$ when σ is any the double transpositions, that $\sigma \cdot \mathbf{b} = \mathbf{b}^{-1}$ when σ is (13), (24), (1234) or (1432) ; and that $(234) \cdot \mathbf{b} = -(1 + \mathbf{b}^{-1})$ and $(243) \cdot \mathbf{b} = -(1 + \mathbf{b})^{-1}$.

The cocycle identity is

$$(2.2) \qquad -\mathbf{b}(a_1, a_2, a_3, a_4)\,\mathbf{b}(a_1, a_4, a_3, a_5) = \mathbf{b}(a_1, a_2, a_3, a_5)$$

2.3. **Triple ratio of a triple of flags.** We refer the reader to [FoGo06, §9.4 p128]. Let $F_i = (p_i, D_i)$, $i = 1, 2, 3$, be a triple of flags in $\mathbb{P} = \mathbb{P}(\mathbb{K}^3)$. The *triple ratio* of the triple (F_1, F_2, F_3) is defined by

$$\text{Tri}(F_1, F_2, F_3) = \frac{\tilde{D}_1(\tilde{p}_2)\tilde{D}_2(\tilde{p}_3)\tilde{D}_3(\tilde{p}_1)}{\tilde{D}_1(\tilde{p}_3)\tilde{D}_2(\tilde{p}_1)\tilde{D}_3(\tilde{p}_2)}$$

where $\tilde{p}_i$ is any vector in $\mathbb{K}^3$ representing p_i and $\tilde{D}_i$ is any linear form in $(\mathbb{K}^3)^*$ representing D_i. It is well defined (in $\mathbb{K} \cup \{\infty\}$) when the triple (F_1, F_2, F_3) is nondegenerated, i.e. when either the numerator or denominator are nonzero (see section 1.1).

Note that $\text{Tri}(F_1, F_2, F_3) = \infty$ if and only if there exists i such that $p_i \in D_{i+1}$ and that $\text{Tri}(F_1, F_2, F_3) = 0$ if and only if there exists i such that $p_i \in D_{i-1}$. In particular, the three flags are pairwise opposite if and only if their triple ratio is not 0 or ∞. The triple ratio is invariant under cyclic permutation of the flags: and reversing the order inverses the triple ratio:

$$\text{Tri}(F_2, F_3, F_1) = \text{Tri}(F_1, F_2, F_3)$$
$$\text{Tri}(F_3, F_2, F_1) = \text{Tri}(F_1, F_2, F_3)^{-1} \ .$$

The triple ratio may be expressed as the following cross ratio on the naturally induced quadruples of lines at p_1 (which is nondegenerated, see section1.1)

(2.3) $$\text{Tri}(F_1, F_2, F_3) = \mathbf{b}(D_1, p_1p_2, p_1p_{23}, p_1p_3)$$

or on the line D_1

$$\text{Tri}(F_1, F_2, F_3) = \mathbf{b}(D_1 \cap D_2, D_1 \cap D_{23}, D_1 \cap D_3, p_1)$$
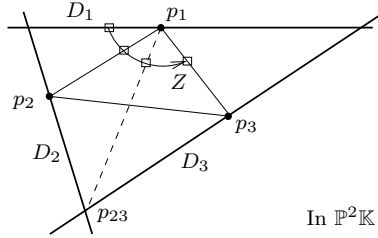

FIGURE 6. The triple ratio $Z = \text{Tri}(F_1, F_2, F_3)$ as a cross ratio.

Generic triples may be characterized by triple ratio: (F_1, F_2, F_3) is generic if and only if $\text{Tri}(F_1, F_2, F_3) \neq \infty, 0, -1$. The triple ratio parametrize the generic triples of flags in the projective plane, more precisely for each $a \in \mathbb{K}_{\neq 0, -1}$ there exists a generic triple of flags in $\mathbb{P}$ with triple ratio a, and $\text{PGL}(\mathbb{K}^3)$ acts 1-transitively on the set of generic triples of flags of given triple ratio (see also Lemma 2.2).

2.4. **FG-invariants of a transverse flag map.** Consider a *flag map*

$$F : \mathcal{F}_\infty(\Sigma) \mapsto \text{Flags}(\mathbb{P}) \ .$$

We denote by p_i (resp. by D_i) the point (resp. the line) of the flag $F_i = F(i)$, for $i \in \mathcal{F}_\infty(\Sigma)$. Let $\widetilde{\mathcal{T}}$ an ideal triangulation of $\widetilde{\Sigma}$. We suppose that F and $\widetilde{\mathcal{T}}$ are *transverse* that is that F sends each triangle in $\widetilde{\mathcal{T}}$ to a generic triple

of flags. We denote by p_{ij} the point $D_i \cap D_j$ (resp. by D_{ij} the line $p_i p_j$) (when defined).

To each triangle τ of $\widetilde{\mathcal{T}}$ with vertices (i, j, k) in $\mathcal{F}_\infty(\Sigma)$, we associate a *triangle invariant*: the triple ratio

$$Z_\tau = \mathrm{Tri}(F_i, F_j, F_k) = \mathbf{b}(D_i, p_i p_j, p_i p_{jk}, p_i p_k)$$

of the triple of flags $F(\tau)$ (where i, j, k are cyclically ordered accordingly to the orientation of the surface). It is well defined and in $\mathbb{K}_{\neq 0, -1}$ as $F(\tau)$ is a generic triple of flags.

To each an oriented edge $e = (k, i)$ in $\widetilde{\mathcal{T}}$, we associate an *edge invariant*: the cross ratio

$$S_e = \mathbf{b}(D_i, p_i p_j, p_i p_k, p_i p_{k\ell}) = \mathbf{b}(p_k, D_k \cap D_\ell, D_k \cap D_i, D_k \cap D_{ij})$$

where i, j, k, ℓ in $\mathcal{F}_\infty(\Sigma)$ are the vertices of the two adjacent triangles $\tau = (i, j, k)$ and $\tau' = (k, \ell, i)$, cyclically ordered accordingly to the orientation of the surface (see figure 7). Since $F(\tau)$ and $F(\tau')$ are generic, this is well defined and in $\mathbb{K}_{\neq 0}$.
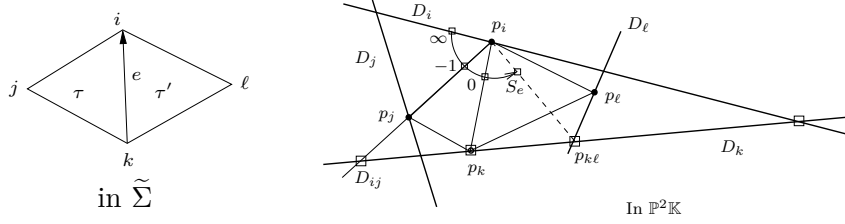

FIGURE 7. The invariant S_e associated with an oriented edge e.

Note that the edge parameters are symmetric with respect to natural duality, as reversing the orientation of e (i.e. applying the half-turn $(ik)(j\ell)$) we get

$$S_{\bar{e}} = \mathbf{b}(D_k, p_k p_\ell, p_k p_i, p_k p_{ij})$$

so exchanging the roles of points and lines correspond to exchange S_e and $S_{\bar{e}}$.

Thus we have a well-defined *FG-invariant*

$$(Z, S) = ((Z_\tau)_\tau, (S_e)_e)$$

in $(\mathbb{K}_{\neq 0, -1})^{T(\widetilde{\mathcal{T}})} \times (\mathbb{K}_{\neq 0})^{\overrightarrow{E}(\widetilde{\mathcal{T}})}$ of the flag map F *with respect to the triangulation* $\widetilde{\mathcal{T}}$.

2.5. **Construction of flag maps from FG-parameters.** We now show that FG-invariants $(Z, S) = ((Z_\tau)_\tau, (S_e)_e)$ in $(\mathbb{K}_{\neq 0, -1})^{T(\widetilde{\mathcal{T}})} \times (\mathbb{K}_{\neq 0})^{\overrightarrow{E}(\widetilde{\mathcal{T}})}$ parametrize $\widetilde{\mathcal{T}}$-transverse flag maps $F : \mathcal{F}_\infty(\Sigma) \to \mathrm{Flags}(\mathbb{P})$ up to the action of $\mathrm{PGL}(\mathbb{K}^3)$.

Fix a base triangle τ_0 in the triangulation $\widetilde{\mathcal{T}}$ with (positively ordered) vertices (o_1, o_2, o_3) in $\mathcal{F}_\infty(\Sigma)$.

Proposition 2.1. [FoGo07] *Let* $(Z, S) = ((Z_\tau)_\tau, (S_e)_e)$ *be a* FG-parameter, *i.e. an element of* $(\mathbb{K}_{\neq 0, -1})^{T(\widetilde{\mathcal{T}})} \times (\mathbb{K}_{\neq 0})^{\overrightarrow{E}(\widetilde{\mathcal{T}})}$. *Fix a generic triple* $f = (F_1, F_2, p_3)$, *where* F_1, F_2 *are two flags in* $\mathbb{P}^2\mathbb{K}$ *and* p_3 *is a point in* $\mathbb{P}^2\mathbb{K}$.

There exists a unique map $F : \mathcal{F}_\infty(\Sigma) \mapsto \text{Flags}(\mathbb{P})$, transverse to $\widetilde{\mathcal{T}}$, such that the FG-invariant of F relatively to $\widetilde{\mathcal{T}}$ is (Z, S), and sending the points o_1, o_2 to the flags F_1, F_2, and the point o_3 to some flag through point p_3.

In order to normalize, we will denote by $F_{Z,S}$ the flag map F with FG-invariant (Z, S) such that the triple $F_{Z,S}(\tau_0) = (F_1, F_2, F_3)$ is in *canonical form*, that is $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $D_1 \cap D_2 = [0 : 0 : 1]$, $p_3 = [1 : 1 : 1]$ is the canonical projective frame.

Proof. Since the dual graph of the triangulation $\widetilde{\mathcal{T}}$ has no cycle (ie, is a tree), existence and unicity of F comes from the following basic facts, by induction on adjacent triangles.

Lemma 2.2. *Let $F_1 = (p_1, D_1)$, $F_2 = (p_2, D_2)$ be two flags in $\mathbb{P}$ and p_3 be a point in $\mathbb{P}$. Suppose that F_1, F_2 and p_3 are in generic position. Let $a \in \mathbb{K}_{\neq 0, -1}$. Then there exists a unique flag $F_3 = (p_3, D_3)$ such that the triple of flags (F_1, F_2, F_3) is generic and $\text{Tri}(F_1, F_2, F_3) = a$.* $\qquad\square$

Lemma 2.3. *Let (F_1, F_2, F_3) be a generic triple of flag. For all S, S' in $\mathbb{K}_{\neq 0}$ and Z' in $\mathbb{K}_{\neq 0, -1}$, there exists a unique flag F_4 such that*

$$S = \mathbf{b}(D_1, p_1 p_2, p_1 p_3, p_1(D_3 \cap D_4))$$

$$S' = \mathbf{b}(D_3, p_3 p_4, p_3 p_1, p_3(D_2 \cap D_1))$$

and the triple of flags (F_1, F_2, F_3) is generic and has triple ratio Z'.

Proof. Since F_1, F_2, and F_3 are in generic position, they define three pairwise distinct points $D_3 \cap D_1$, $D_3 \cap (p_1 p_2)$, and p_3 on the line D_3. So there exists a unique point p on D_3 such that $\mathbf{b}(D_3 \cap D_1, D_3 \cap (p_1 p_2), p_3, p) = S$.

Similarly, we have three pairwise distinct lines D_3, $p_3(D_2 \cap D_1)$, $p_3 p_1$ through point p_3, hence there exists a unique line Δ through p_3 such that $\mathbf{b}(D_3, \Delta, p_3 p_1, p_3(D_2 \cap D_1)) = S'$, and $p_1 \notin \Delta$ as $S' \neq \infty$.

Since $S \neq 0, \infty$, we have $p \neq p_3$ and $p \notin D_1$, hence we have three pairwise distinct lines $D_1, p_1 p_3, p_1 p$ at p_1, and there exists a unique line Δ' through p_1 satisfying $\mathbf{b}(D_1, p_1 p_3, p_1 p, \Delta') = Z'$, and $p_3 \notin \Delta'$ as $Z' \neq -1$. We have $\Delta \neq \Delta'$ (else $p_1 \in \Delta$) so Δ and Δ' intersects in a unique point p_4 with $p_4 \notin D_1, D_3$, and $p_4 \notin p_1 p_3$. Then $p \neq p_4$ (else $p \in \Delta'$ and $\Delta' = p_1 p$ and $Z' = 0$) so we may define $D_4 = p_4 p$, and then $D_4 \neq D_1, D_3$. We have $p_3 \notin D_4$ as $D_4 \cap D_3 = p \neq p_3$. Since $\Delta' = p_1 p_4$ is different from $p_1 p$ (since $Z' \neq 0$), we have that $p_1 \notin D_4$. Since $p = D_4 \cap D_3$ is different from $D_1 \cap D_3$, we have that $D_1 \cap D_3$ is not on D_4. Therefore the triple (F_1, F_3, F_4) is generic and its triple ratio is $\mathbf{b}(D_1, p_1 p_3, p_1 p, p_1 p_4) = Z'$ as $p_1 p_4 = \Delta'$. $\qquad\square$

$\qquad\square$

2.5.1. *Equivariance and construction of representations.* We now suppose that $\widetilde{\mathcal{T}}$ is the lift of an ideal triangulation $\mathcal{T}$ of Σ and that $(\widetilde{Z}, \widetilde{S})$ is a FG-parameter on $\widetilde{\mathcal{T}}$ invariant under $\Gamma = \pi_1(\Sigma)$, i.e. lifting a FG-parameter (Z, S) on $\mathcal{T}$. We denote $F_{Z,S} = F_{\widetilde{Z}, \widetilde{S}}$. We now show that, since $\text{PGL}(\mathbb{K}^3)$ acts 1-transitively on generic triples of flags of given triple ratio, by rigidity of the construction, we have an associated *holonomy* representation.

Proposition 2.4. *Let* $(Z, S) = ((Z_\tau)_\tau, (S_e)_e)$ *in* $(\mathbb{K}_{\neq 0, -1})^{T(\mathcal{T})} \times (\mathbb{K}_{\neq 0})^{\overrightarrow{E}(\mathcal{T})}$, *and let* $F = F_{Z,S}$. *There exists a unique representation* $\rho : \Gamma \to \mathrm{PGL}(\mathbb{K}^3)$ *such that* F *is* ρ-equivariant, i.e. $\rho(\gamma)F_{Z,S}(i) = F(\gamma i)$ *for all* $\gamma \in \Gamma$, $i \in \mathcal{F}_\infty(\Sigma)$. *We will denote* $\rho = \rho_{Z,S}$ *and call it the representation with FG-parameter* (Z, S).

In particular $F_{Z,S}(i)$ is a flag fixed by $\rho_{Z,S}(\gamma_i)$. Note that different choices of (Z, S) may lead to the same representation $\rho_{Z,S}$.

Proof. Let $\gamma \in \Gamma$. The triples of flags $F(\gamma\tau_0)$ and $F(\tau_0)$ have same triple ratio $\tilde{Z}_{\gamma\tau_0} = \tilde{Z}_{\tau_0} \neq -1$, so there exists a unique g in $\mathrm{PGL}(\mathbb{K}^3)$ such that $gF(\tau_0) = F(\gamma\tau_0)$. We set then $\rho(\gamma) = g$. The maps $\rho(\gamma) \circ F$ and $F \circ \gamma$ from $\mathcal{F}_\infty(\Sigma)$ to $\mathrm{Flags}(\mathbb{P})$ have same FG-invariant $(\tilde{Z}, \tilde{S}) = (\tilde{Z}, \tilde{S}) \circ \gamma : T(\widetilde{\mathcal{T}}) \cup \overrightarrow{E}(\widetilde{\mathcal{T}}) \to \mathbb{K}$ with respect to $\widetilde{\mathcal{T}}$, and send the base triangle τ_0 to the same generic triple of flags, hence they coincide by Proposition 2.1. The fact that ρ is a morphism follows then from 1-transitivity on generic triples of flags, since:

$$\rho(\gamma_1\gamma_2)F(\tau_0) = F(\gamma_1\gamma_2\tau_0) = \rho(\gamma_1)F(\gamma_2\tau_0) = \rho(\gamma_1)\rho(\gamma_2)F(\tau_0) \ . \qquad \square$$

2.6. **Other edge invariants and relation with** [FoGo07]**.** Note that our edge invariants S_e differ sligthly from those of [FoGo07]. We here describe the relationship in detail. We use the setting of section 2.4.

Let i, j, k, ℓ in $\mathcal{F}_\infty(\Sigma)$ be the vertices of two adjacent triangles $\tau = (i, j, k)$ and $\tau' = (k, \ell, i)$ with common edge $e = (k, i)$. The associated invariants X, Y, Z, W of [FoGo07] are in our settings $X = Z_\tau$, $Y = Z_{\tau'}$, $Z = Z_e$, and $W = Z_{\bar{e}}$, where Z_e denotes the following cross-ratio

$$Z_e = \mathbf{b}(D_i, p_i p_j, p_i p_k, p_i p_\ell) \ .$$

The edge invariant Z_e is not symmetric under duality, yet exchanging the roles of points and lines provide another natural invariant

$$Z_e^* = \mathbf{b}(p_i, D_i \cap D_j, D_i \cap D_k, D_i \cap D_\ell) \ .$$

Our edge invariants S_e are then easily related to the original Z_e by (using the cocycle identity):

$$(2.4) \qquad \begin{aligned} Z_e &= S_e(1 + Z_{\tau'}) \\ Z_e^* &= S_{\bar{e}}(1 + Z_{\tau'}^{-1}) \end{aligned} \ .$$

In particular, when $\mathbb{K}$ is an ordered field, then if the triangle invariants are positive, our edge invariants are positive if and only if the usual edge invariants are positive.

Note that the relation linking usual FG-invariants of two adjacent triangles

$$(2.5) \qquad Z_e^* = Z_{\bar{e}} \frac{1}{1 + Z_\tau}(1 + Z_{\tau'}^{-1})$$

(compare [FoGo07, 2.5.3]) follows from (2.4) and from the autoduality of the S_e, since reversing the edge e we get

$$S_e = Z_e(1 + Z_{\tau'})^{-1} = Z_{\bar{e}}^*(1 + Z_\tau^{-1})^{-1}$$

$$S_{\bar{e}} = Z_{\bar{e}}(1 + Z_\tau)^{-1} = Z_e^*(1 + Z_{\tau'}^{-1})^{-1} \ .$$

3. THE A_2-COMPLEX K ASSOCIATED WITH A LEFT-SHIFTING (z, s)

3.1. $(\mathbb{A}, W)$-complexes and $\frac{1}{3}$-translation surfaces. In this section, we introduce the notion of W-translation surfaces, generalizing translation and half-translation surfaces, and the more general notion of $(\mathbb{A}, W)$-complexes. Natural examples are subcomplexes of Euclidean buildings with model flat $(\mathbb{A}, W)$. We show that, like Euclidean buildings, these spaces are naturally endowed with a $\mathfrak{C}$-valued metric and associated $\mathfrak{C}$-distance (where $\mathfrak{C}$ is a standard fixed Weyl chamber in $\mathbb{A}$).

3.1.1. W-surfaces. Let $\mathbb{A}$ be a Euclidean vector plane and let W be a finite subgroup of isometries of $\mathbb{A}$. A W-*translation surface* consists of a compact surface M possibly with boundary, a finite set of interior points $M_0 \subset M$ (*singularities*) and a $(W_{aff}, \mathbb{A})$-structure on $M - M_0$ i.e. an atlas of charts $\phi_\mu : U_\mu \to \mathbb{A}$ with transition maps in $W_{aff} = W \ltimes \mathbb{A}$. This atlas induces in particular a flat metric on $M - M_0$, and we require that each singular point $x \in M_0$ has a neighborhood U such that $U - \{x\}$ is isometric to a punctured cone.

For $W = \{\text{id}\}$ (resp. for $W = \{\pm \text{id}\}$) it corresponds to the classic notion of *translation surface* (resp. of *half-translation surface*) (see for example [Mas06], [Yoc10]).

By analogy, we will call a $\frac{1}{3}$-*translation surface* a W-translation surface with W the subgroup of rotations of angle in $\frac{2\pi}{3}\mathbb{Z}$.

3.1.2. $(\mathbb{A}, W)$-complexes. In this section, $(\mathbb{A}, W)$ is a finite reflection group of dimension two. We recall that W_{aff} is the subgroup of affine isomorphisms of $\mathbb{A}$ with linear part in W.

Intuitively speaking, a $(\mathbb{A}, W)$-*complex* (or W-*complex*, or A_2-*complex* when W is of type A_2) is a space K obtained by gluing polygons of $\mathbb{A}$ along boundary segments by elements of W_{aff}.

We now give a precise definition of $(\mathbb{A}, W)$-simplicial complexes following the definition of Euclidean simplicial complexes in [BrHa, I.7.2].

Definition 3.1. $((\mathbb{A}, W)$- simplicial complex) Let $\{P^\mu, \ \mu \in \mathcal{M}\}$ be a family of affine simplices $P^\mu \subset \mathbb{A}$. Let $E = \sqcup_{\mu \in \mathcal{M}} P^\mu \times \{\mu\}$ denote their disjoint union. Let $\simeq$ be an equivalence relation on E and let $K = E/\simeq$ denote the quotient space. Let $\phi : E \to K$ denote the corresponding projection, define $\phi_\mu : P^\mu \to K$ by $\phi_\mu(\alpha) = \phi(\alpha, \mu)$, and denote by $K^\mu \subset K$ the image $\phi_\mu(P^\mu)$.

The space K is called a $(\mathbb{A}, W)$-*simplicial complex* if

(i) for every $\mu \in \mathcal{M}$, the map ϕ_μ is injective.
(ii) If $K^\mu \cap K^{\mu'} \neq \emptyset$, then there is an element $w_{\mu,\mu'}$ of W_{aff} such that for all $\alpha \in P^\mu$ and $\alpha' \in P^{\mu'}$ we have $\phi(\alpha, \mu) = \phi(\alpha', \mu')$ if and only if $\alpha' = w_{\mu,\mu'}(\alpha)$, and $P^{\mu,\mu'} = P^\mu \cap w_{\mu,\mu'}^{-1}(P^{\mu'})$ is a face of P^μ.

In particular, K is a Euclidean simplicial complex of dimension 2. We will suppose from now on that K is connected and that the set of isometry classes of simplices of K is finite. We denote by d the associated metric, which is a complete geodesic length metric (see [BrHa, I.7]). We denote by $\Sigma_x K$ the geometric link of K at a point x, which is a spherical 1-dimensional

complex (hence a metric graph) endowed with the angular length metric $\lhd$ (see [BrHa, I.7.15]).

From now on, we will suppose that K has non positive curvature, that is for all points $x \in K$ each injective loop in the link $\Sigma_x K$ has length at least 2π. If K is simply connected, (K, d) is then a CAT(0) metric space (see Theorem I.5.4 and Lemma I.5.6 of [BrHa]).

3.1.3. $\mathfrak{C}$-*distance.* Germs of non trivial segments at a point $x \in K$ have a well-defined projection in $\partial\overline{\mathfrak{C}}$ (their (*type (of direction)*)). In particular the notions of regular and singular directions still make sense in $\Sigma_x K$. Note that a geodesic segment is not necessarily of constant type of direction, unlike in Euclidean buildings. The $\mathfrak{C}$-*length* $\ell^{\mathfrak{C}}(I)$ of a segment $I = [x, y]$ contained in a simplex K^{μ} of K is defined as the $\mathfrak{C}$-length in $\mathbb{A}$ of the segment $\phi_{\mu}^{-1}(I)$ (note that it does not depend on the choice of μ, because the transition maps are in W_{aff}). The $\mathfrak{C}$-*length* of a piecewise affine path $\sigma : [0, s] \to K$ in X is defined by $\ell^{\mathfrak{C}}(\sigma) = \sum_n \ell^{\mathfrak{C}}([x_n, x_{n+1}])$ for one (any) subdivision $t_0 = 0 < t_1 < \cdots < t_N = s$ of $[0, s]$ such that the restriction of σ to $[t_n, t_{n+1}]$ is an affine segment $[x_n, x_{n+1}]$ conatined in some simplex of K. It is a vector in the closed Weyl chamber $\overline{\mathfrak{C}}$. It is invariant under subdivisions of the simplicial complex K. When K is simply connected (hence CAT(0)), we define the $\mathfrak{C}$-*distance* from x to y in K as the $\mathfrak{C}$-length $d^{\mathfrak{C}}(x, y) = \ell^{\mathfrak{C}}(\sigma)$ of the geodesic σ from x to y. We then have

$$d^{\mathfrak{C}}(y, x) = d^{\mathfrak{C}}(x, y)^{\mathrm{opp}}$$

and

$$\left\| d^{\mathfrak{C}}(x, y) \right\| \leq d(x, y)$$

Remark 3.2. Note that, unlike in Euclidean buildings, the inequality may well be strict. Thus the $\mathfrak{C}$-distance is no longer a refinement of the distance d. A basic example is given by non convex subsets K of $\mathbb{A}$.

3.1.4. $\mathfrak{C}$-*Length of automorphisms.* An *automorphism g* of K is a bijection preserving $d^{\mathfrak{C}}$. In particular it preserves the distance d. The $\mathfrak{C}$-*length* of g of K translating some geodesic σ is defined by

$$\ell^{\mathfrak{C}}(g) = d^{\mathfrak{C}}(x, gx)$$

for one (any) x on σ (it does not depend on the choice of σ as two different translated geodesics bound a flat strip, and may be developped as parallel geodesics in $\mathbb{A}$).

Note that, in contrast to the case of Euclidean buildings, the $\mathfrak{C}$-length do no longer refine the *Euclidean length*

$$\ell_{euc}(g) = \{= \{d(x, gx), \ x \in X\} \, .$$

We have

$$\left\| \ell^{\mathfrak{C}}(g) \right\| \leq \ell_{euc}(g)$$

but the inequality may be strict.

3.2. **Abstract geometric FG-parameters and Left Shift.** Let $\mathcal{T}$ be an ideal triangulation of a punctured surface Σ, with set of triangles T and set of oriented edges $\overrightarrow{E}$. Consider an *geometric FG-parameter* on $\mathcal{T}$, i.e. an element $(z, s) = ((z_\tau)_\tau, (s_e)_e)$ in $\mathbb{R}^T \times \mathbb{R}^{\overrightarrow{E}}$. We now introduce the class of abstract geometric FG-parameters (z, s) to which we are going to associate an A_2-complex K. Let e be an oriented edge in $\mathcal{T}$, with left and right adjacent triangles τ and τ'.

Definition 3.3. We say that (z, s) is *left-shifting on edge e* if we have $s_e > -z_\tau^-, -z_{\tau'}^+$ and $s_{\overline{e}} > -z_\tau^+, -z_{\tau'}^-$.

Remark 3.4. Then we are in one (and only one) of the three following cases:
- (i) $s_e > 0$ and $s_{\overline{e}} > 0$
- (ii) $z_\tau < 0$, $z_{\tau'} > 0$, $s_{\overline{e}} > 0$ and $z_\tau, -z_{\tau'} < s_e \leq 0$;
- (iii) $z_\tau > 0$, $z_{\tau'} < 0$, $s_e > 0$ and $z_{\tau'}, -z_\tau < s_{\overline{e}} \leq 0$;

We say that (z, s) is *left-shifting* if it is left-shifting on edge e for all e, and we denote this property by (L). Note that the subset O_L of left-shifting (z, s) in $\mathbb{R}^T \times \mathbb{R}^{\overrightarrow{E}}$ is an non empty open cone (in fact a finite union of open convex polyhedral cones).

3.3. **Construction of the A_2-complex K associated with (z, s).** Consider a left-shifting geometric FG-parameter $(z, s) = ((z_\tau)_\tau, (s_e)_e)$ in $\mathbb{R}^T \times \mathbb{R}^{\overrightarrow{E}}$ (see Definition 3.3). Lift (z, s) on the universal cover $\widetilde{\mathcal{T}}$ in a Γ-invariant left-shifting geometric FG-parameter, again denoted by (z, s).

For each marked triangle $\tau = (i, j, k)$ of the triangulation $\widetilde{\mathcal{T}}$ let $P^\tau \subset \mathbb{A}$ be the singular equilateral triangle with vertices $\alpha_i^\tau = 0$, $\alpha_j^\tau = (-z_\tau^-, -z_\tau^+)$ and $\alpha_k^\tau = (-z_\tau^+, -z_\tau^-)$ (in simple roots coordinates). Note that P^τ lies in the chamber $-\overline{\mathfrak{C}}$.
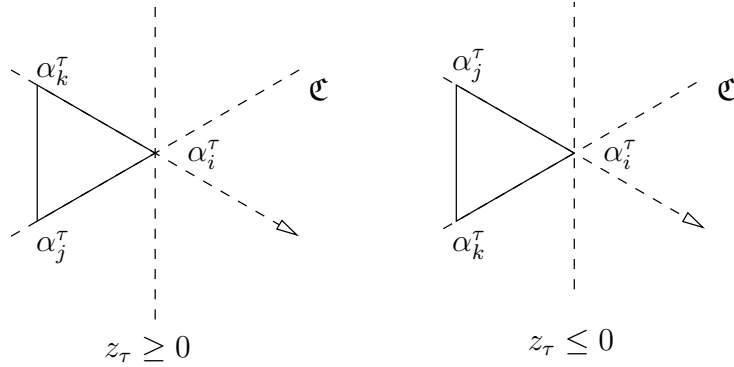

FIGURE 8. Singular triangle P^τ in $\mathbb{A}$.

For each oriented edge $e = (k, i)$ of $\widetilde{\mathcal{T}}$ let $P^e \subset \mathbb{A}$ be either, when $s_e, s_{\overline{e}} \geq 0$, the closed segment from 0 to the point $(s_{\overline{e}}, s_e)$, or, when $s_e < 0$ or $s_{\overline{e}} < 0$, the parallelogram given (in simple roots coordinates) by $P^e = [0, s_{\overline{e}}] \times [0, s_e]$ (intersection of two Weyl chambers of opposite direction).

We now describe formally how $\widetilde{K}$ is constructed, gluing the polygons P^m. We define $\widetilde{K} = \sqcup_{m \in \overrightarrow{\mathcal{M}}} P^m \times \{m\} / \sim$, where $\overrightarrow{\mathcal{M}} = \overrightarrow{T}(\widetilde{\mathcal{T}}) \cup \overrightarrow{E}(\widetilde{\mathcal{T}})$

and $\sim$ is the equivalence relation generated by the following identifications: For every oriented edge $e = (k,i)$ of $\widetilde{\mathcal{T}}$, with positively oriented adjacent triangle $\tau = (i,j,k)$, Remark 3.4 implies that the convex polygons P^τ and P^e intersects on the subsegment (maybe reduced to a point) $[\beta_{ki}^\tau, \alpha_i^\tau]$ of $[\alpha_k^\tau, \alpha_i^\tau]$, with $\beta_{ki}^\tau = (\min(0, s_{\overline{e}}), \min(0, s_e))$.

We then glue $P^\tau \times \{\tau\}$ along $P^e \times \{e\}$ along this segment (i.e. by $(\alpha, \tau) \sim (\alpha, e)$ for $\alpha \in P^\tau \cap P^e$).
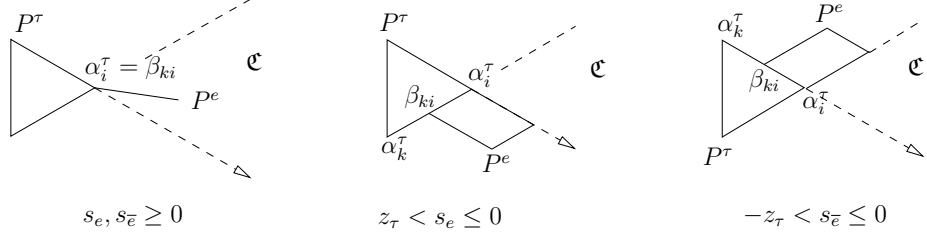

$$s_e, s_{\overline{e}} \geq 0 \qquad\qquad z_\tau < s_e \leq 0 \qquad\qquad -z_\tau < s_{\overline{e}} \leq 0$$

FIGURE 9. Gluings.

If τ' is a permutation of a marked triangle $\tau = (i,j,k)$ in $\widetilde{\mathcal{T}}$, we identify $P^{\tau'} \times \{\tau'\}$ with $P^\tau \times \{\tau\}$ by the unique affine isomorphism $w_{\tau,\tau'} : P^\tau \xrightarrow{\sim} P^{\tau'}$ sending α_s^τ to $\alpha_s^{\tau'}$ for $s = i,j,k$, which is in W_{aff}.

If $\overline{e} = (i,k)$ is the opposite edge of $e = (k,i)$, then we identify $P^{\overline{e}} \times \{\overline{e}\}$ with $P^e \times \{e\}$ by the unique affine isomorphism $w_{e,\overline{e}} : P^e \xrightarrow{\sim} P^{\overline{e}}$ with linear part $w^{\mathrm{opp}} \in W$ (which sends 0 to $(s_e, s_{\overline{e}})$).

We denote by $\phi : \sqcup_m P^m \to \widetilde{K}$ the canonical projection. We denote by $\widetilde{K}^m$ the image $\phi(P^m)$ in $\widetilde{K}$. The *canonical charts* are the restrictions $\phi_m : P^m \to \widetilde{K}^m$ of ϕ to P^m.

We thus obtain a two dimensional $(\mathbb{A}, W)$-complex $\widetilde{K}$ endowed with a free and cocompact action of Γ by automorphisms. We denote by $d^{\mathfrak{C}}$ the natural $\mathfrak{C}$-valued distance on $\widetilde{K}$ (see Section 3.1.3). The quotient K of $\widetilde{K}$ under Γ is the A_2-*complex associated with FG-parameter* (z,s) on the triangulation $\mathcal{T}$. It is a finite 2-dimensional complex homotopy equivalent to Σ. We denote by K^m the image of $\widetilde{K}^m$ in K.

We remember the special points in the above construction for later use: For each marked triangle $\tau = (i,j,k)$ of $\widetilde{\mathcal{T}}$ we denote by $a_i(\tau)$ and b_{ki} the points of $\widetilde{K}$ given by α_i^τ and β_{ki}^τ. If $z_\tau = 0$, then the corresponding singular triangle $\widetilde{K}^\tau$ is reduced to a point a_τ, and $\widetilde{K}$ is locally the union of three non trivial edges at a_τ.

Note that the A_2-complex $\widetilde{K}$ is naturally *oriented*, in the sense that the space of directions at each point $\Sigma_x \widetilde{K}$ and the boundary at infinity $\partial_\infty \widetilde{K}$ inherit a natural cyclic order from the orientation of the surface Σ.

3.4. **Particular cases: from trees to surfaces.**

3.4.1. *Tree.* A case of special interest is when all singular flat triangles K^τ are reduced to a point. The corresponding condition on (z,s) in $\mathbb{R}^T \times \mathbb{R}^{\overrightarrow{E}}$ is

$$(T) \begin{cases} z_\tau = 0 \text{ for all triangles } \tau \text{ of } \mathcal{T} \\ s_e > 0 \text{ for all oriented edges } e \text{ of } \mathcal{T} \end{cases}$$

Then $\widetilde{K}$ is a 3-valent ribbon tree isomorphic to the dual tree of the triangulation $\widetilde{\mathcal{T}}$, and K is a graph isomorphic to the dual graph of the triangulation $\mathcal{T}$. Both are endowed with a A_2-*structure* or $\mathfrak{C}$-*metric*, i.e. a $\mathfrak{C}$-valued function $e \mapsto \ell^{\mathfrak{C}}(e)$ on oriented edges satisfying $\ell^{\mathfrak{C}}(\bar{e}) = \ell^{\mathfrak{C}}(e)^{\mathrm{opp}}$.

3.4.2. *Tree of triangles.* Another - more general - particular case of interest is when is when all the K^e are segments. Then $\widetilde{K}$ is a "tree of triangles", obtained from the dual tree of the triangulation by replacing vertices by triangles. The corresponding condition on the left-shifting FG-parameter (z,s) in $\mathbb{R}^T \times \mathbb{R}^{\vec{E}}$ is

$$(\mathrm{TT})\ s_e \geq 0 \text{ for all oriented edge } e$$

Note that (L) is automatic if $s_e > 0$ for all e, and that (TT) implies (S).

3.4.3. *Surface.* At the other end of the spectrum, another particular case of special interest is when

$$(\mathrm{Sf})\ s_{\bar{e}} < 0 \text{ or } s_e < 0 \text{ for all oriented edge } e$$

Then $\widetilde{K}$ is a surface, homeomorphic to $\widetilde{\Sigma}$ (the thickening of the ribbon tree dual to the triangulation), hence K is a $\frac{1}{3}$-translation surface homeomorphic to Σ (see §3.1.1), with piecewise singular geodesic boundary and no interior singularities.

Remark 3.5. The subset of left-shifting (z,s) in $\mathbb{R}^T \times \mathbb{R}^{\vec{E}}$ satisfying (Sf) is not empty if and only if the triangulation $\mathcal{T}$ is 2-colourable (since z_τ and $z_{\tau'}$ have then opposite sign for adjacent triangles τ and τ'). It is a finite union of non empty open convex polyhedral cones, one for each 2-coloration of $\mathcal{T}$ (choice of prescribed signs for the triangle parameters).

4. NICE INVARIANT COMPLEXES FOR ACTIONS ON BUILDINGS

4.1. **Geometric FG-invariants for actions on buildings.** In this section, we introduce an analog of FG-invariants for actions on Euclidean buildings X of type A_2. These invariants take values in $\mathbb{R}$ and are defined by geometric cross ratios in the projective plane at infinity of X. In the algebraic case (i.e. when X is the Euclidean building of $\mathrm{PGL}(\mathbb{K}^3)$ for some ultrametric field $\mathbb{K}$), these *geometric FG-invariants* are obtained from the $\mathbb{K}$-valued usual FG-invariants by taking logarithms of absolute values. Note that the geometric invariants are substantially weaker than the algebraic invariants (since we take absolute values). The principal advantage is that geometric FG-invariants still make sense when the building X is exotic (non algebraic), whereas the usual FG-invariants are not defined anymore.

We recall that the set $\partial_F X$ of chambers at infinity (Furstenberg boundary) of the Euclidean building X is identified with the set $\mathrm{Flags}(\mathbb{P})$ of flags in the projective plane $\mathbb{P}$ at infinity of X, and that $\boldsymbol{\beta}$ denotes the $\mathbb{R}$-valued cross ratio on $\mathbb{P}$ induced by X (see Section 1.4.5).

Let $F : \mathcal{F}_\infty(\Sigma) \to \partial_F X$ be a *flag map*. We denote by p_i (resp. by D_i) the point (resp. the line) of the flag $F_i = F(i)$ for every $i \in \mathcal{F}_\infty(\Sigma)$. We suppose that the map F is equivariant under an action ρ of $\Gamma = \pi_1(\Sigma)$ on X. Let $\mathcal{T}$ be an ideal triangulation of Σ, and $\widetilde{\mathcal{T}}$ be the lift of $\mathcal{T}$ to $\widetilde{\Sigma}$.

We suppose that F is *transverse to* $\mathcal{T}$ i.e. sends each triangle of $\widetilde{\mathcal{T}}$ on a generic triple of ideal chambers. We recall that we denote by p_{ij} the point $D_i \cap D_j$ (resp. by D_{ij} the line $p_i p_j$) (when defined).

4.1.1. *Triangle invariants.* To each marked triangle $\tau = (i, j, k)$ of the triangulation $\widetilde{\mathcal{T}}$ we associate the *geometric triple ratio* (see [Par15a]) of the generic triple of chambers $F(\tau) = (F_i, F_j, F_k)$, which is the following triple of geometric cross ratios, taking values in $\mathbb{R}$, obtained from permutations of the four lines D_i, $p_i p_j$, $p_i p_{jk}$, $p_i p_k$ in $\mathbb{P}$ (cyclically permuting the three last ones) (see [Par15a] for details)

$$\begin{cases} z_\tau = \mathrm{tri}_1(F_i, F_j, F_k) &:= \boldsymbol{\beta}(D_i, p_i p_j, p_i p_{jk}, p_i p_k) \\ z'_\tau = \mathrm{tri}_2(F_i, F_j, F_k) &:= \boldsymbol{\beta}(D_i, p_i p_k, p_i p_j, p_i p_{jk}) \\ z''_\tau = \mathrm{tri}_3(F_i, F_j, F_k) &:= \boldsymbol{\beta}(D_i, p_i p_{jk}, p_i p_k, p_i p_j) \end{cases} .$$

We recall from [Par15a] the following basic properties. Each of z_τ, z'_τ and z''_τ is invariant under cyclic permutation of τ, and reversing the order gives $z_{\overline{\tau}} = -z_\tau$, $z'_{\overline{\tau}} = -z''_\tau$ (denoting $\overline{\tau} = (k, j, i)$). We have $z_\tau + z'_\tau + z''_\tau = 0$, and moreover the triple $(z_\tau, z'_\tau, z''_\tau)$ is of the form $(0, z, -z)$, $(-z, 0, z)$ or $(z, -z, 0)$ with $z \geq 0$. Note that, if $z'_\tau \leq 0$, then $z'_\tau = -z_\tau^-$ and $z''_\tau = z_\tau^+$.

(by the properties of the geometric cross ratio under 3-cyclic permutation, see Prop. 1.1).

In the algebraic case, in terms of the algebraic triangle FG-invariant (triple ratio) $Z_\tau = \mathrm{Tri}(F_i, F_j, F_k)$ in $\mathbb{K}_{\neq 0, -1}$, we have, by the symmetries of the usual algebraic cross ratio under 3-cyclic permutations,

$$\text{(4.1)} \qquad \begin{array}{rcl} z_\tau &=& \log|Z_\tau| \\ z'_\tau &=& \log\left|(1 + Z_\tau)^{-1}\right| \\ z''_\tau &=& \log\left|1 + Z_\tau^{-1}\right| . \end{array}$$

4.1.2. *Edge invariants.* To each oriented edge $e = (k, i)$ between two adjacent triangles $\tau = (i, j, k)$ and $\tau' = (k, \ell, i)$, where i, j, k, ℓ are cyclically ordered accordingly to orientation of the surface, we associate the triple of geometric cross ratios at the point p_i in $\mathbb{P}$ associated with the four lines $D_i, p_i p_j, p_i p_k, p_i p_{k\ell}$ by cyclic permutation of the three last ones:

$$\begin{cases} s_e &= \boldsymbol{\beta}(D_i, p_i p_j, p_i p_k, p_i p_{k\ell}) \\ s'_e &= \boldsymbol{\beta}(D_i, p_i p_{k\ell}, p_i p_j, p_i p_k) \\ s''_e &= \boldsymbol{\beta}(D_i, p_i p_k, p_i p_{k\ell}, p_i p_j) \end{cases} .$$

As for triangle invariants, we have $s_e + s'_e + s''_e = 0$ and moreover the triple $(s_\tau, s'_\tau, s''_\tau)$ is in $\mathbb{R}^+(0, 1, -1)$, $\mathbb{R}^+(-1, 0, 1)$ or $\mathbb{R}^+(1, -1, 0)$.

In the algebraic case, the link with the algebraic edge invariants $S_e \in \mathbb{K}_{\neq 0}$ (defined in §2.4) is:

$$\text{(4.2)} \qquad \begin{array}{rcl} s_e &=& \log|S_e| \\ s'_e &=& \log\left|(1 + S_e)^{-1}\right| \\ s''_e &=& \log\left|1 + S_e^{-1}\right| . \end{array}$$

As F is ρ-equivariant, the triangle and edge invariants are invariant under the action of Γ on $\widetilde{\mathcal{T}}$, hence induce well-defined invariants associated to triangles and oriented edges of $\mathcal{T}$, we will call the *geometric FG-invariants of F relatively to $\mathcal{T}$*.
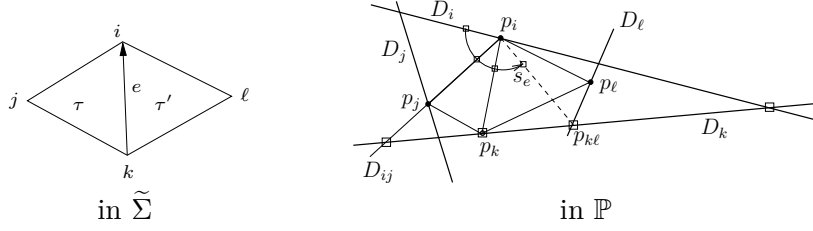
FIGURE 10. The edge invariant s_e associated with an oriented edge e.

4.2. **Main result.** We refer the reader to Sections 3.2 and 3.3 for the notion of left-shifting (L) geometric FG-parameter (z, s), and the associated A_2-complex K.

Theorem 4.1. *Let ρ be an action of $\Gamma = \pi_1(\Sigma)$ on X, and $F : \mathcal{F}_\infty(\Sigma) \to \partial_F X$ be a ρ-equivariant map. Let $\mathcal{T}$ be an ideal triangulation of Σ. Suppose that F is transverse to $\mathcal{T}$. Let z_τ, z'_τ, z''_τ, with $\tau \in T$, and s_e, s'_e, s''_e, with e in $\overrightarrow{E}$, be the geometric FG-invariants of F relatively to $\mathcal{T}$. Suppose that*

(FT) *for each triangle τ in $\mathcal{T}$, we have $z'_\tau \leq 0$,*
  (L) *(z, s) is left-shifting.*

Let K be the A_2-complex of FG-parameter $(z, s) = ((z_\tau)_\tau, (s_e)_e)$. Then there exists a ρ-equivariant map $\Psi : \widetilde{K} \to X$, locally preserving the $\mathfrak{C}$-distance $d^{\mathfrak{C}}$.

Theorem 4.2. *Under the hypotheses and notations of Theorem 4.1, suppose furthermore that*

(FE) *for each oriented edge e in $\mathcal{T}$, we have $s'_e \leq 0$;*
  (S) *for each triangle τ in $\mathcal{T}$ and every pair of edges e_1, e_2 of τ (oriented after τ), we have $-s_{e_1} - s_{e_2} < z_\tau^-$ and $-s_{\overline{e_1}} - s_{\overline{e_2}} < z_\tau^+$.*

Then the map $\Psi : \widetilde{K} \to X$ preserves globally the $\mathfrak{C}$-distance $d^{\mathfrak{C}}$. In particular

(i) *for all $\gamma \in \Gamma$*

$$\ell^{\mathfrak{C}}(\rho(\gamma)) = \ell^{\mathfrak{C}}(\gamma, K)$$

(in particular the length spectrum of ρ depends only on (z, s)) and for usual lengths

$$\ell_{euc}(\rho(\gamma)) = \left\| (\ell^{\mathfrak{C}}(\gamma, K) \right\|,$$

$$\ell_H(\rho(\gamma)) = N_H(\ell^{\mathfrak{C}}(\gamma, K)) .$$

(ii) *The map Ψ is bilipschitz. The action ρ is undistorted, faithfull and proper (hence discrete).*

Note that in general we do not have $\ell_{euc}(\rho(\gamma)) = \ell_{euc}(\gamma, K)$.

Remark 4.3. Hypothesis (S) means geometrically that, in the A_2-complex K, a singular segment entering a triangle K^τ from one adjacent edge cell K^e does not extend outside K^τ (see left side of the figure 11). We will then say that (z, s) is *edge-separating*.
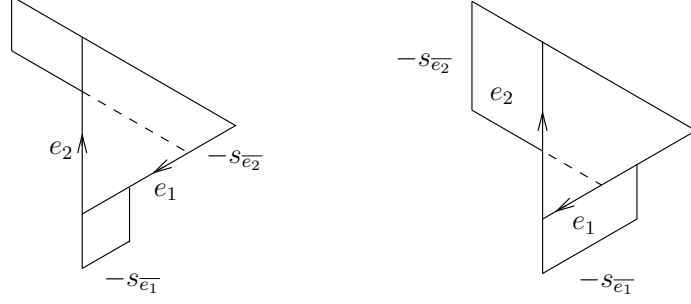
FIGURE 11. Edge-separating (on the left) vs Non edge-separating (on the right) (in the case where $z_\tau > 0$)

4.3. **Proof of Theorem 4.1.** Let $\rho : \Gamma \to \mathrm{Aut}(X)$ be an action, and $F : \mathcal{F}_\infty(\Sigma) \to \partial_F X$ be a ρ-equivariant map. Suppose that F is transverse to the ideal triangulation $\mathcal{T}$ of Σ. For each pair (i,j) of distinct points in the Farey set $\mathcal{F}_\infty(\Sigma)$ of Σ, we denote by A_{ij} the flat in X joining F_i and F_j. Suppose first that (FT) holds, that is: $z'_\tau \le 0$ for each triangle τ in $\mathcal{T}$.

Let $\tau = (i,j,k)$ be a triangle in $\widetilde{\mathcal{T}}$. Since $z'_\tau = \mathrm{tri}_2(F_i, F_j, F_k) \le 0$, we have the following properties for the triple (F_i, F_j, F_k) which are proved in [Par15a] (Theorem 2), and depicted in Figure 12.

Theorem 4.4. *The intersection of the two flats $A(p_i, p_j, p_k)$ and $A(D_i, D_j, D_k)$ is a flat singular triangle $\Delta = \Delta_\tau$ with vertices $v_i = v_i(\tau)$, $v_j = v_j(\tau)$ and $v_k = v_k(\tau)$ such that:*

  (i) The Weyl chamber from v_i to F_i is $A_{ij} \cap A_{ik}$;

  (ii) In any marked flat $f_{ij} : \mathbb{A} \to A_{ij}$ sending $\partial\mathfrak{C}$ to F_j, we have in simple roots coordinates

$$\overrightarrow{v_i v_j} = (z_\tau^+, z_\tau^-) \; ;$$

  (iii) When Δ is not reduced to a point (i.e. when $z_\tau \ne 0$), then Δ and F_i define opposite chambers $\Sigma_{v_i} \Delta$ and $\Sigma_{v_i} F_i$ at v_i.

We now study the behaviour of two adjacent triangles, in particular how edge invariants measure the shift between Δ_τ, $\Delta_{\tau'}$ along the common flat. Let $\tau = (i,j,k)$ and $\tau' = (k,\ell,i)$, be a pair of adjacent triangles in $\widetilde{\mathcal{T}}$ (where (i,j,k,ℓ) are positively ordered), and denote by e the common edge (k,i). Denote by $v_i = v_i(\tau)$, $v_k = v_k(\tau)$, $v'_i = v_i(\tau')$ and $v'_k = v_k(\tau')$ the particular points in the flat A_{ki} joining F_k to F_i defined by each of the two adjacent triples. Let $f_e : \mathbb{A} \to A_{ki}$ be a marking of the flat A_{ki} sending $\partial\mathfrak{C}$ to F_i. By Theorem 4.4 and the invariance by cyclic permutation, in the marked flat f_e, we have in simple roots coordinates

$$\overrightarrow{v_k v_i} = (z_\tau^+, z_\tau^-) \text{ and } \overrightarrow{v'_k v'_i} = (z_{\tau'}^-, z_{\tau'}^+)$$

in particular $\overrightarrow{v_k v_i}$ and $\overrightarrow{v'_k v'_i}$ belong to $\overline{\mathfrak{C}}$.

Proposition 4.5 (Geometric interpretation of edge parameters)**.** *In any marked flat $f_e : \mathbb{A} \to A_{ki}$ sending $\partial\mathfrak{C}$ to F_i, we have*
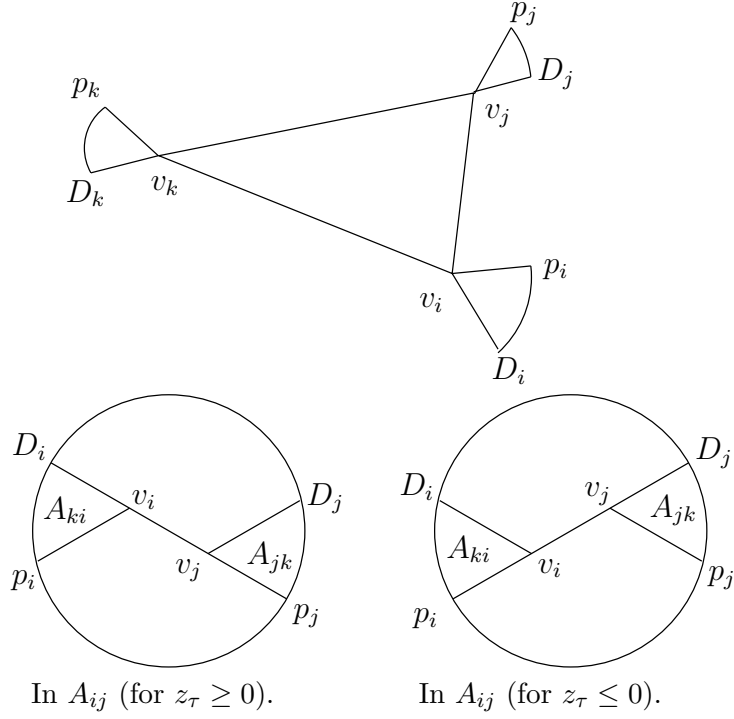
$$\overrightarrow{v_i v'_k} = (s_{\overline{e}}, s_e)$$

FIGURE 12. The flat singular triangle $\Delta_\tau = (v_i, v_j, v_k)$ associated with τ.

in the basis of simple roots of $\mathbb{A}$.

Proof. We project in the transverse tree at infinity X_{p_i} in direction p_i by $\pi_{p_i} : X \to X_{p_i}$. We denote by o and o' the respective projections of $p_i p_j$ and $p_i p_{k\ell}$ (seen as points of $\partial_\infty X_{p_i}$) on the line from D_i to $p_i p_k$ in X_{p_i}. Then we have $\pi_{p_i}(v_i) = o$ and $\pi_{p_i}(v'_k) = o'$ by Lemma 17 of [Par15a]. Thus we have (by (1.2) and (1.4))

$$
\begin{aligned}
\varphi_2(\overrightarrow{v_i v'_k}) &= B_{D_i}\big(\pi_{p_i}(v_i), \pi_{p_i}(v'_k)\big) \\
&= B_{D_i}(o, o') \\
&= \boldsymbol{\beta}(D_i, p_i p_j, p_i p_k, p_i p_{k\ell}) \\
&= s_e .
\end{aligned}
$$

Similarly, projecting in the transverse tree X_{D_i} and denoting by o_*, o'_* the respective projections of $D_i \cap D_j$ and $D_i \cap D_{k\ell}$ (seen as points of $\partial_\infty X_{D_i}$) on the line from p_i to $D_i \cap D_k$ in the tree X_{D_i}, we have

$$
\begin{aligned}
\varphi_1(\overrightarrow{v_i v'_k}) &= B_{p_i}\big(\pi_{D_i}(v_i), \pi_{D_i}(v'_k)\big) \\
&= B_{p_i}(o_*, o'_*) \\
&= \boldsymbol{\beta}(p_i, D_i \cap D_j, D_i \cap D_k, D_i \cap D_{k\ell}) \\
&= \boldsymbol{\beta}(D_k, p_k p_\ell, p_k p_i, p_k p_{ij}) = s_{\overline{e}} .
\end{aligned}
$$

$\square$

In particular, we have the following geometric interpretation in the building X of the hypothesis "left-shifting on edge e".

Corollary 4.6. *The three following assertions are equivalent*

    (i) *(z, s) is left-shifting on edge e ;*

    (ii) *In any marked flat $\mathbb{A} \to A_{ki}$ sending $\partial\mathfrak{C}$ to F_i, the vectors $\overrightarrow{v_i v_i'}$ and $\overrightarrow{v_k v_k'}$ are in $\mathfrak{C}$;*

    (iii) *v_i is in the open Weyl chamber from v_i' to F_i, and v_k' is in the open Weyl chamber from v_k to F_i.*

The following lemma establishes that, if (z, s) is left-shifting on edge e then the associated adjacent singular triangles Δ_τ, $\Delta_{\tau'}$ lie in a common flat.

Lemma 4.7. *Let $\tau = (i, j, k)$ and $\tau' = (k, \ell, i)$ be a pair of adjacent triangles in $\widetilde{\mathcal{T}}$ (where (i, j, k, ℓ) are positively ordered), and denote by e the common edge (k, i). Suppose that (z, s) is left-shifting on edge e. Let C be any Weyl chamber with tip v_i containing Δ_τ, and C' be any Weyl chamber with tip v_k' containing $\Delta_{\tau'}$. There exists a marked flat $f : \mathbb{A} \to X$ such that $f(0) = v_i$, $f(\alpha) = v_k'$ with $\alpha = (s_{\overline{e}}, s_e)$ in simple roots coordinates, $f(-\overline{\mathfrak{C}}) = C$ and $f(\alpha + \overline{\mathfrak{C}}) = C'$.*

Proof. Let c, c' be the boundaries at infinity of C and C'. Let $f_e : \mathbb{A} \to A_{ki}$ be the marked flat sending $\partial\mathfrak{C}$ to F_i and 0 to v_i. Let $f_e^{-1}(v_k) = \alpha_k$, $f_e^{-1}(v_i') = \alpha_i'$ and $f_e^{-1}(v_k') = \alpha_k'$. By Proposition 4.5 and Theorem 4.4, we have that $\alpha_k \in -\overline{\mathfrak{C}}$, $\alpha_i' \in \mathfrak{C}$, $\alpha_k' = (s_{\overline{e}}, s_e)$ is in $\alpha_k + \mathfrak{C}$, and $[0, \alpha_k']$ is contained in $(\alpha_k + \overline{\mathfrak{C}}) \cap (\alpha_i' - \overline{\mathfrak{C}})$. Since F_i and C are opposite at v_i (Theorem 4.4), there exists a marked flat $f_1 : \mathbb{A} \to A_1$ sending $-\overline{\mathfrak{C}}$ to C and $\partial\mathfrak{C}$ to F_i. Since f_1 and f_e both sends 0 to v_i and $\partial\mathfrak{C}$ to F_i, we have $f_1 = f_e$ on the convex subset $f_e^{-1}(f_1(\mathbb{A}))$, which contains $\overline{\mathfrak{C}}$. Since $v_k = f_e(\alpha_k)$ belongs to Δ_τ, hence to $A_1 = f_1(\mathbb{A})$, it implies that $f_1 = f_e$ on $\alpha_k + \overline{\mathfrak{C}}$. Since $\alpha_k' \in \alpha_k + \mathfrak{C}$, f_1 and f_e coincide on a germ of the Weyl chamber $\alpha_k' - \mathfrak{C}$ at α_k'. Then the Weyl chambers $C'' = f_1(\alpha_k' - \mathfrak{C})$ and $f_e(\alpha_k' - \mathfrak{C})$ define the same chamber $\Sigma_{v_k'} c = \Sigma_{v_k'} F_k$ in the space of directions at v_k'. Therefore C'' and C' are opposite at v_k' by Theorem 4.4. Hence there exists a marked flat $f : \mathbb{A} \to A$ sending $\alpha_k' + \overline{\mathfrak{C}}$ to C' and $-\partial\mathfrak{C}$ to c. Since $v_i' = f_e(\alpha_i') = f_1(\alpha_i')$ belongs to A and f and f_1 are very strongly asymptotic on $-\mathfrak{C}$, we have $f = f_1$ on $\alpha_i' - \overline{\mathfrak{C}}$. In particular, $f(-\overline{\mathfrak{C}}) = f_1(-\overline{\mathfrak{C}}) = C$. Moreover $f = f_e$ on $(\alpha_k + \overline{\mathfrak{C}}) \cap (\alpha_i' - \overline{\mathfrak{C}})$, which contains $[0, \alpha_k']$. $\qquad\square$
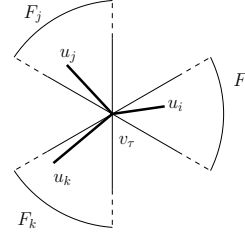
From now on, we suppose that (z, s) is left-shifting. The next lemma formalizes the construction of the map Ψ, and is a straigthforward consequence of Theorem 4.4 and Proposition 4.5. We refer to Section 3.3 for the definition of charts of $\widetilde{K}$, and we recall that $a_i(\tau)$ is the i-vertex of the singular triangle associated with τ in the associated A_2-complex $\widetilde{K}$.

Lemma 4.8. *There exists a unique ρ-equivariant map $\Psi : \widetilde{K} \to X$ such that*

    • *The map Ψ sends $a_i(\tau)$ to $v_i(\tau)$ for all marked triangle $\tau = (i, j, k)$ of $\widetilde{\mathcal{T}}$;*

    • *For every chart $\phi_m : P^m \to \widetilde{K}^m$ of $\widetilde{K}$, the map $\Psi \circ \phi_m : P^m \to X$ is the restriction of a marked flat.* $\qquad\square$

We now check that Ψ is a local $\mathfrak{C}$-isometry. Let x be a point in $\widetilde{K}$. Then either there is a neighbourhood of x contained in some $\widetilde{K}^\tau \cup \widetilde{K}^e$ with e adjacent to τ, on which Ψ is a $\mathfrak{C}$-isometry by Lemma 4.7, or x is the vertex a_τ of a singular triangle $\widetilde{K}^\tau$ reduced to a point (i.e. with invariant $z_\tau = 0$).

In that case, denote by (i,j,k) the vertices of τ and by e_s the oriented edge of τ with terminal vertex s, for $s = i,j,k$. A neighbourhood of x in $\widetilde{K}$ is then given by the union of the three segments $\widetilde{K}^{e_s}$, $s = i,j,k$. The image by Ψ of $\widetilde{K}^{e_s}$ is then a non trivial segment $[v_\tau, u_s]$, contained in the Weyl chamber C_s with vertex v_τ and boundary F_s. The chambers C_s are pairwise opposite at v_τ by Theorem 4.4. Hence Ψ is a local $\mathfrak{C}$-isometry on the union of the three segments $\widetilde{K}^{e_s}$. □

4.4. **Proof of Theorem 4.2.** Let x, x' be two points of $\widetilde{K}$. Let σ be the unique geodesic from x to x' in $\widetilde{K}$. We are going to prove that the image $\eta = \Psi \circ \sigma$ of σ by Ψ is a $\mathfrak{C}$-geodesic path in X, using the criterion in Proposition 1.7. Then we will have $d^{\mathfrak{C}}(\Psi(x), \Psi(x')) = \ell^{\mathfrak{C}}(\eta)$, which is equal to $\ell^{\mathfrak{C}}(\sigma)$ since Ψ preserves the $\mathfrak{C}$-length of paths, hence equal to $d^{\mathfrak{C}}(x, x')$ by definition of the $\mathfrak{C}$-distance $d^{\mathfrak{C}}$ in $\widetilde{K}$, which concludes.

Let $t_0 = 0 < t_1 < \cdots < t_N = 1$ be a minimal subdivision of $[0,1]$ such that $\sigma_{|[t_n,t_{n+1}]}$ has constant type of direction in $\partial\overline{\mathfrak{C}}$, and let $x_n = \sigma(t_n)$ and $y_n = \Psi(x_n)$. For $0 < n < N$, since $\sphericalangle_{x_n}(x_{n-1}, x_{n+1}) > \pi$, the point x_n is a singularity of $\widetilde{K}$, hence by construction of $\widetilde{K}$ it is a boundary point of the form $x_n = b_e$ for some oriented edge e of $\widetilde{\mathcal{T}}$. Suppose that for some $0 < n < N$ the (constant) type of the segment $[x_n, x_{n+1}]$ is singular. We have to prove that the directions $\Sigma_{y_n} y_{n+2}$ and $\Sigma_{y_n} y_{n-1}$ are $\mathfrak{C}$-opposite, i.e. contained in two opposite closed chambers at y_n.
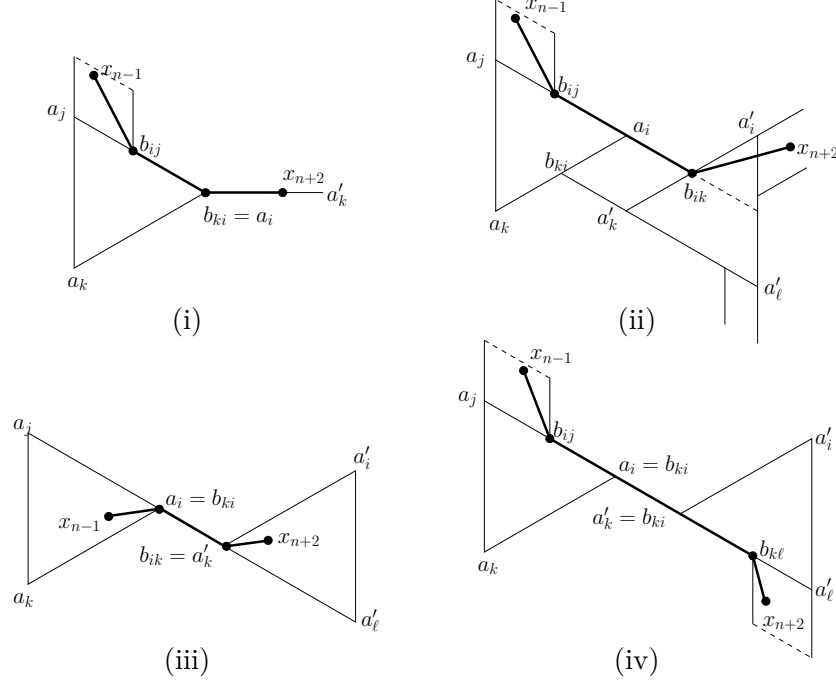
We are first going to show that the edge-separating hypothesis (S) allows us to reduce our study to the case of two adjacent triangles.

Lemma 4.9. *There exist two adjacent triangles $\tau = (i,j,k)$ and $\tau' = (k,\ell,i)$ in $\widetilde{\mathcal{T}}$ such that the segment $[x_n, x_{n+1}]$ is contained in $\widetilde{K}^\tau \cup \widetilde{K}^{ki} \cup \widetilde{K}^{\tau'}$, and, up to exchanging x_n and x_{n+1}, denoting $a_s = a_s(\tau)$ and $a'_s = a_s(\tau')$, we are in one of the following cases*

    (i) $b_{ki} = a_i$, and $x_n = b_{ij}$ and $x_{n+1} = b_{ki}$;
    (ii) $x_n = b_{ij}$ and $x_{n+1} = b_{ik}$;
    (iii) $b_{ki} = a_i$ and $b_{ik} = a'_i$, and $x_n = b_{ki}$ and $x_{n+1} = b_{ik}$;
    (iv) $b_{ki} = a_i$ and $b_{ik} = a'_i$, and $x_n = b_{ij}$ and $x_{n+1} = b_{k\ell}$.

Proof. Since $\sphericalangle_{x_n}(x_{n-1}, x_{n+1}) > \pi$, the singular direction $\Sigma_{x_n} x_{n+1}$ must be in the boundary of $\widetilde{K}$ at x_n. Similarly the singular direction $\Sigma_{x_{n+1}} x_n$ must be in the boundary of $\widetilde{K}$ at x_{n+1}.

Denote by (i,j) the oriented edge such that $x_n = b_{ij}$, and let $\tau = (i,j,k)$ be the (marked) left adjacent triangle in $\widetilde{\mathcal{T}}$.

FIGURE 13. Singular segment in a geodesic in $\widetilde{K}$: the four cases.

Suppose first that the segment from $x_n = b_{ij}$ to x_{n+1} starts in direction of the point a_i. Then it contains $[b_{ij}, a_i]$.

If $x_{n+1} = a_i$, then $a_i = b_{ki}$, and we are done. If $x_{n+1} \neq a_i$, the segment $[b_{ij}, a_i]$ extends by $[a_i, b_{ik}]$ in a constant type segment, and $[x_n, x_{n+1}]$ contains the segment $[b_{ij}, b_{ik}]$.

If $x_{n+1} \neq b_{ik}$, then we now show that, by hypothesis (S), we must have $b_{ik} = a'_k$ and $x_{n+1} = b_{k\ell}$: The constant type geodesic ray r in $\widetilde{K}^{\tau'}$ from b_{ik} parallel to the side $[a'_k, a'_\ell]$ hits the boundary of $\widetilde{K}^{\tau'}$ at the point b of $[a'_i, a'_\ell]$ at distance $d(b_{ik}, a'_k) = d_{ik} = \max(-s_{ki}, -s_{ik})$ from a'_ℓ, and cannot be extended outside $\widetilde{K}^{\tau'}$ since b is on $[a'_\ell, b_{\ell i}[$ since $d_{ik} + d_{\ell i} < |z_{\tau'}|$ by (S) (see remark 4.3). There is only one singular point that may then be on r, which is $b_{k\ell}$ in the case where $b_{ik} = a'_k$.

Suppose now that $\Sigma_{x_n} x_{n+1} = \Sigma_{b_{ij}} a''_i$, where $a''_i = a_i(\tau'')$, with $\tau'' = (j, i, h)$ the triangle adjacent to τ along edge (i, j).

If have $x_{n+1} = a''_i = b_{ji}$, then $x_n = b_{ij} = a_j$, and we are in case (iii), up to replacing the pair of adjacent triangles τ, τ'' by the pair τ', τ.

If $x_{n+1} \neq a''_i$, then x_{n+1} must be the next singular point on the same side of the adjacent triangle cell $\widetilde{K}^{\tau''}$ (since the ray from $a''_i = a_i(\tau'')$ to b_{ih} to no extend outside $\widetilde{K}^{\tau''}$). We are then reduced to the previous case $x_n = b_{ij}$, $x_{n+1} = b_{ik}$ by exchanging the roles of x_n and x_{n+1}.

If $\Sigma_{x_n} x_{n+1}$ is neither $\Sigma_{b_{ij}} a_i$ nor $\Sigma_{b_{ij}} a''_i$, then there is a third boundary direction in $\widetilde{K}$ at b_{ij}, which means that $b_{ij} = a_j$ and $\Sigma_{x_n} x_{n+1} = \Sigma_{a_j} a_k$. Then as $[a_j, a_k]$ is not extendable, we must have $x_{n+1} = b_{jk}$. We are then

reduced to the previous case $x_{n+1} = b_{ij}$, $x_{n+1} = b_{ki}$ by exchanging the roles of x_n and x_{n+1}. $\qquad\square$

Since Ψ is a local $\mathfrak{C}$-isometry (Section 4.3), the path η is a local $\mathfrak{C}$-geodesic in X, and its restriction to $[t_n, t_{n+1}]$ is is the affine segment $[y_n, y_{n+1}]$ (since it is of constant type of direction in $\overline{\mathfrak{C}}$).

Case (i): $x_n = b_{ij}$ **and** $x_{n+1} = b_{ki}$. Then $b_{ki} = a_i$. We then have $y_{n+1} = \Psi(a_i) = v_i$, and $\Sigma_{y_{n+1}} y_{n+2}$ is in $\Sigma_{v_{ki}} \Psi(\widetilde{K}^{ki})$, hence in the closed chamber $\Sigma_{v_{ki}} F_i$. Since $v_{ki} = v_i$ is in the closed Weyl chamber from v_{ij} to F_i by Theorem 4.4, we have that $\Sigma_{y_{n+1}} y_{n+2}$ is in $\Sigma_{v_{ij}} F_i$. Since v_{ij} is in the flat A_{ij}, it proves that $\Sigma_{y_n} y_{n+2}$ is $\mathfrak{C}$-opposite to $\Sigma_{y_n} y_{n-1}$.

Case (ii): $x_n = b_{ij}$ **and** $x_{n+1} = b_{ik}$. At $x_{n+1} = b_{ik}$, the direction $\Sigma_{x_{n+1}} x_{n+2}$ is in the unique closed chamber of $\Sigma_{b_{ik}} \widetilde{K}$ containing $\Sigma_{b_{ik}} a'_i$, where $\tau' = (k, \ell, i)$ is the adjacent triangle in $\widetilde{\mathcal{T}}$. In the building X, we then have $y_n = v_{ij}$, $y_{n+1} = v_{ik}$, We now prove that we then have $\Sigma_{y_n} y_{n+2} \in \Sigma_{v_{ij}} F_i$. Let C be a closed Weyl chamber with tip $y_n = v_{ij}$ containing a germ at $y_{n+1} = v_{ik}$ of the segment $[y_{n+1}, y_{n+2}]$. Then C contains a germ at v_{ik} of the segment $[v_{ik}, v'_i]$. Let $C_i = C(v_{ij}, F_i)$ be the closed Weyl chamber from v_{ij} to F_i. The closed Weyl chambers $C(v_{ik}, F_i)$, and $C(v_{ik}, F_k)$ are opposite at v_{ik} (because $v_{ik} \in A_{ik}$), and respectively contain v'_i and v_{ij}, therefore C_i contains the segment $[v_{ik}, v'_i]$. Since $[v_{ij}, v_{ik}]$ and $[v_{ik}, v'_i]$ are singular segments of different type of direction in $\partial \mathfrak{C}$, we have then $\Sigma_{v_{ij}} C = \Sigma_{v_{ij}} C_i$, hence $\Sigma_{y_n} y_{n+2} \in \Sigma_{v_{ij}} F_i$.

Since $\Sigma_{v_{ij}} F_j$ and $\Sigma_{v_{ij}} F_i$ are opposite closed chambers of $\Sigma_{v_{ij}} X$ (because $v_{ij} \in A_{ij}$), it proves that $\Sigma_{y_n} y_{n+2}$ is $\mathfrak{C}$-opposite to $\Sigma_{y_n} y_{n-1}$.

Case (iii): $x_n = b_{ki} = a_i$ **and** $x_{n+1} = b_{ik} = a'_k$. In the building X we have $y_n = v_i$, $y_{n+1} = v'_k$, $\Sigma_{y_n} y_{n-1} \in \Delta_\tau$, and $\Sigma_{y_{n+1}} y_{n+2} \in \Delta_{\tau'}$. Lemma 4.7 then implies that there then exists two opposite Weyl chamber with tip v_i containing respectively Δ_τ and $[v_i, v'_k] \cup \Delta_{\tau'}$, so $\Sigma_{y_n} y_{n+2}$ is $\mathfrak{C}$-opposite to $\Sigma_{y_n} y_{n-1}$ at $y_n = v_i$.

Case (iv): $x_n = b_{ij}$ **and** $x_{n+1} = b_{k\ell}$. Then we have $b_{ik} = a'_k$, i.e. $s_e = 0$. In the building X, we then have $y_n = v_{ij}$, $y_{n+1} = v_{k\ell}$, and in the spherical building $\Sigma_{v_{ij}} X$ of directions at $y_n = v_{ij}$, we have that $\Sigma_{y_n} y_{n+1}$ belongs to the chamber $\Sigma_{v_{ij}} F_j$, and $\Sigma_{y_{n+1}} y_{n+2}$ belongs to the chamber $\Sigma_{v_{k\ell}} F_\ell$. Since $s'_e \leq 0$, the following Lemma 4.10 implies that the ideal chambers F_j and F_ℓ are then opposite at v_i, so, since v_i is on the singular segment $]v_{ij}, v_{k\ell}[$, it implies that $\Sigma_{y_n} y_{n+2}$ is $\mathfrak{C}$-opposite to $\Sigma_{y_n} y_{n-1}$ as needed.

Lemma 4.10. *Let $\tau = (i, j, k)$ and $\tau' = (k, \ell, i)$, be a pair of adjacent triangles in $\widetilde{\mathcal{T}}$ (where (i, j, k, ℓ) are positively ordered), and denote by e the common edge (k, i). Denote $v_i = v_i(\tau)$, $v_k = v_k(\tau)$, $v'_i = v_i(\tau')$ and $v'_k = v_k(\tau')$. Suppose that (z, s) is left-shifting on edge e and $s_e = 0$. Then*

    (i) There is a geodesic from D_j to p_ℓ through v_j, v_i, v'_k, and v'_ℓ ;

    (ii) F_j and F_ℓ are opposite at v_i if and only if

$$s'_e = \boldsymbol{\beta}(D_i, p_i p_{k\ell}, p_i p_j, p_i p_k) \leq 0$$

Proof of Lemma 4.10. Since (z, s) is left-shifting on edge e and $s_e = 0$, we must have $z_\tau \leq 0$, and $s_{\bar{e}} > 0$ Lemma 4.7 then implies that the path
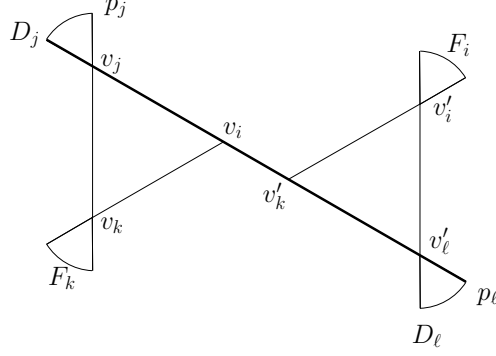
FIGURE 14. Adjacent triples with $s_e = 0$, $s_{\bar{e}} > 0$, $s'_e < 0$.

$(v_j, v_i, v'_k, v'_\ell)$ is a geodesic segment of singular type 1. It extends in a geodesic σ from D_j to p_ℓ, because Δ_τ is opposite to F_j at v_j and $\Delta_{\tau'}$ is opposite to F_ℓ at v'_ℓ (Theorem 4.4), and (i) is proven.

By point (i) and Proposition 4.5, the directions D_j and p_ℓ are opposite at $x = v_i$, and we have $\Sigma_x\, p_\ell = \Sigma_x\, v'_k = \Sigma_x\, p_i$. Thus F_j and F_ℓ are opposite at x. if and only if p_j and D_ℓ opposite at x, i.e. $\Sigma_x\, p_j \notin \Sigma_x\, D_\ell$.

We now prove that $\Sigma_x\, p_j \in \Sigma_x\, D_\ell$ if and only if $\Sigma_x(p_i p_j) = \Sigma_x(p_i p_{k\ell})$. First observe that $\Sigma_x\, p_i$ is different from $\Sigma_x\, p_j$, as x is on the flat $A(p_i, p_j, p_k)$. We also have $\Sigma_x\, p_i \neq \Sigma_x\, p_{k\ell}$, since $\Sigma_x\, p_{k\ell} \in \Sigma_x\, D_k$ and $\Sigma_x\, p_i \notin \Sigma_x\, D_k$ since x is in the flat $A(F_k, F_i)$. Then $\Sigma_x\, p_j \in \Sigma_x\, D_\ell$ if and only if $\Sigma_x\, p_j \oplus \Sigma_x\, p_\ell = \Sigma_x\, D_\ell$ (since $\Sigma_x\, p_j \neq \Sigma_x\, p_\ell$). We have $\Sigma_x\, p_j \oplus \Sigma_x\, p_\ell = \Sigma_x\, p_j \oplus \Sigma_x\, p_i = \Sigma_x(p_i p_j)$ (since $\Sigma_x\, p_j \neq \Sigma_x\, p_i$). On the other hand, since $\Sigma_x\, p_i \neq \Sigma_x\, p_{k\ell}$, we have $\Sigma_x\, D_\ell = \Sigma_x\, p_i \oplus \Sigma_x\, p_{k\ell} = \Sigma_x(p_i p_{k\ell})$, and we are done.

Projecting in the transverse tree at infinity X_{p_i} we now show that $\Sigma_x(p_i p_j) \neq \Sigma_x(p_i p_{k\ell})$ is equivalent to $\boldsymbol{\beta}(D_i, p_i p_{k\ell}, p_i p_j, p_i p_k) = s'_e \leq 0$. Indeed, since the
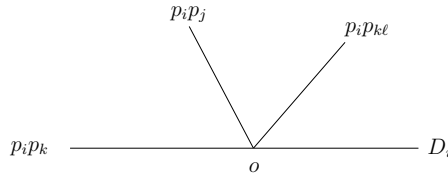

FIGURE 15. In the transverse tree at infinity X_{p_i}.

projection of x is the center o of the ideal tripod D_i, $p_i p_j$, $p_i p_k$ (by Lemma 17 of [Par15a]), the directions $\Sigma_x(p_i p_j)$ and $\Sigma_x(p_i p_{k\ell})$ are distinct if and only if the two geodesic rays in X_{p_i} from o to the ideal points $p_i p_j$ and $p_i p_{k\ell}$ have distinct germs at o.

This is equivalent to $\boldsymbol{\beta}(D_i, p_i p_{k\ell}, p_i p_j, p_i p_k) = s'_e \leq 0$, which concludes. $\qquad\square$

We have proven that in all cases $\Sigma_{y_n}\, y_{n+2}$ is $\mathfrak{C}$-opposite to $\Sigma_{y_n}\, y_{n-1}$. Therefore the piecewise affine path $(y_0, y_1, \ldots, y_N)$ is a global $\mathfrak{C}$-geodesic in X by Proposition 1.7, which concludes the proof of Theorem 4.2.

5. Degenerations of representations and convex $\mathbb{RP}^2$-structures

In this section, we use Theorem 4.2 to describe a large family of degenerations of convex $\mathbb{RP}^2$-structures on Σ, corresponding to a part of the boundary of the moduli space of convex $\mathbb{RP}^2$-structures on Σ constructed in [Par11]. Let $\mathbb{K}$ be any valued field. Starting from §5.4, the field $\mathbb{K}$ will be supposed to be either equal to $\mathbb{R}$ or $\mathbb{C}$ or ultrametric.

5.1. **Background on asymptotic cones.** In this section, we gather definitions and tools about the various notions of ultralimits and asymptotic cones that will be used in what follows. We first fix notations about usual ultralimits of metric spaces (see for example [KlLe97], or [Par11, §2.3] for more details). Then we briefly recall various notions of asymptotic cones of algebraic objects introduced in [Par11, §3]: asymptotic cones of valued fields, normed vector spaces, linear group, ultralimits of representations, and their links. Finally we introduce the notion of *asymptotic cones of projective spaces* and establish some basic properties of asymptotic cones in projective geometry.

Fix a (non principal) ultrafilter ω on $\mathbb{N}$, and a *scaling sequence* $(\lambda_n)_{n\in\mathbb{N}}$, that is a sequence of real numbers such that $\lambda_n \geq 1$ and $\lambda_n \to \infty$.

A point x_ω in a Hausdorff topological space E is the ω-*limit* of a sequence $(x_n)_n$ in E if it is its limit with respect to the filter ω. We will then denote $\lim_\omega x_n = x_\omega$. Note that $\lim_\omega x_n$ is then a cluster value of the sequence $(x_n)_n$. Recall that any sequence contained in a compact (Hausdorff) space has a (unique) ω-limit. The ω-limits of sequences of real numbers are taken in the compact space $[-\infty, +\infty]$.

Given a sequence of pointed metric spaces $(X_n, d_n, o_n)_{n\in\mathbb{N}}$, a sequence $(x_n)_n$ in $\prod_n X_n$ is called ω-*bounded* when $\lim_\omega d_n(o_n, x_n) < \infty$.

The *ultralimit* of $(X_n, d_n, o_n)_{n\in\mathbb{N}}$ is the quotient X_ω of the subspace of ω-bounded sequences in $\prod_n X_n$ by the pseudo-distance d_ω given by

$$d_\omega((x_n),(y_n)) = \lim_\omega d_\omega(x_n, y_n)\ .$$

It is a complete metric space (X_ω, d_ω). The class in X_ω of a ω-bounded sequence (x_n) will be called its *ultralimit* and be denoted by $\mathrm{ulim}_\omega x_n$. Given a sequence $Y_n \subset X_n$, we denote by $\mathrm{ulim}_\omega Y_n$ the subset of X_ω consisting of ultralimits of ω-bounded sequences $(x_n)_n$ such that $x_n \in Y_n$ for all n, and call it the *ultralimit* of the sequence $(Y_n)_n$.

Let $(\mathbb{K}_\omega, |\ |^\omega)$ be the asymptotic cone of the valued field $\mathbb{K}$ with respect to the scaling sequence (λ_n), that is the ultralimit of the sequence of valued fields $(\mathbb{K}, |\ |^{1/\lambda_n})$ (base points are at 0, see [Par11, §3.3]). It is an ultrametric field. Note that its absolute value $|\ |^\omega$ takes all values in $\mathbb{R}_{\geq 0}$. Given a sequence (a_n) in $\mathbb{K}$, we denote $\mathrm{ulim}_\omega a_n = \infty$ when $\lim_\omega |a_n|^{1/\lambda_n} = \infty$, so that every sequence in $\mathbb{K}$ has a well defined ultralimit in $\mathbb{K}_\omega \cup \{\infty\}$.

Denote by η the canonical norm on $V = \mathbb{K}^N$. Let (V_ω, η_ω) be the asymptotic cone of the normed vector space (V, η) with respect to the scaling sequence (λ_n), i.e. the ultralimit of the sequence of normed vector spaces $(V, \eta^{1/\lambda_n})$ (see [Par11, §3.4]). It is a normed vector space over the valued

field $\mathbb{K}_\omega$, canonically isomorphic to $\mathbb{K}_\omega^N$, with canonical basis the ultralimit $\mathbf{e}^\omega = (e_i^\omega)$ of the canonical basis $\mathbf{e} = (e_i)$ of $\mathbb{K}^N$.

Denote by N the norm on $\mathrm{End}(V)$ associated with η. The ultralimit $(\mathrm{End}(V)_\omega, N_\omega)$ of the sequence of normed algebra $(\mathrm{End}(V), N^{1/\lambda_n})$ is a normed algebra over the valued field $\mathbb{K}_\omega$, (see [Par11, §3.5]).

We now describe the asymptotic cone of the linear group $\mathrm{GL}(V)$. Let $\mathrm{GL}(V)_\omega$ be the subgroup of invertible elements of $\mathrm{End}(V)_\omega$. Note that the ultralimit of a sequence $(u_n)_n$ in $\mathrm{GL}(V)$ which is ω-bounded in $\mathrm{End}(V)$ (that is $\lim_\omega N(u_n)^{1/\lambda_n} < \infty$) may be not invertible, so the definition of $\mathrm{GL}(V)_\omega$ in [Par11] is incorrect (definition 3.16) (with no incidence on the remaining of the paper). The following proposition describes the invertible elements in $\mathrm{End}(V)_\omega$.

Proposition 5.1. *Let $u_\omega = \mathrm{ulim}_\omega u_n$ be an element of $\mathrm{End}(V)_\omega$. Then u_ω is invertible in $\mathrm{End}(V)_\omega$ if and only if u_n is in $\mathrm{GL}(V)$ for ω-almost all n and (u_n^{-1}) is ω-bounded in $\mathrm{End}(V)$, i.e. $\lim_\omega N(u_n^{-1})^{1/\lambda_n} < \infty$. Then $u_\omega^{-1} = \mathrm{ulim}_\omega u_n^{-1}$.* $\qquad\square$

Proof. If (u_n^{-1}) is ω-bounded in $\mathrm{End}(V)$, then clearly

$$(\mathrm{ulim}_\omega u_n) \circ (\mathrm{ulim}_\omega(u_n^{-1})) = \mathrm{ulim}_\omega(u_n \circ u_n^{-1}) = 1$$

hence u_ω is invertible with inverse $\mathrm{ulim}_\omega(u_n^{-1})$.

Conversely, suppose that u_ω is invertible in $\mathrm{End}(V)_\omega$, and let $u_\omega' = \mathrm{ulim}_\omega u_n'$ be its inverse. Then $1 - u_\omega \circ u_\omega' = 0$ in $\mathrm{End}(V)_\omega$, hence $\lim_\omega N(\mathrm{id} - u_n \circ u_n')^{1/\lambda_n} = 0$. Then for ω-almost all n we have $N(\mathrm{id} - u_n \circ u_n') < 1$, so $a_n = u_n \circ u_n'$ is invertible in $\mathrm{GL}(V)$ with $N(a_n^{-1}) \leq (1 - N(\mathrm{id} - a_n))^{-1}$. Then u_n is invertible with inverse $u_n^{-1} = u_n' \circ a_n^{-1}$. Since $N(u_n^{-1}) \leq N(u_n')(1 - N(a_n - \mathrm{id}))^{-1}$, we have $\lim_\omega N(u_n^{-1})^{\frac{1}{\lambda_n}} < \infty$, that is (u_n^{-1}) is ω-bounded. Then $u_\omega' = \mathrm{ulim}_\omega u_n^{-1}$ by uniqueness of inverses. $\qquad\square$

A sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{GL}(V)$ will be called ω-*bounded (in* $\mathrm{GL}(V)$*)* if

$$\lim_\omega N(u_n)^{1/\lambda_n} < \infty \text{ and } \lim_\omega N(u_n^{-1})^{1/\lambda_n} < \infty .$$

A ω-bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{End}(V)$ induces an endomorphism u^ω of V_ω defined by

$$u^\omega(\mathrm{ulim}_\omega v_n) = \mathrm{ulim}_\omega u_n(v_n)$$

for all ω-bounded sequence (v_n) in V. This endomorphism depends only on the ultralimit u_ω in $\mathrm{End}(V)_\omega$ of the sequence (u_n). The following results allows us to identify $\mathrm{End}(V)_\omega$ with $\mathrm{End}(V_\omega)$, and $\mathrm{GL}(V)_\omega$ with $\mathrm{GL}(V_\omega)$.

Proposition 5.2. [Par11, Corollaire 3.18] *The map*

$$\begin{array}{ccc} \mathrm{End}(V)_\omega & \to & \mathrm{End}(V_\omega) \\ u_\omega & \mapsto & u^\omega \end{array}$$

is an isomorphism of $\mathbb{K}_\omega$-normed algebras identifying $\mathrm{GL}(V)_\omega$ with $\mathrm{GL}(V_\omega)$.

5.2. **Ultralimits of projective spaces.** One verifies easily that the ultralimit of any sequence of vector subspaces of V (of fixed dimension) is a vector subspace of V_ω (of same dimension). Then any sequence of points in the projective space $\mathbb{P}V$ has a well defined *ultralimit* in the projective space $\mathbb{P}V_\omega$. This induces a canonical identification of $\mathbb{P}V_\omega$ with the ultralimit of the metric spaces $(\mathbb{P}V, d^{1/\lambda_n})$, where d is the distance on $\mathbb{P}V$ induced by the norm η.

Let $p_i \in \mathbb{P}V$, $i = 0, \ldots, N$ be the *canonical projective frame* of $\mathbb{P}V$, which is defined by $p_i = [e_i]$ for $i = 1, \ldots, N$ and $p_0 = [e_1 + \cdots + e_N]$. Let p_i^ω be the ultralimit of the constant sequence $(p_i)_{n \in \mathbb{N}}$. Then $(p_i^\omega)_{i=0,\ldots,N}$ is the canonical projective frame of $\mathbb{P}V_\omega$.

A sequence $(g_n)_n$ in $\mathrm{PGL}(V)$ is ω-*bounded* if it has a ω-bounded lift $(u_n)_n$ in $\mathrm{GL}(V)$. Then the ultralimit $g_\omega \in \mathrm{PGL}(V_\omega)$ of (g_n) is well defined by

$$g_\omega(\mathrm{ulim}_\omega\, p_n) = \mathrm{ulim}_\omega\, g_n(p_n)$$

and it coincides with the class in $\mathrm{PGL}(V_\omega)$ of the ultralimit $u_\omega \in \mathrm{GL}(V_\omega)$ of the sequence u_n.

Here is a useful criterion to see if a sequence $(g_n)_n$ in $\mathrm{PGL}(V)$ is ω-bounded, in terms of the action on the projective space.

Proposition 5.3. *Let $(g_n)_{n \in \mathbb{N}}$ be a sequence in $\mathrm{PGL}(V)$. let $(q_i^n)_{0 \le i \le N}$ be the image by g_n of the canonical projective frame $(p_i)_{0 \le i \le N}$. Denote by q_i^ω the ultralimit of the sequence $(q_i^n)_{n \in \mathbb{N}}$ in $\mathbb{P}V_\omega$. The following assertions are equivalent:*

(i) The points q_i^ω form a projective frame of $\mathbb{P}V_\omega$;
(ii) The sequence $(g_n)_n$ is ω-bounded in $\mathrm{PGL}(V)$,

Then the ultralimit of $(g_n)_{n \in \mathbb{N}}$ in $\mathrm{PGL}(V_\omega)$ is the unique map $g_\omega \in \mathrm{PGL}(V_\omega)$ sending $(p_i^\omega)_{0 \le i \le N}$ to $(q_i^\omega)_{0 \le i \le N}$. $\qquad\square$

Proof. Suppose that the points q_i^ω form a projective frame of $\mathbb{P}V_\omega$, and let g_ω be the projective map in $\mathrm{PGL}(V_\omega)$ sending the frame (p_i^ω) to the frame $(q_i^\omega)_i$. Let u_ω be a lift of g_ω in $\mathrm{GL}(V_\omega)$. There exists a ω-bounded sequence $(u_n)_{n \in \mathbb{N}}$ in $\mathrm{GL}(V)$ with ultralimit u_ω (Propositions 5.2 and 5.1).

For each fixed i, let v_i^n be the image of e_i by u_n. Then $(v_i^n)_n$ is a ω-bounded sequence in V and its ultralimit is $v_i^\omega = u_\omega(e_i)$, which is a non zero vector in V_ω representing the point q_i^ω of $\mathbb{P}V_\omega$.

Let w_i^n be a vector in q_i^n (seen as a line of V) at minimum distance from v_i^n. Then $\eta(w_i^n - v_i^n)^{1/\lambda_n} \le d(u_n(p_i^n), q_i^n)^{1/\lambda_n}$. Since the sequence $(v_i^n)_n$ is ω-bounded, and we have $\lim_\omega d(u_n(p_i^n), q_i^n)^{1/\lambda_n} = 0$, it follows that the sequence $(w_i^n)_n$ is ω-bounded with ultralimit $w_i^\omega = v_i^\omega$.

Let $h_n \in \mathrm{GL}(V)$ be the linear map sending the canonical basis $(e_i)_{i=1\ldots N}$ to the basis $(w_i^n)_{i=1\ldots N}$, which is a lift in $\mathrm{GL}(V)$ of g_n. The sequence $(h_n)_n$ is ω-bounded in $\mathrm{End}(V)$, and its ultralimit in $\mathrm{End}(V_\omega)$ is u_ω (since it sends e_i^ω to v_i^ω). Since $u_\omega = \mathrm{ulim}_\omega h_n$ is invertible in $\mathrm{End}(V_\omega)$, the sequence $(h_n)^{-1}$ is ω-bounded, as wanted. $\qquad\square$

We recall that the ultralimit $\rho_\omega : \Gamma \to \mathrm{PGL}(V_\omega)$ of a sequence of representations $\rho_n : \Gamma \to \mathrm{PGL}(V)$ is well defined when ρ_n is ω-*bounded*, that is when for all $\gamma \in \Gamma$ (or just for a generating set), the sequence $(\rho_n(\gamma))_{n \in \mathbb{N}}$ is ω-bounded in $\mathrm{PGL}(V)$ (see §5.2). It is then defined by $\rho_\omega(\gamma) = \mathrm{ulim}_\omega \rho_n(\gamma)$.

The cross ratio is easily seen to behave well under ultralimit.

Proposition 5.4. *For $n \in \mathbb{N}$, let p_1^n, p_2^n, p_3^n, p_4^n be four points in $\mathbb{P}(V)$ in a common line D^n. Let $p_i^\omega = \text{ulim}_\omega\, p_i^n$ be the ultralimit in $\mathbb{P}(V_\omega)$ of the sequence $(p_i^n)_{n \in \mathbb{N}}$. Suppose that the quadruple $(p_1^\omega, p_2^\omega, p_3^\omega, p_4^\omega)$ is nondegenerated, i.e. has no triple point. Then $(p_1^n, p_2^n, p_3^n, p_4^n)$ is nondegenerated for ω-almost all n, and*

$$\mathbf{b}(p_1^\omega, p_2^\omega, p_3^\omega, p_4^\omega) = \text{ulim}_\omega\, \mathbf{b}(p_1^n, p_2^n, p_3^n, p_4^n)$$

in $\mathbb{K}_\omega \cup \{\infty\}$. $\hfill\square$

5.3. **Asymptotic cones and Fock-Goncharov parameters.** In this section, we show that FG-parametrization of representations behaves well with respect to ultralimits, that is the two constructions commute. We use the hypotheses and notations of Section 2, from which we recall that, for $(Z, S) = ((Z_\tau)_\tau, (S_e)_e)$ in $(\mathbb{K}_{\neq 0,-1})^T \times (\mathbb{K}_{\neq 0})^{\overrightarrow{E}}$ the $\mathcal{T}$-transverse map with FG-parameter (Z, S) is denoted by $F_{Z,S} : \mathcal{F}_\infty(\Sigma) \to \text{Flags}(\mathbb{P})$ and $\rho_{Z,S}$ denotes the associated representation from Γ to $\text{PGL}(\mathbb{K}^3)$. We use the notations and hypotheses of the previous sections for ultralimits and asymptotic cones.

Proposition 5.5. *Let $((Z^n, S^n))_n$ be a sequence in $(\mathbb{K}_{\neq 0,-1})^T \times (\mathbb{K}_{\neq 0})^{\overrightarrow{E}}$ and let $Z^n = (Z_\tau^n)_\tau$ and $S^n = (S_e^n)_e$. Denote by $F^\omega : \mathcal{F}_\infty(\Sigma) \to \text{Flags}(\mathbb{K}_\omega^3)$ the ultralimit of the sequence of maps $F_{Z^n,S^n} : \mathcal{F}_\infty(\Sigma) \to \text{Flags}(\mathbb{P})$. For each triangle τ and oriented edge e of $\mathcal{T}$, denote by $Z_\tau^\omega = \text{ulim}_\omega\, Z_\tau^n\ S_e^\omega = \text{ulim}_\omega\, S_e^n$ the ultralimits in $\mathbb{K}_\omega \cup \{\infty\}$ of the sequence $(Z_\tau^n)_n$ and $(S_e^n)_n$. Suppose that $Z_\tau^\omega \notin \{\infty, -1, 0\}$ for all triangle τ of $\mathcal{T}$, and $S_e^\omega \notin \{\infty, 0\}$ for all oriented edge e of $\mathcal{T}$. Then*

(i) *$F^\omega = F_{Z^\omega,S^\omega}$;*
(ii) *The ultralimit $\rho_\omega : \Gamma \to \text{PGL}_3(\mathbb{K}_\omega)$ of the sequence of representations ρ_{Z^n,S^n} is well defined and $\rho_\omega = \rho_{Z^\omega,S^\omega}$.*

Proof. Denote $F^n = F_{Z^n,S^n}$. Note that, for each $i \in \mathcal{F}_\infty(\Sigma)$ the ultralimit of the sequence of flags $F^n(i) = F_i^n = (p_i^n, D_i^n)$ is a well-defined flag $F_i^\omega = (p_i^\omega, D_i^\omega)$ in $\text{Flags}(\mathbb{K}_\omega^3)$. The ultralimit $F^\omega : \mathcal{F}_\infty(\Sigma) \to \text{Flags}(\mathbb{K}_\omega^3)$ of the maps F^n is thus always well defined. We first prove that $F^\omega = F_{Z^\omega,S^\omega}$. Since the canonical basis of $\mathbb{K}_\omega^3$ is the ultralimit of the canonical basis of $\mathbb{K}^3$, it is clear that the image $(F_1^\omega, F_2^\omega, F_3^\omega)$ of the base triangle τ_0 by F^ω remains in canonical form, i.e. $p_1^\omega = [1:0:0]$, $p_2^\omega = [0:1:0]$, $D_1^\omega \cap D_2^\omega = [0:0:1]$, $p_3^\omega = [1:1:1]$ is the canonical projective frame. So it is enough to prove the two next lemmas, ensuring that F^ω is $\widetilde{\mathcal{T}}$-transverse and of FG-invariant Z^ω by induction on adjacent triangles, following the construction of the map F_{Z^ω,S^ω} in Section 2.5.

Lemma 5.6. *Let τ be a marked triangle in $\widetilde{\mathcal{T}}$ with ordered vertices (i, j, k) in $\mathcal{F}_\infty(\Sigma)$. Suppose that F_i^ω, F_j^ω and p_k^ω are in generic position. Then the triple of flags $(F_i^\omega, F_j^\omega, F_k^\omega)$ is generic and its triple ratio is Z_τ^ω.*

Proof. Denote by p_{ij}^n the point $D_i^n \cap D_j^n$ and by $p_{ij}^\omega = \text{ulim}_\omega\, p_{ij}^n$ its ultralimit. Denote by D_{ki}^n the line $p_k^n p_i^n$, by D_{ki}^n the line $p_k^n p_i^n$, and by $D_{ki}^\omega, D_{kj}^\omega$ their ultralimits. Since F_i^ω, F_j^ω and p_k^ω are in generic position, the points $p_i^\omega, p_j^\omega,$

p_k^ω and $p_{ij}^\omega = D_i^\omega \cap D_j^\omega$ are pairwise distinct and $p_k^\omega p_i^\omega = D_{ki}^\omega$, $p_k^\omega p_i^\omega = D_{kj}^\omega$ and $p_k^\omega p_{ij}^\omega = \mathrm{ulim}_\omega\, p_k^n p_{ij}^n$ are three distinct lines. We have

$$\mathrm{Tri}(F_i^n, F_j^n, F_k^n) = \mathbf{b}(D_{kj}^n, p_k^n p_{ij}^n, D_{ki}^n, D_k^n) = Z_\tau^n$$

and by hypothesis $Z_\tau^\omega = \mathrm{ulim}_\omega\, Z_\tau^n$ is distinct from $\infty, 0, -1$. Hence by Proposition 5.4 taking ultralimits, the line D_k^ω is distinct from the three lines D_{ki}^ω, D_{kj}^ω and $p_k^\omega p_{ij}^\omega$, so the triple of flags $(F_i^\omega, F_j^\omega, F_k^\omega)$ is generic, and $\mathrm{Tri}(F_i^\omega, F_j^\omega, F_k^\omega) = \mathbf{b}(D_{kj}^\omega, p_k^\omega p_{ij}^\omega, D_{ki}^\omega, D_k^\omega) = Z^\omega$. □

Lemma 5.7. *Let τ be a marked triangle in $\widetilde{\mathcal{T}}$ with ordered vertices (i, j, k), and $\tau' = (k, \ell, j)$ be the adjacent triangle. Suppose that the triple of flags $F^\omega(\tau)$ is generic. Then $F^\omega(\tau')$ is generic and*

$$\mathbf{b}(D_i^\omega, p_i^\omega p_j^\omega, p_i^\omega p_k^\omega, p_i^\omega(D_k^\omega \cap D_\ell^\omega)) = S_e^\omega$$

$$\mathbf{b}(D_k^\omega, p_k^\omega p_\ell^\omega, p_k^\omega p_i^\omega, p_k^\omega(D_i^\omega \cap D_j^\omega)) = S_{\overline{e}}^\omega .$$

□

Proof. Denote $p^\omega = \mathrm{ulim}(D_k^n \cap D_\ell^n)$. Then $p^\omega \in D_k^\omega$ and $p^\omega \in D_\ell^\omega$. Since F_k^ω, F_i^ω and F_j^ω are in generic position, we have $D_k^\omega \cap D_i^\omega = \mathrm{ulim}(D_k^n \cap D_i^n)$, $D_k^\omega \cap (p_i^\omega \oplus p_j^\omega) = \mathrm{ulim}(D_k^n \cap (p_i^n \oplus p_j^n))$ and these two points are distinct and distinct from p_k^ω. It follows then from Proposition 5.4 that the cross ratio $\mathbf{b}(D_k^\omega \cap D_i^\omega, D_k^\omega \cap (p_i^\omega \oplus p_j^\omega), p_k^\omega, p^\omega)$ is the ultralimit of $\mathbf{b}(D_k^n \cap D_i^n, D_k^n \cap (p_i^n \oplus p_j^n), p_k^n, p_{k\ell}^n) = S_e^n$, which is S_e^ω. Since $S_e^\omega \neq 0, \infty$, it follows that the point p^ω (which is on the line D_k^ω) is distinct from the two points p_k^ω, $D_i^\omega \cap D_k^\omega$.

Similarly the three lines $p_k^\omega p_i^\omega$, $p_k^\omega p_j^\omega$ and D_k^ω are paiwise distinct, hence the ultralimit Δ^ω of the line $p_k^n p_\ell^n$ satisfies $\mathbf{b}(p_k^\omega p_i^\omega, p_k^\omega p_j^\omega, D_k^\omega, \Delta^\omega) = S_{\overline{e}}^\omega$. The line Δ^ω passes through p_k^ω and is distinct from the lines D_k^ω and $p_k^\omega p_i^\omega$, since $S_{\overline{e}}^\omega \neq 0, \infty$. In particular $p_i^\omega \notin \Delta^\omega$, so $p_\ell^\omega \neq p_i^\omega$.

We have three pairwise distinct lines D_i^ω, $p_i^\omega p_k^\omega$ and $p_i^\omega p^\omega$, hence the cross ratio $\mathbf{b}(D_i^\omega, p_i^\omega p_k^\omega, p_i^\omega p^\omega, p_i^\omega p_k^\omega)$ is the ultralimit of $\mathbf{b}(D_i^n, p_i^n p_k^n, p_i^n p_{k\ell}^n, p_i^n p_\ell^n) = Z_{\tau'}^n$, which is $Z_{\tau'}^\omega$. Since $Z_{\tau'}^\omega \neq \infty$, we have $p_\ell^\omega \notin D_i^\omega$. Since $Z_{\tau'}^\omega \neq -1$, we have $p_i^\omega p_\ell^\omega \neq p_i^\omega p_k^\omega$, so $p_\ell^\omega \notin D_k^\omega$, in particular $p_\ell^\omega \neq p^\omega$, and $p_\ell^\omega \notin p_i^\omega p_k^\omega$. So F_i^ω, F_k^ω and p_ℓ^ω are in generic position. Since $p_\ell^\omega, p^\omega \in D_\ell^\omega$ and $p_\ell^\omega \neq p^\omega$, we have $D_\ell^\omega = p_\ell^\omega p^\omega$. Since $Z_{\tau'}^\omega \neq 0$, we have $p_i^\omega p_\ell^\omega \neq p_i^\omega p^\omega$, so the line $D_\ell^\omega = p_\ell^\omega p^\omega$ do not contain p_i^ω. We also have $p_k^\omega \notin D_\ell^\omega$ (since $p^\omega \neq p_k^\omega$) and D_ℓ^ω do not pass through $D_i^\omega \cap D_k^\omega$ (since $p^\omega \neq D_i^\omega \cap D_k^\omega$). The triple of flags $(F_k^\omega, F_\ell^\omega, F_i^\omega)$ is then generic and of triple ratio $\mathbf{b}(D_i^\omega, p_i^\omega p_k^\omega, p_i^\omega(D_k^\omega \cap D_\ell^\omega), p_i^\omega p_\ell^\omega) = Z_{\tau'}^\omega$ as $p^\omega = D_\ell^\omega \cap D_k^\omega$. □

We may now conclude the proof of Proposition 5.5. Let $\gamma \in \Gamma$. Then $\rho_n(\gamma)$ sends the canonical projective frame F_1, F_2, p_3 to the frame $F^n(\gamma o_1)$, $F^n(\gamma o_2)$, $p^n(\gamma o_3)$ whose ultralimit is $F^\omega(\gamma o_1)$, $F^\omega(\gamma o_2)$, $p^\omega(\gamma o_3)$, which is generic hence a projective frame in $\mathbb{P}(\mathbb{K}_\omega^3)$. Then by Proposition 5.3 $\rho_n(\gamma)$ is ω-bounded and its ultralimit $\rho_\omega(\gamma) = \mathrm{ulim}_\omega \rho_n(\gamma)$ sends the canonical frame $F_1^\omega, F_2^\omega, p_3^\omega$ to $F^\omega(\gamma o_1)$, $F^\omega(\gamma o_2)$, $p^\omega(\gamma o_3)$ hence the flags $F^\omega(\tau_0)$ to $F^\omega(\gamma \tau_0)$ (since they have the same triple ratio). So $\rho_\omega(\gamma) = \rho_{Z^\omega, S^\omega}(\gamma)$ as wanted. □

5.4. **Main result.** We suppose now that $\mathbb{K}$ is either equal to $\mathbb{R}$ or $\mathbb{C}$ or ultrametric. We are now able to describe a large family of degenerations of representations of Γ in $\mathrm{PGL}(\mathbb{K}^3)$ as (length spectra of) A_2-complexes of the form $K_{(z,s)}$ using degenerations of FG-parameters.

We denote by X the CAT(0) metric space (symmetric space or Euclidean building) associated with $\mathrm{PGL}_3(\mathbb{K})$.

Theorem 5.8. *Let $(Z^n, S^n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{K}_{\neq 0, -1}^T \times \mathbb{K}_{\neq 0}^{\vec{E}}$. Let $\rho_n : \Gamma \to \mathrm{PGL}_3(\mathbb{K})$ be the representation of FG-parameter $(Z^n, S^n) = ((Z^n_\tau)_\tau, (S^n_e)_e)$.*

Let $z^n_\tau = \log |Z^n_\tau|$, $z^n_e = \log |S^n_e|$ and $z^n = (z^n_\tau)_\tau$, $s^n = (s^n_e)_e$. Consider a sequence of real numbers $\lambda_n \geq 1$ going to $+\infty$, such that the sequence $\frac{1}{\lambda_n}(z^n, s^n)$ converges to a nonzero (z, s) in $\mathbb{R}^T \times \mathbb{R}^{\vec{E}}$. Suppose that:

(FT') For each triangle τ of $\mathcal{T}$, $\liminf \frac{1}{\lambda_n} \log |Z^n_\tau + 1| \geq 0$;

(FE') For each oriented edge e in $\mathcal{T}$, $\liminf \frac{1}{\lambda_n} \log |S^n_e + 1| \geq 0$;

(L) (z, s) is left-shifting,

(S) (z, s) is edge-separating.

Let K be the A_2-complex of FG-parameter (z, s). Then the renormalized $\mathfrak{C}$-length spectrum of ρ_n converges to the $\mathfrak{C}$-length spectrum of K: for all $\gamma \in \Gamma$ we have

$$\frac{1}{\lambda_n} \ell^{\mathfrak{C}}(\rho_n(\gamma)) \to \ell^{\mathfrak{C}}(\gamma, K)$$

in $\overline{\mathfrak{C}}$.

In particular, the usual Euclidean length spectrum of ρ_n converges to the Euclidean norm of the $\mathfrak{C}$-length spectrum of K:

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \ell_{euc}(\rho_n(\gamma)) = \left\| \ell^{\mathfrak{C}}(\gamma, K) \right\|$$

and the analogous claim holds for the Hilbert length:

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \ell_H(\rho_n(\gamma)) = N_H(\ell^{\mathfrak{C}}(\gamma, K))$$

for all $\gamma \in \Gamma$.

Remark. The hypotheses (FT') and (FE') are automatic for $\mathbb{K} = \mathbb{R}$ (and more generally for $\mathbb{K}$ ordered) and positive FG-parameters (since for positive $a \in \mathbb{K}$ we then have $|a + 1| \geq |1| = 1$).

Proof of Theorem 5.8. The idea of the proof is first to pass to the ultra-limit in an appropriate asymptotic cone associated with the scaling sequence $(\lambda_n)_n$, and then to apply Theorem 4.2 to show that the ultralimit representation preserves a $\mathfrak{C}$-geodesic copy of the A_2-complex K in the associated Euclidean building, hence has same marked $\mathfrak{C}$-length spectrum, and to use the continuity properties of $\mathfrak{C}$-length spectrum with respect to asymptotic cones of [Par11] to conclude.

Let $(\mathbb{K}_\omega, | \ |^\omega)$ be the asymptotic cone of the valued field $\mathbb{K}$ with respect to the scaling sequence $(\lambda_n)_n$ (see Section 5.1). We first check that the ultralimits behave well. For all $\tau \in T$, by hypothesis $\lim_\omega \frac{1}{\lambda_n} \log |Z^n_\tau| = z_\tau < +\infty$, hence the ultralimit Z^ω_τ of the sequence Z^n_τ in $\mathbb{K}_\omega$ is well defined, and $\lim_\omega \frac{1}{\lambda_n} \log |Z^n_\tau| = z_\tau > -\infty$ hence $Z^\omega_\tau \neq 0$. Similarly, the ultralimit S^ω_e of the sequence S^n_e in $\mathbb{K}_\omega$ is well defined and non zero as $|S^\omega_e|^\omega = \exp s_e$. For

all triangle τ, we also have $|Z_\tau^\omega + 1|^\omega = \lim_\omega |Z_\tau^n + 1|^{1/\lambda_n} \geq 1$, in particular $Z_\tau^\omega \neq -1$. Then by Proposition 5.5 the ultralimit $\rho_\omega : \Gamma \to \mathrm{PGL}(\mathbb{K}_\omega^3)$ of the sequence of representations ρ_n is well defined and is the representation $\rho_{Z^\omega, S^\omega}$ associated with the FG-parameter $(Z^\omega, S^\omega) = ((Z_\tau^\omega)_\tau, (S_e^\omega)_e)$.

The FG-parameter (Z^ω, S^ω) clearly satisfies the hypotheses of Theorem 4.2. Hence Theorem 4.2 applies, and ρ_ω, acting on the Euclidean building X_ω associated with $\mathrm{PGL}_3(\mathbb{K}_\omega)$, preserves an equivariant $\mathfrak{C}$-geodesically embedded copy of the A_2-complex K, hence the length spectra coincide:

$$\ell^\mathfrak{C}(\rho_\omega(\gamma)) = \ell^\mathfrak{C}(\gamma, K) \ \text{ for all } \gamma \text{ in } \Gamma \ .$$

Now fix γ in Γ. Since X_ω is the asymptotic cone of the metric space X for the rescaling sequence λ_n (see Theorem 3.21 of [Par11]) and by continuity properties of the $\mathfrak{C}$-length with respect to asymptotic cones (by Theorem 3.21 and Proposition 4.4 of [Par11]), the sequence $\frac{1}{\lambda_n}\ell^\mathfrak{C}(\rho_n(\gamma))$ has ultralimit $\ell^\mathfrak{C}(\rho_\omega(\gamma))$ in $\overline{\mathfrak{C}}$.

This proves that the sequence $\frac{1}{\lambda_n}\ell^\mathfrak{C}(\rho_n(\gamma))$ converges (in the usual sense) to $\ell^\mathfrak{C}(\gamma, K)$ in $\overline{\mathfrak{C}}$, since every subsequence of the sequence $\frac{1}{\lambda_n}\ell^\mathfrak{C}(\rho_n(\gamma))$ has $\ell^\mathfrak{C}(\gamma, K)$ as cluster value in $\overline{\mathfrak{C}}$. $\qquad\square$

We now suppose that $\mathbb{K} = \mathbb{R}$ and we apply this result to describe a part of the compactification of the moduli space of representations constructed in [Par11]. We first recall briefly the compactification. Denote $G = \mathrm{PGL}_3(\mathbb{R})$. Let $\mathcal{X}(\Gamma, G) = \mathrm{Hom}(\Gamma, G)//G$ be the biggest Hausdorff quotient of $\mathrm{Hom}(\Gamma, G)$ under G, which identifies with the locally compact subspace of $\mathrm{Hom}(\Gamma, G)/G$ consisting of completely reducible (i.e. semisimple) representations (see Section 5.1 of [Par11] for more details). The space $\overline{\mathfrak{C}}^\Gamma$ of functions from Γ to $\overline{\mathfrak{C}}$ is endowed with the product topology, and let $\mathbb{P}\overline{\mathfrak{C}}^\Gamma$ denote the quotient space of $\overline{\mathfrak{C}}^\Gamma - \{0\}$ by $\mathbb{R}_{>0}$. In [Par11] we constructed a metrizable compactification $\widetilde{\mathcal{X}}(\Gamma, G)$ of $\mathcal{X}(\Gamma, G)$, with boundary contained in $\mathbb{P}\overline{\mathfrak{C}}^\Gamma$ and endowed with a natural action of the modular group $\mathrm{Out}(\Gamma)$, with following sequential characterization: a sequence $[\rho_n]_n$ in $\mathcal{X}(\Gamma, G)$ converges in $\widetilde{\mathcal{X}}(\Gamma, G)$ to a boundary point $[w]$ in $\mathbb{P}\overline{\mathfrak{C}}^\Gamma$ if and only if the two following conditions are satisfied

(i) $[\rho_n]_n$ eventually gets out of any compact subset of $\mathcal{X}(\Gamma, G)$;

(ii) $[\ell^\mathfrak{C} \circ \rho_n]$ converges to $[w]$ in $\mathbb{P}\overrightarrow{\mathfrak{C}}^\Gamma$.

(see Section 5.3 of [Par11] for more details).

Let $\mathcal{M} = T \cup \overrightarrow{E}$ and denote by $\mathbb{P}^+\mathbb{R}^\mathcal{M}$ the space of rays in $\mathbb{R}^\mathcal{M}$, that is the quotient of $\mathbb{R}^\mathcal{M} - \{0\}$ by $\mathbb{R}_{>0}$, which is the standard sphere of dimension $8|\chi(\Sigma)| - 1$, and let $\mathbb{P}^+ : \mathbb{R}^\mathcal{M} - \{0\} \to \mathbb{P}^+\mathbb{R}^\mathcal{M}$ be the corresponding projection. The FG-parameters space $\mathbb{R}^\mathcal{M}$ is endowed with the standard compactification as a closed ball $\widetilde{\mathbb{R}^\mathcal{M}}$ with boundary $\partial_\infty \mathbb{R}^\mathcal{M} = \mathbb{P}^+\mathbb{R}^\mathcal{M}$.

Denote by $O_L \subset \mathbb{R}^\mathcal{M}$ the subset of left-shifting (z, s), and by $O_{LS} \subset O_L$ the subset of left-shifting and edge-separating (z, s), which are non empty open cones.

Since, for all left-shifting (z, s), the $\mathfrak{C}$-length spectrum $\ell_K^\mathfrak{C} : \gamma \mapsto \ell^\mathfrak{C}(\gamma, K)$ of the A_2-complex K is not identically zero (that is differs from $0 \in \overline{\mathfrak{C}}^\Gamma$), Theorem 5.8 above implies the following result.

Corollary 5.9. *The FG-parametrization map*

$$\varphi : \begin{array}{ccc} \mathbb{R}^{\mathcal{M}} & \to & \widetilde{\mathcal{X}}(\Gamma, G) \\ (z, s) & \mapsto & [\rho_{\exp(z), \exp(s)}] \end{array}$$

extends continuously to the open subset $\mathbb{P}^{+}O_{LS}$ of $\partial_{\infty}\mathbb{R}^{\mathcal{M}}$ by the restriction of the map

$$\begin{array}{ccc} \mathbb{P}^{+}O_{L} & \to & \mathbb{P}\overline{\mathfrak{C}}^{\Gamma} \\ [(z,s)] & \mapsto & [\ell_{K}^{\mathfrak{C}}] \, . \end{array} \qquad\qquad \Box$$

Note that the image by φ of the open cone O_{LS} is contained in the space $\mathcal{P}(\Sigma)$ of convex projective structures on Σ with principal geodesic boundary (see [Go90, FoGo07]).

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