Contents

Introduction
Generalities on triple products
Statement of the problem
Arithmetical nearly holomorphic modular forms
Siegel-Eisenstein series
Algebraic differential operators of Maass and Shimura

L-critical values of the $L$-function $L(f_1 \otimes f_2 \otimes f_3, s)$ (Theorem A)

Scheme of the Proof
Boecherer’s higher twist
Ibukiyama’s differential operator
Algebraic linear form
Evaluation of $p$-adic integrals
Criterion of admissibility

Why study $L$-values attached to modular forms?

A popular proceedure in Number Theory is the following:

Construct a generating function $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]]$ of an arithmetical function $n \mapsto a_n$.

For example $a_n = \rho(n)$

Example 1 [Chaud70]:

(Hardy-Ramanujan)

Good bases, finite dimensions, many relations and identities

Values of $L$-functions, periods, congruences, . . .

Compute $f$ via modular forms, for example $\sum_{n=0}^{\infty} p(n) q^n = (\Delta/q)^{-1/24}$

A number (solution)

Let $p$ be a prime, $p \nmid N$. We view $f_j \in \mathbb{Q}(q) \otimes \mathbb{C}[q]$ via a fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{C}_p$, $\mathbb{C}_p = \mathbb{C}^\times$ is Tate’s field.

Let $\chi$ denote a variable Dirichlet character $\mod Np^\nu$, $\nu \geq 0$.

We view $k_j$ as a variable weight in the weight space $X = X_{Np^\nu} = \text{Hom}_\text{cont}(Y, \mathbb{C}_p)$, $Y = (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times \ni (y_0, y_p)$.

The space $X$ is a $p$-adic analytic space first used in Serre’s [Se73] *Formes modulaires et fonctions zêta $p$-adiques*. Denote by $(k, \chi) \in X$ the homomorphism $\chi(y_0, y_p) : \chi(y_0) \chi(y_p \mod p^\nu) y_p^k$.

We write simply $k_j$ for the couple $(k_j, \psi_j) \in X$.

Our data: three primitive cusp eigenforms

$$f_j(z) = \sum_{n=1}^{\infty} a_{nj} q^n \in S_{k_j}(N_j, \psi_j), \quad (j = 1, 2, 3) \quad (1.1)$$

of weights $k_1, k_2, k_3$, of conductors $N_1, N_2, N_3$, and of nebentypus characters $\psi_j \mod N_j$, $N := \text{LCM}(N_1, N_2, N_3)$.

Let $p$ be a prime, $p \nmid N$. We view $f_j \in \mathbb{Q}(q) \otimes \mathbb{C}_p[q]$ via a fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{C}_p$, $\mathbb{C}_p = \mathbb{C}^\times$ is Tate’s field.

Let $\chi$ denote a variable Dirichlet character $\mod Np^\nu$, $\nu \geq 0$.

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We write simply $k_j$ for the couple $(k_j, \psi_j) \in X$. 

Other examples: Birch and Swinnerton-Dyer conjecture, ... $L$-values attached to modular forms
The purpose of this talk is to describe a four variable $p$-adic $L$ function attached to Garrett’s triple product of three Coleman’s families

\[ k_j \mapsto \left\{ f_{j,k} = \sum_{n=1}^\infty a_n j(k_j) q^n \right\} \]

of cusp eigenforms of three constant slopes $\sigma_j = \text{ord}_p(\alpha_p(j(k_j))) \geq 0$ where $\alpha_p(j(k_j))$, $\alpha_p^{(2)}(j(k_j))$ are the Satake parameters given as inverse roots of the Hecke $p$-polynomial

\[ 1 - a_p jX - \psi_j(p) p^{-j} X^2 = (1 - \alpha_p(j(p))X)(1 - \alpha_p^{(2)}(p)X). \]

We assume that $\text{ord}_p(\alpha_p(j(k_j))) \leq \text{ord}_p(\alpha_p^{(2)}(j(k_j)))$.

This extends a previous result: (see [PaTV], Invent. Math. v. 154, N3 (2003)) where a two variable $p$-adic $L$-function was constructed interpolating on all $k$ a function $(k,s) \mapsto L^*(f_k, s, \chi)$

\[ \sigma > 0 \]

for such a family.

**Coleman proved:**

- The operator $U$ acts as a completely continuous operator on each $\mathcal{A}$-submodule $\mathcal{M}^j(Np^2; \mathcal{A}) \subset \mathcal{A}[[q]]$ (i.e. $U$ is a limit of finite-dimensional operators)
- there is a version of the Riesz theory:

\[ \exists \text{ there exists } \mathcal{F} \text{ such that } \mathcal{F}(g) \in \mathcal{C}_p[[q]] \text{ as classical cusp eigenforms for all } k \text{ in a neigbourhood } \mathcal{B} \subset X \text{ (see in [CoPB])} \]

1) The Fourier coefficients $a_n(k)$ of $f_k$ and one of the Satake $p$-parameters $\alpha_p(k) := \alpha_p^{(1)}(k)$ are given by certain $p$-adic analytic functions $k \mapsto a_n(k)$ for $(n, p) = 1$

2) The slope is constant and positive: $\text{ord}(\alpha(p)) = \sigma > 0$

The existence of families of slope $\sigma > 0$ was established in [CoPB].

R. Coleman gave an example with $p = 7$, $f = \Delta$, $k = 12$

\[ a_7 = \tau(7) = -7 \cdot 2392, s = 1. \]

A program in PARI for computing such families is contained in [CST98] (see also the Web-page of W. Stein, http://modular.fas.harvard.edu/)

**A family of slope $\sigma > 0$ of cusp eigenforms $f_k$ of weight $k \geq 2$:**

\[ k \mapsto f_k = \sum_{n=1}^\infty a_n(k) q^n \in \mathbb{C}[[q]] \subset C_p[[q]] \]

A model example of a $p$-adic family (not cusp and $\sigma = 0$):

Eisenstein series

\[ a_n(k) = \sum_{d | p} d^k, f_k = E_k \]

The triple product with a Dirichlet character $\chi$ is defined as the following complex $L$-function (an Euler product of degree eight):

\[ L(f_1 \otimes f_2 \otimes f_3, s, \chi) = \prod_{p \nmid N} L((f_1 \otimes f_2 \otimes f_3)_p, \chi(p)p^{-s}). \]

where $L((f_1 \otimes f_2 \otimes f_3)_p, X)^{-1} = \prod_{\eta}(1 - \alpha_{p,1}^{(\eta)}\alpha_{p,2}^{(\eta)}\alpha_{p,3}^{(\eta)} X)$

\[ \det \left( 1 - X \begin{pmatrix} \alpha_{p,1}^{(1)} & 0 & 0 \\ 0 & \alpha_{p,2}^{(1)} & 0 \\ 0 & 0 & \alpha_{p,3}^{(1)} \end{pmatrix} \otimes \begin{pmatrix} \alpha_{p,1}^{(2)} & 0 & 0 \\ 0 & \alpha_{p,2}^{(2)} & 0 \\ 0 & 0 & \alpha_{p,3}^{(2)} \end{pmatrix} \otimes \begin{pmatrix} \alpha_{p,1}^{(3)} & 0 & 0 \\ 0 & \alpha_{p,2}^{(3)} & 0 \\ 0 & 0 & \alpha_{p,3}^{(3)} \end{pmatrix} \right) \]

\[ = \prod_{\eta}(1 - \alpha_{p,1}^{(\eta)}\alpha_{p,2}^{(\eta)}\alpha_{p,3}^{(\eta)} X)\left(1 - \alpha_{p,1}^{(1)}\alpha_{p,2}^{(1)}\alpha_{p,3}^{(1)} X\right)\left(1 - \alpha_{p,1}^{(2)}\alpha_{p,2}^{(2)}\alpha_{p,3}^{(2)} X\right)\left(1 - \alpha_{p,1}^{(3)}\alpha_{p,2}^{(3)}\alpha_{p,3}^{(3)} X\right). \]

product taken over all 8 maps $\eta : \{1, 2, 3\} \rightarrow \{1, 2\}.$
Critical values and functional equation

We use the corresponding normalized $L$ function (see [De79], [Co], [Co-PeRei]), which has the form:

$$
\Lambda(f_1 \otimes f_2 \otimes f_3, s, \chi) = \Gamma_C(s-k_3+1)\Gamma_C(s-k_2+1)\Gamma_C(s-k_1+1)\mathcal{L}(f_1 \otimes f_2 \otimes f_3, s, \chi),
$$

where $\Gamma_C(s) = (2\pi)^{-s}\Gamma(s)$. The Gamma-factor determines the critical values $s = k_1, \ldots, k_2 + k_3 - 2$ of $\Lambda(s)$, which we explicitly evaluate (like in the classical formula $\zeta(2) = \pi^2/6$). A functional equation of $\Lambda(s)$ has the form:

$$
s \mapsto k_1 + k_2 + k_3 - 2 - s.
$$

Our method includes:

- a version of Garrett’s integral representation for the triple $L$-functions of the form: for $r = 0, \ldots, k_2 + k_3 - k_1 - 2$,

$$
\Lambda(f_{j_1} \otimes f_{j_2} \otimes f_{j_3}, k_2 + k_3 - r, \chi) =
\int \int \int \int \tilde{f}_{j_1}(z_1) \tilde{f}_{j_2}(z_2) \tilde{f}_{j_3}(z_3) \mathcal{E}(z_1, z_2, z_3; -r, \chi) \prod_j \left( \frac{dx_j dy_j}{y_j} \right)
$$

where $\tilde{f}_{j_a} := f_{j_a}^0$ is an eigenfunction of $U_p$, $\mathcal{M}_{j_a}(N\mathbb{P}^2, \psi)$, $\mathcal{E}(z_1, z_2, z_3; -r, \chi) \in \mathcal{M}_T(N^2 \mathbb{P}^2, \psi_1) \otimes \mathcal{M}_T(N^2 \mathbb{P}^2, \psi_2) \otimes \mathcal{M}_T(N^2 \mathbb{P}^2, \psi_3)$ is the triple modular form of triple weight $(k_1, k_2, k_3)$, and of fixed triple Nebentypus character $(\psi_1, \psi_2, \psi_3)$, obtained from a nearly holomorphic Siegel-Eisenstein series $F_{x,r} = G^*(z, -r; k, (N\mathbb{P}^2)^3, \psi)$, of degree 3, of weight $k = k_2 + k_3 - k_1$, and the Nebentypus character $\psi = \chi \psi_1 \psi_2 \psi_3$.

Given three $p$-adic analytic families $f_j$ of slope $\sigma_j \geq 0$, to construct a four-variable $p$-adic $L$-function attached to Garrett’s triple product of these families

we show that this function interpolates the special values

$$(s, k_1, k_2, k_3) \mapsto \Lambda(f_{j_1} \otimes f_{j_2} \otimes f_{j_3}, s, \chi)$$

at critical points $s = k_1, \ldots, k_2 + k_3 - 2$ for balanced weights $k_1 \leq k_2 + k_3 - 2$; we prove that these values are algebraic numbers after dividing by certain “periods”). However the construction uses directly modular forms, and not the $L$-values in question, and a comparison of special values of two functions is done after the construction.

We obtain $\mathcal{E}(z_1, z_2, z_3; -r, \chi)$ from a Siegel-Eisenstein series by applying to $F_{y, r}$, Boecherer’s higher twist (see (10.19)) and Ibukiyama’s differential operator (see (10.20)).

These operations act explicitly on the Fourier expansions. Then one uses:

- The theory of $p$-adic integration with values in Serre’s type $A$-modules $\mathcal{M}_T(A)$ of triple arithmetical nearly holomorphic modular forms over $p$-adic Banach algebras $A$. Explicit Fourier coefficients $a_{j_a}(r, T) \in \mathbb{Q}[R, T]$ of $\mathcal{E}(z, -r, \chi)$ are given by special polynomials of matrices $T = (t_{ij})$, $R = \{R_j\}$ and of $\mathcal{E}(z, -r, \chi)$ of the form (with $\hat{\beta} \in \mathbb{Z}_p^* \cap \mathbb{Q}$) i.e. the coefficients of $a_{j_a}(r, T)$ by some elementry $p$-adic measures $\int_Y \chi y^\beta d\mu_T \in A$. Here $A = A(\mathbb{Z})$ is a certain $p$-adic Banach algebra of functions on an open analytic subspace $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset \mathbb{X}^3$ in the product of three copies of the weight space $X = \text{Hom}_{\text{cont}}(Y, \mathbb{C}_p^*)$.

These measures on the group $\mathbb{Y} = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^*$ produce the coefficients of $a_{j_a, r}$ of $\mathcal{E}(z, -r, \chi)$ of $\mathcal{M}_T(A)$ for all $p$-adic weights $r \times \mathbb{X}$, given by

$$
\int_Y x(y)dy \in A \text{ (an interpolation from } x = \chi y^\beta \text{ to all } x \in X).
$$
The spectral theory of triple Atkin’s operator $U = U_{p,T}$ allows to evaluate the integral using at each weight $(k_1, k_2, k_3)$ the equality $\langle \ell^0, \mathcal{E}(-r, \chi) \rangle = \langle \ell^0, \pi_\chi(\mathcal{E}(-r, \chi)) \rangle$ with the projection $\pi_\chi$ of $\mathcal{M}_T(A)$ to the $\lambda$-part $\mathcal{M}_T(A)^\lambda$, defined by:

$\ker \pi_\chi := \bigcap_{n \geq 1} \text{im } (U_T - \lambda)^n$, $\text{im } \pi_\chi := \bigcup_{n \geq 1} \ker (U_T - \lambda)^n$.

We prove that $U$ is a completely continuous $\mathcal{A}$-linear operator on a certain Coleman’s submodule $\mathcal{M}(A)^\lambda$ of Serre’s type module $\mathcal{M}(A)$. Then the projection $\pi_\chi$ exists (on this submodule) due to general results of Serre and Coleman, see [CoPB], [SePB]. We show that there exists an element $\tilde{\mathcal{E}}(-r, \chi) \in \mathcal{M}(A)^\lambda$ such that at each weight $(k_1, k_2, k_3)$ the equality holds:

$\langle \ell^0, \mathcal{E}(-r, \chi) \rangle = \langle \ell^0, \pi_\chi(\tilde{\mathcal{E}}(-r, \chi)) \rangle$, and the product can be expressed through certain coefficients the series $\tilde{\mathcal{E}}(-r, \chi)$ which are the same as those of $\mathcal{E}(-r, \chi)$.

Using the evaluation map and the Mellin transform

We obtain the measure $\mu = \ell_T(\tilde{\mathcal{E}}^\lambda)$ with values in $\mathcal{A}$ on the profinite group $Y$.

- Construct an analytic function $\mathcal{L}_\mu : X \to \mathcal{A} = \mathcal{A}(\mathcal{B})$ as the $p$-adic Mellin transform $\mathcal{L}_\mu(x) = \int_Y x(y) d\mu(y) \in \mathcal{A}, x \in X$.

- Solution: the function in question $\mathcal{L}_\mu(x, s)$ is given by evaluation of $\mathcal{L}_\mu(x)$ at $s = (s_1, s_2, s_3) \in \mathcal{B}$: this is a $p$-adic analytic function in four variables

$$(x, s) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$$

$$\mathcal{L}_\mu(x, s) := ev_q(\mathcal{L}_\mu(x)) \quad (x \in X, s \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3, \mathcal{L}_\mu(x) \in \mathcal{A}).$$

Key point: modular admissible measures

Let us write for simplicity: $\mathcal{E}(-r, \chi)$ for $\mathcal{E}(\tilde{\mathcal{E}}(-r, \chi))$ $\mathcal{M}_T(A)$ instead of $\mathcal{M}_T(A)^\lambda$ (Coleman’s submodule).

One defines admissible $p$-adic measures $\tilde{\mathcal{E}}^\lambda$ with values in Banach $\mathcal{A}$-modules $\mathcal{M}_T(A)$ which are locally free of finite rank, using the test functions: $\int_Y \lambda y^r \tilde{\mathcal{E}}^\lambda = \pi_\chi(\mathcal{E}(-r, \chi))$.

- Passage from values in modular forms to scalar values: apply an algebraic $\mathcal{A}$-linear form $\mathcal{M}_T(A)^\lambda$ to the constructed measure $\tilde{\mathcal{E}}^\lambda$ (in modular forms), and the evaluation maps $\mathcal{A} \otimes_p \mathbb{C}_p$ for any $p$-adic triple weights $s \in X^3$.

The linear form $\ell_T$ is an algebraic version of the Petersson product (a geometric meaning of $\ell_T$: the first coordinate in an (orthogonal) $\mathcal{A}$-basis of eigenfunctions of all Hecke operators $T_q$ for $q \mid Np$, with the first basis element $f_0 \in \mathcal{M}^\lambda(A)$).

Final step: comparison between $\mathbb{C}$ and $\mathbb{C}_p$

- We check an equality relating the values of the constructed analytic function $\mathcal{L}_\mu(x, s)$ at the arithmetical characters $x = y_q^r \chi \in X$, and at triple weights $s = (k_1, k_2, k_3) \in \mathcal{B}$, with the normalized critical special values

$L^*(f_1, k_1 \otimes f_2, k_2 \otimes f_3, k_3, k_2 + k_3 - 2 - r, \chi) \quad (r = 0, \ldots, k_2 + k_3 - k_1 - 2)$

for certain Dirichlet characters $\chi \mod Np^\nu$, $\nu \geq 1$. These are algebraic numbers, embedded into $\mathbb{C}_p = \overline{\mathbb{Q}_p}$ (the Tate field of $p$-adic numbers). The normalisation of $L^*$ includes at the same time Gauss sums, Petersson scalar products, powers of $\pi$, the product $\lambda_p(k_1, k_2, k_3)$, and a certain finite Euler product.
Arithmetical nearly holomorphic modular forms (the elliptic case)

Let $\mathcal{A}$ be a commutative ring (a subring of $\mathbb{C}$ or $\mathbb{C}_p$)

Arithmetical nearly holomorphic modular forms (in the sense of Shimura, [ShiAr]) are certain formal series

$$ g = \sum_{n=0}^{\infty} a(n; R)q^n \in \mathcal{A}\langle q \rangle[R], $$

with the property that for $\mathcal{A} = \mathbb{C}$, $z = x + iy \in \mathbb{H}$, $R = (4\pi y)^{-1}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}$ of a given weight $k$ and Dirichlet character $\psi$. The coefficients $a(n; R)$ are polynomials in $\mathcal{A}[R]$. If $\deg_R a(n; R) \leq r$ for all $n$, we call $g$ nearly holomorphic of type $r$ (it is annihilated by $(\frac{2}{m})^{r+1}$, see [ShiAr]).

Triple arithmetical modular forms

Let $\mathcal{A}$ be a commutative ring. The tensor product over $\mathcal{A}$

$$ M_{k,r,T}(N,\psi,\mathcal{A}) := M_{k_1,r}(N,\psi_1,\mathcal{A}) \otimes M_{k_2,r}(N,\psi_2,\mathcal{A}) \otimes M_{k_3,r}(N,\psi_3,\mathcal{A}) $$

consists of triple arithmetical modular forms as certain formal series of the form

$$ g = \sum_{n_1,n_2,n_3=0} a(n_1, n_2, n_3; R_1, R_2, R_3)q_1^{n_1}q_2^{n_2}q_3^{n_3} \in \mathcal{A}\langle q_1, q_2, q_3 \rangle[R_1, R_2, R_3], $$

where $z_j = x_j + iy_j \in \mathbb{H}$, $R_j = (4\pi y_j)^{-1}$, with the property that for $\mathcal{A} = \mathbb{C}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}$ of a given weight $(k_1, k_2, k_3)$ and character $(\psi_1, \psi_2, \psi_3)$, $j = 1, 2, 3$. The coefficients $a(n_1, n_2, n_3; R_1, R_2, R_3)$ are polynomials in $\mathcal{A}[R_1, R_2, R_3]$. Examples of such modular forms come from the restriction to the diagonal of Siegel modular forms of degree 3.

Siegel modular groups

Let $J_{2m} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}$. The symplectic group

$$ \text{Sp}_m(\mathbb{R}) = \{ g \in \text{GL}_{2m}(\mathbb{R}) | g \cdot J_{2m} \tilde{g} = J_{2m} \}, $$

acts on the Siegel upper half plane

$$ \mathbb{H}_m = \{ z = z + \bar{z} \in M_m(\mathbb{C}) | \text{Im} z > 0 \} $$

by $g(z) = (az + b)(cz + d)^{-1}$, where we use the bloc notation $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_{2m}(\mathbb{R})$. We use the congruence subgroup

$$ \Gamma_0^n(N) = \{ \gamma \in \text{Sp}_m(\mathbb{Z}) | \gamma \equiv (0_n)^\ast \mod N \} \subset \text{Sp}_m(\mathbb{Z}). $$
A Siegel modular form

\[ f \in \mathcal{M}_k(\Gamma_0(N), \chi) \text{ of degree } m > 1, \text{ weight } k \text{ and a Dirichlet character } \chi \mod N \text{ is a holomorphic function } f : \mathbb{H}_m \to \mathbb{C} \text{ such that for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ one has} \]

\[ f(\gamma(z)) = \chi(\det d) \det(cz + d)^k f(z). \]

The Fourier expansion of \( f \) uses the symbol \( q^T = \exp(2\pi i \text{tr}(Tz)) \)

\[ = \prod_{i,j=1}^{m} q_i^T \prod_{i<j} q_i^T q_j^T \subset \mathbb{C}[q_{ii}, \ldots, q_{mm}], \quad q_{ij} = \exp(2\pi i \sqrt{-1} z_{ij}), \quad \text{and } T \text{ in the semi-group} \]

\[ B_m = \{ T = ^tT \geq 0 | T \text{ half-integral} \} : \]

\[ f(z) = \sum_{T \in B_m} a(T) q^T \in \mathbb{C}[q^{B_m}](\text{a formal } q\text{-expansion } \in \mathbb{C}[q^{B_m}]). \]

Arithmetical Siegel modular forms

Consider a commutative ring \( A \), the formal variables \( q = (q_{ij}); j = 1, \ldots, m \), \( R = (R_{ij}); j = 1, \ldots, m \), and the ring of formal Fourier series

\[ A[q^{B_m}][R_{ij}] = \left\{ f = \sum_{T \in B_m} a(T, R) q^T \bigg| a(T, R) \in A[R_{ij}] \right\} \quad \text{(5.5)} \]

(over the complex numbers this notation corresponds to \( q^T = \exp(2\pi i \text{tr}(Tz)) \), \( R = (4\pi \text{Im}(z))^{-1} \).

The formal Fourier expansion of a nearly holomorphic Siegel modular form \( f \) with coefficients in \( A \) is a certain element of \( A[q^{B_m}][R_{ij}] \). We call \( f \) arithmetical in the sense of Shimura [ShiAr], if \( A = \bar{\mathbb{Q}} \).

Siegel-Eisenstein series

Example ([Nag2], p.408)

\[ E_4^{(2)}(z) = 1 + 240q_{11} + 240q_{22} + 2160q_{11}^2 + (240q_{12}^2 + 13440q_{12}^{-1} \cdot 30240 + 13440q_{12} + 240q_{12}^2)q_{11}q_{22} + 2160q_{22}^2 + \cdots \]

\[ E_6^{(2)}(z) = 1 - 504q_{11} - 504q_{22} - 16632q_{11}^2 + (-540q_{12}^2 + 44352q_{12}^{-1} \cdot 166320 + 44352q_{12}^2)q_{11}q_{22} - 16632q_{22}^2 + \cdots \]

Maass differential operator

Let us consider the Maass differential operator (see [Maa]) \( \Delta_m \) of degree \( m \), acting on complex \( \mathcal{C}^\infty \)-functions on \( \mathbb{H}_m \) by:

\[ \Delta_m = \text{det}(\tilde{\partial}_j), \quad \tilde{\partial}_j = 2^{-1}(1 + \delta_j \partial/\partial q_j), \quad \text{(5.6)} \]

its algebraic version is the Ramanujan operator of degree \( m \):

\[ \Theta_m := \text{det}(\frac{1}{2\pi i} \tilde{\partial}_j) = \text{det}(\partial_j) = \frac{1}{(2\pi i)^m} \Delta_m, \quad \text{(5.7)} \]

where \( \Theta_m(q^T) = \text{det}(T)q^T \).
Shimura differential operator

The Shimura differential operator (see [Shi76, ShiAr]):

\[ \delta_k f(z) = \det(R)^{k+1-\kappa} \Theta_m \left[ \det(R)^{\kappa-1-k} f \right], \]

where \( R = (4\pi y)^{-1} \), acts on arithmetic nearly holomorphic Siegel modular forms, and the composition is defined

\[ \delta_k^{(r)} = \delta_{k+2r-2} \circ \cdots \circ \delta_k : \tilde{M}_k^m(N, \psi; \mathbb{C}) \to \tilde{M}_{k+2m}^m(N, \psi; \mathbb{C}), \]

where

\[ \delta_k f(z) = \left( \frac{-1}{4\pi} \right)^m \det(y)^{-1} \det(z-z')^{\kappa-k} \Delta_m \left[ \det(z-z')^{\kappa+1-k} f(z) \right]. \]

Universal polynomials \( Q(R; \mathcal{T}; k, r) \)

Let \( f = \sum_{T \in \mathcal{B}_m} c(T) q^T \in M_\mu^m(N, \psi) \) be a formal holomorphic Fourier expansion. One shows that \( \delta_k^{(r)} f \) is given by

\[ \delta_k^{(r)} f = \sum_{T \in \mathcal{B}_m} Q(R; T; k, r) c(T) q^T. \]

Universal polynomials (continued)

Here we use a universal polynomial (5.9) which can be defined for all \( k \in \mathbb{C} \), and it expresses the action of the Shimura operator on the exponential (of degree \( m \)):

\[ \delta_k^{(r)} (q^T) = Q(R; T; k, r) q^T. \]

If \( m = 1, r \) arbitrary (see [Shi76]),

\[ \delta_k^{(r)} = \sum_{j=0}^r (-1)^{r-j} \frac{\Gamma(k+r)}{\Gamma(k+j)} R^{-j} \rho_j, \]

\[ Q(R; k; k, r) = \sum_{j=0}^r (-1)^{r-j} \frac{\Gamma(k+r)}{\Gamma(k+j)} R^{-j} \rho_j. \]
In general (see [CourPa], Theorem 3.14) one has:

\[ Q(R, \mathcal{J}) = Q(R, \mathcal{J}; k, r) \]

\[ = \sum_{t=0}^{r} \binom{r}{t} \det(\mathcal{J})^{r-t} \sum_{|L| \leq mt-t} R_L(\kappa - k - r)Q_L(R, \mathcal{J}). \]

\[ Q_L(R, \mathcal{J}) = \text{tr} \left( \rho_{m-\kappa}(R)\rho_{\kappa}(\mathcal{J}) \right) \cdot \ldots \cdot \text{tr} \left( \rho_{m-\kappa}(R)\rho_{\kappa}(\mathcal{J}) \right). \]

In (5.9), \( L \) goes over all the multi-indices \( 0 \leq l_1 \leq \ldots \leq l_r \leq m \), such that \( |L| = l_1 + \ldots + l_r \leq mt - t \), and \( R_L(\beta) \in \mathbb{Z}[1/2][\beta] \) in (5.9) are polynomials in \( \beta \) of degree \( (mt - |L|) \) (used with \( \beta = \kappa - k - r \)).

Note the differentiation rule of degree \( m \) (see [Sh83], p.466):

\[ \Delta(f_g) = \sum_{t=0}^{m} \text{tr} \left( \rho_t(\partial f_g/\partial z) \cdot \rho_{m-\kappa}(R)\rho_{\kappa}(\mathcal{J}) \right). \]

Example (Siegel-Eisenstein series of odd degree and higher level)

\[ G^\ast(z, s; k, \psi, N) \]

\[ = \det(\chi)^{\frac{m}{2}} \sum_{c,d} \psi(\det(c)) \det(cz + d)^{\kappa - s} \det(az + b)^{-2s} \cdot \Gamma(k,s)L_N(k + 2s, \psi) \left( \prod_{i=1}^{[m/2]} L_N(2k + 4s - 2i, \psi^2) \right). \]

\( (c, d) \) runs over all "non-associated coprime symmetric pairs" with \( \det(c) \) coprime to \( N \), \( \kappa = (m + 1)/2 \), and for \( m \) odd the \( \Gamma \)-factor has the form:

\[ \Gamma(k,s) = \frac{\pi^{m+1}}{2^{m+2}k^{m+1}} \prod_{p | m+1} p^{m+1}. \]

We use this series with \( \psi = \chi^2 \psi_1^2 \psi_2^2 \), \( k = k_2 + k_3 + 1 \geq 2 \), \( m = 3 \),

\[ \kappa = \frac{m+1}{2} = 2, \frac{[m/2]}{2} = 1. \]

Theorem 5.3 [Siegel, Shimura [Sh83], P. Feit [Fei86]]

Let \( m \) be an odd integer such that \( 2k + m > m \), and \( N > 1 \) be an integer, then:

For an integer \( s \) such that \( s = -r, 0 \leq r \leq k - \kappa \), there is the following Fourier expansion

\[ G^\ast(z, -r) = G^\ast(z, -r; k, \psi, N) = \sum_{A_m \ni 3 \geq 0} a(\mathcal{J}, R)q^T, \]

where for \( s > (m + 2 - 2k)/4 \) in (5.11) the only non-zero terms occur for positive definite \( \mathcal{T} > 0 \).
Main Theorem (on $p$-adic analytic function in four variables)

1) The function $L_F: (s, k_1, k_2, k_3) \mapsto \left\langle \overline{\rho}, \mathcal{E}(-r, \chi) \right\rangle$ depends $p$-adically on four variables

$$(x \cdot y_p^* k_1, k_2, k_3) \in X \times B_1 \times B_2 \times B_3.$$ 

2) Comparison of complex and $p$-adic values: for all $(k_1, k_2, k_3)$ in an affinoid neighborhood $B = B_1 \times B_2 \times B_3 \subset X^3$, satisfying $k_1 \leq k_2 + k_3 - 2$: the values at $s = k_2 + k_3 - 2 - r$ coincide with the normalized critical special values

$$L^*(f_{k_1} \otimes f_{k_2} \otimes f_{k_3}, k_2 + k_3 - 2 - r, \chi) \quad (r = 0, \ldots, k_2 + k_3 - k_1 - 2),$$ 

(6.14)

for Dirichlet characters $\chi \mod Np^\nu$, $\nu \geq 1$, such that all three corresponding Dirichlet characters $\chi_j$ have $Np$-complete conductors:

3) Dependence on $x \in X$: let $H = [2\operatorname{ord}_\varphi(\lambda)] + 1$. For any fixed $(k_1, k_2, k_3) \in B$ and $x = \chi \cdot y_p^*$ the function

$$x \mapsto \left\langle \overline{\rho}, \mathcal{E}(-r, \chi) \right\rangle \overline{\left\langle \rho, f_0 \right\rangle}$$

extends to a $p$-adic analytic function of type $o(\log^H(\cdot))$ of the variable $x \in X$.

Outline of the proof

1) At each classical weight $(k_1, k_2, k_3)$ let us use the equality

$$\left\langle \overline{\rho}, \mathcal{E}(-r, \chi) \right\rangle = \left\langle \rho, \pi_\lambda(\mathcal{E}(-r, \chi)) \right\rangle$$

deduced from the definition of the projector $\pi_\lambda$:

$\ker \pi_\lambda := \bigcap_{n \geq 1} \operatorname{Im} (U_T - \lambda I)^n$, $\operatorname{Im} \pi_\lambda := \bigcup_{n \geq 1} \ker (U_T - \lambda I)^n$.

Notice that the coefficients of $\mathcal{E}(-r, \chi) \in \mathcal{M}(A)$ depend $p$-adically on $(k_1, k_2, k_3) \in B = B_1 \times B_2 \times B_3$, where

$$A = \mathcal{A}(B_1 \times B_2 \times B_3)$$

is the $p$-adic Banach algebra of rigid-analytic functions on $B$.

Interpolation to all $p$-adic weights:

- At each classical weight $(k_1, k_2, k_3)$ the scalar product $\left\langle \overline{\rho}, \mathcal{E}(-r, \chi) \right\rangle$ is given by the first coordinate of $\pi_\lambda(\mathcal{E}(-r, \chi))$ with respect to an orthogonal basis of $\mathcal{M}(A)$ containing $f_0$ with respect to Hida’s algebraic Petersson product $(\overline{\rho}, h)_\varphi := \left\langle \overline{\rho}^v \left( \begin{smallmatrix} 0 & -1 \\ Np & 0 \end{smallmatrix} \right), h \right\rangle$, see [H100].

Let us extend the linear form $\ell(h) := \left\langle \overline{\rho}, h \right\rangle$ (defined for classical weights), to Coleman’s type submodule of overconvergent families $h \in \mathcal{M}(A)^1 \subset \mathcal{M}(A)$ as the first coordinate of $h$ with respect to some $A$-basis of eigenfunctions of all (triple) Hecke operators $T_q$ for $q \mid Np$, having the first basis vector $f_0 \in \mathcal{M}(A)^1$.

The linear form $\ell$ can be characterized as a normalized eigenfunction of the adjoint Atkin’s operator, acting on the dual $A$-module of $\mathcal{M}(A)^1$: $\ell(f_0) = 1$.

In order to extend $\ell$ to $h = \mathcal{E}(-r, \chi)$, we need to choose a certain representative of $\mathcal{E}(-r, \chi) \in A$-submodule $\mathcal{M}(A)^1$, which is locally free of finite rank.
A representative of $E(-r, \chi)$ in the (locally free of finite rank $A$-submodule) $M^\lambda(A)^\dagger$

Choose a (local) basis $\ell^1, \ldots, \ell^n$ given by some triple Fourier coefficients of the dual (locally free of finite rank) $A$-module $M^\lambda(A)^\dagger$.

Then define

$$\ell = \beta_1\ell^1 + \cdots + \beta_n\ell^n,$$

where $\beta_i = \ell(\ell_i) \in A$, and $\ell_i$ denotes the dual basis of $M^\lambda(A)^\dagger$:

$$\ell(\ell_i) = \delta_{ij}.$$

At each $p$-adic weight $(k_1, k_2, k_3) \in B$ let us define

$$\ell(E(-r, \chi)) := \beta_1\ell^1(E(-r, \chi)) + \cdots + \beta_n\ell^n(E(-r, \chi)) (\text{belongs to } A),$$

where $\beta_i = \ell(\ell_i) \in A$, and $\ell^i(E(-r, \chi)) \in A$ are certain Fourier coefficients of the series $E(-r, \chi)$.

---

**Conclusion**

There exists an element

$$\tilde{E}(-r, \chi) \in M^\lambda(A)^\dagger \subseteq M(A)^\dagger$$

such that

$$\ell(E(-r, \chi)) = \ell(\tilde{E}(-r, \chi))$$

(at each weight $(k_1, k_2, k_3)$). In fact, let us define

$$\tilde{E}(-r, \chi) := E(-r, \chi)^1 + \cdots + E(-r, \chi)^n \ell(E(-r, \chi))$$

$$= \beta_1\ell^1(E(-r, \chi)) + \cdots + \beta_n\ell^n(E(-r, \chi))$$

Thus, the dependence of $\ell(E(-r, \chi)) \in A$ on $(k_1, k_2, k_3) \in X^3$ is $p$-adic analytic.

In order to prove the remaining statements 2), 3), the dependence on $x = \chi \cdot y$ is studied in the next section.

---

**Distributions and measures with values in Banach modules**

A $\mathcal{D}$-module $\mathcal{D}$ on $Y$ with values in $V$ is an $A$-linear form

$$\mathcal{D} : \mathcal{O}_{\text{loc-const}}(Y, A) \to V, \quad \varphi \mapsto \mathcal{D}(\varphi) = \int_Y \varphi d\mathcal{D}.$$  

b) A measure $\mu$ on $Y$ with values in $V$ is a continuous $A$-linear form

$$\mu : \mathcal{O}(Y, A) \to V, \quad \varphi \mapsto \int_Y \varphi d\mu.$$  

The integral $\int_Y \varphi d\mu$ can be defined for any continuous function $\varphi$, and any bounded distribution $\mu$, using the Riemann sums.
Admissible measures

Let $h$ be a positive integer. A more delicate notion of an $h$-admissible measure was introduced in [Am-V] by Y. Amice, J. Vélu (see also [MTT], [V]):

Definition

a) For $h \in \mathbb{N}$, $h \geq 1$ let $\mathcal{P}_h(Y, A)$ denote the $A$-module of locally polynomial functions of degree $< h$ of the variable $y_p : Y \rightarrow \mathbb{Z}_p^\times \hookrightarrow A^\times$; in particular,

$$\mathcal{P}_1(Y, A) = \mathcal{O}_{\text{loc-const}}(Y, A)$$

(the $A$-submodule of locally constant functions). Let also denote $\mathcal{O}^\lambda_{\text{loc-an}}(Y, A)$ the $A$-module of locally analytic functions, so that

$$\mathcal{P}_1(Y, A) \subset \mathcal{P}_h(Y, A) \subset \mathcal{O}^\lambda_{\text{loc-an}}(Y, A) \subset \mathcal{O}(Y, A).$$

b) Let $V$ be a normed $A$-module with the norm $| \cdot |_{p, V}$. For a given positive integer $h$ an $h$-admissible measure on $Y$ with values in $V$ is an $A$-module homomorphism

$$\Phi : \mathcal{P}_h(Y, A) \rightarrow V$$

such that for fixed $a \in Y$ and for $\nu \rightarrow \infty$ the following growth condition is satisfied:

$$\left| \int_{\mathcal{P}_h(Y, A)} (y_p - a_p)^\nu d\Phi \right|_{p, V} = o(p^{-\nu(h'-h)})$$

(7.16)

for all $h' = 0, 1, \ldots, h - 1, a_p := y_p(a)$

The condition (7.16) allows to integrate the locally-analytic functions on $Y$ along $\Phi$ using Taylor’s expansions! This means: there exists a unique extension of $\Phi$ to $\mathcal{O}^\lambda_{\text{loc-an}}(Y, A) \rightarrow V$.

Using the canonical projection $\pi_\chi$

We construct our $H$-admissible measure $\tilde{\Phi}_\chi : \mathcal{P}_H(Y, A) \rightarrow \mathcal{M}(A)$ out of a sequence of distributions $\Phi_r : \mathcal{P}_1(Y, A) \rightarrow \mathcal{M}(A)$ defined on local monomials $y^r_p$ of each degree $r$ by the rule

$$\int_Y \chi y_p^r d\tilde{\Phi}_\chi = \pi_\chi(\tilde{E}(-r, \chi)), \text{ where } \tilde{E}(-r, \chi) \in M = \mathcal{M}(A).$$

Here $\tilde{E}(-r, \chi)$ takes values in an $A$-module

$$M = \mathcal{M}(A) \subset A[q_1, q_2, q_3][R_1, R_2, R_3]$$

of nearly holomorphic (overconvergent) triple modular forms over $A$ (for $0 \leq r \leq H - 1$, $H = [2\text{ord}_p \lambda_0] + 1$), and the formal series $\tilde{E}(-r, \chi)$ was constructed in the proof of 1) of Main Theorem.

Functions $f_j$ and $f_j^0$

Recall that for any primitive cusp eigenform $f_j = \sum_{n=1}^\infty a_n(f)q^n$, there is an eigenfunction $f_j^0 = \sum_{n=1}^\infty a_n(f_j^0)q^n \in \mathcal{O}(q)$ of $U = U_p$ with the eigenvalue $\alpha = \alpha_{p, j}^{(1)} \in \overline{\mathbb{Q}}(U(f_0) = \alpha_0)$ given by

$$f_j^0 = f_j - \alpha_{p, j}^{(2)} f_j \big| V_p = f_j - \alpha_{p, j}^{(2)} p^{-k/2} f_j \left( \begin{array}{c} p^0 \\ 0 \end{array} \right) \right.$$ (7.17)

$$\sum_{n=1}^\infty a_n(f_j^0)n^{-s} = \sum_{n=1}^\infty a_n(f_j)n^{-s} (1 - \alpha_{p, j}^{(1)} p^{-s})^{-1}.$$ (7.18)

Moreover, there is an eigenfunction $f_j^0$ of $U^*_p$ given by

$$f_j^0 = f_j^0 \left( \begin{array}{cc} 0 & -1 \\ Np & 0 \end{array} \right), \text{ where } f_j^0 = \sum_{n=1}^\infty \alpha(n, f_0) q^n.$$ (7.18)

Therefore, $U_T(f_1 \otimes f_2 \otimes f_3) = \mathcal{M}(f_1 \otimes f_2 \otimes f_3).$
Triple products of Coleman’s families

Critical values of the $L$-function $L(f_1 \otimes f_2 \otimes f_3, \chi)$ (Theorem A)

Theorem A (algebraic properties of the triple product)

Assume that $k_2 + k_3 - k_1 \geq 2$, then for all pairs $(\chi, r)$ such that the corresponding Dirichlet characters $\chi_j$ have $N_p$-complete conductors, and $0 \leq r < k_2 + k_3 - k_1 - 2$, we have that

$$\Lambda(f_1^p \otimes f_2^p \otimes f_3^p, k_2 + k_3 - 2 - r, \psi_1 \psi_2 \chi) \in \mathbb{C}$$

where

$$\langle f_1^p \otimes f_2^p \otimes f_3^p \rangle_T = \langle f_1^p, f_1^p \rangle_N \langle f_2^p, f_2^p \rangle_N \langle f_3^p, f_3^p \rangle_N = \langle \rho_f \rangle_N (\rho_{f_2} \rho_{f_3})_{N}(\rho_{f_1} \rho_{f_3} \rho_{f_1})_{N}.$$

Recall: the $p$-adic weight space and the Mellin transform

The $p$-adic weight space is the group $X = \text{Hom}_{\text{cont}} (Y, \mathbb{C}^*)$ of (continuous) $p$-adic characters of the commutative profinite group $Y = \lim_{\text{v}} (\mathbb{Z}/N_p^r \mathbb{Z})^*$. The group $X$ is isomorphic to a finite union of discs $U = \{ z \in \mathbb{C}_p \mid |z|_p < 1 \}$. A $p$-adic $L$-function $L(\rho) : X \to \mathbb{C}_p$ is a certain meromorphic function on $X$. Such a function usually come from a $p$-adic measure $\mu$ on $Y$ (bounded or admissible in the sense of Amice-Vélu, see [Am-V]). The $p$-adic Mellin transform of $\mu$ is given for all $x \in X$ by

$$L(\rho)(x) = \int_{Y_{N_p}} x(y) d\mu(y), \quad L(\rho) : X \to \mathbb{C}_p$$

Theorem B (continued)

(i) for all pairs $(r, \chi)$ such that $\chi \in \chi_{N_{p}}^{\text{cont}}$, and all three corresponding Dirichlet characters $\chi_j$ have $N_p$-complete conductor $(j = 1, 2, 3)$, and $r \in \mathbb{Z}$ is an integer with $0 \leq r \leq k_2 + k_3 - k_1 - 2$, the following equality holds:

$$\mathcal{D}(\rho)(\chi x_{N_p}, f_1 \otimes f_2 \otimes f_3) = i_p \left( \frac{(\psi_1 \psi_2)(2)}{\Gamma(\chi_1) \Gamma(\chi_2) \Gamma(\chi_3)} \right)^{4(2) + 2 - r} \Lambda(f_1^p \otimes f_2^p \otimes f_3^p, k_2 + k_3 - 2 - r, \psi_1 \psi_2 \chi)$$

where $v = \text{ord}_p(\chi_0)$, $G(\chi)$ denotes the Gauß sum of a primitive Dirichlet character $\chi_0$ attached to $\chi$ (modulo the conductor of $\chi_0$),

Theorem B (continued)

Under the assumptions as above there exist a $\mathbb{C}_p$-valued measure $\tilde{\mu}_f^\lambda$, $\tilde{\mu}_g^\lambda$ on $Y_{N_p}$, and a $\mathbb{C}_p$-analytic function $\mathcal{D}(\rho)(x, f_1 \otimes f_2 \otimes f_3) : X_{N_p} \to \mathbb{C}_p$, given for all $x \in X_{N_p}$ by the integral $\mathcal{D}(\rho)(x, f_1 \otimes f_2 \otimes f_3) = \int_{Y_{N_p}} x(y) d\tilde{\mu}_f^\lambda \otimes \tilde{\mu}_g^\lambda (y)$, and having the following properties:
Theorem B (continued)

(ii) if \( \text{ord}_p \lambda_p = 0 \) then the holomorphic function in (i) is a bounded \( \mathbb{C}_p \)-analytic function;

(iii) in the general case (but assuming that \( \lambda_p \neq 0 \)) the holomorphic function in (i) belongs to the type \( o(\log(x^H)) \) with \( H = [2\text{ord}_p \lambda_p] + 1 \) and it can be represented as the Mellin transform of Coleman's families

Theorems B-D

Theorem C (on \( p \)-adic measures for families of triple products)

Put \( H = [2\text{ord}_p \lambda] + 1 \). There exists a sequence of distributions \( \Phi_\rho \) on \( Y \) with values in \( M = M(A) \) giving an \( H \)-admissible measure \( \tilde{\Phi}^\lambda \) with values in \( M^* \subset M \) with the following properties:

There exists an \( A \)-linear form \( \ell = f_1 \otimes f_2 \otimes f_3 : M(A)^\lambda \to A \) (given by (10.21), such that the composition

\[ \tilde{\mu} = \tilde{\mu}_{f_1 \otimes f_2 \otimes f_3, \lambda} := \ell \otimes f_1 \otimes f_2 \otimes f_3(\tilde{\Phi}^\lambda) \]

is an \( H \)-admissible measure with values in \( A \), and for all \( (k_1, k_2, k_3) \) in the affinoid neighborhood \( B = B_1 \times B_2 \times B_3 \), as above, satisfying \( k_1 \leq k_2 + k_3 - 2 \)

we have that the evaluated integrals

\[ \text{ev}_{(k_1, k_2, k_3)}\left( (f_1 \otimes f_2 \otimes f_3, \lambda)(\tilde{\Phi}^\lambda)(y \chi) \right) \]

on the arithmetical characters \( y \chi \) coincide with the critical special values

\[ \Lambda^*(f_1 \otimes f_2 \otimes f_3, k_1 + k_2 + k_3 - 2 - r, \chi) \]

for \( r = 0, \ldots, k_2 + k_3 - k_1 - 2 \), and for all Dirichlet characters \( \chi \mod N \), with \( N \geq 1 \), with all three corresponding Dirichlet characters \( \chi \) given by (6.15), having \( N \)-complete conductors. Here the normalisation of \( \Lambda^* \) includes at the same time certain Gauss sums, Petersson scalar products, powers of \( \pi \) and of \( \lambda(k_1, k_2, k_3) \), and a certain finite Euler product.

The \( p \)-adic Mellin transform and four variable \( p \)-adic analytic functions

Any \( h \)-admissible measure \( \tilde{\mu} \) on \( Y \) with values in a \( p \)-adic Banach algebra \( A \) can be characterized its Mellin transform \( \mathcal{L}_{\tilde{\mu}}(x) \)

\[ \mathcal{L}_{\tilde{\mu}} : X \to A, \text{ defined by } \mathcal{L}_{\tilde{\mu}}(x) = \int_Y x(y) \mu(y), \text{ where } x \in X, \mathcal{L}_{\tilde{\mu}}(x) \in A. \]

Key property of \( h \)-admissible measures \( \tilde{\mu} \): its Mellin transform \( \mathcal{L}_{\tilde{\mu}} \) is analytic with values in \( A \).

Let \( A = A(B) = A_1 \circ A_2 \circ A_3 = A(B_1) \circ A(B_2) \circ A(B_3) \) denote again the Banach algebra \( A \) where \( B \) is an affinoid neighbourhood around \( (k_1, k_2, k_3) \) in \( X^3 \) (with a given integer \( k \) and Dirichlet character \( \psi \mod N \)).
Theorem D (on \( p \)-adic analytic function in four variables)

Put \( H = [2\text{ord}_p(\lambda)] + 1 \). There exists a \( p \)-adic analytic function in four variables \((x, s) \in X \times B_1 \times B_2 \times B_3 \subset X \times X \times X \times X:\)

\[
\mathcal{L}_\lambda : (x, s) \mapsto \text{ev}_Q(\mathcal{L}_\lambda(x)) \quad (x \in X, \quad \mathcal{L}_\lambda(x) \in A).
\]

with values in \( \mathbb{C}_p \), such that for all \((k_1, k_2, k_3)\) in the affinoid neighborhood as above \( B = B_1 \times B_2 \times B_3 \), satisfying \( k_1 \leq k_2 + k_3 - 2 \), we have that the special values \( \mathcal{L}_\lambda(x, s) \) at the arithmetical characters \( x = \chi^r \), and \( s = (k_1, k_2, k_3) \in B \) coincide with the normalized critical special values

\[
L^r(f_{s_1} \otimes f_{s_2} \otimes f_{s_3}, \chi^r) \quad (r = 0, \ldots, k_2 + k_3 - k_1 - 2),
\]

for Dirichlet characters \( \chi \mod Np^r, r \geq 1 \), such that all three corresponding Dirichlet characters \( \chi_i \) given by (6.15), have \( Np \)-complete conductors where the same normalisation of \( L^r \) as in Theorem C.

Theorem D (continued)

Moreover, for any fixed \( s = (k_1, k_2, k_3) \in B \) the function

\[
x \mapsto \mathcal{L}_\lambda(x, s)
\]

is \( p \)-adic analytic of type \( o(\log^H(\cdot)) \).

Indeed, we obtain the function in question \( \mathcal{L}_\lambda(x, s) \) by evaluation at \( s = ((s_1, \psi_1), (s_2, \psi_2), (s_3, \psi_3)) \in B \): this is a \( p \)-adic analytic function in four variables \((x, s) \in X \times B_1 \times B_2 \times B_3 \subset X \times X \times X \times X:\)

\[
\mathcal{L}_\lambda(x, s) := \text{ev}_Q(\mathcal{L}_\lambda(x)) \quad (x \in X, \ s \in B_1 \times B_2 \times B_3, \ \mathcal{L}_\lambda(x) \in A).
\]

This is a joint work in progress with S. Boecherer, we use:

1) the higher twists of the Siegel-Eisenstein series, introduced in [Boe-Schm],

2) Ibukiyama’s differential operators (see [Ibu], [BSY]).

Boecherer’s Higher Twist

1) We define the higher twist of the series \( F_{x, r} = \sum_{\tau} q_{\lambda, x, r}(R, \tau)q^{\tau} \) by some Dirichlet characters \( \chi_1, \chi_2, \chi_3 \) as the following formal nearly holomorphic Fourier expansion:

\[
F_{x, r} = \sum_{\tau} \chi_1(t_{12})\chi_2(t_{13})\chi_3(t_{23})q_{\lambda, x, r}(R, \tau)q^{\tau}. \tag{10.19}
\]

The series (10.19) is a Siegel modular form of some higher level, but it has additional symmetries with respect to symplectic embedding \( \gamma : \Gamma_0(Np^{2\nu}) \times \Gamma_0(Np^{2\nu}) \times \Gamma_0(Np^{2\nu}) \rightarrow \text{Sp}_3: \) its triple Nebentypus character does not depend on \( \chi \mod Np^r \), and is equal to \( (\psi_1, \psi_2, \psi_3) \), if we choose Dirichlet characters as in (6.15):

\[
\begin{align*}
\chi_1 \mod Np^r &= \chi, \quad \chi_2 \mod Np^r = \psi_2\psi_3, \\
\chi_3 \mod Np^r &= \psi_1\psi_3, \quad \psi = \chi_1^2 \psi_1 \psi_2 \psi_3,
\end{align*}
\]

Ibukiyama’s differential operator

2) We use an algebraic version of Ibukiyama’s differential operator, which generalizes the algebraic “pull-back”: it transforms a nearly holomorphic Siegel modular form of weight \( k \) to a nearly holomorphic triple modular form of weight \( (k_1, k_2, k_3) \) \((k = k_2 + k_3 - k_1)\).

On a holomorphic Siegel modular form \( F = \sum_{\tau} a(\tau)q^{\tau} \), this operator has the form

\[
\mathcal{L}_{k, \lambda}^\ast(F) = \sum_{\tau} \mathcal{P}(k_1, k_2, k_3, 0, \tau)_{\lambda} a(\tau)q_1^{k_1}q_2^{k_2}q_3^{k_3}, \tag{10.20}
\]

where \( \lambda = k_1 - k_3 \geq k_2 - k_1 \geq 0, \) and \( \mathcal{P}(k_1, k_2, k_3; r; \tau) \) is certain Ibukiyama’s polynomial, equal to \((t_{12}t_{23}t_{31})^{\lambda}(t_{23}t_{31}t_{12})^{r}\), if \( r = 0 \).
Triple products of Coleman’s families

Scheme of the Proof

Algebraic linear form

3) From $\mathcal{M}_F^\lambda(A)$ to $A$: we use a $\mathbb{Q}$-valued linear forms of type

$$L: h \mapsto \frac{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, h \rangle}{\langle f_1^0, f_1^0 \rangle\langle f_2^0, f_2^0 \rangle\langle f_3^0, f_3^0 \rangle}$$

where $f_j^0$ is the eigenfunction (7.17) of the conjugate Atkin’s operator $U_p^*$, and $f_j^0$ is the eigenfunction (7.18) of $U_p$. The linear form $L$ is defined on the finite dimensional $\mathbb{Q}$-vector characteristic subspace

$$h \in \mathcal{M}_k(\mathbb{Q})^\lambda \subset \mathcal{M}_{k_1,r^*}(Np, \psi_1; \mathbb{Q}) \otimes \mathcal{M}_{k_2,r^*}(Np, \psi_2; \mathbb{Q}) \otimes \mathcal{M}_{k_3,r^*}(Np, \psi_3; \mathbb{Q}).$$

This map is then extended to an $A$-linear map

$$\ell = \ell_{f_1 \otimes f_2 \otimes f_3, \lambda}: \mathcal{M}(A)^\lambda \to A; \quad (10.21)$$

on the locally free $A$-module of finite rank $\mathcal{M}(A)^\lambda$.

L-values and $p$-adic integrals

4) We show that for any appropriate Dirichlet character $\chi \mod Np^\nu$ the integral

$$\mu^\lambda(\chi) = L(\pi_{\lambda}(\Phi_r(\chi))) \in A$$

evaluated at $(k_1, k_2, k_3) \in \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, coincides (up to a normalisation) with the special $L$-value

$$L^*(f_{1,k_1}^0 \otimes f_{2,k_2}^0 \otimes f_{3,k_3}^0, k_2 + k_3 - 2 - r, \psi_1 \psi_2 \chi)$$

under the above assumptions on $\chi$ and $r$.

A general integral representation of Garrett’s type

The basic idea how a Dirichlet character $\chi$ is incorporated in the integral representation [Ga87, BoeSP] is somewhat similar to the one used in [Boe-Schm], but (surprisingly) more complicated to carry out.

Note however that the existence of a $A$-valued admissible measure $\tilde{\mu}^\lambda = \ell(\Phi^\lambda)$ established at stage 4), does not depend on this technical computation.

In order to control the denominators of the modular forms

$$\pi_{\lambda}(\tilde{e}(-r, \chi)) \in \mathcal{M}^\lambda(A) =: \Phi_r(\chi),$$

used in the construction (the admissibility condition) we use the following result.
Theorem (Criterion of admissibility)

Let $\alpha \in A^*$, $0 < |\alpha|_p < 1$ and suppose that there exists a positive integer $h$ such that the following conditions are satisfied:

1) for all $r = 0, 1, \cdots, h - 1$ with $h = \lfloor \text{ord}_p \alpha \rfloor + 1$, and $v \geq 1$,

$$\Phi_r(a + (Np^v)) \in M(Np^{rv}) \quad \text{(the level condition)} \quad (11.22)$$

2) the following congruence for the coefficients holds: for all $t = 0, 1, \cdots, h - 1$

$$U^{rv} \sum_{r=0}^{t} \left( \frac{1}{t} \right) (-ap)^{t-r} \Phi_r(a + (Np^v)) \equiv 0 \mod p^v \quad (11.23)$$

\text{(the divisibility condition)}

Then the linear form given by $\tilde{\Phi}_\alpha(\delta \alpha + (Np^v)) := \pi_{\alpha}(\Phi_r(a + (Np^v)))$ on local monomials (for all $r = 0, 1, \cdots, h - 1$), is an $h$-admissible measure:

$$\tilde{\Phi}: \mathcal{A}(Y, Q) \rightarrow \mathcal{M} \subset \mathcal{M}$$

and one controls the denominators of the modular forms of all levels $v$ by the relation:

$$\pi_{\alpha,v}(h) = U^{-v} \pi_{\alpha,0}(U^v h) =: \pi_{\alpha}(h) \quad (11.24)$$

The equality (11.24) can be used as the definition of $\pi_{\alpha}$ at any level $Np^v$.

The growth condition (see (7.16)) for $\pi_{\alpha}(\Phi_r)$ is deduced from the congruences (11.23) between modular forms, using the relation (11.24).

Proof uses the commutative diagram:

$$
\begin{array}{ccc}
\mathcal{M}(Np^{v+1}, \psi; A) & \xrightarrow{\pi_{\alpha,v}} & \mathcal{M}(Np^{v+1}, \psi; A) \\
U^v & \downarrow & \downarrow \\
\mathcal{M}(Np, \psi; A) & \xrightarrow{\pi_{\alpha,0}} & \mathcal{M}(Np, \psi; A) = \mathcal{M}(Np^{v+1}, \psi; A).
\end{array}
$$

The existence of the projectors $\pi_{\alpha,v}$ comes from Coleman’s Theorem A.4.3 [CoPB].

On the right: $U$ acts on the locally free $A$-module $\mathcal{M}(Np^{v+1}, A)$ via the matrix:

$$\begin{pmatrix}
\alpha & \cdots & \cdots & * \\
0 & \alpha & \cdots & * \\
0 & 0 & \ddots & \cdots \\
0 & 0 & \cdots & \alpha
\end{pmatrix}$$

where $\alpha \in A^*$

$\implies U^v$ is an isomorphism over $A$.

References

Trials products of Coleman's families

References


Gorsse B., Robert G., Computing the Petersson product $(f_0^0, f_0)$. Prépublication de l’Institut Fourier, N°564, (2004).


Jory–Hugue, F., Unicité des $h$-mesures admissibles $p$-adiques données par des valeurs de fonctions $L$ sur les caractères, Prépublication de l’Institut Fourier (Grenoble), N°764, 1-33, 2005


Lang S., Introduction to Modular Forms, Springer Verlag, 1976


Puydt, J., *Valeurs spéciales de fonctions L de formes modulaires adéliques*, Thèse de Doctorat, Institut Fourier (Grenoble), 19 décembre 2003


