

PARI-GP Reference Card

(PARI-GP version 2.6.1)

Note: optional arguments are surrounded by braces {}.
To start the calculator, type its name in the terminal: **gp**
To exit **gp**, type **quit**, **\q**, or **<C-D>** at prompt.

Help

describe function *?function*
extended description *??keyword*
list of relevant help topics *???pattern*

Input/Output

previous result, the result before *%, %', %''*, etc.
n-th result since startup *%n*
separate multiple statements on line ;
extend statement on additional lines \
extend statements on several lines {*seq1*; *seq2*;}
comment /* ... */
one-line comment, rest of line ignored \\ ...

Metacommands & Defaults

set default *d* to *val* **default**{*{d}*, {*val*}, {*flag*}}
toggle timer on/off **#**
print time for last result **##**
print defaults **\d**
set debug level to *n* **\g n**
set memory debug level to *n* **\gm n**
set output mode (raw=0, default=1) **\o n**
set *n* significant digits **\p n**
set *n* terms in series **\ps n**
quit GP **\q**
print the list of PARI types **\t**
print the list of user-defined functions **\u**
read file into GP **\r filename**

Debugger / break loop

get out of break loop **break** or **<C-D>**
go up *n* frames **dbg_up**{*{n}*}
examine object *o* **dbg_x**(*o*)

PARI Types & Input Formats

t_INT/t_REAL. Integers, Reals *±n*, *±n.ddd*
t_INTMOD. Integers modulo *m* **Mod**(*n, m*)
t_FRAC. Rational Numbers *n/m*
t_FFELT. Elt in finite field F_q **ffgen**(*q*)
t_COMPLEX. Complex Numbers *x + y * I*
t_PADIC. *p*-adic Numbers *x + 0(p^k)*
t_QUAD. Quadratic Numbers *x + y * quadgen*(*D*)
t_POLMOD. Polynomials modulo *g* **Mod**(*f, g*)
t_POL. Polynomials *a * x^n + ... + b*
t_SER. Power Series *f + 0(x^k)*
t_QFI/t_QFR. Imag/Real bin. quad. forms **Qfb**(*a, b, c, {d}*)
t_RFRAC. Rational Functions *f/g*
t_VEC/t_COL. Row/Column Vectors *[x, y, z]*, *[x, y, z]~*
t_MAT. Matrices *[x, y, z; u, v]*
t_LIST. Lists **List**(*[x, y, z]*)
t_STR. Strings "abc"

Reserved Variable Names

$\pi = 3.14\dots$, $\gamma = 0.57\dots$, $C = 0.91\dots$ **Pi**, Euler, Catalan
square root of -1 **I**
big-oh notation **O**

Information about an Object

PARI type of object *x* **type**(*x*)
length of *x* / size of *x* in memory **#x**, **sizebyte**(*x*)
real or *p*-adic precision of *x* **precision**(*x*), **padicprec**

Operators

basic operations **+**, **-**, *****, **/**, **^**
i=i+1, **i=i-1**, **i=i*j**, ... **i++**, **i--**, **i*=j**, ...
euclidean quotient, remainder *x\y*, *x\y*, *x%y*, **divrem**(*x, y*)
shift *x* left or right *n* bits *x<<n*, *x>>n* or **shift**(*x, ±n*)
comparison operators **<=**, **<**, **>=**, **>**, **==**, **!=**, **===**, **lex**, **cmp**
boolean operators (or, and, not) **||**, **&&**, **!**
bit operations **bitand**, **bitneg**, **bitor**, **bitxor**
sign of $x = -1, 0, 1$ **sign**(*x*)
maximum/minimum of *x* and *y* **max**, **min**(*x, y*)
integer or real factorial of *x* *x!* or **factorial**(*x*)
derivative of *f* w.r.t. *x* *f'*
apply differential operator **diffop**
restore *x* as a formal variable *x='x*
simultaneous assignment $x \leftarrow v_1, y \leftarrow v_2$ **[x, y] = v**

Select Components

n-th component of *x* **component**(*x, n*)
n-th component of vector/list *x* *x[n]*
components *a, a + 1, ..., b* of vector *x* *x[a..b]*
(*m, n*)-th component of matrix *x* *x[m, n]*
row *m* or column *n* of matrix *x* *x[m,]*, *x[, n]*
numerator/denominator of *x* **numerator**(*x*), **denominator**

Conversions

to vector, matrix, set, list, string **Col/Vec, Mat, Set, List, Str**
create PARI object (*x mod y*) **Mod**(*x, y*)
make *x* a polynomial of *v* **Pol**(*x, {v}*)
as **Pol/Vec**, starting with constant term **Polrev**, **Vecrev**
make *x* a power series of *v* **Ser**(*x, {v}*)
string from bytes / from format+args **Strchr**, **Strprintf**
convert *x* to simplest possible type **simplify**(*x*)
object *x* with precision *n* **precision**(*x, n*)

Conjugates and Lifts

conjugate of a number *x* **conj**(*x*)
conjugate vector of algebraic number *x* **conjvec**(*x*)
norm of *x*, product with conjugate **norm**(*x*)
square of L^2 norm of vector *x* **norml2**(*x*)
lift of *x* from Mods **lift**, **centerlift**(*x*)

Lists, Sets & Sorting

sort *x* by *k*-th component **vecsort**(*x, {k}, {fl = 0}*)
min. *m* of *x* ($m = x[i]$), max. **vecmin**(*x, {&i}*), **vecmax**
does *y* belong to *x*, sorted wrt. *f* **vecsearch**(*x, y, {f}*)
Sets (= row vector of strings with strictly increasing entries)
intersection of sets *x* and *y* **setintersect**(*x, y*)
set of elements in *x* not belonging to *y* **setminus**(*x, y*)
union of sets *x* and *y* **setunion**(*x, y*)
does *y* belong to the set *x* **setsearch**(*x, y, {flag}*)
is *x* a set? **setisset**(*x*)

Lists. create empty list: $L = \text{List}()$

append *x* to list *L* **listput**(*L, x, {i}*)
remove *i*-th component from list *L* **listpop**(*L, {i}*)
insert *x* in list *L* at position *i* **listinsert**(*L, x, i*)
sort the list *L* in place **listsort**(*L, {flag}*)

Programming

Functions and closures

fun(vars) = my(local vars); *seq*
fun = (vars) -> my(local vars); *seq*

Control Statements (*X*: formal parameter in expression *seq*)

eval. *seq* for $a \leq X \leq b$ **for**($X = a, b, seq$)
eval. *seq* for *X* dividing *n* **fordiv**(*n, X, seq*)
eval. *seq* for primes $a \leq X \leq b$ **forprime**($X = a, b, seq$)
eval. *seq* for $a \leq X \leq b$ stepping *s* **forstep**($X = a, b, s, seq$)
multivariable **for** **forvec**($X = v, seq$)
loop over partitions of *n* **forpart**($p = n, seq$)
loop over vectors $v, q(v) \leq B, q > 0$ **forqfvec**(v, q, b, seq)
loop over subgrps *H* of abelian grp *G* **forsubgroup**($H = G$)
evaluate *seq* until $a \neq 0$ **until**(*a, seq*)
while $a \neq 0$, evaluate *seq* **while**(*a, seq*)
exit *n* innermost enclosing loops **break**{*{n}*}
start new iteration of *n*-th enclosing loop **next**{*{n}*}
return *x* from current subroutine **return**{*{x}*}
raise an exception **error**()
if $a \neq 0$, evaluate *seq1*, else *seq2* **if**(*a, {seq1}, {seq2}*)
try *seq1*, evaluate *seq2* on error **iferr**(*seq1, E, seq2*)
select from *v* according to *f* **select**(*f, v*)
apply *f* to all entries in *v* **apply**(*f, v*)

Input/Output

print with/without **\n**, **T_EX** format **print**, **print1**, **printtex**
formatted printing **printf**()
write *args* to file **write**, **writel**, **writetex**(*file, args*)
write *x* in binary format **writebin**(*file, x*)
read file into GP **read**{*{file}*}
read file, return as vector of lines **readvec**{*{file}*}
read a string from keyboard **input**()

Interface with User and System

allocates a new stack of *s* bytes **allocatemem**{*{s}*}
alias *old* to *new* **alias**(*new, old*)
install function from library **install**(*f, code, {gpf}, {lib}*)
execute system command *a* **system**(*a*)
as above, feed result to GP **extern**(*a*)
as above, return GP string **externstr**(*a*)
get \$VAR from environment **getenv**("VAR")
measure time in ms. **gettime**()
timeout command after *s* seconds **alarm**(*s, expr*)

Iterations, Sums & Products

numerical integration **intnum**($X = a, b, expr, {flag}$)
sum *expr* over divisors of *n* **sumdiv**(*n, X, expr*)
sumdiv, with *expr* multiplicative **sumdivmult**(*n, X, expr*)
sum $X = a$ to $X = b$, initialized at *x* **sum**($X = a, b, expr, {x}$)
sum of series *expr* **suminf**($X = a, expr$)
sum of alternating/positive series **sumalt**, **sumpos**
sum of series using **intnum** **sumnum**
product $a \leq X \leq b$, initialized at *x* **prod**($X = a, b, expr, {x}$)
product over primes $a \leq X \leq b$ **prodeuler**($X = a, b, expr$)
infinite product $a \leq X \leq \infty$ **prodinf**($X = a, expr$)
real root of *expr* between *a* and *b* **solve**($X = a, b, expr$)

Random Numbers

random integer/prime in $[0, N[$ **random**(*N*), **randomprime**
get/set random seed **getrand**, **setrand**(*s*)

Vectors & Matrices

dimensions of matrix x	<code>matsize(x)</code>
concatenation of x and y	<code>concat(x, {y})</code>
extract components of x	<code>vecextract(x, y, {z})</code>
transpose of vector or matrix x	<code>mattranspose(x)</code> or <code>x-matadjoin(x)</code>
adjoint of the matrix x	<code>matadjoin(x)</code>
eigenvectors/values of matrix x	<code>mateigen(x)</code>
characteristic/minimal polynomial of x	<code>charpoly(x)</code> , <code>minpoly</code>
trace/determinant of matrix x	<code>trace(x)</code> , <code>matdet</code>
Frobenius form of x	<code>matfrobenius(x)</code>
QR decomposition	<code>matqr(x)</code>

Constructors & Special Matrices

row vec. of $expr$ eval'd at $1 \leq i \leq n$	<code>vector(n, {i}, {expr})</code>
col. vec. of $expr$ eval'd at $1 \leq i \leq n$	<code>vectorv(n, {i}, {expr})</code>
matrix $1 \leq i \leq m, 1 \leq j \leq n$	<code>matrix(m, n, {i}, {j}, {expr})</code>
define matrix by blocks	<code>matconcat(B)</code>
diagonal matrix with diagonal x	<code>matdiagonal(x)</code>
$n \times n$ identity matrix	<code>matid(n)</code>
Hessenberg form of square matrix x	<code>mathess(x)</code>
$n \times n$ Hilbert matrix $H_{ij} = (i + j - 1)^{-1}$	<code>mathilbert(n)</code>
companion matrix to polynomial x	<code>matcompanion(x)</code>
Sylvester matrix of x	<code>polsylvestermatrix(x)</code>

Gaussian elimination

kernel of matrix x	<code>matker(x, {flag})</code>
intersection of column spaces of x and y	<code>matintersect(x, y)</code>
solve $M * X = B$ (M invertible)	<code>matsolve(M, B)</code>
as solve, modulo D (col. vector)	<code>matsolvemod(M, D, B)</code>
one sol of $M * X = B$	<code>matinverseimage(M, B)</code>
basis for image of matrix x	<code>matimage(x)</code>
supplement columns of x to get basis	<code>mataugment(x)</code>
rows, cols to extract invertible matrix	<code>matindexrank(x)</code>
rank of the matrix x	<code>matrank(x)</code>

Lattices & Quadratic Forms

upper triangular Hermite Normal Form	<code>mathnf(x)</code>
HNF of x where d is a multiple of $\det(x)$	<code>mathnfmod(x, d)</code>
elementary divisors of x	<code>matsnf(x)</code>
LLL-algorithm applied to columns of x	<code>qflll(x, {flag})</code>
like <code>qflll</code> , x is Gram matrix of lattice	<code>qflllgram(x, {flag})</code>
LLL-reduced basis for kernel of x	<code>matkerint(x)</code>
\mathbf{Z} -lattice \longleftrightarrow \mathbf{Q} -vector space	<code>matrixqx(x, p)</code>
signature of quad form $t_y * x * y$	<code>qfsign(x)</code>
decomp into squares of $t_y * x * y$	<code>qfgaussred(x)</code>
eigenvals/eigenvecs for real symmetric x	<code>qfjacobi(x)</code>
find up to m sols of $t_y * x * y \leq b$	<code>qfminim(x, b, m)</code>
perfection rank of x	<code>qfperfection(x)</code>
$v, v[i] :=$ number of sols of $t_y * x * y = i$	<code>qfrep(x, B, {flag})</code>
automorphism group of q	<code>qfauto(q)</code>
find isomorphism between q and Q	<code>qfiso(q, Q)</code>

Formal & p-adic Series

truncate power series or p -adic number	<code>truncate(x)</code>
valuation of x at p	<code>valuation(x, p)</code>

Dirichlet and Power Series

Taylor expansion around 0 of f w.r.t. x	<code>taylor(f, x)</code>
$\sum a_k b_k t^k$ from $\sum a_k t^k$ and $\sum b_k t^k$	<code>serconvol(a, b)</code>
$f = \sum a_k t^k$ from $\sum (a_k/k!) t^k$	<code>serlaplace(f)</code>
reverse power series F so $F(f(x)) = x$	<code>serreverse(f)</code>
Dirichlet series multiplication / division	<code>dirmul, dirdiv(x, y)</code>
Dirichlet Euler product (b terms)	<code>direuler(p = a, b, expr)</code>

PARI-GP Reference Card

(PARI-GP version 2.6.1)

Polynomials & Rational Functions

degree of f	<code>poldegree(f)</code>
coeff. of degree n of f , leading coeff.	<code>polcoeff(f, n)</code> , <code>pollead</code>
gcd of coefficients of f	<code>content(f)</code>
replace x by y	<code>subst(f, x, y)</code>
evaluate f replacing vars by their value	<code>eval(f)</code>
replace polynomial expr. $T(x)$ by y in f	<code>substpol(f, T, y)</code>
replace x_1, \dots, x_n by y_1, \dots, y_n in f	<code>substvec(f, x, y)</code>
discriminant of polynomial f	<code>poldisc(f)</code>
resultant $R = \text{Res}_v(f, g)$	<code>polresultant(f, g, {v})</code>
$[u, v, R], xu + yv = \text{Res}_v(f, g)$	<code>polresultantext(x, y, {v})</code>
derivative of f w.r.t. x	<code>deriv(f, {x})</code>
formal integral of f w.r.t. x	<code>intformal(f, {x})</code>
formal sum of f w.r.t. x	<code>sumformal(f, {x})</code>
reciprocal poly $x^{\deg f} f(1/x)$	<code>polrecip(f)</code>
interpol. pol. eval. at a	<code>polinterpolate(X, {Y}, {a}, {&e})</code>
initialize t for Thue equation solver	<code>thueinit(f)</code>
solve Thue equation $f(x, y) = a$	<code>thue(t, a, {sol})</code>

Roots and Factorization

number of real roots of $f, a < x \leq b$	<code>polsturm(f, {a}, {b})</code>
complex roots of f	<code>polroots(f)</code>
symmetric powers of roots of f up to n	<code>polsym(f, n)</code>
factor f	<code>factor(f, {lim})</code>
factor f mod p / roots	<code>factormod(f, p)</code> , <code>polrootsmod</code>
factor f over \mathbf{F}_{p^a} / roots	<code>factorff(f, p, a)</code> , <code>polrootsff</code>
factor f over \mathbf{Q}_p / roots	<code>factorpadic(f, p, r)</code> , <code>polrootspadic</code>
find irreducible $T \in \mathbf{F}_p[x], \deg T = n$	<code>ffinit(p, n, {x})</code>
$\#\{\text{monic irred. } T \in \mathbf{F}_q[x], \deg T = n\}$	<code>ffnbirred(q, n)</code>
p -adic root of f cong. to a mod p	<code>padicappr(f, a)</code>
Newton polygon of f for prime p	<code>newtonpoly(f, p)</code>
extensions of \mathbf{Q}_p of degree N	<code>padicfields(p, N)</code>

Special Polynomials

n -th cyclotomic polynomial in var. v	<code>polcyclo(n, {v})</code>
d -th degree subfield of $\mathbf{Q}(\zeta_n)$	<code>polsubcyclo(n, d, {v})</code>
$P_n, T_n/U_n, H_n$	<code>pollegendre, polchebyshev, polhermite</code>

Transcendental and p -adic Functions

real, imaginary part of x	<code>real(x)</code> , <code>imag(x)</code>
absolute value, argument of x	<code>abs(x)</code> , <code>arg(x)</code>
square/nth root of x	<code>sqrt(x)</code> , <code>sqrtn(x, n, {&z})</code>
trig functions	<code>sin, cos, tan, cotan</code>
inverse trig functions	<code>asin, acos, atan</code>
hyperbolic functions	<code>sinh, cosh, tanh</code>
inverse hyperbolic functions	<code>asinh, acosh, atanh</code>
exponential / natural log of x	<code>exp, log</code>
Euler Γ function, $\log \Gamma, \Gamma'/\Gamma$	<code>gamma, lngamma, psi</code>
incomplete gamma function ($y = \Gamma(s)$)	<code>incgam(s, x, {y})</code>
exponential integral $\int_x^\infty e^{-t}/t dt$	<code>eint1(x)</code>
error function $2/\sqrt{\pi} \int_x^\infty e^{-t^2} dt$	<code>erfc(x)</code>
dilogarithm of x	<code>dilog(x)</code>
m -th polylogarithm of x	<code>polylog(m, x, {flag})</code>
U -confluent hypergeometric function	<code>hyperu(a, b, u)</code>
Bessel $J_n(x), J_{n+1/2}(x)$	<code>besselj(n, x)</code> , <code>besseljh(n, x)</code>
Bessel $I_\nu, K_\nu, H_\nu^1, H_\nu^2, N_\nu$	<code>(bessel)i, k, h1, h2, n</code>
Lambert $W: x$ s.t. $xe^x = y$	<code>lambertw(y)</code>
Teichmuller character of p -adic x	<code>teichmuller(x)</code>

Elementary Arithmetic Functions

vector of binary digits of $ x $	<code>binary(x)</code>
bit number n of integer x	<code>bittest(x, n)</code>
Hamming weight of integer x	<code>hammingweight(x)</code>
ceiling/floor/fractional part	<code>ceil, floor, frac</code>
round x to nearest integer	<code>round(x, {&e})</code>
truncate x	<code>truncate(x, {&e})</code>
gcd/LCM of x and y	<code>gcd(x, y)</code> , <code>lcm(x, y)</code>
gcd of entries of a vector/matrix	<code>content(x)</code>

Primes and Factorization

add primes in v to prime table	<code>addprimes(v)</code>
Chebyshev $\pi(x), n$ -th prime p_n	<code>primepi(x)</code> , <code>prime(n)</code>
vector of first n primes	<code>primes(n)</code>
smallest prime $\geq x$	<code>nextprime(x)</code>
largest prime $\leq x$	<code>preprime(x)</code>
factorization of x	<code>factor(x, {lim})</code>
$n = df^2, d$ squarefree/fundamental	<code>core(n, {fl}), coredisc</code>
recover x from its factorization	<code>factorback(f, {e})</code>

Divisors

number of prime divisors $\omega(n) / \Omega(n)$	<code>omega(n)</code> , <code>bigomega</code>
divisors of n / number of divisors $\tau(n)$	<code>divisors(n)</code> , <code>numdiv</code>
sum of (k -th powers of) divisors of n	<code>sigma(n, {k})</code>

Special Functions and Numbers

binomial coefficient $\binom{x}{y}$	<code>binomial(x, y)</code>
Bernoulli number B_n as real/rational	<code>bernreal(n)</code> , <code>bernfrac</code>
Bernoulli polynomial $B_n(x)$	<code>bernpol(n, {x})</code>
n -th Fibonacci number	<code>fibonacci(n)</code>
Stirling numbers $s(n, k)$ and $S(n, k)$	<code>stirling(n, k, {flag})</code>
number of partitions of n	<code>numbpart(n)</code>
Möbius μ -function	<code>moebius(x)</code>
Hilbert symbol of x and y (at p)	<code>hilbert(x, y, {p})</code>
Kronecker-Legendre symbol $\left(\frac{x}{y}\right)$	<code>kronecker(x, y)</code>
Dedekind sum $s(h, k)$	<code>sumdedekind(h, k)</code>

Multiplicative groups $(\mathbf{Z}/N\mathbf{Z})^*, \mathbf{F}_q^*$

Euler ϕ -function	<code>eulerphi(x)</code>
multiplicative order of x (divides o)	<code>znorder(x, {o})</code> , <code>fforder</code>
primitive root mod q / x .mod	<code>znprimroot(q)</code> , <code>ffprimroot(x)</code>
structure of $(\mathbf{Z}/n\mathbf{Z})^*$	<code>znstar(n)</code>
discrete logarithm of x in base g	<code>znlog(x, g, {o})</code> , <code>fflog</code>

Miscellaneous

integer square / n -th root of x	<code>sqrtint(x)</code> , <code>sqrtnint(x, n)</code>
solve $z \equiv x$ and $z \equiv y$	<code>chinese(x, y)</code>
minimal u, v so $xu + yv = \text{gcd}(x, y)$	<code>gcdext(x, y)</code>
continued fraction of x	<code>contfrac(x, {b}, {lmax})</code>
last convergent of continued fraction x	<code>contfracpnqn(x)</code>
rational approximation to x	<code>bestappr(x, k)</code> , <code>bestapprPade</code>

True-False Tests

is x the disc. of a quadratic field?	<code>isfundamental(x)</code>
is x a prime?	<code>isprime(x)</code>
is x a strong pseudo-prime?	<code>ispseudoprime(x)</code>
is x square-free?	<code>issquarefree(x)</code>
is x a square?	<code>issquare(x, {&n})</code>
is x a perfect power?	<code>ispower(x, {k}, {&n})</code>
is pol irreducible?	<code>polisirreducible(pol)</code>

Based on an earlier version by Joseph H. Silverman
September 2013 v2.27. Copyright © 2013 K. Belabas

Permission is granted to make and distribute copies of this card provided the copyright and this permission notice are preserved on all copies.

Send comments and corrections to (Karim.Belabas@math.u-bordeaux.fr)

PARI-GP Reference Card (2)

(PARI-GP version 2.6.1)

Elliptic Curves

Elliptic curve initially given by 5-tuple $v = [a_1, a_2, a_3, a_4, a_6]$.

Initialize *ell* struct `E = ellinit(v, {Domain})`

Points are `[x,y]`, the origin is `[0]`. Struct members accessed as *E.member*:

- All domains: `E.a1,a2,a3,a4,a6, b2,b4,b6,b8, c4,c6, disc, j`
 - *E* defined over **R** or **C**
 - x*-coords. of points of order 2 `E.roots`
 - periods / quasi-periods `E.omega, E.eta`
 - volume of complex lattice `E.area`
 - *E* defined over **Q_p**
 - residual characteristic `E.p`
 - If $|j|_p > 1$: Tate's $[u^2, u, q, [a, b]]$ `E.tate`
 - *E* defined over **F_q**
 - characteristic `E.p`
 - $\#E(\mathbf{F}_q)$ /cyclic structure/generators `E.no, E.cyc, E.gen`
 - *E* defined over **Q**
 - generators of $E(\mathbf{Q})$ (require `elldata`) `E.gen`
 - $[a_1, a_2, a_3, a_4, a_6]$ from *j*-invariant `ellfromj(j)`
 - change curve *E* using $v = [u, r, s, t]$ `ellchangecurve(E, v)`
 - change point *z* using $v = [u, r, s, t]$ `ellchangept(E, v)`
 - add points $P + Q / P - Q$ `elladd(E, P, Q), ellsub(E, P, Q)`
 - negate point `ellneg(E, P)`
 - compute $n \cdot z$ `ellmul(E, z, n)`
 - n*-division polynomial $f_n(x)$ `elldivpol(E, n, {x})`
 - check if *z* is on *E* `ellisoncurve(E, z)`
 - order of torsion point *z* `ellorder(E, z)`
 - y*-coordinates of point(s) for *x* `ellordinate(E, x)`
 - point $[\wp(z), \wp'(z)]$ corresp. to *z* `ellztopoint(E, z)`
 - complex *z* such that $p = [\wp(z), \wp'(z)]$ `ellpointtoz(E, p)`
- Curves over finite fields, Pairings**
- random point on *E* `random(E)`
 - $\#E(\mathbf{F}_q)$ `ellcard(E)`
 - structure $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$ of $E(\mathbf{F}_q)$ `ellgroup(E)`
 - Weil pairing of *m*-torsion pts *x, y* `ellweilpairing(E, x, y, m)`
 - Tate pairing of *x, y*; *x m*-torsion `elltatepairing(E, x, y, m)`
 - Discrete log, find *n* s.t. $P = [n]Q$ `elllog(E, P, Q, {ord})`
- Curves over Q and the L-function**
- canonical bilinear form taken at z_1, z_2 `ellbil(E, z1, z2)`
 - canonical height of *z* `ellheight(E, z, {flag})`
 - height regulator matrix for pts in *x* `ellheightmatrix(E, x)`
 - cond, min mod, Tamagawa num $[N, v, c]$ `ellglobalred(E)`
 - reduction of $y^2 + Qy = P$ (genus 2) `genus2red(Q, P, {p})`
 - Kodaira type of *p*-fiber of *E* `elllocalred(E, p)`
 - minimal model of E/\mathbf{Q} `ellminimalmodel(E, {&v})`
 - p*-th coeff a_p of *L*-function, *p* prime `ellap(E, p)`
 - k*-th coeff a_k of *L*-function `ellak(E, k)`
 - vector of first *n* a_k 's in *L*-function `elllan(E, n)`
 - $L(E, s)$ `elllseries(E, s)`
 - $L^{(r)}(E, 1)$ `ellL1(E, r)`
 - return a Heegner point on *E* of rank 1 `ellheegner(E)`
 - order of vanishing at 1 `ellanalyticrank(E, {eps})`
 - root number for $L(E, \cdot)$ at *p* `ellrootno(E, {p})`
 - torsion subgroup with generators `elltors(E)`
 - modular parametrization of *E* `elltaniyama(E)`

Elldata package, Cremona's database:

db code \leftrightarrow `[conductor, class, index]` `ellconvertname(s)`
generators of Mordell-Weil group `ellgenerators(E)`
look up *E* in database `ellidentify(E)`
all curves matching criterion `ellsearch(N)`
loop over curves with cond. from *a* to *b* `forell(E, a, b, seq)`

Elliptic & Modular Functions

$w = [\omega_1, \omega_2]$ or *ell* struct (`E.omega`), $\tau = \omega_1/\omega_2$.
arithmetic-geometric mean `agm(x, y)`
elliptic *j*-function $1/q + 744 + \dots$ `ellj(x)`
Weierstrass $\sigma/\wp/\zeta$ function `ellsigma(w, z), ellwp, ellzeta`
periods/quasi-periods `ellperiods(E, {flag}), elleta(w)`
 $(2i\pi/\omega_2)^k E_k(\tau)$ `elleisnum(w, k, {flag})`
modified Dedekind η func. $\prod(1 - q^n)$ `eta(x, {flag})`
Jacobi sine theta function `theta(q, z)`
k-th derivative at $z=0$ of $\theta(q, z)$ `thetanullk(q, k)`
Weber's *f* functions `weber(x, {flag})`
Riemann's zeta $\zeta(s) = \sum n^{-s}$ `zeta(s)`

Binary Quadratic Forms

create $ax^2 + bxy + cy^2$ (distance *d*) `Qfb(a, b, c, {d})`
reduce x ($s = \sqrt{D}$, $l = [s]$) `qfbred(x, {flag}, {D}, {l}, {s})`
composition of forms $x*y$ or `qfbnucomp(x, y, l)`
n-th power of form x^n or `qfbnupow(x, n)`
composition without reduction `qfbcomprow(x, y)`
n-th power without reduction `qfbpowrow(x, n)`
prime form of disc. *x* above prime *p* `qfbprimeform(x, p)`
class number of disc. *x* `qfbclassno(x)`
Hurwitz class number of disc. *x* `qfbhclassno(x)`
Solve $Q(x, y) = p$ in integers, *p* prime `qfbsolve(Q, p)`

Quadratic Fields

quadratic number $\omega = \sqrt{x}$ or $(1 + \sqrt{x})/2$ `quadgen(x)`
minimal polynomial of ω `quadpoly(x)`
discriminant of $\mathbf{Q}(\sqrt{D})$ `quaddisc(x)`
regulator of real quadratic field `quadregulator(x)`
fundamental unit in real $\mathbf{Q}(x)$ `quadunit(x)`
class group of $\mathbf{Q}(\sqrt{D})$ `quadclassunit(D, {flag}, {t})`
Hilbert class field of $\mathbf{Q}(\sqrt{D})$ `quadhilbert(D, {flag})`
ray class field modulo *f* of $\mathbf{Q}(\sqrt{D})$ `quadray(D, f, {flag})`

General Number Fields: Initializations

A number field *K* is given by a monic irreducible $f \in \mathbf{Z}[X]$.

init number field structure *nf* `nfinit(f, {flag})`

nf members:

polynomial defining *nf*, $f(\theta) = 0$ `nf.pol`
number of real/complex places `nf.r1/r2/sign`
discriminant of *nf* `nf.disc`
 T_2 matrix `nf.t2`
vector of roots of *f* `nf.roots`
integral basis of \mathbf{Z}_K as powers of θ `nf.zk`
different `nf.diff`
codifferent `nf.codiff`
index `nf.index`
recompute *nf* using current precision `nfnewprec(nf)`
init relative *rnf* given by $g = 0$ over *K* `rnfinit(nf, g)`
init *bnf* structure `bnfinit(f, {flag})`

bnf members: same as *nf*, plus

underlying *nf* `bnf.nf`
classgroup `bnf.clgp`
regulator `bnf.reg`
fundamental units `bnf.fu`
torsion units `bnf.tu`
compute a *bnf* from small *bnf* `bnfinit(sbnf)`
add *S*-class group and units, yield *bnf s* `bnfsunit(nf, S)`
init class field structure *bnr* `bnrinit(bnf, m, {flag})`
bnr members: same as *bnf*, plus
underlying *bnf* `bnr.bnf`
big ideal structure `bnr.bid`
modulus `bnr.mod`
structure of $(\mathbf{Z}_K/m)^*$ `bnr.zkst`

Basic Number Field Arithmetic (nf)

Elements are `t_INT, t_FRAC, t_POL, t_POLMOD, or t_COL` (on integral basis *nf.zk*). Basic operations (prefix `nfelt`): `(nfelt)add, mul, pow, div, diveuc, mod, divrem, val, trace, norm`
express *x* on integer basis `nfalgtobasis(nf, x)`
express element *x* as a polmod `nfbastoalg(nf, x)`
reverse polmod $a = A(X) \bmod T(X)$ `modreverse(a)`
integral basis of field def. by $f = 0$ `nfbasis(f)`
field discriminant of field $f = 0$ `nfdisc(f)`
smallest poly defining $f = 0$ (slow) `polredabs(f, {flag})`
small poly defining $f = 0$ (fast) `polredbest(f, {flag})`
are fields $f = 0$ and $g = 0$ isomorphic? `nfisom(f, g)`
is field $f = 0$ a subfield of $g = 0$? `nfisincl(f, g)`
compositum of $f = 0, g = 0$ `polcompositum(f, g, {flag})`
subfields (of degree *d*) of *nf* `nfsubfields(nf, {d})`
roots of unity in *nf* `nfrootsof1(nf)`
roots of *g* belonging to *nf* `nfroots({nf}, g)`
factor *g* in *nf* `nfactor(nf, g)`
factor *g* mod prime *pr* in *nf* `nfactormod(nf, g, pr)`
conjugates of a root θ of *nf* `nfgaloisconj(nf, {flag})`
apply Galois automorphism *s* to *x* `nfgaloisapply(nf, s, x)`
quadratic Hilbert symbol (at *p*) `nfhilbert(nf, a, b, {p})`

Linear and algebraic relations

poly of degree $\leq k$ with root $x \in \mathbf{C}$ `algdep(x, k)`
alg. dep. with pol. coeffs for series *s* `seralgdep(s, x, y)`
small linear rel. on coords of vector *x* `lindep(x)`

Dedekind Zeta Function ζ_K , Hecke *L* series

ζ_K as Dirichlet series, $N(I) < b$ `dirzetak(nf, b)`
init *nfz* for field $f = 0$ `zetakinit(f)`
compute $\zeta_K(s)$ `zetak(nfz, s, {flag})`
Artin root number of *K* `bnrrootnumber(bnr, chi, {flag})`
 $L(1, \chi)$, for all χ trivial on *H* `bnrL1(bnr, {H}, {flag})`

Class Groups & Units (bnf, bnr)

$a_1, \{a_2\}, \{a_3\}$ usually *bnr, subgp* or *bnf, module, {subgp}*
remove GRH assumption from *bnf* `bnfcertify(bnf)`
expo. of ideal *x* on class gp `bnfisprincipal(bnf, x, {flag})`
expo. of ideal *x* on ray class gp `bnrisprincipal(bnr, x, {flag})`
expo. of *x* on fund. units `bnfisunit(bnf, x)`
as above for *S*-units `bnfissunit(bnfs, x)`
signs of real embeddings of *bnf.fu* `bnfsignunit(bnf)`
narrow class group `bnfnarrow(bnf)`

Class Field Theory

ray class number for mod. m `bnrclassno(bnf, m)`
discriminant of class field ext `bnrdisc(a1, {a2}, {a3})`
ray class numbers, l list of mods `bnrclassnolist(bnf, l)`
discriminants of class fields `bnrdiscclist(bnf, l, {arch}, {flag})`
decode output from `bnrdiscclist` `bnfdecodemodule(nf, fa)`
is modulus the conductor? `bnrisconductor(a1, {a2}, {a3})`
conductor of character chi `bnrconductorofchar(bnr, chi)`
conductor of extension `bnrconductor(a1, {a2}, {a3}, {flag})`
conductor of extension def. by g `rnfconductor(bnf, g)`
Artin group of ext. def'd by g `rnfnormgroup(bnr, g)`
subgroups of bnr , index $\leq b$ `subgrouplist(bnr, b, {flag})`
rel. eq. for class field def'd by sub `rnfkummer(bnr, sub, {d})`
same, using Stark units (real field) `bnrstark(bnr, sub, {flag})`

Ideals: elements, primes, or matrix of generators in HNF

is id an ideal in nf ? `nfisideal(nf, id)`
is x principal in bnf ? `bnfisprincipal(bnf, x)`
give $[a, b]$, s.t. $a\mathbf{Z}_K + b\mathbf{Z}_K = x$ `idealtwoelt(nf, x, {a})`
put ideal a ($a\mathbf{Z}_K + b\mathbf{Z}_K$) in HNF form `idealhnf(nf, a, {b})`
norm of ideal x `idealnrm(nf, x)`
minimum of ideal x (direction v) `idealmin(nf, x, v)`
LLL-reduce the ideal x (direction v) `idealred(nf, x, {v})`

Ideal Operations

add ideals x and y `idealadd(nf, x, y)`
multiply ideals x and y `idealmul(nf, x, y, {flag})`
intersection of ideals x and y `idealintersect(nf, x, y, {flag})`
 n -th power of ideal x `idealpow(nf, x, n, {flag})`
inverse of ideal x `idealinv(nf, x)`
divide ideal x by y `idealdiv(nf, x, y, {flag})`
Find $(a, b) \in x \times y$, $a + b = 1$ `idealaddtoone(nf, x, {y})`
coprime integral A, B such that $x = A/B$ `idealnumden(nf, x)`

Primes and Multiplicative Structure

factor ideal x in nf `idealfactor(nf, x)`
expand ideal factorization in nf `idealfactorback(nf, f, e)`
decomposition of prime p in nf `idealprimedec(nf, p)`
valuation of x at prime ideal pr `idealval(nf, x, pr)`
weak approximation theorem in nf `idealchinese(nf, x, y)`
give $bid = \text{structure of } (\mathbf{Z}_K/id)^*$ `idealstar(nf, id, {flag})`
discrete log of x in $(\mathbf{Z}_K/bid)^*$ `ideallog(nf, x, bid)`
idealstar of all ideals of norm $\leq b$ `ideallist(nf, b, {flag})`
add Archimedean places `ideallistarch(nf, b, {ar}, {flag})`
init `prmod` structure `nfmodprinit(nf, pr)`
kernel of matrix M in $(\mathbf{Z}_K/pr)^*$ `nfkermodpr(nf, M, prmod)`
solve $Mx = B$ in $(\mathbf{Z}_K/pr)^*$ `nfsvlmodpr(nf, M, B, prmod)`

Galois theory over \mathbf{Q}

Galois group of field $\mathbf{Q}[x]/(f)$ `polgalois(f)`
initializes a Galois group structure G `galoisinit(pol, {den})`
action of p in `nfgaloisconj` form `galoispermtopol(G, {p})`
identify as abstract group `galoisidentify(G)`
export a group for GAP/MAGMA `galoisexport(G, {flag})`
subgroups of the Galois group G `galoissubgroups(G)`
is subgroup H normal? `galoisisnormal(G, H)`
subfields from subgroups `galoissubfields(G, {flag}, {v})`
fixed field `galoisfixedfield(G, perm, {flag}, {v})`
Frobenius at maximal ideal P `idealfrobenius(nf, G, P)`
ramification groups at P `idealramgroups(nf, G, P)`

PARI-GP Reference Card (2)

(PARI-GP version 2.6.1)

is G abelian? `galoisisabelian(G, {flag})`
abelian number fields/ \mathbf{Q} `galoissubcyclo(N, H, {flag}, {v})`
query the `galpol` package `galoisgetpol(a, b, {s})`

Relative Number Fields (rnf)

Extension L/K is defined by $T \in K[x]$.
absolute equation of L `rnfequation(nf, T, {flag})`
is L/K abelian? `rnfisabelian(nf, T)`
relative `nfgalgtobasis` `rnfalgtobasis(rnf, x)`
relative `nfbasistoalg` `rnfbasistoalg(rnf, x)`
relative `idealhnf` `rnfidealhnf(rnf, x)`
relative `idealmul` `rnfidealmul(rnf, x, y)`
relative `idealtwoelt` `rnfidealtwoelt(rnf, x)`

Lifts and Push-downs

absolute \rightarrow relative repres. for x `rnfeltabstorel(rnf, x)`
relative \rightarrow absolute repres. for x `rnfeltreltoabs(rnf, x)`
lift x to the relative field `rnfeltup(rnf, x)`
push x down to the base field `rnfeltdown(rnf, x)`
idem for x ideal: (`rnfideal`)`reltoabs`, `abstorel`, `up`, `down`

Norms

absolute norm of ideal x `rnfidealnrmabs(rnf, x)`
relative norm of ideal x `rnfidealnrmrel(rnf, x)`
solutions of $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$ `bnfisintnorm(bnf, x)`
is $x \in \mathbf{Q}$ a norm from K ? `bnfisnorm(bnf, x, {flag})`
initialize T for norm eq. solver `rnfnorminit(K, pol, {flag})`
is $a \in K$ a norm from L ? `rnfnorm(T, a, {flag})`

Maximal order \mathbf{Z}_L as a \mathbf{Z}_K -module

relative `polred` `rnfpolred(nf, T)`
relative `polredabs` `rnfpolredabs(nf, T)`
characteristic poly. of a mod T `rnfcharpoly(nf, T, a, {v})`
relative Dedekind criterion, prime pr `rnfdedekind(nf, T, pr)`
discriminant of relative extension `rnfdisc(nf, T)`
pseudo-basis of \mathbf{Z}_L `rnfpsseudobasis(nf, T)`
General \mathbf{Z}_K -modules: $M = [\text{matrix, vec. of ideals}] \subset L$
relative HNF / SNF `nfhnf(nf, M)`, `nfnfnf`
reduced basis for M `rnfilllgram(nf, T, M)`
determinant of pseudo-matrix M `rnfdet(nf, M)`
Steinitz class of M `rnfstteinitz(nf, M)`
 \mathbf{Z}_K -basis of M if \mathbf{Z}_K -free, or 0 `rnfnfnfbasis(bnf, M)`
 n -basis of M , or $(n+1)$ -generating set `rnfbasis(bnf, M)`
is M a free \mathbf{Z}_K -module? `rnfisfree(bnf, M)`

Graphic Functions

crude graph of $expr$ between a and b `plot(X = a, b, expr)`
High-resolution plot (immediate plot)
plot $expr$ between a and b `plotoh(X = a, b, expr, {flag}, {n})`
plot points given by lists lx, ly `plotthraw(lx, ly, {flag})`
terminal dimensions `plotsizes()`

Rectwindow Functions

init window w , with size x, y `plotinit(w, x, y)`
erase window w `plotkill(w)`
copy w to w_2 with offset (dx, dy) `plotcopy(w, w_2, dx, dy)`
clips contents of w `plotclip(w)`
scale coordinates in w `plotscale(w, x1, x2, y1, y2)`
plotoh in w `plotrecth(w, X = a, b, expr, {flag}, {n})`
plotthraw in w `plotrectthraw(w, data, {flag})`
draw window w_1 at $(x_1, y_1), \dots$ `plotdraw([[w1, x1, y1], ...])`

Low-level Rectwindow Functions

set current drawing color in w to c `plotcolor(w, c)`
current position of cursor in w `plotcursor(w)`
write s at cursor's position `plotstring(w, s)`
move cursor to (x, y) `plotmove(w, x, y)`
move cursor to $(x + dx, y + dy)$ `plotrmove(w, dx, dy)`
draw a box to (x_2, y_2) `plotbox(w, x_2, y_2)`
draw a box to $(x + dx, y + dy)$ `plotrbox(w, dx, dy)`
draw polygon `plotlines(w, lx, ly, {flag})`
draw points `plotpoints(w, lx, ly)`
draw line to $(x + dx, y + dy)$ `plotrline(w, dx, dy)`
draw point $(x + dx, y + dy)$ `plotrpoint(w, dx, dy)`
draw point $(x + dx, y + dy)$ `plotrpoint(w, dx, dy)`

Postscript Functions

as `plotoh` `psplotoh(X = a, b, expr, {flag}, {n})`
as `plotthraw` `psplotthraw(lx, ly, {flag})`
as `plotdraw` `psdraw([[w1, x1, y1], ...])`

Based on an earlier version by Joseph H. Silverman
September 2013 v2.27. Copyright © 2013 K. Belabas

Permission is granted to make and distribute copies of this card provided the copyright and this permission notice are preserved on all copies.

Send comments and corrections to Karim.Belabas@math.u-bordeaux.fr