Local and functional methods in arithmetic and in the information transmission theory; p-adic and nearly holomorphic modular forms.

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Algebraic geometry and number theory,
on the occasion of 60-th birthdays of
M.A.Tsfasman and S.G.Vladut
Independent Moscow University

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It is a great priviledge and luck for me, to know Mikhail Anatolievich TSFASMAN and Serguei Georgevich VLADUTS since 1971,

entering the same year Mechmat Faculty of MSU.
We studied together with Vadim SCHECHTMAN, Dima LOGACHEV, Misha VISHIK, Igor SKORNYAKOV, Igor ARTAMKIN, Misha KIFER, and other wonderful people, see the list of our year in http://vanteev.narod.ru/Spisok1971.htm.
Starting from the first semester we often passed examinations together, sometimes in advance.
We attended excellent seminars of E.M.LANDIS, E.S.GOLOD, Yu.A. BACHTURIN and A.Yu. OLSHANSKY, and even we already tried the seminars of I.M.GELFAND, Yu.I.MANIN, A.A.KIRILLOV, V.I.ARNOLD and I.R.SHAFAREVICH.

New amazing worlds of numbers, functions and varieties has opened to us in these seminars, and also, in regular meeting of the Moscow Mathematical Society, where crowds of mathematicians from all parts of the city were gathering. Many interesting things were learned in corridors and staircases.
These mathematical domains turned out to be closely related by analogies.
MANY THANKS TO MISHA AND SEREZHA FOR OUR JOINT DICOVERY OF THIS BIG MATHEMATICAL WORLD!
CONGRATULATIONS WITH THEIR 60 ANNIVERSARIES! BEST WISHES OF GOOD HEALTH AND MANY NEW ACHIEVEMENTS! NEW GOOD STUDENTS, BOOKS, AND MUCH ENTHUSIASM IN MATHEMATICS AND LIFE!

The choice the Chair of Higher Algebra in 1973

- An exceptional influence on us in this period was the two-year course "Galois Theory" by I.R.SHAFAREVICH, who explained also the representations of finite groups, group cohomology, extensions, crystallographic groups, ...
- In 1973-75, in the joint seminar of Yu.I.MANIN and A.A.KIRILLOV, p-adic zeta functions were treated, especially p-adic integration, Serre's p-adic modular forms, and Galois representations, with participation of Neal Koblitz, V.Drinfeld, I.I.Piatetsky-Shapiro, V.Berkovich, P.Kurchanov, Yu.Zarhin. Each time we learned "WHAT, HOW and WHY", and we often use all these things until now.
- Algebraic-geometric codes came to us later, in 1980, thanks to a talk by V.D.GOPPA in the Seminar of the Chair of Higher Algebra, organized by A.I.KOSTRIKIN. A great enthusiasm and participation of Yu.I.MANIN influenced very active research in this area, thanks of his vision of analogies between functions on algebraic curves and codes, such as parity check vs. sum of residues, genus vs. code distance, Riemann-Roch theorem vs. duality for codes etc.

Student years and first mathematical works

There were much joint activity during our student years, including travels to Mozhaisk region, to Yaroslavl' mathematical schools, to several places at the Black See coast, to Kiev (!) for a one-month military education, ...
There were many jokes, much humor, musics, books, graduation diploma work, first mathematical publications, ...
Then came postgraduate study, PhD thesis, ...
Direction of the first PhD students was much inspired by the example of direction by our supervisor Yu.I.MANIN.
We were much interested mutually in our mathematical results and, later on, of our PhD students.
In Moscow (see details below...)
In France and worldwide (see details below...)
Especially, Alexey ZYKIN, is one of the best hopes of our tradition.
All these young mathematicians and PhD students belogn to the mathematical school founded by our great teachers
Igor Rostislavovich SHAFAREVICH,
Yuri Ivanovich MANIN, and goes back to
Pafnutij Lvovich CHEBYSHEV.
I am very proud that both M.A.Tsfasman and S.G.Vladut, and also myself, belogn to this mathematical school, which is well presented as the 4th Mathematical Genealogy Tree, see [4thGT]

Analogies between numbers and functions

The ideas of using these fruitful analogies and various other tools come to us largerly from I. R. SHAFAREVICH and Y. I. MANIN. Let us quote Manin's Introduction to "Periods of cusp forms and p-adic Hecke series" (1973) :

> *. . . Elementary questions about congruences and equations have found themselves becoming interwoven in an intricate and rich complex of constructions drawn from abstract harmonic analysis, topology, highly technical ramifications of homological algebra, algebraic geometry, measure theory, logic, and so on – corresponding to the spirit of Gödel's theorem on the incompleteness of the techniques of elementary arithmetic and on our capabilities of recognizing even those truths which we are in a position to "prove". . .*
> *A new "synthetic" number theory, taking in the legacy of the "analytic" theory, is possibly taking shape under our very eyes . . .*

Examples of analogies between numbers and functions

- Proof of Mordell's Conjecture and Shafarevich conjectures by G.Faltings (1983) developping ideas of I.R.Shafarevich (Stockholm, 1962), A.N.Parshin, Yu.G.Zarhin, ...
- Grothendieck's theory of motives, and L-functions attached to them, giving analogy of a "cell decomposition" of an algebraic variety
- Proof of Weil Conjectures and Ramanujan Conjecture by Deligne (1974) using the ℓ-adic étale cohomology

Applications to Information Transmission Theory.

- Fast multiplication (Fast Fourier Transform) "integers as polynomials over a base like 2^n"
- Geometric codes (Information transmission) "code words as functions"
- Drinfeld modules and Drinfeld modular varieties "Cryptology schemes based on elliptic curves and Drinfeld modules", coming from an analogy with the Chebyshev polynomials, the unique polynomials satisfying $T_n(\cos(\vartheta)) = \cos(n\vartheta)$.

Domains of my PhD students considered by M.A.Tsfasman and S.G.Vladut

1. Arithmetic of global function fields, periods, duality and endomorphisms for Drinfeld modules (N.Doubovitskaya, I.Potemine, 1991-1997)
2. Complex and p-adic L-functions attached to modular forms and algebraic varieties (motives) Andrzhej Dabrowski (1992)
3. Minkowski-Hasse principle and Brauer groups: local to global (I.Chipchakov, 1989).
4. Distributions, congruences between modular forms and combinatirics of differential operators of Maass-Shimura and Rankin-Cohen (Michel Courtieu, Fabienne Jory, My Vinh Quang, 1990-2000)
5. p-adic Jacobi forms and L-values (P.Guerzhoy, 1991)
6. L-functions, special values and the method of canonical projection (Julien Puydt, Anh-Tuan Do, 2003-2014).
7. The idea of composition in the cryptology based on global function fields (A.Gewirtz, K.Vankov, 2004-2008 ...).
8. Hecke algebras and their use in coding theory and cryptography via geometric codes and modular forms (Kirill Vankov, 2008)

According to PhD students (in Moscow)

Natalia DOUBOVITSKAYA, Transcendency and duality for modules over polyomial ring of positive characteristic, MSU, 1990

MY Vinh Quang, Non archimedean analysis and convolutions of Hilbert modular forms, MSU, 1990

Pavel GUERZHOY, Arithmetic families of modular forms and Jacobi forms, with applications to diophantine geometry, MSU, 1991

Andrzej DĄBROWSKI, Non archimedean L functions of motives and automorphic forms, MSU, 1992

Igor POTEMINE, Arithmetic of global fields and geometry of Drinfeld modular schemes, UJF, 1997 (started in Moscow and completed in Grenoble)

According to PhD students (in France)

Fabienne JORY-HUGUE, Families of modular symbols and p-adic L-functions, UJF, 1998

Michel COURTIEU, Families of operators on Siegel modular formes, UJF 2000

Julien PUYDT, Special values and L functions of adelic modular forms, UJF, 2003

Alexandre GEWIRTZ (co-direction with Franck LEPREVOST), Arithmetic of function fields and applications to algorithmic and cryptology,

Kirill VANKOV, Hecke algebras, generating series and applications, UJF 2008

Anh-Tuan DO, Admissible measures of Siegel modular forms of arbitrary genus, UJF 2014

Distinuished books (I mention only two of them)

Serge G. Vladut, Kronecker's Jugendtraum and Modular Functions, [Vladut91]. From the preface
The arithmetical theory of modular functions is a result ofthe synthesis of numerous ideas of many outstanding mathematicians...
The principal intention of this book is to follow the development of one of the main sources of this theory, namely, complex multiplication of elliptic functions, and to describe the present state of the theory. Complex multiplication theory is usually associated with the name of Kronecker, who contributed much to it, and whose "liebster Jugendtraum" (the most beloved dream of his youth) was to see it completed ... can be found in earlier papers by Gauss, Abel, and Eisenstein.
... $\longrightarrow$ Construction of good error-correcting codes, p.366

Algebraic Geometric Codes: Basic Notions (Mathematical Surveys and Monographs) by Michael Tsfasman, Serge Vladut, Dmitry Nogin, [TsVlNo]. From the preface
" ... devoted to the theory of algebraic geometric codes, a subject formed on the border of several domains of mathematics. On one side there are such classical areas as algebraic geometry and number theory; on the other, information transmission theory, combinatorics, finite geometries, dense packings, etc. The authors give a unique perspective on the subject. ... this book constantly looks for interpretations that connect coding theory to algebraic geometry and number theory. ... in the last chapter the authors explain relations between all of these: the theory of algebraic geometric codes".

Splendid texts to teach Geometric Coding Theory, [TsVlNo] and previous texts

A favorite topic, and a great source of motivations is

Theorem (Ihara-Tsfasman-Zink-Drinfeld-Vladut)

Let $g = g(X)$ be the genus of X, $n = \mathrm{Card}\, X(\mathbb{F}_q))$. Then
(a)

$$\liminf_X \frac{g}{n} \geq \frac{1}{\sqrt{q}-1}. \qquad (1)$$

(b) *If q is a square, the bound (1) is exact*

$$\liminf_X \frac{g}{n} = \frac{1}{\sqrt{q}-1}.$$

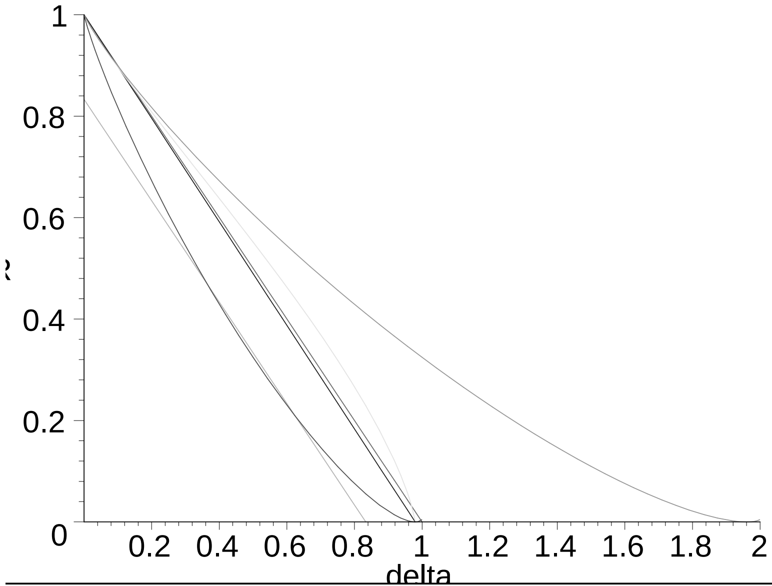
An important consequence is a construction of excellent families of codes $C(D, D_0) = Ev_{X(\mathbb{F}_q)}\mathcal{L}(D)$ of divisor D, $\deg D < n$, disjoint with $D_0 = \sum_i x_i$, $X(\mathbb{F}_q) = \{x_1, \dots, x_n\}$, via evaluation mapping.
Let $R = \frac{k}{n}$, $\delta = \frac{d}{n}$ be the relative speed (relative distance) of transmission with the code $C(D, D_0)$.

Corollary

If q is a square, then $\limsup R \geq 1 - \frac{1}{\sqrt{q}-1} - \delta$. ∎

It suffices to substitute $\liminf_X \frac{g}{n} = \frac{1}{\sqrt{q}-1}$ in the Riemann-Roch inequality $k + d \geq n - g + 1$ in the form $R + \delta \geq 1 - \frac{g}{n} + \frac{1}{n}$, and pass to the limit.

For $q = 49$



Real-analytic and p-adic modular forms

A nice example of a bridge between real-analytic $\leftrightarrow$ p-adic worlds comes from p-adic modular forms invented by J.-P.Serre [Se73] as p-adic limits of q-expansions of modular forms with rational coefficients for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, see also [SeCP]. The ring $\mathcal{M}_p$ of such forms contains $\mathcal{M} = \oplus_{k\geq 0}\mathcal{M}_k(\Gamma, \mathbb{Z}) = \mathbb{Z}[E_4, E_6]$, and it contains $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$. On the other hand,

$$\tilde{E}_2 = -\frac{3}{\pi y} + E_2 = -12S + E_2, \text{ where } S = \frac{1}{4\pi y},$$

is a nearly holomorphic modular form (its coefficients are polynomials of S over $\mathbb{Q}$). Let $\mathcal{N}$ be the ring of such forms. Then

$$\tilde{E}_2|_{S=0} = E_2,$$

and it was proved by J.-P.Serre that E_2 is a p-adic modular form. Elements of the ring $\mathcal{M}^\sharp = \mathcal{N}|_{S=0}$ are quasimodular forms. These phenomena are quite general and can be used in computations and proofs.

Given a quasimodular form f, find a nearly holomorphic form $\tilde{f}$ such that $\tilde{f}|_{S=0} = f$.

If we use the structure theorem as the polynomial ring [MaRo5],

$$\mathcal{N} = \mathbb{Q}[\tilde{E}_2, E_4, E_6], \quad \mathcal{M}^\sharp = \mathbb{Q}[E_2, E_4, E_6],$$

the answer is $\tilde{f} = P(\tilde{E}_2, E_4, E_6)$ for any $f = P(E_2, E_4, E_6)$.
In general, consider the derivative

$$D = \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq},$$

then D acts on $\mathcal{M}^\sharp$. In general, $\tilde{f}$ can be recovered from the transformation law of f (see [MaRo5]): if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$(cz+d)^{-k} f(\gamma z) = \sum_{t=0} \left(\frac{c}{cz+d}\right)^t f_t(z) = P_{z,f}(X) \text{ then}$$

$$\tilde{f} = P_{z,f}\left(\frac{1}{2iy}\right) = P_{z,f}(-2i\pi S), \text{ where } S = \frac{1}{4\pi y}, y = \frac{1}{4\pi S}.$$

Examples : (a) Serre's p-adic modular form E_2 is quasi-modular in the following sense

The transformation law of $f = E_2$ is

$$(cz+d)^{-2}E_2(\gamma(z)) = E_2(z) + \frac{6}{i\pi}\left(\frac{c}{cz+d}\right) = P_{z,E_2}\left(\frac{c}{cz+d}\right),$$

where $P_{z,E_2}(X) = E_2(z) + \dfrac{6}{i\pi}X$ is a polynomial with holomorphic coefficients for $X = \dfrac{c}{cz+d}$.
The same polynomial $P_{z,E_2}(X)$ gives the corresponding nearly holomorphic function $\tilde{E}_2$ as its value at $X = -2i\pi S$:

$$\tilde{E}_2 = -\frac{3}{\pi y} + E_2 = -12S + E_2 = P_{z,E_2}\left(\frac{1}{2iy}\right) = P_{z,E_2}(-2i\pi S).$$

This function has rational coefficients of q and S, so it is "algebraic".

Examples : (b) Derivative of a modular form is not a modular form, but it is quasi-modular (Don Zagier, 1994)

If $f = D^r g$ where $g \in \mathcal{M}_l(\Gamma)$ is a holomorphic modular form of weight ℓ, then the transformation law of weight $k = \ell + 2r$ is

$$(cz+d)^{-k} D^r g(\gamma z) = \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(t+\ell)} \left(\frac{1}{2\pi i} \cdot \frac{c}{cz+d}\right)^{r-t} D^t g(z)$$

using p.59 of [MFDO], or [Kuz75].
In this way we get SHIMURA's differential operator

$$\delta_\ell^r g = \widetilde{D^r g} = \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(r-t+\ell)} (2\pi i)^{-t} (-2\pi i S)^t D^{r-t} g(z)$$
$$= \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(r-t+\ell)} (-S)^t D^{r-t} g(z), \text{ where } S = \frac{1}{4\pi y},$$

which preserves the rationality of the coefficients of S and q. It comes from the above transformation law of $D^r g$ with $X = \dfrac{c}{cz+d}$ replaced by $-2\pi i S$:

$$P_{z,D^r g}(X) = \sum_{t=0} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(t+\ell)} (X/2\pi i)^{r-t} D^t g(z).$$

A conceptual explanation of this real-analytic $\leftrightarrow$ p-adic bridge comes from the algebraic GAUSS-MANIN connection (due to GROTHENDIECK in higher dimensions, see [Gr66], [KaOd68]), and expressed via δ_ℓ^r in the Siegel case in [Ha81], p.111).

Using algebraic and p-adic modular forms in computations

There are several methods to compute various L-values attached to Siegel modular forms using Petersson products of holomorphic and nearly-holomorphic Siegel modular forms :
the Rankin-Selberg method,
the doubling method (pull-back method).
A well-known example is the standard zeta function $D(s, f, \chi)$ of a Siegel cusp eigenform $f \in \mathcal{S}_k^n(\Gamma)$ of genus n (with local factors of degree $2n+1$) and χ a Dirichlet character.
Theorem (the case of even genus n (Courtieu-A.P.), via the Rankin-Selberg method) gives a p-adic interpolation of the normailzed critical values $D^*(s, f, \chi)$ using Andrianov-Kalinin integral representation of these values $1+n-k \leq s \leq k-n$ through the Petersson product $\langle f, \theta_{T_0}\delta^r E\rangle$ where δ^r is a certain composition of Maass-Shimura differential operators, θ_{T_0} a theta-series of weight $n/2$, attached to a fixed $n \times n$ matrix T_0.
Theorem (the general case (by Boecherer-Schmidt), via the doubling method) uses Boecherer–Garrett–Shimura identity (a pull-back formula)

A pull-back formula

allows to compute the critical values through certain double Petersson product by integrating over $z \in \mathbb{H}_n$ the identity:

$$\Lambda(l+2s,\chi)D(l+2s-n,f,\chi)f = \langle f(w), E^{2n}_{l,\nu,\chi,s}(\mathrm{diag}[z,w])\rangle_w.$$

Here $k = l+\nu$, $\nu \geq 0$, $\Lambda(l+2s,\chi)$ is a product of special values of Dirichlet L-functions and Γ-functions, $E^{2n}_{l,\nu,\chi,s}$ a higher twist of a Siegel-Eisenstein series on $(z,w) \in \mathbb{H}_n \times \mathbb{H}_n$ (see [Boe85], [Boe-Schm]).

A p-adic construction uses congruences for the L-values, expressed through the Fourier coefficients of the Siegel modular forms and nearly-modular forms.

We indicate a new approach of computing the Petersson products and L-values, using an injection of algebraic nearly holomorphic modular forms into p-adic modular forms.

Applications to families of Siegel modular forms are given.

As a consequence, explicit two-parameter families are constructed.

A recent discovery by Takashi Ichikawa (Saga University), [Ich12], J. reine angew. Math., [Ich13]

allows to inject nearly-holomorphic arithmetical (vector valued) Siegel modular forms into p-adic modular forms.

Via the Fourier expansions, the image of this injection is represented by certain quasimodular holomorphic forms like $E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$, with algebraic Fourier expansions.

This description provides many advantages, both computational and theoretical, in the study of algebraic parts of Petersson products and L-values, which we would like to develop here.

This work is related to a recent preprint [BoeNa13] by S. Boecherer and Shoyu Nagaoka where it is shown that Siegel modular forms of level $\Gamma_0(p^m)$ are p-adic modular forms. Moreover they show that derivatives of such Siegel modular forms are p-adic. Parts of these results are also valid for vector-valued modular forms.

Arithmetical nearly-holomorphic Siegel modular forms, see [MMJ05]

Nearly-holomorphic Siegel modular forms over a subfield k of $\mathbb{C}$ are certain $\mathbb{C}^d$-valued smooth functions f of $Z = X + \sqrt{-1}Y \in \mathbb{H}_n$ given by the following expression

$$f(Z) = \sum_T P_T(S) q^T,$$

where T runs through the set B_n of all half-integral semi-positive matricies, $S = (4\pi Y)^{-1}$ a symmetric matrix, $q^T = \exp(2\pi\sqrt{-1}\mathrm{tr}(TZ))$, $P_T(S)$ are vectors of degree d whose entries are polynomials over k of the entries of S.

Formal Fourier expansions over a commutative ring A

Algebraically we may use the notation

$$q^T = \prod_{i=1}^{n} q_{ii}^{T_{ii}} \prod_{i<j} q_{ij}^{2T_{ij}} \in A[\![q_{11}, \dots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$$

(with $q^T = \exp(2\pi i \mathrm{tr}(TZ))$, $q_{ij} = \exp(2\pi(\sqrt{-1}Z_{i,j}))$ for $A = \mathbb{C}$). The elements q^T form a multiplicative semi-group so that $q^{T_1} \cdot q^{T_2} = q^{T_1+T_2}$, and one may consider f as a formal q-expansion over an arbitrary ring A via elements of the semi - group algebra $A[\![q^{B_n}]\!]$.

Algebraic definition: $f \in S_e(Sym^2(A^n), A[\![q^{B_n}]\!]^d)$, where S_e denotes the A-polynomial mappings of degree e on symmetric matricies $S \in Sym^2(A^n)$ of order n with vector values in $A[\![q^{B_n}]\!]^d$.

Computing the Petersson products

The Petersson product of a given modular form
$f(Z) = \sum_T a_T q^T \in \mathcal{M} \subset \mathcal{M}_\rho(\bar{\mathbb{Q}})$ by another modular form
$h(Z) = \sum_T b_T q^T \in \mathcal{M} \subset \mathcal{M}_{\rho^*}(\bar{\mathbb{Q}})$ produces a linear form

$$\ell_f : h \mapsto \frac{\langle f, h\rangle}{\langle f, f\rangle}$$

defined over a subfield $k \subset \bar{\mathbb{Q}}$. Thus ℓ_f can be expressed through the Fourier coefficients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coefficients:

$$\ell_{T_i} : h \mapsto b_{T_i} \quad (i = 1,\ n).$$

It follows that $\ell_f(h) = \sum_i \gamma_i b_{T_i},$ where $\gamma_i \in k$.

Complex and p-adic L-functions

There exist two kinds of L-functions

- Complex L-functions (Euler products) on $\mathbb{C} = \mathrm{Hom}(\mathbb{R}_+^*, \mathbb{C}^*)$.
- p-adic L-functions (Mellin transforms L_μ of p-adic measures)

Both are used in order to obtain a number (L-value) from an automorphic form. Usually such a number is algebraic (after normalization) via the embeddings

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \quad \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p.$$

How to define and to compute p-adic L-functions?

The Mellin transform of a p-adic distribution μ on $\mathbb{Z}_p^*$ gives an analytic function on the group of p-adic characters

$$x \mapsto L_\mu(x) = \int_{\mathbb{Z}_p^*} x(y)\,d\mu, \quad x \in X_{\mathbb{Z}_p^*} = \mathrm{Hom}_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*).$$

A general idea is to construct p-adic measures directly from Fourier coefficients of modular forms proving Kummer-type congruences for L-values. Here is a new method to construct p-adic L-functions via quasimodulat forms:

How to prove Kummer-type congruences using the Fourier coefficients?

Suppose that we are given some L-function $L_f^*(s,\chi)$ attached to a Siegel modular form f and assume that for infinitely many "critical pairs" (s_j,χ_j) one has an integral representation $\boxed{L_f^*(s_j,\chi_j)=\langle f,h_j\rangle}$ with all $h_j=\sum_T b_{j,T}q^T\in\mathcal{M}$ in a certain finite-dimensional space $\mathcal{M}$ containing f and defined over $\bar{\mathbb{Q}}$.
We want to prove the following Kummer-type congruences:

$$\forall x\in\mathbb{Z}_p^*\sum_j\beta_j\chi_jx^{k_j}\equiv 0 \mod p^N \Longrightarrow \sum_j\beta_j\frac{L_f^*(s_j,\chi_j)}{\langle f,f\rangle}\equiv 0 \mod p^N.$$

for any choice of $\beta_j\in\bar{\mathbb{Q}}$, $k_j=\begin{cases}s_j-s_0 & \text{if } s_0=\min_j s_j \text{ or}\\ k_j=s_0-s_j & \text{if } s_0=\max_j s_j.\end{cases}$

Using the above expression for $\ell_f(h_j)=\sum_i\gamma_{i,j}b_{j,T_i}$, the above congruences reduce to

$$\sum_{i,j}\gamma_{i,j}\beta_jb_{j,T_i}\equiv 0 \mod p^N.$$

Reduction to a finite dimensional case

In order to prove the congruences

$$\sum_{i,j} \gamma_{i,j}\beta_j b_{j,T_i} \equiv 0 \mod p^N.$$

in general we use the functions h_j which belong only to a certain infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{M} = \mathcal{M}(\overline{\mathbb{Q}})$

$$\mathcal{M}(\overline{\mathbb{Q}}) := \bigcup_{m\geq 0} \mathcal{M}_{\rho^*}(Np^m, \overline{\mathbb{Q}}).$$

Starting from the functions h_j, we use their caracteristic projection $\pi = \pi^\alpha$ on the characteristic subspace $\mathcal{M}^\alpha$ (of generalized eigenvectors) associated to a non-zero eigenvalue α Atkin's U-operator on f which turns out to be of fixed finite dimension so that for all j, $\pi^\alpha(h_j) \in \mathcal{M}^\alpha$.

From holomorphic to nearly holomorphic and p-adic modular forms, see [MMJ02]

Next we explain, how to treat the functions h_j which belong to a certain infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{N} \subset \mathcal{N}_\rho(\overline{\mathbb{Q}})$ (of *nearly holomorphic modular forms*).

Usually, h_j can be expressed through the functions $\delta^{k_j}(\varphi_0(\chi_j))$ for a certain non-negative power k_j of the Maass-Shimura-type differential operator applied to a holomorphic form $\varphi_0(\chi_j)$.

Then the idea is to proceed in two steps:

1) to pass from the infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{N} = \mathcal{N}(\overline{\mathbb{Q}})$ of nearly holomorphic modular forms,

$$\mathcal{N}(\overline{\mathbb{Q}}) := \bigcup_{m \geq 0} \mathcal{N}_{k,r}(Np^m, \overline{\mathbb{Q}}) \text{ (of the depth } r).$$

to a fixed finite dimensional characteristic subspace $\mathcal{N}^\alpha \subset \mathcal{N}(Np)$ of U_p in the same way as for the holomorphic forms.

This step controls Petersson products using conjugate f^0 of an eigenfunction f_0 of $U(p)$:

$$\langle f^0, h \rangle = \alpha^{-m} \langle f^0, |U(p)^m h \rangle = \langle f^0, \pi^\alpha(h) \rangle.$$

From holomorphic to nearly holomorphic and p-adic modular forms (continued)

2) To apply Ichikawa's mapping $\iota_p : \mathcal{N}(Np) \to \mathcal{M}_p(Np)$ to a certain space $\mathcal{M}_p(Np)$ of p-adic Siegel modular forms. Assume algebraically,

$$h_j = \sum_T b_{j,T}(S) q^T \mapsto \kappa(h_j) = \sum_T b_{j,T}(0) q^T,$$

which is also a certain Siegel quasi-modular form. Under this mapping, computation become much easier, as the action of δ^{k_j} becomes simply a k_j-power of the Ramanujan Θ-operator

$$\Theta : \sum_T b_T q^T \mapsto \sum_T \det(T) b_T q^T,$$

in the scalar-valued case. In the vector-valued case such operators were studied in [BoeNa13].
After this step, proving the Kummer-type congruences reduces to those for the Fourier coefficients the quasimodular forms $\kappa(h_j(\chi_j))$ which can be explicitely evaluated using the Θ-operator.

Computing with Siegel modular forms over a ring A

There are several types of Siegel modular forms (vector-valued, nearly-holomorphic, quasi-modular, p-adic). Consider modular forms over a ring $A = \mathbb{C}, \mathbb{C}_p, \Lambda = \mathbb{Z}_p[\![T]\!], \cdots$ as certain formal Fourier expansions over A.

Let us fix the congruence subgroup Γ of a nearly holomorphic modular form $f \in \mathcal{N}_\rho$ and its depth r as the maximal S-degree of the poynomial Fourier Fourier coefficients $a_T(S)$ of a nearly holomorphic form

$$f = \sum_T a_T(S) q^T \in \mathcal{N}_\rho(A),$$

over R, and denote by $\mathcal{N}_{\rho,r}(\Gamma, A)$ the A-module of all such forms. This module is often locally-free of finite rank, that is, it becomes a finite-dimensional F-vector space over the fraction field $F = Frac(A)$.

Types of modular forms

- $\mathcal{M}_\rho$ (holomorphic vector-valued Siegel modular forms attached to an algebraic representation $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$)
- $\mathcal{N}_\rho$ (nearly holomorphic vector-valued Siegel modular forms attached to ρ over a number field $k \subset \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$))
- $\mathcal{M}^\sharp_\rho$ (quasi-modular vector-valued forms attached to ρ)
- $\mathcal{M}^\flat_\rho$ (algebraic p-adic vector-valued forms attached to ρ over a number field $k \subset \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$)

Definitions and interrelations:

- $\mathcal{M}^\sharp_{\rho,r} = \kappa(\mathcal{N}_\rho) \subset \mathcal{R}^d_{n,\infty}$, where $\kappa : f \mapsto f|_{S=0} = \sum_T P_T(0)q^T$, where $\mathcal{R}_{n,\infty} = \mathbb{C}[\![q_{11}, \dots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$.
- $\mathcal{M}^\flat_{\rho,r}(R,\Gamma) = F_c(\iota_p(\mathcal{N}_{\rho,r}(R,\Gamma))) \subset \mathcal{R}^d_{n,p}$, where $\mathcal{R}_{n,p} = \mathbb{C}_p[\![q_{11}, \dots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$.

Let us fix the level Γ, the depth r, and a subring R of $\bar{\mathbb{Q}}$, then all the R-modules $\mathcal{M}_\rho(R,\Gamma)$, $\mathcal{N}_{\rho,r}(R,\Gamma)$, $\mathcal{M}^\sharp_{\rho,r}(R,\Gamma)$, $\mathcal{M}^\flat_{\rho,r}(R,\Gamma)$ are then locally free of finite rank.
In interesting cases, there is an inclusion $\mathcal{M}^\sharp_{\rho,r}(R,\Gamma) \hookrightarrow \mathcal{M}^\flat_{\rho,r}(R,\Gamma)$.
If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $k = 2$, $P = E_2$ is a p-adic modular form, see [Se73], p.211.
Question: Prove it in general! (after discussions with S.Boecherer and T.Ichikawa)

Computing with families of Siegel modular forms

Let $\Lambda = \mathbb{Z}_p[\![T]\!]$ be the Iwasawa algebra, and consider Serre's ring

$$\mathcal{R}_{n,\Lambda} = \Lambda[\![q_{11}, \ldots, q_{nn}]\!][q_{ij}^{\pm 1}]_{i,j=1,\cdots,n}.$$

For any pair (k, χ) as above consider the homomorphisms:

$$\kappa_{k,\chi} : \Lambda \to \mathbb{C}_p, \mathcal{R}^d_{n,\Lambda} \mapsto \mathcal{R}^d_{n,\mathbb{C}_p}, \text{ where } T \mapsto \chi(1+p)(1+p)^k - 1.$$

Definition (families of Siegel modular forms)

Let $\mathbf{f} \in \mathcal{R}^d_{n,\Lambda}$ such that for infinitely many pairs (k, χ) as above,

$$\kappa_{k,\chi}(\mathbf{f}) \in \mathcal{M}_{\rho_k}((i_p(\bar{\mathbb{Q}})) \overset{F_c}{\hookrightarrow} \mathcal{R}^d_{n,\mathbb{C}_p}$$

is the Fourier expansion at c of a Siegel modular form over $\bar{\mathbb{Q}}$.
All such $\mathbf{f}$ generate the Λ-submodule $\mathcal{M}_{\rho_k}(\Lambda) \subset \mathcal{R}^d_{n,\Lambda}$ of Λ-adic Siegel modular forms of weight ρ.
In the same way, the Λ-submodule $\mathcal{M}^\sharp_{\rho_k}(\Lambda) \subset \mathcal{R}_{n,\Lambda}$ of Λ-adic Siegel quasi-modular forms is defined.

Examples of families of Siegel modular forms

can be constructed via differential operators of Maass $\Delta = \det(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial z_{ij}})$, so that $\Delta q^T = \det(T)q^T$. Shimura's operator $\delta_k f(Z) = (-4\pi)^{-n}\det(Z-\bar{Z})^{\frac{1+n}{2}-k}\Delta(\det(Z-\bar{Z})^{k-\frac{1+n}{2}+1}f(Z)$ acts on q^T using $\rho_r : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(\wedge^r\mathbb{C}^n)$ and its adjoint ρ_r^*:

$$\delta_k(q^T) = \sum_{l=0}^{n}(-1)^{n-l}c_{n-l}(k+1-\frac{1+n}{2})\mathrm{tr}({}^t\rho_{n-l}(S)\rho_l^*(T))q^T,$$

where $c_{n-l}(s) = s(s-\frac{1}{2})\cdots(s-\frac{n-l-1}{2})$, $S = (2\pi i(\bar{z}-z))^{-1}$.

- Nearly holomorphic Λ-adic Siegel-Eisenstein series as in [PaSE] can be produced from the pairs $(-s,\chi)$: if s is a nonpositive integer such that $k+2s > n+1$,

$$E_k(Z,s,\chi) = \prod_{i=0}^{-s-1} c_n(k+2s+2i)^{-1}\delta_{k+2s}^{(-s)}(E_{k+2s}(Z,0,\chi)).$$

Examples of families of Siegel modular forms (continued)

- Ichikawa's construction: quasi-holomorphic (and p-adic) Siegel - Eisenstein series obtained in [Ich13] using the injection ι_p

$$\iota_p(\pi^{ns}E_k(Z,s,\chi)) = \prod_{i=0}^{-s-1} c_n(k+2s+2i)^{-1} \sum_T \det(T)^{-s} b_{k+2s}(T)q^T,$$

  where
  $E_{k+2s}(Z,0,\chi) = \sum_T b_{k+2s}(T)q^T, k+2s > n+1, s \in \mathbb{Z}.$
- A two-variable family is for the parameters $(k+2s, s), k+2s > n+1, s \in \mathbb{Z}$ will be now constructed.

Normalized Siegel-Eisenstein series of two variables

Let us start with an explicit family described in [Ike01], [PaSE], [Pa91] as follows

$$\mathcal{E}_k^n = E_k^n(z)2^{n/2}\zeta(1-k)\prod_{i=1}^{[n/2]}\zeta(1-2k+2i) = \sum_T a_T(\mathcal{E}_k^n)q^T,$$

where for any non-degenerate matrice T of quadratic character ψ_T:

$$a_T(\mathcal{E}_k^n)$$
$$= 2^{-\frac{n}{2}}\det T^{k-\frac{n+1}{2}}M_T(k)\times\begin{cases}L(1-k+\frac{n}{2},\psi_T)C_T^{\frac{n}{2}-k+(1/2)}, & n \text{ even},\\ 1, & n \text{ odd},\end{cases}$$

($C_T=$ cond(ψ_T), $M_T(k)$ a finite Euler product over $\ell|\det(2T)$). Starting from the holomorphic series of weight $k>n+1$ and $s=0$, let us move to all points $(k+2s,s), k+2s>n+1, s\in\mathbb{Z}, s\le 0$. Then Ichikawa's construction is applicable and it provides a two-variable family.

Examples of families of Siegel modular forms (continued)

- Ikeda-type families of cusp forms of even genus [PaIsr11] (reported in Luminy, May 2011). Start from a p-adic family

$$\boldsymbol{\varphi} = \{\varphi_{2k}\} : 2k \mapsto \varphi_{2k} = \sum_{n=1}^{\infty} a_n(2k)q^n \in \overline{\mathbb{Q}}[\![q]\!] \subset \mathbb{C}_p[\![q]\!],$$

  where the Fourier coefficients $a_n(2k)$ of the normalized cusp Hecke eigenform φ_{2k} and one of the Satake p-parameters $\alpha(2k) := \alpha_p(2k)$ are given by certain p-adic analytic functions $k \mapsto a_n(2k)$ for $(n, p) = 1$. The Fourier expansions of the modular forms $F = F_{2n}(\varphi_{2k})$ can be explicitly evaluated where $L(F_{2n}(\varphi), St, s) = \zeta(s) \prod_{i=1}^{2n} L(\varphi, s + k + n - i)$. This sequence provide an example of a p-adic family of Siegel modular forms.
- Ikeda-Myawaki-type families of cusp forms of $n = 3$, [PaIsr11] (reported in Luminy, May 2011).
- Families of Klingen-Eisenstein series extended in [JA13] from $n = 2$ to a general case (reported in Journées Arithmétiques, Grenoble, July 2013).

Instead of conclusion: why study L-values attached to modular forms?

Zeta functions, L-functions and modular forms often give answers to questions in Number Theory as follows:

Construct a generating function $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]]$ of an arithmetical function $n \mapsto a_n$, for example $a_n = p(n)$ $\rightsquigarrow$ Compute f via modular forms, for example $\sum_{n=0}^{\infty} p(n) q^n = (\Delta/q)^{-1/24}$ $\rightsquigarrow$ A number (solution)

Example (Hardy-Ramanujan)

$$p(n) = \frac{e^{\pi\sqrt{2/3(n-1/24)}}}{4\sqrt{3}\lambda_n^2} + O(e^{\pi\sqrt{2/3(n-1/24)}}/\lambda_n^3),$$
$$\lambda_n = \sqrt{n-1/24},$$

$\uparrow$ Good bases, finite dimensions, many relations and identities ...

$\uparrow$ Values of L-functions, (complex and p-adic), periods, congruences, ...

Other examples: Birch and Swinnerton-Dyer conjecture, L-values attached to modular forms, Wiles's proof of Fermat's Last Theorem,...(see [Ma-Pa05]. Also, in talks for Zeta Functions I-IV in J.-V. Poncelet Laboratory, UMI 2615 of CNRS in Moscow).

Thank you!

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