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Preface

These lecture notes are based on a course in the theory of partitions given by Bruce Berndt at the University of Illinois at Urbana-Champaign in the spring of 2014. The notes are not in polished form and need to be reorganized. The lecturer had intended to give more applications to partitions from Ramanujan's lost notebook, in particular, from mock theta functions. However, time did not permit the inclusion of these topics.

Because the notes were prepared somewhat hastily while the course was being offered, misprints are likely abundant. Please send any comments and corrections to Bruce Berndt, berndt@illinois.edu .

Chapter 1

Elementary Approaches to Partitions and Their Generating Functions

1.1. Texts in the Theory of Partitions and Basic Notation

The texts that have been devoted to the theory of partitions are generally well written. The contents of each of these books moderately overlaps with these notes. However, these notes contain material that is not found in any of these texts. Two of them are difficult to obtain, namely, *Classical Partition Identities and Basic Hypergeometric Series* by W. Chu and L. Di Claudio [**47**], and *Partition Theory* by A.K. Agarwal, Padmavathamma, and M. V. Subbarao [**1**]. The most famous and broadest of books devoted to partition theory is *The Theory of Partitions* by G. E. Andrews [**10**], and the most elementary text is *Integer Partitions* by Andrews and K. Eriksson [**19**]. The latter is suitable for bright high school students and undergraduates, because it requires little background. The author's book, *Number Theory in the Spirit of Ramanujan* [**28**], as the title indicates, is a broad introduction to some of contributions of Ramanujan to number theory, and in particular, to some of his discoveries about partitions. The emphasis in this book, not unlike the book by Chu and Di Claudio, is partition theory from the viewpoint of q-series. Not surprisingly, there is some overlap with the lecturer's book [**28**] and these notes. As in [**28**], contributions of Ramanujan are emphasized in these notes. Also, like in [**28**], q-series are at the heart of our approach. Entries from Ramanujan's lost notebook [**91**] will also be discussed.

Little background in q-series is needed to read these notes. Most of our needs in q-series are developed *ab initio*, although in a few instances we may quote results from [**10**], [**28**], or G. Gasper and M. Rahman's 'bible' of the subject [**58**]. You may not know what a q-series is, and indeed a general definition of a q-series may be difficult to give, but nonetheless we shall give an admittedly vague definition below.

For any nonnegative integer n and complex number a, set

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k). \tag{1.1.1}$$

If $n=0$, the empty product above is interpreted to be equal to 1. For any complex numbers a and q, $|q|<1$, define

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n = \prod_{k=0}^{\infty}(1-aq^k).$$

If the base q is understood, then we often abbreviate the notation by writing $(a)_n := (a;q)_n$ and $(a)_\infty := (a;q)_\infty$. In later chapters, it will be convenient to use another abbreviated notation

$$(a_1,a_2,\dots,a_m;q)_n := (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n. \tag{1.1.2}$$

Similarly,

$$(a_1,a_2,\dots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty. \tag{1.1.3}$$

We also occasionally use the notation

$$[a]_\infty := [a;q]_\infty := (a;q)_\infty(q/a;q)_\infty \tag{1.1.4}$$

and

$$[a_1,a_2,\dots,a_m]_\infty := [a_1,a_2,\dots,a_m;q]_\infty := [a_1;q]_\infty[a_2;q]_\infty\cdots[a;q_m]_\infty. \tag{1.1.5}$$

A q-series generally will have summands containing various products of the sort $(a;q)_n$. For example,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2} \qquad \text{and} \qquad \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}$$

As we shall see later in this chapter, the first series is a generating function for the partition function, while the latter is one of Ramanujan's mock theta functions. Theta functions frequently appear in the theory of q-series. They do not contain q-products in their summands, but because of their ubiquitous appearances in the theory of q-series, we consider theta functions as q-series as well. We use Ramanujan's general definition of a theta function, namely,

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab|<1. \tag{1.1.6}$$

In particular, use is often made of the Jacobi triple product identity [**26**, p. 35, Entry 19], [**28**, Theorem 1.3.3, p. 10]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{1.1.7}$$

for which several proofs will be given in the sequel.

Further useful properties are given in the following two lemmas from Ramanujan's second notebook [**90**], [**26**, p. 34, Entry 18(iv); p. 48, Entry 31].

Lemma 1.1.1. *We have*

$$f(a,b) = f(b,a), \tag{1.1.8}$$

$$f(1,a) = 2f(a,a^3), \tag{1.1.9}$$

$$f(-1,a) = 0, \tag{1.1.10}$$

and, if n *is any integer,*

$$f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}). \tag{1.1.11}$$

A proof of Lemma 1.1.1 is left for the exercises.

Lemma 1.1.2. *Let* $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ *and* $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$. *Then, for any positive integer* n,

$$f(a,b) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{1.1.12}$$

Ramanujan actually wrote Lemma 1.1.2 in the form

$$\begin{aligned} f(U_1,V_1) = f(U_n,V_n) &+ U_1 f\left(\frac{V_{n-1}}{U_1}, \frac{U_{n+1}}{U_1}\right) + V_1 f\left(\frac{U_{n-1}}{V_1}, \frac{V_{n+1}}{V_1}\right) \\ &+ U_2 f\left(\frac{V_{n-2}}{U_2}, \frac{U_{n+2}}{U_2}\right) + V_2 f\left(\frac{U_{n-2}}{V_2}, \frac{V_{n+2}}{V_2}\right) + \cdots, \end{aligned} \tag{1.1.13}$$

where the sum on the right-hand side contains n terms. However, by (1.1.11), for $r \geq 1$,

$$V_r f\left(\frac{U_{n-r}}{V_r}, \frac{V_{n+r}}{V_r}\right) = U_{n-r} f\left(\frac{U_{n-r}^2 V_{n+r}}{V_r^3}, \frac{V_r}{U_{n-r}}\right) = U_{n-r} f\left(\frac{U_{2n-r}}{U_{n-r}}, \frac{V_r}{U_{n-r}}\right).$$

Thus, we see that the sums on the right sides of (1.1.12) and (1.1.13) agree.

Proof of Lemma 1.1.2. Using the definitions of U_n and V_n, we see that

$$\begin{aligned} \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right) &= \sum_{k=-\infty}^{\infty}\sum_{r=0}^{n-1} U_r \left(\frac{U_{n+r}}{U_r}\right)^{k(k+1)/2} \left(\frac{V_{n-r}}{U_r}\right)^{k(k-1)/2} \\ &= \sum_{k=-\infty}^{\infty}\sum_{r=0}^{n-1} U_r^{1-k^2} U_{n+r}^{k(k+1)/2} V_{n-r}^{k(k-1)/2} \\ &= \sum_{k=-\infty}^{\infty}\sum_{r=0}^{n-1} a^{(nk+r)(nk+r+1)/2} b^{(nk+r)(nk+r-1)/2} \\ &= \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2} \\ &= f(a,b). \end{aligned}$$

□

1.2. Definitions and Generating Functions

Definition 1.2.1. *The ordinary or unrestricted partition function $p(n)$ is the number of representations of the positive integer n as a sum of positive integers, where different orders of the summands are not considered to be distinct.*

Thus, $p(4) = 5$, because there are five ways to write 4 as a sum of positive integers, namely, 4, 3+1, 2+2, 2+1+1, 1+1+1+1. A generating function for $p(n)$ can be given by, for $|q| < 1$,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \sum_{n_1=0}^{\infty} q^{1\cdot n_1} \sum_{n_2=0}^{\infty} q^{2n_2} \cdots \sum_{n_k=0}^{\infty} q^{kn_k} \cdots, \tag{1.2.1}$$

where we make the convention that $p(0) = 1$, and where we have expanded each factor $1/(1 - q^k)$, $k \geq 1$, in the denominator of $1/(q;q)_\infty$ into a geometric series. Thus, the first series on the far right side generates the number of 1's in a particular partition of n, say; the second series generates the number of 2's in a partition of n, etc. It is clear that every partition of n can be achieved, and achieved uniquely, in the product of geometric series on the right side of (1.2.1). This formal argument can easily be made rigorous.

Definition 1.2.2. *Let $p_m(n)$ denote the number of partitions of n into parts that are not larger than m.*

It is clear that

$$\frac{1}{(q;q)_m} = \sum_{n=0}^{\infty} p_m(n)q^n. \tag{1.2.2}$$

We now put these observations in a theorem, originally due to Euler.

Theorem 1.2.3. *For $|q| < 1$,*

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{1.2.3}$$

Proof. Observe that

$$\frac{1}{(q;q)_\infty} - \frac{1}{(q;q)_m} = \sum_{n=m+1}^{\infty} p_m^*(n)q^n \leq \sum_{n=m+1}^{\infty} p(n)q^n, \tag{1.2.4}$$

where $p_m^*(n)$ denotes the number of partitions of n into parts, such that at least one of the parts is greater than or equal to $m+1$. Now let $m \to \infty$ in (1.2.4). Because the series on the far right-hand side of (1.2.4) is a convergent series, by the dominated convergence theorem, the first series on the right-hand side of (1.2.4) tends to 0 as $m \to \infty$. We also remark that the infinite series on the left-hand side of (1.2.1) or (1.2.3) converges for $|q| < 1$, because $1/(q;q)_\infty$ is analytic for $|q| < 1$. □

In the sequel, we derive generating functions for several types of partitions. Rigorous proofs can generally be developed along the same lines as our proof above for Theorem

1.2.3. To avoid repetitions of the same kind of argument, we shall forego such proofs in the future.

Definition 1.2.4. *Let S be any subset of the natural numbers. Then $p(S, m, n)$ denotes the number of partitions of n into exactly m parts taken from S. If $S = \mathbb{N}$, then we delete $\mathbb{N}$ from our notation. In particular, $p(m, n)$ is the number of partitions of n into precisely m parts.*

We now introduce the concept of a Ferrers graph, which originated with N. M. Ferrers in the 1850's. The idea was exploited by J. J. Sylvester, who was always grateful to Ferrers for his graphical interpretation of a partition. Arrange the parts in a partition of n in decreasing order, and graphically represent a part j by j dots, with the first dot in each part flush at the left margin. We provide an example.

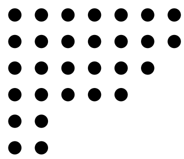

Figure 1. Ferrers graph of $29 = 7 + 7 + 6 + 5 + 2 + 2$

Definition 1.2.5. *The Durfee square for a partition of n is the largest square of nodes in the upper left-hand corner of the Ferrers diagram of the partition; if the number of such nodes on a side is s, then we say that we have a Durfee square of side s.*

Thus, for the partition $7 + 7 + 6 + 5 + 2 + 2$, the Durfee square has size 4. The idea of a Durfee square is due to W. P. Durfee, who was a a student of J. J. Sylvester.

Theorem 1.2.6. *Recall that $p_m(n)$ denotes the number of partitions of n into parts with largest part less than or equal to m, and that $p(m, n)$ is defined in Definition 1.2.4. Then*

$$p(m, n) = p_m(n), \tag{1.2.5}$$

which implies that the partitions of n in which the the largest is precisely m are equinumerous with the partitions of n into exactly m parts.

Proof. Consider the Ferrers graph of n, such as the one in Figure 1, giving a partition of $p(6, 29)$. Instead of reading the partition from top to bottom, read the Ferrers graph from left to right. Thus, in Figure 1, this partition, namely, $29 = 6+6+4+4+4+3+2$, is counted by $p_6(29)$. Every such partition counted by $p(m, n)$ is thus uniquely associated with a partition counted by $p_m(n)$. Clearly, this establishes a bijection for partitions counted by $p(m, n)$ with those counted by $p_m(n)$. Hence, (1.2.5) immediately follows. $\square$

As a corollary of Theorem 1.2.6, we see that the number of partitions of n in which the number of parts is less than or equal to m is equal to the number of partitions of n in which the largest part is no more than m. As another immediate corollary, we note that the partitions of n in which the largest is precisely m are equinumerous with the partitions of n into exactly m parts. If we let $p(r,m,n)$ equal the number of partitions of n into exactly m parts with the largest being r, then we have shown that $p(r,m,n) = p(m,r,n)$.

Definition 1.2.7. *The conjugate of a partition of n represented by a Ferrers graph is the partition that one obtains by reading the graph from left to right.*

Definition 1.2.8. *A partition of n is self-conjugate if its conjugate is identical with that same partition.*

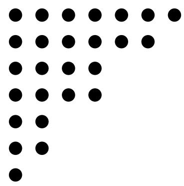

Figure 2. Ferrers graph of the self-conjugate partition $7+6+4+4+2+2+1$

If we enumerate the nodes in each right angle beginning with the outside and proceeding inward, we easily see that each right angle has an odd number of nodes, and, moreover these odd numbers are distinct. Thus, we obtain a partition into distinct odd numbers. In Figure 2, the partition $13+9+3+1$ is generated. Conversely, if we have a partition into distinct odd parts, then we can generate a unique self-conjugate partition. We easily see that we have a bijection between self-conjugate partitions and the set of partitions into odd, distinct integers.

Theorem 1.2.9. *The set of self-conjugate partitions of a positive integer n is equinumerous with the set of partitions into distinct odd parts.*

We now use the Ferrers graph in Table 2 to help us determine the generating function for self-conjugate partitions, or a generating function for partitions into odd, distinct parts, where we keep track of the number of odd, distinct parts.

Theorem 1.2.10 (Euler)**.** *We have*

$$(-xq;q^2)_\infty = \sum_{j=0}^{\infty} \frac{x^j q^{j^2}}{(xq^2;q^2)_j}. \tag{1.2.6}$$

Proof. Observe that the left side of (1.2.6) is the generating function of a positive integer n into, say s, odd, distinct parts. This leads us to examine the Ferrers graph of a self-conjugate partition, as in Figure 2, because of the one-to-one correspondence with such partitions and those partitions into distinct odd parts. As in Table 2, we identify the largest Durfee square of side s. But now note that s is identical to the number of odd, distinct parts. We know that this partition is also a self-conjugate partition, and so the partition π_1, reading from top to bottom, below the Durfee square is identical to the partition π_2, reading from left to right, to the right of the Durfee square. Each part in each partition is $\leq s$. Now consider the union of π_1 and π_2 formed by doubling either

π_1 or π_2. In our picture, both of these partitions are given by $2+2+1$, and do the union is $4+4+2$. We obtain a partition into even parts, with each part $\leq 2s$. The generating function for such partitions is $1/(q^2;q^2)_s$, but if we also want to keep track of the number of parts in π_1, then our generating function is $1/(xq^2;q^2)_s$. Thus, the generating function for all such partitions into odd distinct parts with Durfee square of side s is $q^{s^2}x^s/(xq^2;q^2)_s$. Summing over all s completes the proof. □

Definition 1.2.11. *Let $Q(n)$ denote the number of partitions of n into distinct parts. More generally, $Q(S,m,n)$ denotes the number of partitions of n into m distinct parts of S. If $S=\mathbb{N}$, we write $Q(\mathbb{N},m,n)=Q(m,n)$, so that $Q(m,n)$ is the number of partitions of n into exactly m distinct parts.*

Theorem 1.2.12. *The generating function for $Q(n)$ is given by*

$$\sum_{n=0}^{\infty} Q(n)q^n = (-q;q)_\infty. \tag{1.2.7}$$

Proof. Observe that $(-q;q)_\infty$ yields only the first two terms of each geometric series on the right-hand side of (1.2.1). Thus, in each partition of n, each integer not larger than n can appear in a particular partition of n at most one time. Theorem 1.2.12 is thus immediate. □

Definition 1.2.13. *Let $p_o(n)$ denote the number of partitions of n into odd parts, and let $p_e(n)$ denote the number of partitions of n into even parts.*

Observe that

$$\frac{1}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} p_o(n)q^n \quad \text{and} \quad \frac{1}{(q^2;q^2)_\infty} = \sum_{n=0}^{\infty} p_e(n)q^n. \tag{1.2.8}$$

Theorem 1.2.14. *For each $n \geq 1$,*

$$Q(n) = p_o(n). \tag{1.2.9}$$

Proof. By Theorem 1.2.12 and (1.2.8),

$$\begin{aligned}\sum_{n=0}^{\infty} Q(n)q^n = (-q;q)_\infty &= \frac{(-q;q)_\infty(q;q)_\infty}{(q;q)_\infty}\\ &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty} = \frac{1}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} p_o(n)q^n.\end{aligned}$$

We now see that (1.2.9) follows. □

Theorem 1.2.14 is due to Euler and illustrates one of the fascinating aspects of the theory of partitions, namely, that the number of partitions of n of one particular type often equals the number of partitions of n of an entirely different type. We mention another famous example in illustration. The first of the two Rogers–Ramanujan identities that we will prove in Chapter 6 has the following combinatorial interpretation. The number of partitions of a positive integer n into parts differing by at least 2 is equal to the number of partitions of n into parts that are congruent to either 1 or 4 modulo 5. For example, the two respective sets of partitions for 8 are

$$8 = 7+1 = 6+2 = 5+3;$$
$$6+1+1 = 4+4 = 4+1+1+1+1 = 1+1+1+1+1+1+1+1.$$

Theorems 1.2.3 and 1.2.12 can easily be generalized.

Theorem 1.2.15. *For $|q|<1$ and $|z|<1/|q|$,*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(S,m,n)z^m q^n = \prod_{n\in S}\frac{1}{1-zq^n}, \tag{1.2.10}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q(S,m,n)z^m q^n = \prod_{n\in S}(1+zq^n). \tag{1.2.11}$$

Proof. Let $S=\{n_1,n_2,\dots\}$. We prove (1.2.10); the proof of (1.2.11) is similar. Observe that

$$\prod_{n\in S}\frac{1}{1-zq^n} = \prod_{j=1}^{\infty}\sum_{m_j=0}^{\infty} z^{m_j}q^{n_j m_j} \tag{1.2.12}$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots z^{m_1+m_2+\cdots}q^{n_1m_1+n_2m_2+\cdots}. \tag{1.2.13}$$

Let π be a partition of n into m_1 n_1's, m_2 n_2's, etc., i.e., $n = n_1m_1+n_2m_2+\cdots$. The number of parts in this partition is thus $m_1+m_2+\cdots := m$, say. Thus, from (1.2.12) and our discussion, we see that (1.2.10) holds. □

We emphasize that the power of z in (1.2.10) and (1.2.11) keeps track of the number of parts in each partition of n.

Definition 1.2.16. *We define $p^{(s)}(S,m,n)$ to be the number of partitions of $n\in S$ into exactly m parts, with each part appearing no more than s times in any partition of n.*

Theorem 1.2.17. *If $zq^n \neq 1$ for each nonnegative integer n, then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p^{(s)}(S,m,n)z^m q^n = \prod_{n\in S}(1+q^nz+q^{2n}z^2+\cdots+q^{sn}z^s)$$
$$= \prod_{n\in S}\frac{1-q^{(s+1)n}z^{s+1}}{1-q^nz}.$$

Proof. Using the definition of $p^{(s)}(S,m,n)$, we find that the first equality of Theorem 1.2.17 holds, while the second equality arises from summing the geometric series. □

Observe that $p^{(1)}(S,m,n) = Q(S,m,n)$. If there are no restrictions on the number of parts m, we drop m from the notation for partition functions. For example, if $S=\mathbb{O}$, the set of all positive odd integers, then $p(\mathbb{O}, n)$ denotes the number of partitions of n into odd summands, or odd parts. As before, it $S = \mathbb{N}$, we delete $\mathbb{N}$ from our notation. In alternative notations, we reformulate Theorem 1.2.14.

Theorem 1.2.18. *If $S = \mathbb{O}$, then*

$$p(\mathbb{O}, n) = Q(n) = p^{(1)}(n). \tag{1.2.14}$$

Second Proof of Theorem 1.2.18. Suppose that we have a partition into odd parts. Merge the odd parts by successively doubling. Repeat this process until all the parts are distinct. Note that this process will terminate, because each merging operation decreases the number of parts by 1, and by construction, the final parts will be distinct. For example, consider

$$3+3+3+1+1+1+1 \to 6+3+2+2 \to 6+3+4.$$

Suppose that we have a partition into distinct parts. We begin a 'halving' operation by successively halving each even part. Clearly, this process must terminate. For example, consider

$$6+4+3 \to 3+3+2+2+3 \to 3+3+3+1+1+1+1.$$

It is easy to see that these operations are inverses of each other, i.e., we have a bijection, and so we have reproved Theorem 1.2.18. □

Recall that in Definition 1.2.5 we defined a Durfee square of side s. In Figure 3 below, we indicate the Durfee square of size 4.

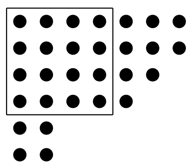

Figure 3. A Ferrers graph with Durfee square of size 4

Using Durfee squares, we can derive another generating function for $p(n)$.

Theorem 1.2.19. *We have*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \sum_{n=0}^{\infty} p(n) q^n. \tag{1.2.15}$$

Proof. Consider the nodes lying below the Durfee square of size s for an arbitrary partition of n. We note that we have the graphical representation of a partition π_1 of m_1, where m_1 equals the number of nodes of the partition to below the Durfee square. Moreover, each part is no larger than s. Now examine the nodes to the right of the Durfee square. Reading from left to right, we have a partition π_2 of a number m_2 with each part $\leq s$. For example, if π is the partition represented in Figure 3, then $s = 4$, π_1 is the partition 2+2, and π_2 is the partition 4+3+2. The number of choices for π_1 is $p(\{1,2,\dots,s\},m_1) =: p_s(m_1)$, and the number of choices for π_2 equals $p(\{1,2,\dots,s\},m_2) =: p_s(m_2)$. Note that $n = s^2+m_1+m_2$. The generating function for all partitions of n, with fixed s is thus

$$\sum_{n=0}^{\infty} q^n \sum_{n=s^2+m_1+m_2} p_s(m_1)p_s(m_2) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} q^{s^2+m_1+m_2} = \frac{q^{s^2}}{(q;q)_s^2}. \tag{1.2.16}$$

We now sum (1.2.16) over s, $0 \le s < \infty$, to generate all partitions. Thus,

$$\sum_{n=0}^{\infty} p(n)q^n = \sum_{s=0}^{\infty} \frac{q^{s^2}}{(q;q)_s^2},$$

which is the same as (1.2.15), and so the proof of Theorem 1.2.19 is complete. □

In view of Theorem 1.2.15, it is natural to ask if there is a generating function for $p(\mathbb{N},m,n)$ generalizing Theorem 1.2.19. We derive such a generating function in the next theorem.

Theorem 1.2.20. *We have*

$$\sum_{m,n=0}^{\infty} p(\mathbb{N},m,n)z^m q^n = \frac{1}{(zq;q)_\infty} = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(zq;q)_n(q;q)_n}. \tag{1.2.17}$$

Proof. The argument is similar to that in the proof of Theorem 1.2.19. Let m_1 denote the number of nodes in a partition below the Durfee square of size s, and let m_2 denote the number of nodes in the partition π_2, with its nodes lying to the right of the Durfee square of size s. Thus, $n = s^2+m_1+m_2$. However, we now want to keep track of the number of parts r in the partition π_1. The number of such partitions is $p(\{1,2,\dots,s\},r,m_1)$, while the number of partitions π_2 is $p(\{1,2,\dots,s\},m_2)$. Note that the number of parts of π is equal to $s+r$. Hence, for fixed s, the generating function for the number of partitions of n is equal to

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^n \sum_{r=0}^{\infty} \sum_{s^2+m_1+m_2=n} z^{r+s} p(\{1,2,\dots,s\},r,m_1)p(\{1,2,\dots,s\},m_2)\\
&= \sum_{r=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} q^{s^2+m_1+m_2} z^{r+s} p_s(r,m_1)p_s(m_2)\\
&= \frac{q^{s^2}z^s}{(zq;q)_s(q;q)_s}.
\end{aligned} \tag{1.2.18}$$

We now sum over s, $0 \le s < \infty$, and use (1.2.10) to deduce that

$$\sum_{s=0}^{\infty} \frac{q^{s^2}z^s}{(zq;q)_s(q;q)_s} = \frac{1}{(zq;q)_\infty}.$$

This completes the proof. □

Definition 1.2.21. *Let $Q_m(n)$ stand for the number of partitions of n into exactly m distinct parts.*

Definition 1.2.22. *The Ferrers triangle for a partition of n is the largest right, isosceles triangle in the upper left-hand corner of the Ferrers graph of n.*

Theorem 1.2.23. *The generating function for $Q_m(n)$ is given by*

$$\sum_{n=0}^{\infty} Q_m(n)q^n = \frac{q^{m(m+1)/2}}{(q;q)_m}. \tag{1.2.19}$$

Proof. Take a partition of n into exactly m distinct parts. We see that the largest Durfee triangle has $\frac{1}{2}m(m+1)$ nodes. We will have a partition of $n - \frac{1}{2}m(m+1)$ into no more than m parts to the right of this Durfee triangle. There are no further restrictions, and so the generating function for all those partitions lying to the right of the Durfee triangle is $1/(q;q)_m$. Thus, for a fixed m, the generating function for such partitions of n with Durfee triangle of side m is equal to

$$\frac{q^{m(m+1)/2}}{(q;q)_m}.$$

If we sum all of these expressions over $0 \le m < \infty$, we obtain (1.2.19), thus completing the proof. □

Our next theorem is an analogue of Theorem 1.2.20 and a refinement of Theorem 1.2.23. Recall that $Q_m(n)$ denotes the number of partitions of n into precisely m distinct parts.

Definition 1.2.24. *The length of a partition $\ell(\pi)$ is the number of parts of π.*

Theorem 1.2.25. *The generating function for $Q_m(n)$ is given by*

$$\begin{aligned}\sum_{m,n=0}^{\infty} Q_m(n)z^m q^n &= \prod_{n=1}^{\infty}(1+zq^n)\\ &= \sum_{s=0}^{\infty} z^s q^{s(3s+1)/2}(zq^{2s+1}+1)\frac{(-zq;q)_s}{(q;q)_s}.\end{aligned} \tag{1.2.20}$$

Proof. Suppose that π is a partition of n with distinct parts and Durfee square of side s. We consider two cases: 1) The lower edge of the Durfee square constitutes a complete part of n. 2) The lower edge of the Durfee square does not constitute a complete part of n. We regard s as fixed.

Case 1. Let π_1 denote the partition below the Durfee square, where we read the parts from top to bottom. Note that π_1 is a partition of n, say, with w, say, distinct parts, each of which is $\le s-1$. Let π_2 denote the partition to the right of the Durfee square, where we read from top to bottom. Let us say that π_2 is a partition of m, say, with exactly $s-1$ distinct parts. We see that $\ell(\pi) = s + \ell(\pi_1)$. The generating function for these partitions is, for s fixed,

$$\begin{aligned}
&\sum_{N=0}^{\infty}\sum_{M=0}^{\infty} z^M q^N \sum_{s^2+m+n=N} Q_{s-1}(m) \sum_{s+w=M} Q(\{1,2,\dots,s-1\},w,n)\\
&= z^s q^{s^2}\sum_{n=0}^{\infty} Q(\{1,2,\dots,s-1\},w,n) z^w q^n \sum_{m=0}^{\infty} Q_{s-1}(m) q^m\\
&= z^s q^{s^2}(-zq;q)_{s-1}\frac{q^{s(s-1)/2}}{(q;q)_{s-1}}, \qquad (1.2.21)
\end{aligned}$$

by Theorems 1.2.15 and 1.2.23.

Case 2. As in Case 1, π_1 is the partition represented below the Durfee square of side s, where we read from top to bottom. The partition π_2 is located to the right of the Durfee square, and we again read the partition from top to bottom. The parts are distinct, and there are exactly s of them. The generating function for these partitions is

$$\begin{aligned}
&\sum_{N=0}^{\infty}\sum_{M=0}^{\infty} z^M q^N \sum_{s^2+m+n=N} Q_{s}(m) \sum_{s+w=M} Q(\{1,2,\dots,s\},w,n)\\
&= z^s q^{s^2}(-zq;q)_{s}\frac{q^{s(s+1)/2}}{(q;q)_{s}}, \qquad (1.2.22)
\end{aligned}$$

by Theorems 1.2.15 and 1.2.23 once again. We now sum over s, $1 \le s < \infty$, in (1.2.21), and sum over s, $0 \le s < \infty$, in (1.2.22). Hence,

$$\begin{aligned}
(-zq;q)_\infty &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Q(\mathbb{N},m,n) z^m q^n\\
&= \sum_{s=1}^{\infty} z^s q^{s^2}(-zq;q)_{s-1}\frac{q^{s(s-1)/2}}{(q;q)_{s-1}}\\
&\quad + \sum_{s=0}^{\infty} z^s q^{s^2}(-zq;q)_{s}\frac{q^{s(s+1)/2}}{(q;q)_{s}}\\
&= \sum_{s=0}^{\infty} z^{s+1} q^{s^2+2s+1}(-zq;q)_{s}\frac{q^{s(s+1)/2}}{(q;q)_{s}}\\
&\quad + \sum_{s=0}^{\infty} z^s q^{s^2+s(s+1)/2}\frac{(-zq;q)_s}{(q;q)_{s}}\\
&= \sum_{s=0}^{\infty} z^s q^{s(3s+1)/2}(zq^{2s+1}+1)\frac{(-zq;q)_s}{(q;q)_{s}},
\end{aligned}$$

which is (1.2.20). □

Corollary 1.2.26 (Euler's Pentagonal Number Theorem)**.** *For* $|q| < 1$*,*

$$(q;q)_\infty = \sum_{s=-\infty}^{\infty} (-1)^s q^{s(3s-1)/2}. \qquad (1.2.23)$$

Proof. Set $z = -1$ in (1.2.20) and replace s by $-s-1$ in the second series in the second step below. Therefore,

$$\begin{aligned}(q;q)_\infty &= \sum_{s=0}^{\infty}(-1)^s q^{s(3s+1)/2}(1-q^{2s+1})\\ &= \sum_{s=0}^{\infty}(-1)^s q^{s(3s+1)/2} - \sum_{s=0}^{\infty}(-1)^s q^{s(3s+1)/2+2s+1}\\ &= \sum_{s=0}^{\infty}(-1)^s q^{s(3s+1)/2} - \sum_{s=-1}^{-\infty}(-1)^s q^{s(3s+1)/2}\\ &= \sum_{s=-\infty}^{\infty}(-1)^s q^{s(3s+1)/2}\\ &= \sum_{s=-\infty}^{\infty}(-1)^s q^{s(3s+1)/2}, \qquad (1.2.24)\end{aligned}$$

where we replaced s by $-s$. □

Definition 1.2.27. *The numbers $s(3s-1)/2$, $1 \leq s < \infty$, are called the* pentagonal numbers, *while the numbers $s(3s-1)/2$, $-\infty < s < \infty$, are called the* generalized pentagonal numbers.

Observe that (1.2.23) tells us that the cardinality of the set of partitions of n with an even number of distinct parts is usually equal to the cardinality of the set of partitions of n with an odd number of distinct parts. We make this observation more precise.

Corollary 1.2.28. *Let $D_e(n)$ denote the number of partitions of n into an even number of distinct parts, and let $D_o(n)$ denote the number of partitions of n into an odd number of distinct parts. Then*

$$D_e(n) - D_o(n) = \begin{cases} 0, & \text{if } n \text{ is not a generalized pentagonal number},\\ (-1)^s & \text{if } n \text{ is a generalized pentagonal number } \tfrac{1}{2}s(3s-1). \end{cases} \qquad (1.2.25)$$

Corollary 1.2.26 leads to a method for numerically calculating values of $p(n)$, as we shall see in the next theorem. This recurrence relation was used by P. A. MacMahon to calculate $p(n)$, $1 \leq n \leq 200$, and by H. Gupta to determine $p(n)$ for $201 \leq n \leq 300$. If one wants to calculate a certain value of $p(n)$ on a computer, then the computer will likely employ Theorem 1.2.29 to do so.

Theorem 1.2.29. *For each positive integer N,*

$$p(N) = \sum_{\substack{k(3k\pm1)/2+n=N\\ 0\leq n<N}} (-1)^{k-1} q^{k(3k\pm1)/2} p(n). \qquad (1.2.26)$$

Proof. From the generating function (1.2.1) for $p(n)$ and the pentagonal number theorem, Corollary 1.2.23, we see that

$$1 = \frac{(q;q)_\infty}{(q;q)_\infty} = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} \sum_{n=0}^{\infty} p(n) q^n. \tag{1.2.27}$$

Equating coefficients of q^N, $N > 0$, in (1.2.27), we find that

$$0 = \sum_{k(3k\pm1)/2+n=N} (-1)^k q^{k(3k\pm1)/2} p(n). \tag{1.2.28}$$

Solving (1.2.28) for $p(N)$, we complete the proof of (1.2.26). □

Instead of using $(q;q)_\infty$, we can take its logarithmic derivative to derive a recurrence formula for $\sigma(n) := \sum_{d|n} d$, as we now demonstrate.

Theorem 1.2.30. *If* $n \in \mathbb{N}$, *then*

$$\sum_{\substack{k(3k\pm1)/2+j=n \\ 1\le j\le n}} (-1)^k \sigma\left(n - \frac{k(3k-1)}{2}\right) = \begin{cases} (-1)^k n, & \text{if } n = \frac{1}{2}k(3k-1), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $F(q) = (q;q)_\infty$. We first observe that

$$q\frac{F'(q)}{F(q)} = -\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j} = -\sum_{j=1}^{\infty} j \sum_{k=1}^{\infty} q^{jk} = -\sum_{n=1}^{\infty} \sigma(n) q^n. \tag{1.2.29}$$

By (1.2.23) and (1.2.29),

$$qF'(q) = F(q)\frac{qF'(q)}{F(q)} = -\sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} \sum_{j=1}^{\infty} \sigma(j) q^j. \tag{1.2.30}$$

On the other hand, by (1.2.23),

$$qF'(q) = \sum_{n=-\infty}^{\infty} (-1)^n \frac{n(3n-1)}{2} q^{n(3n-1)/2}. \tag{1.2.31}$$

If we equate coefficients of q^n, $n \ge 1$, in (1.2.30) and (1.2.31), we conclude that

$$\sum_{k=-\infty}^{\infty} (-1)^k \sigma\left(n - \frac{k(3k-1)}{2}\right) = \begin{cases} (-1)^k n, & \text{if } n = \frac{1}{2}k(3k-1), \\ 0, & \text{otherwise,} \end{cases}$$

and this completes the proof. □

The next result falls under the same purview as the two preceding theorems.

Theorem 1.2.31. *For* $n \ge 1$,

$$\sigma(n) = \sum_{k=-\infty}^{\infty} (-1)^{k+1} \frac{k(3k+1)}{2} p\left(n - \frac{k(3k+1)}{2}\right). \tag{1.2.32}$$

Proof. Employing (1.2.29) and (1.2.30), we find that

$$-\sum_{n=1}^{\infty}\sigma(n)q^n = \frac{qF'(q)}{F(q)} = \frac{1}{F(q)}\cdot qF'(q)$$
$$= \frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^k\frac{k(3k+1)}{2}q^{k(3k+1)/2}.$$

Equating coefficients of q^n on both sides above, we complete the proof. □

We now offer a bijective proof of Corollary 1.2.26, or Theorem 1.2.28, which is due to F. Franklin in 1881. *Franklin's Bijection* has been employed numerous times in bijective proofs in the theory of partitions.

Second Proof of Corollaries 1.2.26 and 1.2.28. Consider the Ferrers graph of a partition of n into distinct parts. Identify the Durfee square of side s. To the right of this square, there will be a Durfee triangle with side less than or equal to s. Reading from top to bottom, we have a partition π_1 with distinct parts below the square, with the largest part being not greater than s. Reading from left to right, we see that we have a partition π_2 in which each part is no larger than s. We consider two cases: 1) Each part of π_2 is not larger than $s-1$. 2) One part of π_2 is equal to s.

Case 1. If the smallest part in the upper region is smaller than any part in the lower region, transfer that smallest part to the lower region. Otherwise, the smallest part in the lower region is transferred to the upper region. In the case of a tie, we transfer the smallest part in the lower region to the upper region. Note that such a transfer changes the parity of the number of parts in each region. Of course, this procedure breaks down if there are not any parts in both regions. As an example, consider the partition $\pi = 8+4+3+2+1$. We observe that the Durfee square has side 3, and the Durfee triangle has side 2. In the lower region, $\pi_1 = 2+1$; in the upper region, $\pi_2 = 1+1+1$. The smallest part of these two partitions is equal to 1 in each instance. Thus, we transfer the part 1 in the lower region to a part 1 in the upper region. The parities of the partitions in each case changes with the transfer. This establishes a one-to-one correspondence between partitions in those cases when at least one of the partitions π_1, π_2 is non-empty. Hence, for such partitions, $D_e(n) - D_o(n) = 0$. We now need to consider those cases when π_1 and π_2 are both empty. In Case 1, the lower side of the Durfee square is equal to s, and this implies that the Durfee triangle has side equal to $s-1$. Thus, the number of nodes in a partition giving such a Ferrers graph is $s^2 + s(s-1)/2 = s(3s-1)/2$. If s is odd, then there is one more odd part than even part. In such a case, $D_e(n) - D_o(n) = -1 = (-1)^s$. If s is even, then the number of parts with even parity equals the number with odd parity, and so there is an 'extra' partition with an even number of distinct parts.

Case 2. The argument is identical to that in Case 1 when at least one of π_1, π_2 is non-empty. Thus, we obtain equal numbers of partitions with an odd number of distinct parts, and with an even number of distinct parts. There are cases, however, when both π_1 and π_2 are empty. In such cases, the side of the Durfee triangle is equal to s. Thus, if s is even, we conclude that $D_e(n) - D_o(n) = 1 = (-1)^s$. If s is odd, then we obtain one additional partition with an odd number of parts, and $D_e(n) - D_o(n) = -1 = (-1)^s$. To illustrate this case, consider the partition $\pi = 9+8+6+3+1$. Here, $s = 3$, which is also equal to the side of the Durfee triangle. We thus obtain one additional partition with an odd number of distinct parts.

Thus, Corollaries 1.2.26 and 1.2.28 have been established. □

1.3. A Proof Due to Sun Kim [72]

We conclude our chapter on elementary methods with a generalization of Theorem 1.2.31. We first give a rather straightforward proof of this generalization giving a formula for a certain divisor function as a sum of certain partition functions. We then give an elegant combinatorial proof of this theorem due to Sun Kim [**72**].

Theorem 1.3.1. *Let $p_{n,m}(N)$ equal the number of partitions of N into parts $\equiv n, m,$ or $0 \pmod{L}$, where $m+n=L$. Let*

$$\sigma_{m,n}(k) = \sum_{\substack{d|k \\ d\equiv m,n,0 \pmod{L}}} d.$$

Then

$$-\sigma_{n,m}(N) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{k^2(m+n)+k(m-n)}{2} p_{m,n}\left(N - \frac{k^2(m+n)+k(m-n)}{2}\right). \tag{1.3.1}$$

First Proof of Theorem 1.3.1. If we logarithmically differentiate the Jacobi triple product representation for $f(-q^m, q^n)$, given in (1.1.7), where m and n are arbitrary nonnegative integers, we find that

$$\begin{aligned}
q\frac{\frac{d}{dq}f(-q^m,-q^n)}{f(-q^m,-q^n)} &= -\sum_{k=1}^{\infty}\frac{(km+n)q^{km+n}}{1-q^{km+n}} \\
&\quad -\sum_{k=1}^{\infty}\frac{(m+kn)q^{m+kn}}{1-q^{m+kn}} - \sum_{k=1}^{\infty}\frac{k(m+n)q^{k(m+n)}}{1-q^{k(m+n)}} \\
&= -\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}(km+n)q^{(km+n)(j+1)} \\
&\quad -\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}(m+kn)q^{(m+kn)(j+1)} - \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}k(m+n)q^{k(m+n)(j+1)} \\
&= -\sum_{k=1}^{\infty}\sum_{\substack{d|k \\ d\equiv n,m,0 \pmod{L}}} dq^k \\
&= -\sum_{k=1}^{\infty}\sigma_{n,m}(k)q^k.
\end{aligned}$$

On the other hand,

$$q\frac{\frac{d}{dq}f(-q^m,-q^n)}{f(-q^m,-q^n)} = q\frac{q}{dq}\sum_{k=-\infty}^{\infty}(-1)^k q^{k^2(m+n)/2+k(m-n)/2}\sum_{j=0}^{\infty}p_{m,n}(j)q^j$$

$$= \sum_{k=-\infty}^{\infty}(-1)^k\frac{k^2(m+n)+k(m-n)}{2}q^{k^2(m+n)/2+k(m-n)/2}\sum_{j=0}^{\infty}p_{m,n}(j)q^j$$

$$= \sum_{N=0}^{\infty}\sum_{k=-\infty}^{\infty}(-1)^k\frac{k^2(m+n)+k(m-n)}{2}p_{m,n}\left(N-\frac{k^2(m+n)+k(m-n)}{2}\right)q^N.$$

If we equate the coefficients of q^N on the far right-hand sides of the two strings of equalities above, we immediately deduce (1.3.1) ☐

We now give a beautiful combinatorial proof of (1.3.1) that is due to Sun Kim [**72**].

Second Proof of Theorem 1.3.1. We begin with some definitions.

Definition 1.3.2. *Let $D_{m,n}(N)$ denote the set of partitions $d_{m,n}(N)$ of N into distinct parts that are $\equiv m, n, 0 \pmod{L}$. Let $P_{m,n}(N)$ denote the set of unrestricted partitions $p_{m,n}(N)$ of N into parts that are $\equiv m, n, 0 \pmod{L}$. Let*

$$A_{m,n}(N) = \{(\pi,\lambda) : |\pi| + |\lambda| = N,\ \pi \in D_{m,n}, \lambda \in P_{m,n}\}.$$

Lastly, let $\ell(\pi)$ denote the number of parts of the partition π; we similarly define $\ell(\lambda)$.

Let

$$B_{m,n}(N) := \sum_{(\pi,\lambda)\in A_{m,n}(N)}(-1)^{\ell(\pi)}|\pi|.$$

Our task is to show that $B_{m,n}(N)$ is equal to both the left and right sides of (1.3.1).

We first attempt to reach the right-hand side of (1.3.1). For brevity, set $b = |\lambda|$, so that $|\pi| = N - b$. Thus, we can write

$$B_{m,n}(N) = \sum_{b=0}^{N}(N-b)p_{m,n}(b)\sum_{\pi\in D_{m,n}(N-b)}(-1)^{\ell(\pi)}. \tag{1.3.2}$$

First, by the Jacobi triple product identity (1.1.7) and the definition of $p_{m,n}(N)$,

$$\frac{1}{f(-q^m,-q^n)} = \frac{1}{(q^m;q^L)_\infty(q^n;q^L)_\infty(q^L;q^L)_\infty} = \sum_{N=0}^{\infty}p_{m,n}(N)q^N.$$

Second, by the definition of $f(-q^m,-q^n)$, the Jacobi triple product identity (1.1.7), and the definition of $d_{m,n}(N)$,

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{(m+n)k^2/2+(m-n)k/2} = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(k+1)m/2+k(k-1)n/2}$$

$$= f(-q^m,-q^n) = (q^m;q^L)_\infty(q^n;q^L)_\infty(q^L;q^L)_\infty = \sum_{N=0}^{\infty}(-1)^{\ell(\pi)}d_{m,n}(N)q^N.$$

Thus, if

$$\pi \in D_{m,n}\left(\frac{(m+n)k^2+(m-n)k}{2}\right),$$

then

$$b = N - \frac{(m+n)k^2+(m-n)k}{2}.$$

Now remember that π is a partition into parts congruent to m, n, or 0 modulo $L = m+n$, and so the number being partitioned is composed of a linear combination of m's, n's, and L's. Now one can think of $(m+n)k^2/2 + (m-n)k/2$ as $k(k+1)/2$ m's and $k(k-1)/2$ n's for a total of k^2 m and n's. But, generally, the partition also contains some L's, each of which contains one m and one n. Hence, we see that

$$(-1)^k = (-1)^{k^2} = (-1)^{\ell(\pi)}.$$

Hence, substituting into (1.3.2) the values we have just determined for b, $N-b$, and $(-1)^{\ell(\pi)}$, we find that

$$B_{m,n}(N) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{(m+n)k^2+(m-n)k}{2} p_{m,n}\left(N - \frac{(m+n)k^2+(m-n)k}{2}\right). \tag{1.3.3}$$

Demonstrating that $B_{m,n}(N)$ equals the left side of (1.3.1) is more difficult. First, we show that

$$\#\{(\pi,\lambda) \in A_{m,n}(\lambda) : \ell(\pi) \text{ is even}\} = \#\{(\pi,\lambda) \in A_{m,n}(\lambda) : \ell(\pi) \text{ is odd}\} \tag{1.3.4}$$

For any partition π^*, define

$$s(\pi^*) = \text{ smallest part of } \pi^*; \quad s(\pi^*) = \infty, \text{ if } \pi^* = \emptyset.$$

If $s(\pi) \le s(\lambda)$, then move $s(\pi)$ to λ. If π' and λ' are the new partitions thus formed, we observe that $s(\lambda') < s(\pi')$ and that $(-1)^{\ell(\pi)} = -(-1)^{\ell(\pi')}$. If $s(\lambda) < s(\pi)$, then move $s(\lambda)$ to π. If π' and λ' are the newly formed partitions, we see that that $s(\pi') \le s(\lambda')$ and that $(-1)^{\ell(\pi)} = -(-1)^{\ell(\pi')}$. We thus have established an involution between the two sets of partitions with $s(\pi) \le s(\lambda)$ and $s(\lambda) < s(\pi)$, with $\ell(\pi)$ changing by 1 with each movement of a smallest part from one set to the other. This then establishes the equality in (1.3.4).

We now subdivide our partitions π according to their smallest parts a. In the case that $s(\pi) > s(\lambda)$, we can think of this situation as arising from moving the smallest part a from a partition π' into one of size $|\pi| - a$ with sign $(-1)^{\ell(\pi)-1}$. Thus,

$$\begin{aligned} B_{m,n}(N) &:= \sum_{(\pi,\lambda)\in A_{m,n}(N)} (-1)^{\ell(\pi)}|\pi| \\ &= \sum_{s(\pi)\le s(\lambda)} (-1)^{\ell(\pi)}|\pi| + \sum_{s(\pi)>s(\lambda)} (-1)^{\ell(\pi)}|\pi| \\ &= \sum_{a=1}^{N} \sum_{\substack{s(\pi)=a \\ s(\pi)\le s(\lambda)}} (-1)^{\ell(\pi)} \left(|\pi| - (|\pi| - a)\right) \end{aligned}$$

$$= \sum_{a=1}^{N} a \sum_{\substack{s(\pi)=a \\ s(\pi)\le s(\lambda)}} (-1)^{\ell(\pi)}. \tag{1.3.5}$$

In view of (1.3.1) and (1.3.5), it now suffices to show that

$$\sum_{\substack{s(\pi)=a \\ s(\pi)\le s(\lambda)}} (-1)^{\ell(\pi)} = \begin{cases} -1, & \text{if } a|N, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3.6}$$

Let $L(\pi^*)$ denote the largest part of a partition π^*, with the convention that $L(\pi^*) = 0$, if $\pi^* = \emptyset$. Consider $(\pi, \lambda) \in A_{m,n}(N)$, with $s(\pi) = a \le s(\lambda)$. Let $\pi = a + \mu$. If $L(\mu) \ge L(\lambda)$ and $\mu \ne \emptyset$, then move $L(\mu)$ to λ. If $L(\mu) < L(\lambda)$, except when $L(\mu) = 0 < L(\lambda) = a$, then move $L(\lambda)$ to μ. We thus obtain a new partition pair (π', λ') with

$$s(\pi') = a \le s(\lambda'), \quad (-1)^{\ell(\pi)} = -(-1)^{\ell(\pi')}.$$

However, this map fails in two cases. First, suppose that $\pi = a$, i.e., $\mu = \emptyset$, *and* $\lambda = \emptyset$. Then there is nothing to move. Second, suppose that $\pi = a$ and $\lambda = a + a + \cdots + a$. Again, in this case, there is nothing to move, for if we did move a from λ, we would not obtain a new partition π' with distinct parts, i.e., we would have two parts equaling a. Thus, in these exceptional cases, $a|N$. Thus,

$$\sum_{\substack{s(\pi)=a \\ s(\pi)\le s(\lambda)}} (-1)^{\ell(\pi)} = \begin{cases} -1, & \text{if } a|N, \\ 0, & \text{otherwise,} \end{cases}$$

i.e., (1.3.6) has been demonstrated. Hence,

$$B_{m,n}(N) := \sum_{(\pi,\lambda)\in A_{m,n}(N)} (-1)^{\ell(\pi)}|\pi| = -\sum_{\substack{a|N \\ a\equiv m,n,L \pmod{L}}} = -\sigma_{m,n}(N). \tag{1.3.7}$$

Hence, (1.3.3) and (1.3.7) taken together yield (1.3.1) to complete the proof. □

1.4. Exercises

1. Prove that the number of partitions of n into parts $\equiv 1, 2 \pmod{3}$ is equal to $p^{(2)}(n)$.

2. Prove that

$$\begin{aligned} &p(n|\text{even (odd) number of odd parts}) \\ &\quad = p(n|\text{distinct parts, with an even (odd) number of odd parts}) \end{aligned} \tag{1.4.1}$$

Example 1.4.1. *Let* $n = 4$. *Those partitions with an even number of odd parts are:* $3+1, 1+1+1+1$. *The partitions of* 4 *with an even number of distinct odd parts are:* $4, 3+1$.

3. Let $k \geq 2$ and $n \geq 1$. Prove that

$$\begin{aligned} &p(n|\text{no part is divisible by } k) \\ &= p(n|\text{there are less than } k \text{ copies of each part}). \end{aligned}$$

Observe that if $k = 2$ in Exercise 3, then its conclusion gives $p(\mathbb{O}, n) = Q(n)$, i.e., we obtain Theorem 1.2.14.

4. Prove Lemma 1.1.1.

Chapter 2

MacMahon's Partition Analysis; Gaussian Binomial Coefficients

2.1. MacMahon's Partition Analysis

In this chapter, we examine an elementary, but powerful, method due to P. A. MacMahon in the study of partitions. (The initials P. A. stand for Percy Alexander.) In his time, he was known as Major MacMahon, because he served as a Major in the British army for 20 years before he became a mathematician.

Recall that in Chapter 1, $p(m, n)$ designates the number of partitions with exactly m parts, while $p_m(n)$ denoted the number of partitions of n with largest part m. However, in Theorem 1.2.6 , we showed that $p(m, n) = p_m(n)$. Since these two numbers are the same, we will confine ourselves to the former notation, leaving the notation $p_m(n)$ free for another use.

Definition 2.1.1. *The number of partitions of n into no more than m parts is to be denoted by $p_m(n)$ in the sequel.*

We put the m parts of a partition π in decreasing order, say $n_1 \geq n_2 \geq \cdots \geq n_m \geq 0$. Note that we allow some of the parts to be equal to 0. Thus, we can write the generating function for $p_m(n)$ in the form

$$\sum_{n=0}^{\infty} p_m(n) q^n = \sum_{n_1 \geq n_2 \geq \cdots \geq n_m \geq 0} q^{n_1+n_2+\cdots+n_m}.$$

Consider the sum

$$\sum_{n_1, n_2, \ldots, n_m \geq 0} q^{n_1+n_2+\cdots+n_m} \lambda_1^{n_1-n_2} \lambda_2^{n_2-n_3} \cdots \lambda_{m-1}^{n_{m-1}-n_m}.$$

To consider only those terms with nonnegative powers of λ_j, $1 \le j \le m-1$, we introduce the restrictions $n_1 \ge n_2,\ n_2 \ge n_3,\ \dots n_{m-1} \ge n_m$. We now define an operator that performs exactly this task.

Definition 2.1.2. *Define an operator* $\underset{\ge}{\Omega}$ *on multiple Laurent series so that it annihilates terms with negative exponents and sets any remaining* λ_j*'s equal to* 1*.*

Note that

$$\begin{aligned}\sum_{n=0}^{\infty} p_m(n)q^n &= \underset{\ge}{\Omega} \sum_{n_1,n_2,\dots,n_m\ge 0} q^{n_1+n_2+\cdots+n_m}\lambda_1^{n_1-n_2}\lambda_2^{n_2-n_3}\cdots\lambda_{m-1}^{n_{m-1}-n_m}\\ &= \underset{\ge}{\Omega} \sum_{n_1=0}^{\infty}(q\lambda_1)^{n_1}\sum_{n_2=0}^{\infty}(q\lambda_2/\lambda_1)^{n_2}\cdots\sum_{n_{m-1}=0}^{\infty}(q\lambda_{m-1}/\lambda_{m-2})^{n_{m-1}}\sum_{n_m=0}^{\infty}(q/\lambda_{m-1})^{n_m}\\ &= \underset{\ge}{\Omega}\frac{1}{(1-q\lambda_1)(1-q\lambda_2/\lambda_1)\cdots(1-q\lambda_{m-1}/\lambda_{m-2})(1-q/\lambda_{m-1})}. \qquad (2.1.1)\end{aligned}$$

We now explicitly calculate the effect of the operator $\underset{\ge}{\Omega}$ on special cases of the identity above.

Lemma 2.1.3.

$$\underset{\ge}{\Omega}\frac{1}{(1-\lambda x)(1-y/\lambda)} = \frac{1}{(1-x)(1-xy)}. \qquad (2.1.2)$$

Proof. Setting $k = n - m$ below, we find that

$$\begin{aligned}\underset{\ge}{\Omega}\frac{1}{(1-\lambda x)(1-y/\lambda)} &= \underset{\ge}{\Omega}\sum_{n=0}^{\infty}(\lambda x)^n\sum_{m=0}^{\infty}(y/\lambda)^m = \sum_{n\ge m\ge 0} x^n y^m\\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} x^{m+k}y^m = \sum_{k=0}^{\infty} x^k \sum_{m=0}^{\infty}(xy)^m\\ &= \frac{1}{(1-x)(1-xy)}.\end{aligned}$$

□

The following lemma is (1.2.2), for which we wrote, "It is clear that ...". It is also a special case of Theorem 1.2.15, equation (1.2.10). To illustrate MacMahon's partition analysis, we will prove it again here.

Lemma 2.1.4. *Recall that* $p_m(n)$ *is defined in Definition* 2.1.1*. Then*

$$\sum_{n=0}^{\infty} p_m(n)q^n = \frac{1}{(q;q)_m}.$$

Proof. We apply Lemma 2.1.3 several times. In the first instance, we replace x by q, λ by λ_1, and y by $\lambda_2 q$; in the second application, we replace x by q^2, λ by λ_2, and y by

$q\lambda_3$, etc. Hence, by (2.1.1) and Lemma 2.1.3,

$$\begin{aligned}\sum_{n=0}^{\infty} p_m(n)q^n &= \underset{\geq}{\Omega} \frac{1}{(1-q\lambda_1)(1-q\lambda_2/\lambda_1)\cdots(1-q\lambda_{m-1}/\lambda_{m-2})(1-q/\lambda_{m-1})} \\ &= \underset{\geq}{\Omega} \frac{1}{(1-q)(1-\lambda_2 q^2)(1-q\lambda_3/\lambda_2)\cdots(1-q\lambda_{m-1}/\lambda_{m-2})(1-q/\lambda_{m-1})} \\ &= \underset{\geq}{\Omega} \frac{1}{(1-q)(1-q^2)(1-q^3\lambda_3)\cdots(1-q\lambda_{m-1}/\lambda_{m-2})(1-q/\lambda_{m-1})} \\ &= \cdots \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)},\end{aligned}$$

and so the proof of Lemma 2.1.4 has been completed. □

Corollary 2.1.5. *The partitions of* n *into no more than* m *parts are equinumerous with the partitions of* n *into parts* $\leq m$.

Lemma 2.1.6. *If* α *is a nonnegative integer, then*

$$\underset{\geq}{\Omega} \frac{\lambda^{-\alpha}}{(1-\lambda x)(1-y/\lambda)} = \frac{x^{\alpha}}{(1-x)(1-xy)}.$$

Proof. Putting $k = n - m - \alpha$ below, we find that

$$\begin{aligned}\underset{\geq}{\Omega} \frac{\lambda^{-\alpha}}{(1-\lambda x)(1-y/\lambda)} &= \underset{\geq}{\Omega} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x^n y^m \lambda^{n-m-\alpha} \\ &= \sum_{m=0}^{\infty} \sum_{n\geq m+\alpha} x^n y^m \\ &= \sum_{k=0}^{\infty} x^{m+k+\alpha} \sum_{m=0}^{\infty} y^m.\end{aligned}$$

Summing the geometric series on the right-hand side above, we complete the proof. □

Definition 2.1.7. *Let* $\Delta(n)$ *denote the number of incongruent triangles with positive integral sides and perimeter* n.

Theorem 2.1.8. *A generating function for* $\Delta(n)$ *is given by*

$$\sum_{n=0}^{\infty} \Delta(n)q^n = \frac{q^3}{(1-q^2)(1-q^3)(1-q^4)}. \tag{2.1.3}$$

Proof. Let $n_1 \geq n_2 \geq n_3$ denote the sides of the triangle. Note that $n_2 + n_3 \geq n_1 + 1$. We make three applications of Lemma 2.1.6. In the first instance, replace λ by λ_1, x by q/λ_3, and y by $q\lambda_2\lambda_3$. In the second, replace λ by λ_2, x by q^2, and y by $q\lambda_3$. In the third application, replace λ by λ_3, x by q^3, y by q, and α by 1. Hence,

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta(n)q^n &= \underset{\geq}{\Omega}\sum_{n_1,n_2,n_3=0}^{\infty} q^{n_1+n_2+n_3}\lambda_1^{n_1-n_2}\lambda_2^{n_2-n_3}\lambda_3^{n_2+n_3-n_1-1}\\
&= \underset{\geq}{\Omega}\frac{\lambda_3^{-1}}{(1-q\lambda_1/\lambda_3)(1-q\lambda_2\lambda_3/\lambda_1)(1-q\lambda_3/\lambda_2)}\\
&= \underset{\geq}{\Omega}\frac{\lambda_3^{-1}}{(1-q/\lambda_3)(1-q^2\lambda_2)(1-q\lambda_3/\lambda_2)}\\
&= \underset{\geq}{\Omega}\frac{\lambda_3^{-1}}{(1-q/\lambda_3)(1-q^2)(1-q^3\lambda_3)}\\
&= \frac{q^3}{(1-q^4)(1-q^2)(1-q^3)}.\end{aligned}$$

□

Corollary 2.1.9. *$\Delta(n)$ is equal to the number of partitions of n into 2's, 3's, and 4's, with at least one 3.*

Theorem 2.1.10. *We have*

$$\underset{\geq}{\Omega}\frac{1}{(1-\lambda x)(1-y_1/\lambda)(1-y_2/\lambda)\cdots(1-y_j/\lambda)} = \frac{1}{(1-x)(1-xy_1)\cdots(1-xy_j)}, \tag{2.1.4}$$

$$\underset{\geq}{\Omega}\frac{1}{(1-\lambda x)(1-\lambda y)(1-z/\lambda)} = \frac{1-xyz}{(1-x)(1-y)(1-xz)(1-yz)}, \tag{2.1.5}$$

$$\underset{\geq}{\Omega}\frac{1}{(1-\lambda x)(1-\lambda y)(1-z/\lambda^2)} = \frac{1+xyz-x^2yz-xy^2z}{(1-x)(1-y)(1-x^2z)(1-y^2z)}. \tag{2.1.6}$$

Proof of (2.1.4). We first observe that the case $j=1$ of (2.1.4) is identical with Lemma 2.1.3. Thus, we shall proceed by induction on j. Suppose that (2.1.4) is true up to and including $j-1$. Next, we check that

$$\frac{1}{(1-y_{j-1}/\lambda)(1-y_j/\lambda)} = \frac{1}{y_{j-1}-y_j}\left(\frac{y_{j-1}}{1-y_{j-1}/\lambda}-\frac{y_j}{1-y_j/\lambda}\right). \tag{2.1.7}$$

Hence, using the foregoing identity and induction, we find that

$$\begin{aligned}&\underset{\geq}{\Omega}\frac{1}{(1-\lambda x)(1-y_1/\lambda)(1-y_2/\lambda)\cdots(1-y_j/\lambda)}\\
&= \frac{1}{y_{j-1}-y_j}\underset{\geq}{\Omega}\left(\frac{y_{j-1}}{(1-\lambda x)(1-y_1/\lambda)\cdots(1-y_{j-2}/\lambda)(1-y_{j-1}/\lambda)}\right.\\
&\quad\left.-\frac{y_j}{(1-\lambda x)(1-y_1/\lambda)\cdots(1-y_{j-2}/\lambda)(1-y_j/\lambda)}\right)\\
&= \frac{1}{y_{j-1}-y_j}\underset{\geq}{\Omega}\left(\frac{y_{j-1}}{(1-x)(1-xy_1)\cdots(1-xy_{j-2})(1-xy_{j-1})}\right.\end{aligned}$$

$$-\frac{y_j}{(1-x)(1-xy_1)\cdots(1-xy_{j-2})(1-xy_j)}\Bigg)$$
$$=\frac{1}{(1-x)(1-xy_1)\cdots(1-xy_j)},$$

and so the proof of (2.1.4) is complete. □

Second Proof of (2.1.5). Using (2.1.7) and Lemma 2.1.3, we find that

$$\begin{aligned}
&\underset{\geq}{\Omega}\frac{1}{(1-\lambda x)(1-\lambda y)(1-z/\lambda)}\\
&=\frac{1}{x-y}\left(\frac{x}{1-x\lambda}-\frac{y}{1-y\lambda}\right)\frac{1}{1-z/\lambda}\\
&=\underset{\geq}{\Omega}\frac{x}{(x-y)(1-x)(1-xz)}-\frac{y}{(x-y)(1-y)(1-yz)}\\
&=\frac{x(1-y)(1-yz)-y(1-x)(1-xz)}{(x-y)(1-x)(1-y)(1-xz)(1-yz)}\\
&=\frac{x+xy^2z-y-x^2yz}{(x-y)(1-x)(1-y)(1-xz)(1-yz)}\\
&=\frac{(x-y)(1-xyz)}{(x-y)(1-x)(1-y)(1-xz)(1-yz)}.
\end{aligned}$$

□

We leave the proof of (2.1.6) as an exercise.

Recall that Theorem 1.2.23 provides a generating function for $Q_m(n)$. We shall use MacMahon's partition analysis to give an alternative proof of Theorem 1.2.23.

Second Proof of Theorem 1.2.23 Let $n = n_1 + n_2 + \cdots + n_m$, and suppose that $n_j \geq n_{j+1}+1$, $1 \leq j \leq m-1$. Also assume that $n_m \geq 1$. Then, with several applications of Lemma 2.1.6 with $\alpha = 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_m(n)q^n &= \underset{\geq}{\Omega}\sum_{n_1,n_2,\ldots,n_m=0}^{\infty} q^{n_1+n_2+\cdots+n_m}\lambda_1^{n_1-n_2-1}\lambda_2^{n_2-n_3-1}\cdots\lambda_{m-1}^{n_{m-1}-n_m-1}\lambda_m^{n_m-1}\\
&=\underset{\geq}{\Omega}\frac{\lambda_1^{-1}\lambda_2^{-1}\cdots\lambda_m^{-1}}{(1-\lambda_1 q)(1-\lambda_2 q/\lambda_1)\cdots(1-\lambda_m q/\lambda_{m-1})}\\
&=\underset{\geq}{\Omega}\frac{q\lambda_2^{-1}\cdots\lambda_m^{-1}}{(1-q)(1-\lambda_2 q^2)(1-\lambda_3 q/\lambda_2)\cdots(1-\lambda_m q/\lambda_{m-1})}\\
&=\underset{\geq}{\Omega}\frac{q\cdot q^2\lambda_3^{-1}\cdots\lambda_m^{-1}}{(1-q)(1-q^2)(1-q^3\lambda_3)(1-\lambda_4 q/\lambda_3)\cdots(1-\lambda_m q/\lambda_{m-1})}\\
&=\cdots\underset{\geq}{\Omega}\frac{q\cdot q^2\cdot q^3\cdots q^{m-1}\lambda_m^{-1}}{(1-q)(1-q^2)\cdots(1-q^{m-1})(1-q^m\lambda_m)}\\
&=\frac{q\cdot q^2\cdot q^3\cdots q^{m-1}\cdot q^m}{(1-q)(1-q^2)\cdots(1-q^{m-1})(1-q^m)},
\end{aligned}$$

where in our last application of Lemma 2.1.6, $y = 0$. Combining the powers of q in the last line above, we complete our second proof of Theorem 1.2.23. ☐

Definition 2.1.11. *Let $Q_m^{(k,\ell)}(n)$ denote the number of partitions of n into exactly m distinct parts, where each part (in descending order) differs from the next by at least k, and where the smallest part is $\geq \ell$.*

Theorem 2.1.12. *For $Q_m^{(k,\ell)}(n)$ as defined above,*

$$\sum_{n=0}^{\infty} Q_m^{(k,\ell)}(n)q^n = \frac{q^{\ell m + km(m-1)/2}}{(q;q)_m}.$$

Note that when $k = \ell = 1$, Theorem 2.1.12 reduces to Theorem 1.2.23.

Proof. We observe that

$$\begin{aligned}\sum_{n=0}^{\infty} Q_m^{(k,\ell)}(n)q^n &= \underset{\geq}{\Omega} \sum_{n_1,n_2,\ldots,n_m=0}^{\infty} q^{n_1+n_2+\cdots+n_m} \lambda_1^{n_1-n_2-k} \lambda_2^{n_2-n_3-k} \cdots \lambda_{m-1}^{n_{m-1}-n_m-k} \lambda_m^{n_m-\ell} \\ &= \underset{\geq}{\Omega} \frac{\lambda_1^{-k}\lambda_2^{-k}\cdots\lambda_m^{-\ell}}{(1-\lambda_1 q)(1-\lambda_2 q/\lambda_1)\cdots(1-\lambda_m q/\lambda_{m-1})}.\end{aligned}$$

The remainder of the proof follows exactly along the same lines as the proof for Theorem 1.2.23 that we gave above, and so we leave it as an exercise for readers. ☐

Definition 2.1.13. *We let $p_m(j,n)$ denote the number of partitions of n into at most m parts, with the largest part being j. Let $Q_m(j,n)$ equal the number of partitions of n into at most m distinct parts, with the largest part being j.*

Theorem 2.1.14. *We have*

$$\sum_{j,n=0}^{\infty} p_m(j,n)z^j q^n = \frac{1}{(zq;q)_m}.$$

Proof. Note that

$$\begin{aligned}\sum_{j,n=0}^{\infty} p_m(j,n)z^j q^n &= \underset{\geq}{\Omega} \sum_{n_1,n_2,\ldots,n_m=0}^{\infty} z^{n_1} q^{n_1+n_2+\cdots+n_m} \lambda_1^{n_1-n_2} \lambda_2^{n_2-n_3} \cdots \lambda_{m-1}^{n_{m-1}-n_m} \\ &= \underset{\geq}{\Omega} \frac{1}{(1-zq\lambda_1)(1-q\lambda_2/\lambda_1)\cdots(1-q\lambda_{m-1}/\lambda_{m-2})(1-q/\lambda_{m-1})}.\end{aligned}$$

We now proceed as in the proof of Theorem 1.2.23, and because the details are similar, we shall forego giving them. ☐

Theorem 2.1.15. *We have*

$$\sum_{j,n=0}^{\infty} Q_m(j,n)z^j q^n = \frac{z^m q^{m(m+1)/2}}{(zq;q)_m}.$$

Proof. We first observe that we can write the generating function for $Q_m(j,n)$ in the form

$$\sum_{j,n=0}^{\infty} Q_m(j,n) z^j q^n$$
$$= \underset{\geq}{\Omega} \sum_{n_1,n_2,\ldots,n_m=0}^{\infty} z^{n_1} q^{n_1+n_2+\cdots+n_m} \lambda_1^{n_1-n_2-1} \lambda_2^{n_2-n_3-1} \cdots \lambda_{m-1}^{n_{m-1}-n_m-1} \lambda_m^{n_m-1}.$$

The remainder of the argument is similar to that given in our second proof of Theorem 1.2.23, earlier in this chapter, and so we omit it. ☐

2.2. Elementary Partition Identities Involving Gaussian Binomial Coefficients

Definition 2.2.1. *Let $p(N,M,n)$ denote the number of partitions of n into at most M parts, each $\leq N$. Let $Q(N,M,n)$ denote the number of partitions of n into exactly M distinct parts, each $\leq N$.*

Definition 2.2.2. *The Gaussian coefficient, or the q-binomial coefficient, or the q-Gaussian polynomial, is defined by*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, & 0 \leq m \leq n, \\ 0, & \textit{otherwise}. \end{cases}$$

Definition 2.2.3. *Define an operator $[z^j]$ by*

$$[z^j] \sum_{n=0}^{\infty} a_n z^n = a_j.$$

Note that

$$\sum_{j=0}^{N} a_j = \sum_{j=0}^{N} [z^j] \sum_{n=0}^{\infty} a_n z^n = [z^N] \sum_{j=0}^{\infty} z^j \sum_{n=0}^{\infty} a_n z^n$$
$$= [z^N] \frac{\sum_{n=0}^{\infty} a_n z^n}{1-z}. \tag{2.2.1}$$

Theorem 2.2.4. *Recall that $p(N,M,n)$ and $Q(N,M,n)$ are defined in Definition 2.2.1. Then*

$$\sum_{n=0}^{\infty} p(N,M,n) q^n = \begin{bmatrix} M+N \\ M \end{bmatrix}_q, \tag{2.2.2}$$
$$\sum_{n=0}^{\infty} Q(N,M,n) q^n = q^{M(M+1)/2} \begin{bmatrix} N \\ M \end{bmatrix}_q. \tag{2.2.3}$$

We are putting the cart before the horse by offering the following proof, because the primary ingredient in the proof is the q-binomial theorem, Theorem 3.1.2, the initial theorem in Chapter 3.

Proof. We prove (2.2.2). First note that

$$[z^h]\sum_{j,n=0}^{\infty} p_M(j,N)z^jq^n = \sum_{n=0}^{\infty} p_M(j,n)q^n \tag{2.2.4}$$

and

$$\sum_{h=0}^{N} p_M(h,n) = p(N,M,n). \tag{2.2.5}$$

Thus, using (2.2.5), (2.2.4), Theorem 2.1.14, (2.2.1), and finally the q-binomial theorem, Theorem 3.1.2, we arrive at

$$\begin{aligned}
\sum_{n=0}^{\infty} p(N,M,n)q^n &= \sum_{n=0}^{\infty}\sum_{h=0}^{N} p_M(h,n)q^n \\
&= \sum_{h=0}^{N}[z^h]\sum_{j,n=0}^{\infty} p_M(j,N)z^jq^n \\
&= \sum_{h=0}^{N}\sum_{n=0}^{\infty} p_M(j,n)q^n \\
&= \sum_{h=0}^{N}[z^h]\frac{1}{(zq;q)_M} \\
&= [z^N]\frac{1}{(z;q)_{M+1}} \\
&= [z^N]\frac{(zq^{M+1};q)_\infty}{(z;q)_\infty} \\
&= [z^N]\sum_{k=0}^{\infty}\frac{(q^{M+1};q)_k}{(q;q)_k}z^k \\
&= [z^N]\sum_{k=0}^{\infty}\frac{(q^{M+1};q)_k(q;q)_M}{(q;q)_k(q;q)_M}z^k \\
&= [z^N]\sum_{k=0}^{\infty}\frac{(q;q)_{M+k}}{(q;q)_k(q;q)_M}z^k \\
&= [z^N]\sum_{k=0}^{\infty}\begin{bmatrix} M+k \\ M \end{bmatrix}_q z^k \\
&= \begin{bmatrix} M+N \\ M \end{bmatrix}_q,
\end{aligned}$$

and this completes the proof of (2.2.2).

We now turn to the proof of (2.2.3). Utilizing analogues of (2.2.4) and (2.2.5), Theorem 2.1.15, (2.2.1), and finally a corollary of the q-binomial theorem, Theorem 3.1.2, we arrive at

$$\begin{aligned}
\sum_{n=0}^{\infty} Q(N,M,n)q^n &= \sum_{h=0}^{N}[z^h] \sum_{j,n=0}^{\infty} Q_M(j,n)z^j q^n \\
&= \sum_{h=0}^{N}[z^h] \frac{z^M q^{M(M+1)/2}}{(zq;q)_M} \\
&= [z^N] \frac{z^M q^{M(M+1)/2}}{(z;q)_{M+1}} \\
&= [z^N] z^M q^{M(M+1)/2} \sum_{k=0}^{\infty} \begin{bmatrix} M+k \\ M \end{bmatrix}_q z^k \\
&= q^{M(M+1)/2} \begin{bmatrix} N \\ M \end{bmatrix}_q ,
\end{aligned}$$

where we chose the $k = (N-M)$th coefficient of the series. □

Theorem 2.2.5. *We have*

$$\sum_{n,M=0}^{\infty} Q(N,M,n)z^M q^n = (-zq;q)_N. \tag{2.2.6}$$

Proof. Applying (2.2.3), we arrive at

$$\begin{aligned}
\sum_{n,M=0}^{\infty} Q(N,M,n)z^M q^n &= \sum_{M=0}^{\infty} z^M \sum_{n=0}^{\infty} Q(N,M,n)q^n \\
&= \sum_{M=0}^{\infty} z^M q^{M(M+1)/2} \begin{bmatrix} N \\ M \end{bmatrix}_q .
\end{aligned} \tag{2.2.7}$$

We need to put the Gaussian binomial coefficients in a different form. To that end,

$$\begin{aligned}
\begin{bmatrix} N \\ M \end{bmatrix}_q &= \frac{(q)_N}{(q)_M (q)_{N-M}} = \frac{(q^{N-M+1})_M}{(q)_m} \\
&= \frac{(1-q^{N-M+1})(1-q^{N-M+2})\cdots(1-q^N)}{(q)_M} \\
&= (-1)^M q^{NM-M(M-1)/2} \frac{(1-q^{-N+M-1})(1-q^{-N+M-2})\cdots(1-q^{-N})}{(q)_M} \\
&= (-1)^M q^{NM-M(M-1)/2} \frac{(q^{-N})_M}{(q)_M}.
\end{aligned}$$

Employing the calculation above in (2.2.7), we deduce that

$$\sum_{n,M=0}^{\infty} Q(N,M,n)z^M q^n = \sum_{M=0}^{\infty} \frac{(q^{-N})_M}{(q)_M}(-zq^{N+1})^M$$
$$= \frac{(-zq)_\infty}{(-zq^{N+1})_\infty} = (-zq)_N,$$

where we applied the q-binomial theorem, Theorem 3.1.2, in the penultimate equality. □

Generally, if the base q is clear, the subscript q is deleted from the Gaussian coefficient.

2.3. Exercises

1. Find a combinatorial proof of Corollary 2.1.9.

2. Prove (2.1.6).

3. Prove that

$$\sum_{m,n=0}^{\infty} Q_m^{(2,1)}(n)z^m q^n = \sum_{m=0}^{\infty} \frac{z^m q^{m^2}}{(q;q)_m}.$$

4. Prove that

$$\sum_{m,n=0}^{\infty} Q_m^{(2,2)}(n)z^m q^n = \sum_{m=0}^{\infty} \frac{z^m q^{m^2+m}}{(q;q)_m}.$$

When $z = 1$, the two functions on the right-hand sides in Exercises 3,4 are the Rogers–Ramanujan functions.

5. Prove the *two* analogues of Pascal's formula for the ordinary binomial coefficients:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}, \tag{2.3.1}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^{m} \begin{bmatrix} n-1 \\ m \end{bmatrix}. \tag{2.3.2}$$

6. Prove that

$$\lim_{q\to 1} \begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m}.$$

Chapter 3

Some Primary Theorems in the Theory of q-Series

3.1. Introduction

As we remarked in Chapter 1, a q-series, sometimes also dubbed an Eulerian series, generally, but not always, has at least one q-product in its summands. There is one class of q-series, called *basic hypergeometric series*, for which an enormous and beautiful theory has been developed. We now define a ${}_r\phi_s$ basic hypergeometric series.

Definition 3.1.1. *If r and s are nonnegative integers and $|q|<1$, then*

$$
\begin{aligned}
{}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z) &\equiv {}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] \\
&= \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(b_1;q)_n (b_2;q)_n \cdots (b_s;q)_n (q;q)_n} \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} z^n, \quad |z|<1. \qquad (3.1.1)
\end{aligned}
$$

Of course, conditions, depending on r, s, and other parameters, need to be imposed for existence and convergence. Note that when $r = s+1$, the expression involving square brackets is identically equal to 1. In these notes, all of the basic hypergeometric series that appear will be instances when $r = s+1$, in which case the series converges for $|z|<1$. We also have convergence on $|z|=1$ if $\operatorname{Re}(b_1 + \cdots + b_s - (a_1 + \cdots + a_r)) > 0$. For a fuller discussion of convergence, consult Gasper and Rahman's text [**58**, pp. 4–5].

The most useful theorem in q-series is the q-binomial theorem.

Theorem 3.1.2 (q-analogue of the binomial theorem)**.** *For $|q|, |z| < 1$,*

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty}. \qquad (3.1.2)$$

Proof. Note that the product on the right side of (3.1.2) converges uniformly on compact subsets of $|z| < 1$ and so represents an analytic function on $|z| < 1$. Thus, we may write

$$F(z) := \frac{(az)_\infty}{(z)_\infty} = \sum_{n=0}^{\infty} A_n z^n, \qquad |z| < 1. \tag{3.1.3}$$

From the product representation in (3.1.3), we can readily verify that

$$(1 - z)F(z) = (1 - az)F(qz). \tag{3.1.4}$$

Equating coefficients of z^n, $n \geq 1$, on both sides of (3.1.4), we find that

$$A_n - A_{n-1} = q^n A_n - aq^{n-1} A_{n-1},$$

or

$$A_n = \frac{1 - aq^{n-1}}{1 - q^n} A_{n-1}, \qquad n \geq 1. \tag{3.1.5}$$

Iterating (3.1.5) and using the value $A_0 = 1$, which is readily apparent from (3.1.3), we deduce that

$$A_n = \frac{(a)_n}{(q)_n}, \qquad n \geq 0. \tag{3.1.6}$$

Using (3.1.6) in (3.1.3), we complete the proof of (3.1.2). □

Second Proof of Theorem 3.1.2. Define

$$f_a(z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} z^k. \tag{3.1.7}$$

Then

$$\begin{aligned} \frac{f_a(z) - f_a(qz)}{z} &= \sum_{k=0}^{\infty} \left(\frac{(a)_k}{(q)_k} z^{k-1} - \frac{(a)_k}{(q)_k} q^k z^{k-1} \right) \\ &= \sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} (1 - q^k) z^{k-1} \\ &= (1 - a) \sum_{k=1}^{\infty} \frac{(aq)_{k-1}}{(q)_{k-1}} z^{k-1} \\ &= (1 - a) f_{aq}(z). \end{aligned} \tag{3.1.8}$$

Rearrange (3.1.8) to achieve the form

$$f_a(z) - f_a(qz) = (1 - a)x f_{aq}(x). \tag{3.1.9}$$

Next, consider

$$\begin{aligned} f_a(z) - f_{aq}(z) &= \sum_{k=0}^{\infty} \frac{(aq)_{k-1}}{(q)_k} \left\{ (1 - a) - (1 - aq^k) \right\} z^k \\ &= -a \sum_{k=1}^{\infty} \frac{(aq)_{k-1}}{(q)_{k-1}} z^k = -az f_{aq}(z). \end{aligned}$$

In other words,

$$f_a(z) = (1 - az)f_{aq}(z). \tag{3.1.10}$$

From (3.1.9) and (3.1.10),

$$f_a(z) - f_a(qz) = \frac{(1-a)z}{1-az} f_a(z),$$

which we can further write in the form

$$f_a(z) = \frac{1-az}{1-z} f_a(qz). \tag{3.1.11}$$

If we iterate (3.1.11) a total of n times, we find that

$$f_a(z) = \frac{(ax)_n}{(x)_n} f_a(q^n z).$$

Letting $n \to \infty$, we conclude that

$$f_a(z) = \frac{(az)_\infty}{(z)_\infty} f_a(0) = \frac{(az)_\infty}{(z)_\infty}, \tag{3.1.12}$$

because, from (3.1.9), $f_a(0) = 0$. Hence, (3.1.2) has been proved. □

We now state two useful corollaries of Theorem 3.1.2.

Corollary 3.1.3. *For* $|q|, |z| < 1$,

$$\sum_{n=0}^{\infty} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty}. \tag{3.1.13}$$

Proof. Equality (3.1.13) is an immediate consequence of (3.1.2) by setting $a = 0$. □

Corollary 3.1.4. *For* $z \in \mathbb{Z}$,

$$\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{(q)_n} = (z)_\infty. \tag{3.1.14}$$

Proof. Replace a by a/b and z by bz in (3.1.2) to find that, for $|bz| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a/b)_n}{(q)_n} (bz)^n = \frac{(az)_\infty}{(bz)_\infty}. \tag{3.1.15}$$

Now let $b \to 0$. We note that

$$\begin{aligned} \lim_{b\to 0} (a/b)_n b^n &= \lim_{b\to 0} \left(1 - \frac{a}{b}\right)\left(1 - \frac{aq}{b}\right) \cdots \left(1 - \frac{aq^{n-1}}{b}\right) b^n \\ &= (-a)^n q^{n(n-1)/2}. \end{aligned} \tag{3.1.16}$$

With the use of (3.1.16), equality (3.1.14) now follows upon setting $a = 1$. □

In letting $b \to 0$ in the proof above, we have swept the details "under the rug." See [**28**, pp. 9–10] for a complete justification of this procedure.

In Chapter 1, we defined Ramanujan's general theta function $f(a,b)$ and stated the Jacobi triple product identity. For convenience, we restate them again here. Let

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{3.1.17}$$

The most useful theorem in the theory of theta functions is indeed the Jacobi triple product identity [**26**, p. 35, Entry 19], [**28**, Theorem 1.3.3, p. 10].

Theorem 3.1.5. *For each theta function $f(a,b)$,*

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{3.1.18}$$

The first proof that we give is based on the q-binomial theorem and a theorem of Rothe.

Theorem 3.1.6 (Rothe)**.** *For any positive integer N,*

$$\sum_{k=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j x^j q^{j(j-1)/2} = (x;q)_N. \tag{3.1.19}$$

Proof. In the q-binomial theorem, Theorem 3.1.2, replace z by xq^N and let $a = q^{-N}$ to find that

$$(x;q)_N = \frac{(x)_\infty}{(xq^N)_\infty} = \sum_{j=0}^{\infty} \frac{(q^{-N})_j}{(q)_j} x^j q^{Nj}. \tag{3.1.20}$$

We leave it to readers to verify that

$$(q^N)_j = (-1)^j q^{-NJ+j(j-1)/2} \frac{(q)_N}{(q)_{N-j}}. \tag{3.1.21}$$

Putting (3.1.21) in (3.1.20), we deduce that

$$(x;q)_N = \sum_{j=0}^{\infty} \frac{(q)_N}{(q)_j(q)_{N-j}} (-1)^j x^j q^{j(j-1)/2} = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j x^j q^{j(j-1)/2}. \tag{3.1.22}$$

Thus, the proof of (3.1.19) is complete. □

Proof of Theorem 3.1.5. In Theorem 3.1.6, let $N = 2n$ and replace the index k by $k+n$ to see that

$$\sum_{k=-n}^{n} \begin{bmatrix} 2n \\ k+n \end{bmatrix} (-1)^{k+n} q^{(k+n)(k+n-1)/2} x^{k+n} = (x;q)_{2n}. \tag{3.1.23}$$

Replace x by xq^{-n} in (3.1.23) and use a calculation almost identical to that in (3.1.22) to find that

$$(xq^{-n};q)_{2n} = (xq^{-n};q)_n(x;q)_n = (-1)^n q^{-n(n+1)/2} x^n (q/x;q)_n(x;q)_n. \tag{3.1.24}$$

Thus, from (3.1.23), with the use of (3.1.24),

$$\sum_{k=-n}^{n} \frac{(q;q)_{2n}(-1)^{k+n} q^{(k+n)(k+n-1)/2} x^{k+n} q^{-n(k+n)}}{(q;q)_{n+k}(q;q)_{n-k}} = (-1)^n q^{-n(n+1)/2} x^n (q/x;q)_n (x;q)_n,$$

which can be put in the more simplified form

$$\sum_{k=-n}^{n} \frac{(q;q)_{2n}(-1)^k q^{k(k-1)/2} x^k}{(q;q)_{n+k}(q;q)_{n-k}} = (q/x;q)_n (x;q)_n. \tag{3.1.25}$$

Letting $n \to \infty$ in (3.1.25), and using Tannery's Theorem or the Dominated Convergence Theorem to justify taking the limit inside the summation sign, we conclude that

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{k(k-1)/2} x^k}{(q;q)_\infty} = (q/x;q)_\infty (x;q)_\infty, \tag{3.1.26}$$

which is easily seen to be equivalent to (3.1.18). □

We offer three corollaries of Theorem 3.1.5. If we set $x = -q$ in (3.1.26) and replace k by $-k$ for the negative indexed terms on the left-hand side, and use Theorem 1.2.10, we deduce the following corollary.

Corollary 3.1.7.

$$\psi(q) := \sum_{k=0}^{\infty} q^{k(k+1)/2} = (-q;q)_\infty^2 (q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{3.1.27}$$

We have introduced Ramanujan's notation $\psi(q)$ on the left side of (3.1.27). If we set $a = b = q$ in (3.1.18), we obtain the following corollary, where we use Ramanujan's notation $\varphi(q)$.

Corollary 3.1.8. *We have*

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = (q;q^2)_\infty^2 (q^2;q^2)_\infty. \tag{3.1.28}$$

If we replace x by $-xq$ in (3.1.26), we see that

$$\sum_{k=0}^{\infty} q^{k(k+1)/2}(x^k + x^{-k-1}) = (-xq;q)_\infty (-1/x;q)_\infty (q;q)_\infty. \tag{3.1.29}$$

Divide both sides of (3.1.29) by $x+1$, and then let $x \to -1$, with the aid of L'Hospital's rule, we deduce the next corollary.

Corollary 3.1.9 (Jacobi's Identity)**.**

$$\sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)/2} = (q;q)_\infty^3. \tag{3.1.30}$$

The second proof that we give is due to J. J. Sylvester and Hathaway.

Second Proof of Theorem 3.1.5. We shall prove (3.1.18) in the form

$$(-zq;q)_\infty(-1/z;q)_\infty(q;q)_\infty = \sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2}. \tag{3.1.31}$$

Set

$$\phi(z) := (-zq;q)_\infty(-1/z;q)_\infty. \tag{3.1.32}$$

It is easily verified that $\phi(z)$ satisfies the functional equation

$$\phi(zq) = \frac{1}{zq}\phi(z). \tag{3.1.33}$$

From its definition (3.1.32), we easily see that $\phi(z)$ is analytic in a deleted neighborhood of the origin, and so we shall write

$$\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad |z| < 1/|q|. \tag{3.1.34}$$

By (3.1.33),

$$\sum_{n=-\infty}^{\infty} a_n z^n = zq \sum_{n=-\infty}^{\infty} a_n (zq)^n = \sum_{n=-\infty}^{\infty} a_{n-1}(zq)^n, \tag{3.1.35}$$

Equating coefficients of z^n on both sides of (3.1.35), we deduce the recurrence relation

$$a_n = q^n a_{n-1}. \tag{3.1.36}$$

Assume now that $n \geq 1$. By successively iterating (3.1.36), we find that

$$a_n = q^{n(n+1)/2} a_0, \quad n \geq 1. \tag{3.1.37}$$

If $n \leq 0$, successive iterations of (3.1.36) yield

$$a_{n-1} = q^{-n(-n+1)/2} a_0,$$

or, with the replacement of n by $n+1$,

$$a_n = q^{n(n+1)/2} a_0, \quad n \leq -1. \tag{3.1.38}$$

Hence, (3.1.37) and (3.1.38) show that for all integers n

$$a_n = q^{n(n+1)/2} a_0, \quad -\infty < n < \infty. \tag{3.1.39}$$

Hence, we have shown that

$$\phi(z) = a_0 \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} z^n. \tag{3.1.40}$$

There remains the task of computing a_0.

Now return to the definition of $\phi(z)$ given in (3.1.34), and note that the constant term a_0 is of course equal to the constant term in the product of the two infinite products on the left-hand side. The constant term arises from multiplying expressions of the type zq^{a_j} by expressions of the type $z^{-1}q^{b_k}$. A typical term will be of the form

$$(zq^{a_1})(zq^{a_2})\cdots(zq^{a_r})(z^{-1}q^{b_1})(z^{-1}q^{b_2})\cdots(z^{-1}q^{b_r}), \tag{3.1.41}$$

where $a_1 > a_2 > \cdots > a_r \geq 0, \quad b_1 > b_2 > \cdots > b_r \geq 0$. To more clearly understand how many of these expressions can arise, we define the Frobenius Symbol.

Definition 3.1.10. *Consider the Ferrers graph of a partition of a positive integer n. Let r be the size of a Durfee square. Form the diagonal of the Durfee square, which will have r nodes. To the right of the diagonal is a graphical representation of a partition of no more than r parts, reading from top to bottom, say $a_1, a_2, \ldots, a_r$. To the left of the diagonal is a graphical representation of another partition of no more than r parts, reading from left to right, namely $b_1, b_2, \ldots, b_r$, say. Thus, $n = r + \sum_{j=1}^{r}(a_j + b_j)$. A matrix representation corresponding to these two partitions can be given by the Frobenius symbol*

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}.$$

As an example, consider the partition $15 = 5 + 4 + 4 + 2$. Here, $r = 3$. To the right of the diagonal is the partition 4+2+1 of 7; below the diagonal is the partition 3+2 of 5. Hence, the Frobenius symbol for our original partition is given by

$$\begin{pmatrix} 4 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix}.$$

We resume our proof of (3.1.18). We see that with each partition of n, there corresponds a unique Frobenius symbol. Conversely, each Frobenius symbol is associated with a unique partition. Moreover, by (3.1.18), (3.1.32), and (3.1.41), each Frobenius symbol corresponds to a unique multiplication of series arising from r terms of the sort $(1 + zq^n)$ and r expressions of the form $(1 + q^{n-1}/z)$. Thus, the constant term a_0 arises from all of the partitions of all nonnegative integers n, i.e.,

$$a_0 = \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{3.1.42}$$

Putting (3.1.42) into (3.1.40) and noting that our goal was to prove (3.1.31), we have completed our proof of Theorem 3.1.5. □

Third Proof of Theorem 3.1.5. We provide a beautiful bijective proof that is due to E. M. Wright [**103**].

We begin by writing (3.1.18), or (3.1.31), in the equivalent form

$$(-zq;q^2)_\infty(-q/z;q^2)_\infty = \frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty} q^{n^2}z^n. \tag{3.1.43}$$

If we furthermore set $x = qz$ and $y = q/z$, so that $q = xy$, then (3.1.43) becomes

$$(-x;xy)_\infty(-y;xy)_\infty = \frac{1}{(xy;xy)_\infty}\sum_{n=-\infty}^{\infty} x^{n(n+1)/2}y^{n(n-1)/2}. \tag{3.1.44}$$

The left-hand side of (3.1.44) is the generating function for $\alpha(n,m)$, the number of bipartite partitions of the forms

$$(a, a-1), (b-1, b), \tag{3.1.45}$$

where a and b are positive integers. Before equating coefficients of $x^n y^m$, $m, n \geq 0$, on both sides of (3.1.44), we slightly rewrite (3.1.44) in the form

$$\sum_{n,m=0}^{\infty} \alpha(n,m) x^n y^m = \sum_{j=0}^{\infty} p(j) x^j y^j \sum_{r=-\infty}^{\infty} x^{r(r+1)/2} y^{r(r-1)/2}. \tag{3.1.46}$$

Note that

$$j + \tfrac{1}{2} r(r+1) = n \quad \text{and} \quad j + \tfrac{1}{2} r(r-1) = m.$$

We want to write j in terms of m and n. In fact, we claim that

$$j = n - \tfrac{1}{2}(n-m)(n-m+1). \tag{3.1.47}$$

Examining the various powers in (3.1.46), we see that for x,

$$n = n - \tfrac{1}{2}(n-m)(n-m+1) + \tfrac{1}{2} r(r+1), \tag{3.1.48}$$

and for y,

$$m = n - \tfrac{1}{2}(n-m)(n-m+1) + \tfrac{1}{2} r(r-1). \tag{3.1.49}$$

From (3.1.48),

$$\tfrac{1}{2} r(r+1) = \tfrac{1}{2}(n-m)(n-m+1), \tag{3.1.50}$$

and from (3.1.49),

$$\begin{aligned} \tfrac{1}{2} r(r-1) &= (m-n) + \tfrac{1}{2}(n-m)(n-m+1) \\ &= (n-m)\tfrac{1}{2}(n-m-1). \end{aligned} \tag{3.1.51}$$

Subtracting (3.1.51) from (3.1.50), we arrive at

$$\begin{aligned} r &= \tfrac{1}{2}(n-m)(n-m+1) = \tfrac{1}{2}(n-m)(n-m-1) \\ &= \tfrac{1}{2}(n-m)\left\{(n-m+1) - (n-m-1)\right\} = n-m. \end{aligned} \tag{3.1.52}$$

We conclude from these calculations that (3.1.47) holds and therefore that

$$\alpha(n,m) = p(n - \tfrac{1}{2}(n-m)(n-m+1)). \tag{3.1.53}$$

Without loss of generality, assume that $n \geq m$. In view of (3.1.52), we can write $r = n - m$. We shall demonstrate that each partition of $(n, n-r)$ into parts of the form given in (3.1.45) corresponds bijectively with the partition

$$n = \sum_{j=1}^{v+r} a_j + \sum_{j=1}^{v} (b_j - 1), \quad 1 \leq a_1 < a_2 < \cdots, \ 1 \leq b_1 < b_2 < \cdots. \tag{3.1.54}$$

Write $k = n - \frac{1}{2} r(r+1)$. Thus, the right-hand side of (3.1.53) is $p(k)$. If $k < 0$, i.e., $n < \frac{1}{2} r(r+1)$, then $p(k) = 0$. Now,

$$n \geq \sum_{j=1}^{r} a_j \geq \sum_{j=1}^{r} j = \tfrac{1}{2} r(r+1) \tag{3.1.55}$$

which contradicts the fact that $n < \frac{1}{2}r(r+1)$. Therefore, there are no solutions to (3.1.54) if $k < 0$. If $k = 0$, then $n = \frac{1}{2}r(r+1)$. Thus, $p(k) = p(0) = 1$, and there exists 1 solution to (3.1.54), namely, $v = 0$, $a_j = j$.

Suppose lastly that $k > 0$. We will accompany our proof with an example: $31 = 9 + 9 + 6 + 4 + 2 + 1$. Above the Ferrers graph of the partition, place a right-angled triangle of r rows, the lowest of which has r nodes. The triangle should be situated such that one side of r nodes is adjacent to the top row of the original Ferrers graph and that the other side of length r is along the left vertical side of the Ferrers graph. In our example, we take $r = 2$. We now have $n = k + \frac{1}{2}r(r+1)$ nodes. Next, draw a diagonal line lying just above the hypotenuse, extending through the Ferrers graph, and dividing the Ferrers graph into two parts. The set of nodes below the diagonal has $r + v$ columns, for some particular v. In our example, $v = 3$. The columns have different numbers of nodes, and so we obtain a partition into distinct parts. In our example, we have $a_1 = 1$, $a_2 = 3$, $a_3 = 4$, $a_4 = 6$, and $a_5 = 8$. To the right of the diagonal, we have a Ferrers graph of v rows. Note that the last row might be empty, but the last two rows cannot both be empty. The partition to the right of the diagonal will have v rows of distinct lengths. We denote the parts of this partition by $b_1 - 1$, $b_2 - 1$, $\dots$, $b_v - 1$, where $1 \le b_1 < b_2 < \cdots$. In our example, $b_1 = 2$, $b_2 = 6$, and $b_3 = 7$. Thus, we compose the partition

$$n = \sum_{j=1}^{v+r} a_j + \sum_{j=1}^{v} (b_j - 1).$$

This process can be reversed. We start with a solution of (3.1.54) and construct a graph, as in our second graph above. We delete $\frac{1}{2}r(r+1)$ rows at the top of the graph, and so obtain a partition of k. The correspondence is one-to-one, and so we have established (3.1.53), which is the arithmetical interpretation of the Jacobi triple product identity. □

3.2. Some Standard Theorems about ${}_2\phi_1(a, b; c; q; z)$

Recall that at the beginning of this chapter, we gave the definition of a general basic hypergeometric series (3.1.1). The theory for the basic hypergeometric function

$${}_2\phi_1(a, b; c; z) := {}_2\phi_1(a, b; c; q, z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n, \quad |z| < 1,$$

is more extensive than those for other pairs of indices, but we emphasize that it cannot be separated from the theories for other values of r and s. E. Heine [**67**] was the first to systematically study ${}_2\phi_1(a, b; c; z)$, although special cases were studied earlier by Cauchy, Euler, Gauss, and others. We now prove a few basic properties that we shall utilize in the sequel. Several theorems in the ${}_2\phi_1$ theory are analogues of fundamental theorems in the theory of the ordinary or Gaussian hypergeometric series ${}_2F_1$.

Theorem 3.2.1 (Heine's Transformation)**.** *For* $|q|, |z|, |b| < 1$*,*

$$ {}_2\phi_1(a, b; c; z) = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} {}_2\phi_1(c/b, z; az; b). \tag{3.2.1} $$

Proof. Using the q-binomial theorem, Theorem (3.1.2), twice, we find that

$$ \begin{aligned} {}_2\phi_1(a, b; c; z) &= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(q)_n} \frac{(cq^n)_\infty}{(bq^n)_\infty} \\ &= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(q)_n} \sum_{m=0}^{\infty} \frac{(c/b)_m (bq^n)^m}{(q)_m} \\ &= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m b^m}{(q)_m} \sum_{n=0}^{\infty} \frac{(a)_n (zq^m)^n}{(q)_n} \\ &= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m b^m}{(q)_m} \frac{(azq^m)_\infty}{(zq^m)_\infty} \\ &= \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m (z)_m}{(q)_m (az)_m} b^m, \end{aligned} $$

which is what we wanted to prove. □

Theorem 3.2.2 (q-analogue of Gauss's Theorem)**.** *If* $|c| < |ab|$ *and* $|q| < 1$*, then*

$$ {}_2\phi_1\left(a, b; c; \frac{c}{ab}\right) = \frac{(c/a)_\infty (c/b)_\infty}{(c/(ab))_\infty (c)_\infty}. \tag{3.2.2} $$

If we set $a = q^{-n}$ in Theorem 3.2.2, we obtain the following famous corollary.

Theorem 3.2.3 (q-analogue of the Chu–VanderMonde Theorem)**.** *For each nonnegative integer* n*,*

$$ {}_2\phi_1\left(q^{-n}, b; c; \frac{cq^n}{b}\right) = \frac{(c/b; q)_n}{(c; q)_n}. \tag{3.2.3} $$

Proof. Applying Heine's transformation followed by the q-analogue of the binomial theorem, we see that

$$ \begin{aligned} {}_2\phi_1\left(a, b; c; \frac{c}{ab}\right) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (q)_n} \left(\frac{c}{ab}\right)^n \\ &= \frac{(b)_\infty (c/b)_\infty}{(c)_\infty (c/(ab))_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_n (c/(ab))_n}{(c/b)_n (q)_n} b^n \\ &= \frac{(b)_\infty (c/b)_\infty}{(c)_\infty (c/(ab))_\infty} \frac{(c/a)_\infty}{(b)_\infty}. \end{aligned} $$

After cancellation, we obtain the desired result. □

Theorem 3.2.4 (Bailey's Theorem)**.** *For* $|q| < \min(1, |b|)$,

$$ {}_2\phi_1(a, b; qa/b; -q/b) = \frac{(aq;q^2)_\infty(-q;q)_\infty(q^2a/b^2;q^2)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty}. \tag{3.2.4} $$

Proof. Applying Heine's transformation with a and b switched and Theorem 3.1.2, we find that

$$ \begin{aligned} {}_2\phi_1(a, b; qa/b; -q/b) &= \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(qa/b;q)_n(q)_n}\left(-\frac{q}{b}\right)^n \\ &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty}\sum_{n=0}^{\infty}\frac{(q/b;q)_n(-q/b;q)_n}{(-q;q)_n(q;q)_n}a^n \\ &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b:q)_\infty}\sum_{n=0}^{\infty}\frac{(q^2/b^2;q^2)_n}{(q^2;q^2)_n}a^n \\ &= \frac{(a;q)_\infty(-q;q)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty}\frac{(q^2a/b^2;q^2)_\infty}{(a;q^2)_\infty} \\ &= \frac{(aq;q^2)_\infty(-q;q)_\infty(q^2a/b^2;q^2)_\infty}{(qa/b;q)_\infty(-q/b;q)_\infty}. \end{aligned} $$

□

Corollary 3.2.5. *We have*

$$ \sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}q^{n(n+1)/2} = (aq;q^2)_\infty(-q;q)_\infty. $$

Proof. Set $b = 1/\beta$ in Theorem 3.2.4 and then let $\beta \to 0$. To that end,

$$ (aq/b)_n \to 1, \quad \frac{(b)_n}{b^n} = (1/\beta)_n\beta^n \to (-1)^n q^{n(n-1)/2}. $$

Then the desired result follows immediately from (3.2.4). □

Theorem 3.2.6 (q-analogue of Euler's Transformation)**.** *For* $|z|, |abz/c| < 1$,

$$ {}_2\phi_1(a, b; c; z) = \frac{(abz/c)_\infty}{(z)_\infty}\,{}_2\phi_1(c/a, c/b; c; abz/c). \tag{3.2.5} $$

Proof. By Heine's transformation (3.2.1),

$$ {}_2\phi_1(a, b; c; z) = \frac{(b)_\infty(az)_\infty}{(c)_\infty(z)_\infty}\,{}_2\phi_1(c/b, z; az; b). $$

With the roles of c/b and z reversed, we apply Heine's transformation a second time to deduce that

$$ {}_2\phi_1(a, b; c; z) = \frac{(b)_\infty(az)_\infty}{(c)_\infty(z)_\infty}\frac{(c/b)_\infty(bz)_\infty}{(az)_\infty(b)_\infty}\,{}_2\phi_1(abz/c, b; bz; c/b). \tag{3.2.6} $$

With the roles of abz/c and b reversed, we apply Heine's transformation a third time to obtain our final identity

$$\begin{aligned} {}_2\phi_1\,(a,b;c;z) &= \frac{(c/b)_\infty(bz)_\infty}{(c)_\infty(z)_\infty}\frac{(abz/c)_\infty(c)_\infty}{(bz)_\infty(c/b)_\infty}{}_2\phi_1\,(c/a,c/b;c;abz/c) \\ &= \frac{(abz/c)_\infty}{(z)_\infty}{}_2\phi_1\,(c/a,c/b;c;abz/c). \end{aligned} \tag{3.2.7}$$

□

Because (3.2.6) is used so frequently in applications, we are going to elevate it from a step in a proof of a q-analogue of Euler's transformation to a corollary.

Corollary 3.2.7 (Second Iterate of Heine's Transformation)**.** *We have*

$$ {}_2\phi_1\,(a,b;c;z) = \frac{(c/b)_\infty(bz)_\infty}{(c)_\infty(z)_\infty}{}_2\phi_1(abz/c,b;bz;c/b). \tag{3.2.8}$$

Our next result is a corollary of Heine's transformation, Theorem 3.2.1.

Theorem 3.2.8. *For* $|z|<1$,

$$\sum_{n=0}^\infty \frac{(z)_n}{(q)_n}a^nq^{n(n+1)/2} = (z)_\infty(-aq)_\infty\sum_{n=0}^\infty\frac{z^n}{(q)_n(-aq)_n}. \tag{3.2.9}$$

Proof. Setting $c=0$ in Heine's transformation, Theorem 3.2.1, we find that

$$\sum_{n=0}^\infty\frac{(a)_n(b)_n}{(q)_n}z^n = \frac{(b)_\infty(az)_\infty}{(z)_\infty}\sum_{n=0}^\infty\frac{(z)_n}{(az)_n(q)_n}b^n.$$

Let $z=c$ and then put $b=z$ above. Thus,

$$\sum_{n=0}^\infty\frac{(a)_n(z)_n}{(q)_n}c^n = \frac{(z)_\infty(ac)_\infty}{(c)_\infty}\sum_{n=0}^\infty\frac{(c)_n}{(ac)_n(q)_n}z^n. \tag{3.2.10}$$

Next, replace a by $-qa/c$ and then let c tend to 0 in (3.2.10). Noting that

$$\lim_{c\to0}(-qa/c)_nc^n = q^{n(n+1)/2}a^n,$$

we readily deduce that 3.2.9 holds. □

Second Proof of Theorem 3.2.8. We begin by writing (3.2.9) in the form

$$\sum_{m=0}^\infty\frac{a^mq^{m(m+1)/2}}{(q)_m}\frac{1}{(q^mz)_\infty} = \sum_{n=0}^\infty\frac{z^n}{(q)_n}(-aq^{n+1})_\infty. \tag{3.2.11}$$

Let $L(n,m,N)$ and $R(n,m,N)$ denote the coefficients of $z^na^mq^N$ on the left- and right-hand sides, respectively, of (3.2.11). A bipartition is an ordered pair of partitions. The two types of partitions need not be of the same type. We show that bipartitions are enumerated on each side of (3.2.11), and that these numbers are equal.

Let P_r^k be the set of partitions consisting of exactly r parts, with each part $\geq k$. Let D_r^k be the set of partitions of r distinct parts, with each part $\geq k$. We see that

$$\frac{a^m q^{m(m+1)/2}}{(q)_m} \tag{3.2.12}$$

enumerates partitions with exactly m distinct parts, i.e., parts in D_m^1. To see this, we first consider a partition of $1/(q)_m$ of less than or equal to m parts and write the parts in decreasing order. Now $1+2+\cdots+m = m(m+1)/2$, and so we take our partition from $1/(q)_m$ and add the parts $m, m-1, \ldots, 1$ to it in decreasing order. We thus have a partition into *exactly* m distinct parts. Note that the power of a is the number of parts in the partition. Thus, (3.2.12) generates the partitions in D_m^1. The terms

$$\frac{1}{(q^m z)_\infty} \tag{3.2.13}$$

generates partitions with each part at least equal to m. The power of z, say n, tells us how many parts we have in a particular partition. Thus, (3.2.13) generates partitions in P_n^m. In summary, $L(n, m, N)$ is equal to the number of bipartite partitions

$$D_m^1 \oplus P_n^m. \tag{3.2.14}$$

On the right-hand side of (3.2.11), consider first

$$(-aq^{n+1})_\infty, \tag{3.2.15}$$

which generates partitions into distinct parts, with $n+1$ being the smallest possible part. The power of a indicates the number of parts. Furthermore,

$$\frac{1}{(q)_n} \tag{3.2.16}$$

generates partitions into less than or equal to n parts. Take each such partition and borrow, in the language of Ferrers' graphs, nm nodes from each partition from the partitions generated from (3.2.16), so that instead of having m parts at least $n+1$ in size, we now have m parts at least 1 in size. So now, when we previously had $\leq n$ parts, by adding m nodes to each of these $\leq n$ parts, we now have exactly n parts, with at least m in each part. Thus, we are then counting partitions generated by

$$D_m^1 \oplus P_n^m. \tag{3.2.17}$$

Note that this transfer of nm nodes does not affect the number of parts, i.e., the power of a indicating the number of parts is the same. The power n of z is also exactly the number of parts in P_n^m. From (3.2.14) and (3.2.17), we complete the proof. □

3.3. The More General ${}_{r+1}\phi_r$

Let us return to Definition 3.1.1 with r replaced by $r+1$ and $s=r$. Thus, we are going to study

$$\begin{aligned}{}_{r+1}\phi_r(a_1,a_2,\dots,a_{r+1};b_1,b_2,\dots,b_r;q,z) &= {}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,b_2,\dots,b_r\end{matrix};q,z\right]\\ &= \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_{r+1};q)_n}{(b_1;q)_n(b_2;q)_n\cdots(b_r;q)_n(q;q)_n}z^n, \quad |z|<1. \end{aligned} \tag{3.3.1}$$

We begin with one of the classic theorems in basic hypergeometric series, due to Pfaff and Saalschütz.

Theorem 3.3.1. *For each positive integer n,*

$$ {}_3\phi_2\left(\begin{matrix}a,b,q^{-n}\\ c,abq^{1-n}/c\end{matrix};q;q\right)=\frac{(c/a)_n(c/b)_n}{(c)_n(c/(ab))_n}. \tag{3.3.2}$$

Proof. By the q-binomial theorem, Theorem 3.1.2,

$$\frac{(abz/c)_\infty}{(z)_\infty}=\sum_{k=0}^{\infty}\frac{(ab/c)_k}{(q)_k}z^k, \quad |z|<1. \tag{3.3.3}$$

Next, by the q-analogue of Euler's transformation, Theorem 3.2.6, and (3.3.3),

$$\begin{aligned}{}_2\phi_1(a,b;c;q;z) &= \frac{(abz/c)_\infty}{(z)_\infty}\sum_{m=0}^{\infty}\frac{(c/a)_m(c/b)_m}{(c)_m(q)_m}\left(\frac{abz}{c}\right)^m\\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(ab/c)_k}{(q)_k}\frac{(c/a)_m(c/b)_m}{(c)_m(q)_m}\left(\frac{ab}{c}\right)^m z^{m+k}. \end{aligned} \tag{3.3.4}$$

Equating coefficients of z^n, $n\ge 0$, on both sides of (3.3.4), we find that

$$\frac{(a)_n(b)_n}{(c)_n(q)_n}=\sum_{m=0}^{n}\frac{(ab/c)_{n-m}}{(q)_{n-m}}\frac{(c/a)_m(c/b)_m}{(c)_m(q)_m}\left(\frac{ab}{c}\right)^m. \tag{3.3.5}$$

It can readily be shown that

$$(a)_{n-m}=\frac{(a)_n}{(q^{1-n}/a)_m}\left(-\frac{q}{a}\right)^m q^{m(m-1)/2-nm}, \tag{3.3.6}$$

which we leave as an exercise. Applying (3.3.6) with a replaced by ab/c and q, respectively, in (3.3.5), and simplifying, we find that

$$\frac{(a)_n(b)_n}{(c)_n(ab/c)_n}=\sum_{m=0}^{n}\frac{(q^{-n})_m(c/a)_m(c/b)_m}{(q^{1-n}c/(ab))_m(c)_m(q)_m}q^m. \tag{3.3.7}$$

If we replace the pair a,b by the pair $c/a, c/b$, respectively, in (3.3.7), we deduce (3.3.2) to complete the proof. □

Our next goal is to utilize the Pfaff–Saalschütz Theorem to derive a general transformation formula that enables us to reduce certain ${}_{r+2}\phi_{r+1}$ series to ${}_{r}\phi_{r-1}$ series. This reduction transformation can be used repeatedly, with the aim of reaching a hypergeometric series that perhaps can be summed in closed form. Our final result can also be viewed as a form of Bailey's Theorem, which is an enormous tool for producing q-series identities.

Return to Theorem 3.3.1 and replace a, b, and c, respectively, by aq^k, $aq/(bc)$, and aq/b. Hence,

$$\begin{aligned}{}_3\phi_2(q^{-k}, aq^k, aq/(bc); aq/b, aq/c; q; q)\\ = \frac{(q^{1-k}/b)_k(c)_k}{(aq/b)_k(cq^{-k}/a)_k} = \frac{(b)_k(c)_k}{(aq/b)_k(aq/c)_k}\left(\frac{aq}{bc}\right)^k, \end{aligned} \tag{3.3.8}$$

after some elementary manipulation. Multiply (3.3.8) by $(q^{-n})_k A_k/(q)_k$, $0 \leq k \leq n$, where there are no restrictions on A_k, $0 \leq k \leq n$. Also multiply both sides by $(bc/(aq))^k$. Now sum both sides from $k=0$ to $k=n$ to arrive at

$$\sum_{k=0}^{n} \frac{(b)_k(c)_k(q^{-n})_k A_k}{(aq/b)_k(aq/c)_k(q)_k} = \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(aq/(bc))_j(aq^k)_j(q^{-k})_j(q^{-n})_k}{(aq/b)_j(aq/c)_j(q)_j(q)_k} q^j \left(\frac{bc}{aq}\right)^k A_k. \tag{3.3.9}$$

Invert the order of summation in (3.3.9), so that in the inner sum $j \leq k \leq n$, and in the outer sum, $0 \leq j \leq n$. Next, let $k = i + j$. Thus, using the more compact notation (1.1.2), we can rewrite (3.3.9) in the form

$$\begin{aligned}&\sum_{k=0}^{n} \frac{(b)_k(c)_k(q^{-n})_k A_k}{(aq/b)_k(aq/c)_k(q)_k}\\ &= \sum_{j=0}^{n}\sum_{i=0}^{n-j} \frac{(aq/(bc), aq^{i+j}, q^{-i-j}; q)_j (q^{-n}; q)_{i+j}}{(aq/b, aq/c, q; q)_j (q; q)_{i+j}} q^j \left(\frac{bc}{aq}\right)^{i+j} A_{i+j}.\end{aligned} \tag{3.3.10}$$

To simplify (3.3.10), we employ the elementary identities

$$(q^{-i-j}; q)_j = (-1)^j (q^{i+1})_j q^{-ij-j(j+1)/2}, \tag{3.3.11}$$

$$\frac{(q^{-i-j}; q)_j}{(q)_{i+j}} = \frac{(-1)^j q^{-ij-j(j+1)/2}}{(q)_i}, \tag{3.3.12}$$

$$(aq^{i+j}; q)_j = \frac{(aq^j; q)_j (aq^{2j}; q)_i}{(aq^j; q)_i}, \tag{3.3.13}$$

$$(q^{-n}; q)_{i+j} = (q^{-n}; q)_j (q^{j-n}; q)_i. \tag{3.3.14}$$

Using (3.3.11)–(3.3.14) in (3.3.10), we deduce the following theorem, which can be regarded as a version of *Bailey's Lemma*.

Theorem 3.3.2. *We have*

$$\sum_{k=0}^{n}\frac{(b,c,q^{-n};q)_k}{(aq/b,aq/c;q;q)_k}A_k=\sum_{j=0}^{n}\frac{(aq/(bc),aq^j,q^{-n};q)_j}{(aq/b,aq/c,q;q)_j}(-1)^j q^{-j(j-1)/2}$$
$$\times\sum_{i=0}^{n-j}\frac{(aq^{2j},q^{j-n};q)_i}{(aq^j,q;q)_i}q^{-ij}\left(\frac{bc}{aq}\right)^{i+j}A_{i+j}. \tag{3.3.15}$$

If we set

$$A_k=\frac{(a,a_1,\ldots,a_r;q)_k}{(b_1,b_2,\ldots,b_{r+1};q)_k}z^k$$

and note that

$$\frac{(aq^j)_j(a)_{i+j}}{(aq^j)_i}=(a)_{2j},$$

then Theorem 3.3.2 takes the following shape.

Theorem 3.3.3. *For any nonnegative integer* r,

$$_{r+4}\phi_{r+3}\left[\begin{matrix}a,b,c,a_1,\ldots,a_r,q^{-n}\\ aq/b,aq/c,b_1,\ldots,b_{r+1}\end{matrix};q;z\right]$$
$$=\sum_{j=0}^{n}\frac{(aq/(bc),a_1,\ldots,a_r,q^{-n};q)_j}{(aq/b,aq/c,q,b_1,\ldots,b_{r+1};q)_j}(a;q)_{2j}\left(-\frac{bcz}{aq}\right)^j q^{-j(j-1)/2}$$
$$\times\, _{r+2}\phi_{r+1}\left[\begin{matrix}aq^{2j},a_1q^j,\ldots,a_rq^j,q^{j-n}\\ b_1q^j,\ldots,b_{r+1}q^j\end{matrix};q;\frac{bcz}{aq^{j+1}}\right]. \tag{3.3.16}$$

The thrust of Theorem 3.3.3 is that we can reduce the evaluation of a $_{r+4}\phi_{r+3}$ to the evaluation of a $_{r+2}\phi_{r+1}$. If this is not possible, then we reiterate (3.3.16) until we find that, for a "small" value of r, we can evaluate the series using one of our standard theorems.

Definition 3.3.4. *In the notation* (3.3.1)*, we say that* $_{r+1}\phi_r$ *is well-poised if*

$$qa_1=a_2b_1=a_3b_2=\cdots=a_{r+1}b_r. \tag{3.3.17}$$

Definition 3.3.5. *In the notation* (3.3.1)*, we say that* $_{r+1}\phi_r$ *is very-well-poised if, in addition to* (3.3.17)*,*

$$a_2=q\sqrt{a_1},\qquad a_3=-q\sqrt{a_1}. \tag{3.3.18}$$

In the next theorem, we evaluate in closed form certain very-well-poised $_4\phi_3$-functions.

Theorem 3.3.6. *If* $\delta_{n,0}$ *denotes the "Kronecker delta" and* n *is a nonnegative integer, then*

$$_4\phi_3\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},q^{-n}\\ \sqrt{a},-\sqrt{a},aq^{n+1}\end{matrix};q;q^n\right]=\delta_{n,0}. \tag{3.3.19}$$

Proof. In Theorem 3.3.3, set

$$b = q\sqrt{a}, c = -q\sqrt{a},\ a_k = b_k, 1 \le k \le r,\ b_{r+1} = aq^{n+1}.$$

Hence,

$$\begin{aligned} &{}_4\phi_3 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq^{n+1} \end{matrix} ; q; z \right] \qquad (3.3.20) \\ &= \sum_{j=0}^{n} \frac{(-q^{-1}, q^{-n}; q)_j (a; q)_{2j}}{(q, \sqrt{a}, -\sqrt{a}, aq^{n+1}; q)_j} (qz)^j q^{-j(j-1)/2}{}_2\phi_1 \left[\begin{matrix} aq^{2j}, q^{j-n} \\ aq^{j+n+1} \end{matrix} ; q; -zq^{1-j} \right]. \end{aligned}$$

Next, let $z = q^n$ and apply Bailey's Theorem 3.2.4 to deduce that

$${}_2\phi_1 \left[\begin{matrix} aq^{2j}, q^{j-n} \\ aq^{j+n+1} \end{matrix} ; q; -q^{1+n-j} \right] = \frac{(-q;q)_\infty (aq^{2j+1}, aq^{2n+2}; q^2)_\infty}{(aq^{1+n+j}, -q^{1+n-j}; q)_\infty}. \qquad (3.3.21)$$

Let R denote the right-hand side of (3.3.20). Putting (3.3.21) in R, we find that

$$R = (-q;q)_\infty \sum_{j=0}^{n} \frac{(-q^{-1}, q^{-n}; q)_j (a;q)_{2j} (aq^{2j+1}, aq^{2n+2}; q^2)_\infty}{(q, \sqrt{a}, -\sqrt{a}, aq^{n+1}; q)_j (aq^{1+n+j}, -q^{1+n-j}; q)_\infty} q^{(n+1)j} q^{-j(j-1)/2}. \qquad (3.3.22)$$

We now record the simplifications, which we leave as exercises,

$$\frac{(a;q)_{2j}}{(\sqrt{a}, -\sqrt{a}; q)_j} = (aq; q^2)_j, \qquad (3.3.23)$$

$$(aq : q^2)_j (aq^{2j+1}; q^2)_\infty = (aq; q^2)_\infty, \qquad (3.3.24)$$

$$(aq^{n+1}; q)_j (aq^{n+j+1}; q)_\infty = (aq^{n+1}; q)_\infty, \qquad (3.3.25)$$

and

$$(-q^{1+n-j}; q)_\infty = (-q^{-n}; q)_j (-q^{n+1}; q)_\infty q^{nj - j(j-1)/2}. \qquad (3.3.26)$$

Utilizing (3.3.23)–(3.3.26) in (3.3.22), we find that

$$\begin{aligned} R &= \frac{(-q;q)_\infty (aq;q^2)_\infty (aq^{2n+2};q^2)_\infty}{(aq^{n+1};q)_\infty (-q^{n+1};q)_\infty} \sum_{j=0}^{n} \frac{(-q^{-1}, q^{-n}; q)_j}{(q, -q^{-n}; q)_j} q^j \\ &= \frac{(-q;q)_n (aq;q)_n}{(\sqrt{a}q;q)_n (-\sqrt{a}q;q)_n} {}_2\phi_1(-q^{-1}, q^{-n}; -q^{-n}; q; q). \qquad (3.3.27) \end{aligned}$$

If $n = 0$, we easily see that $R = 1$. If $n > 0$, we use the q-analogue of the Chu–VanderMonde Theorem, Theorem 3.2.3, with $b = -q^{-1}$ and $c = -q^{-n}$ to conclude that

$${}_2\phi_1(-q^{-1}, q^{-n}; -q^{-n}; q; q) = \frac{(q^{-n+1}; q)_n}{(-q^{-n}; q)_n} = 0. \qquad (3.3.28)$$

Recalling that R denotes the right-hand side of (3.3.20), that R is given by (3.3.27), and that R is evaluated by (3.3.28) for $n > 0$ and in the line above for $n = 0$, we complete the proof of Theorem 3.3.6. □

In the next theorem, we sum a very-well-poised ${}_6\phi_5$.

Theorem 3.3.7. *For each nonnegative integer n,*

$$ {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{n+1} \end{matrix}; q; \frac{aq^{n+1}}{bc}\right] = \frac{(aq, aq/(bc); q)_n}{(aq/b, aq/c; q)_n}. \tag{3.3.29} $$

Proof. Let

$$ a_1 = q\sqrt{a}, a_2 = -q\sqrt{a}, b_1 = \sqrt{a}, b_2 = -\sqrt{a},\ a_k = b + k, 3 \le k \le r,\ b_{r+1} = aq^{n+1} $$

in Theorem 3.3.3 to find that

$$ \begin{aligned} &{}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{n+1} \end{matrix}; q; z\right] \\ &= \sum_{j=0}^{n} \frac{(aq/(bc), q\sqrt{a}, -q\sqrt{a}, q^{-n}; q)_j (a;q)_{2j}}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{n+1}; q)_j} \left(-\frac{bcz}{aq}\right)^j q^{-j(j-1)/2} \\ &\quad \times {}_4\phi_3\left[\begin{matrix} aq^{2j}, q^{j+1}\sqrt{a}, -q^{j+1}\sqrt{a}, q^{j-n} \\ q^j\sqrt{a}, -q^j\sqrt{a}, aq^{j+n+1} \end{matrix}; q; \frac{bcz}{aq^{j+1}}\right]. \end{aligned} \tag{3.3.30} $$

Now let $z = aq^{n+1}/(bc)$ in (3.3.30) and apply Theorem 3.3.6 with a replaced by aq^{2j} and n replaced by $n-j$. Thus, in the sum on j in (3.3.30), all of the terms equal 0 except the term with $j = n$. Hence,

$$ \begin{aligned} &{}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{n+1} \end{matrix}; q; z\right] \\ &= \frac{(aq/(bc), q\sqrt{a}, -q\sqrt{a}, q^{-n}; q)_n (a;q)_{2n}}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{n+1}; q)_n} (-1)^n q^{n^2} q^{-n(n-1)/2}. \end{aligned} \tag{3.3.31} $$

Since

$$ \frac{(q\sqrt{a}, -q\sqrt{a}; q)_n}{(\sqrt{a}, -\sqrt{a}; q)_n} = \frac{1 - aq^{2n}}{1 - a}, \tag{3.3.32} $$

$$ \frac{(a;q)_{2n}(1 - aq^{2n})}{(1-a)(q, aq^{n+1}; q)_n} = \frac{(aq;q)_n}{(q;q)_n}, $$

and

$$ (q^{-n}; q)_n = (-1)^n (q;q)_n q^{-n(n+1)/2}, $$

we find from (3.3.31) that

$$ {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{n+1} \end{matrix}; q; z\right] = \frac{(aq, aq/(bc); q)_n}{(aq/b, aq/c; q)_n}, $$

which completes the proof of Theorem 3.3.7. □

The next theorem is Watson's transformation formula for a very-well-poised terminating ${}_8\phi_7$. It is one of the most useful theorems in q-series, especially in the theory of partitions.

Theorem 3.3.8 (q-analogue of Whipple's Theorem)**.** *For each nonnegative integer n,*

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n}\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}\end{matrix}; q; \frac{a^2q^{n+2}}{bcde}\right]\\
&= \frac{(aq, aq/(de); q)_n}{(aq/d, aq/e; q)_n}\,{}_4\phi_3\left[\begin{matrix}q^{-n}, d, e, aq/(bc)\\ aq/b, aq/c, deq^{-n}/a\end{matrix}; q; q\right]. \qquad (3.3.33)
\end{aligned}
$$

Proof. With an appeal to Theorem 3.3.3, we set

$$
\begin{aligned}
&a_1 = q\sqrt{a}, a_2 = -q\sqrt{a}, a_3 = d, a_4 = e, b_1 = \sqrt{a}, b_2 = -\sqrt{a},\\
&b_3 = aq/d, b_4 = aq/e,\ a_k = b_k, 5 \le k \le r.
\end{aligned}
$$

Also, set

$$z = \frac{a^2q^{2+n}}{bcde}.$$

Hence,

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n}\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}\end{matrix}; q; \frac{a^2q^{n+2}}{bcde}\right] \qquad (3.3.34)\\
&= \sum_{j=0}^{n} \frac{(aq/(bc), q\sqrt{a}, -q\sqrt{a}, d, e, q^{-n}; q)_j (a; q)_{2j}}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q)_j}\left(-\frac{aq^{n+1}}{de}\right)^j q^{j(j-1)/2}\\
&\times {}_6\phi_5\left[\begin{matrix}aq^{2j}, q^{j+1}\sqrt{a}, -q^{j+1}\sqrt{a}, dq^j, eq^j, q^{j-n}\\ q^j\sqrt{a}, -q^j\sqrt{a}, aq^{j+1}/d, aq^{j+1}/e, aq^{j+n+1}\end{matrix}; q; \frac{aq^{n+1-j}}{de}\right].
\end{aligned}
$$

We now use Theorem 3.3.7 with a replaced by aq^{2j}, $b = dq^j$, $c = eq^j$, and n replaced by $n - j$ to evaluate the ${}_6\phi_5$ above. Consequently,

$$
\begin{aligned}
&{}_6\phi_5\left[\begin{matrix}aq^{2j}, q^{j+1}\sqrt{a}, -q^{j+1}\sqrt{a}, dq^j, eq^j, q^{j-n}\\ q^j\sqrt{a}, -q^j\sqrt{a}, aq^{j+1}/d, aq^{j+1}/e, aq^{j+n+1}\end{matrix}; q; \frac{aq^{n+1-j}}{de}\right]\\
&= \frac{\left(aq^{2j+1}, \dfrac{aq^{2j+1}}{deq^{2j}}\right)_{n-j}}{\left(\dfrac{aq^{2j+1}}{dq^j}, \dfrac{aq^{2j+1}}{eq^j}\right)_{n-j}} = \frac{\left(aq^{2j+1}, \dfrac{aq}{de}\right)_{n-j}}{\left(\dfrac{aq^{j+1}}{d}, \dfrac{aq^{j+1}}{e}\right)_{n-j}}\\
&= \frac{(aq^{j+1})_n (aq/(de))_n (dq^{-n-j}/a)_j (eq^{-n-j}/a)_j}{(q^{-n-2j}/a)_j (q^{-n}de/a)_j (aq^{j+1}/d)_n (aq^{j+1}/e)_n}, \qquad (3.3.35)
\end{aligned}
$$

where we made two applications of (3.3.6). First, using (3.3.32), we readily find that

$$\frac{(q\sqrt{a}, -q\sqrt{a}; q)_j}{(\sqrt{a}, -\sqrt{a}; q)_j}(a; q)_{2j}(aq^{2j+1}; q)_n = (aq; q)_{n+2j}. \qquad (3.3.36)$$

Second, a brief calculation shows that

$$(q^{-n-2j}/a; q)_j = (aq^{n+j+1}; q)_j (-1)^j q^{-nj-j(3j+1)/2} a^{-j}. \qquad (3.3.37)$$

Third, we readily see that

$$\frac{(aq; q)_{n+2j}}{(aq^{n+j+1}; q)_j (aq^{n+1}; q)_j} = (aq; q)_n. \qquad (3.3.38)$$

Fourth,

$$\frac{(dq^{-n-j}/a;q)_j}{(aq/d;q)_j(aq^{j+1}/d;q)_n}=\frac{(-1)^j d^j a^{-j} q^{-nj-j(j+1)/2}}{(aq/d;q)_n}. \tag{3.3.39}$$

Similarly,

$$\frac{(eq^{-n-j}/a;q)_j}{(aq/e;q)_j(aq^{j+1}/e;q)_n}=\frac{(-1)^j e^j a^{-j} q^{-nj-j(j+1)/2}}{(aq/e;q)_n}. \tag{3.3.40}$$

Hence, using (3.3.35) and (3.3.36)–(3.3.40) in (3.3.34), after an enormous amount of simplification, we finally find that

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,d,e,q^{-n}\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e,aq^{n+1}\end{matrix};q;\frac{a^2q^{n+2}}{bcde}\right]\\ &\qquad=\frac{(aq;q)_n(aq/(de);q)_n}{(aq/d;q)_n(aq/e;q)_n}\sum_{j=0}^{n}\frac{(aq/bc,d,e,q^{-n};q)_j}{(q,aq/b,aq/c,q^{-n}de/a;q)_j}q^j.\end{aligned}$$

Thus, the proof of Theorem 3.3.8 is complete. □

Definition 3.3.9. *If $qa_1a_2\cdots a_{r+1}=b_1b_2\cdots b_r$, then we say that ${}_{r+1}\phi_r$ is balanced or Saalschützian.*

Note that the ${}_4\phi_3$ in Watson's transformation is balanced.

Watson's ${}_8\phi_7$ transformation is not found in any of Ramanujan's published papers, his earlier notebooks, or his lost notebook. However, the following corollary is Entry 7 in Chapter 16 of his second notebook [**90**], [**26**, p. 16].

Corollary 3.3.10. *We have*

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(a)_k(d/b)_k(d/c)_k(d/q)_k(1-dq^{2k-1})}{(b)_k(c)_k(d/a)_k(q)_k(1-d/q)}\left(\frac{bc}{a}\right)^k q^{k(k-1)}\\ &\qquad=\frac{(a)_\infty(d)_\infty}{(b)_\infty(c)_\infty}\sum_{k=0}^{\infty}\frac{(b/a)_k(c/a)_k}{(d/a)_k(q)_k}a^k.\end{aligned} \tag{3.3.41}$$

Proof. We let $n\to\infty$ in Theorem 3.3.8. On the left-hand side of (3.3.33), we observe that

$$\lim_{n\to\infty}\frac{(q^{-n})_kq^{nk}}{(aq^{n+1})_k}=(-1)^kq^{k(k-1)/2},$$

while on the right-hand side of (3.3.33), we see that

$$\lim_{n\to\infty}\frac{(q^{-n})_k}{(deq^{-n}/a)_k}=\left(\frac{a}{de}\right)^k.$$

Thus, so far, we have shown from (3.3.33) that

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(c)_k(d)_k(e)_k(1-aq^{2k})}{(aq/b)_k(aq/c)_k(aq/d)_k(aq/e)_k(1-a)(q)_k}\left(-\frac{a^2}{bcde}\right)^k q^{k(k+3)/2}\\ &\qquad=\frac{(aq,aq/(de))_\infty}{(aq/d,aq/e)_\infty}{}_3\phi_2\left[\begin{matrix}aq/(bc),d,e\\ aq/b,aq/c\end{matrix};\frac{aq}{de}\right].\end{aligned} \tag{3.3.42}$$

Next, we let $c \to \infty$. Observing that

$$\lim_{c\to\infty} \frac{(c)_k c^{-k}}{(aq/c)_k} = (-1)^k q^{k(k-1)/2},$$

we see that (3.3.42) reduces to

$$\sum_{k=0}^{\infty} \frac{(a)_k(b)_k(d)_k(e)_k(1-aq^{2k})}{(aq/b)_k(aq/d)_k(aq/e)_k(1-a)(q)_k} \left(\frac{a^2}{bde}\right)^k q^{k(k+1)}$$
$$= \frac{(aq, aq/(de))_\infty}{(aq/d, aq/e)_\infty} \sum_{k=0}^{\infty} \frac{(d)_k(e)_k}{(aq/b)_k(q)_k} \left(\frac{aq}{de}\right)^k. \quad (3.3.43)$$

Now replace a, b, d, and e by d/q, a, d/b, and d/c, respectively. Hence, (3.3.43) and the q-analogue of Euler's transformation, Theorem 3.2.6, yield

$$\sum_{k=0}^{\infty} \frac{(a)_k(d/b)_k(d/c)_k(d/q)_k(1-dq^{2k-1})}{(b)_k(c)_k(d/a)_k(q)_k(1-d/q)} \left(\frac{bc}{a}\right)^k q^{k(k-1)}$$
$$= \frac{(d)_\infty(bc/d)_\infty}{(b)_\infty(c)_\infty} \sum_{k=0}^{\infty} \frac{(d/b)_k(d/c)_k}{(d/a)_k(q)_k} \left(\frac{bc}{d}\right)^k$$
$$= \frac{(d)_\infty(bc/d)_\infty}{(b)_\infty(c)_\infty} \frac{(a)_\infty}{(bc/d)_\infty} \sum_{k=0}^{\infty} \frac{(b/a)_k(c/a)_k}{(d/a)_k(q)_k} a^k. \quad (3.3.44)$$

After minor simplification, we complete the proof. □

3.4. Exercises

1. Prove that q-binomial coefficients are polynomials in q.

2. For each positive integer N, establish a result to H.A. Rothe, namely,

$$\sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q (-1)^k q^{k(k-1)/2} x^k = (x;q)_N.$$

3. For each positive integer N and $|x| < 1$, prove that

$$\sum_{k=0}^{\infty} \begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q x^k = \frac{1}{(x;q)_N}.$$

4. Verify (3.3.6).

5. Verify (3.3.11)–(3.3.14).

6. Verify (3.3.23)–(3.3.24).

7. Reverse the order of summation in the q-analogue of the Chu–VanderMonde Theorem, Theorem 3.2.3, to show that

$$ {}_2\phi_1(q^{-n}, b; c; q; q) = \frac{(c/b;q)_n}{(c;q)_n} b^n. \tag{3.4.1} $$

Chapter 4

Partition Identities Arising from q-Series Identities

4.1. Theorems Arising from Basic q-Series Theorems

We begin with a famous identity due to Lebesgue.

Theorem 4.1.1. *For $b \in \mathbb{C}$,*

$$\sum_{n=0}^{\infty} \frac{(-bq;q)_n}{(q;q)_n} q^{n(n+1)/2} = \frac{(-bq^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{4.1.1}$$

First Proof of Theorem 4.1.1. Employing Theorem 3.2.8 with $a = 1$ and $z = -bq$, Corollary 3.1.3, and Euler's identity,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-bq;q)_n}{(q;q)_n} q^{n(n+1)/2} &= (-bq;q)_\infty (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(-bq)^n}{(q;q)_n(-q;q)_n} \\
&= (-bq;q)_\infty (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(-bq)^n}{(q^2;q^2)_n} \\
&= (-bq;q)_\infty (-q;q)_\infty \frac{1}{(-bq;q^2)_\infty} \\
&= \frac{(-bq^2;q^2)_\infty}{(q;q^2)_\infty}.
\end{aligned}$$

□

Second Proof of Theorem 4.1.1. Replacing b by $1/b$ in Bailey's Theorem 3.2.4, we find that

$$ {}_2\phi(a,1/b;qab;-qb)=\frac{(aq;q^2)_\infty(-q;q)_\infty(q^2ab^2;q^2)_\infty}{(qab;q)_\infty(-qb;q)_\infty}. \tag{4.1.2}$$

Now,

$$\lim_{b\to 0}(1/b;q)_n(-qb)^n=q^{n(n+1)/2}. \tag{4.1.3}$$

Letting b tend to 0 on both sides of (4.1.2) and using (4.1.3), we find that

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}q^{n(n+1)/2}=\frac{(aq;q^2)_\infty}{(q;q^2)_\infty}.$$

Letting $a=-bq$ and using Euler's theorem, we complete the proof. □

The following beautiful result is due to N. J. Fine [**53**]; its proof can also be found in Andrews' text [**10**, pp. 26–27]. Observe that Theorem 4.1.2 is a refinement of Euler's theorem, Theorem 1.2.3.

Theorem 4.1.2. *The number of partitions of a positive integer n into distinct parts with largest part k equals the number of partitions of n into odd parts, such that $2k+1$ equals the largest part plus twice the number of parts.*

We interpret Theorem 4.1.2 as an equivalent q-series identity.

Theorem 4.1.3. *We have*

$$\sum_{j=0}^{\infty}(-q;q)_j(tq)^{j+1}=\sum_{j=0}^{\infty}\frac{t^{j+1}q^{2j+1}}{(tq;q^2)_{j+1}}. \tag{4.1.4}$$

Proof. Write

$$\sum_{j=0}^{\infty}(-q;q)_j(tq)^{j+1}=\sum_{j=0}^{\infty}(1+q)(1+q^2)\cdots(1+q^j)q^{j+1}t^{j+1}.$$

Let $k=j+1$. We readily see that the series above generates partitions of n into distinct parts, with largest part k. Let us say that there are $A_k(n)$ such partitions. On the other hand,

$$\sum_{j=0}^{\infty}\frac{t^{j+1}q^{2j+1}}{(tq;q^2)_{j+1}}=\sum_{j=0}^{\infty}t^{j+1}q^{2j+1}\sum_{n_1=0}^{\infty}(tq)^{n_1}\sum_{n_2=0}^{\infty}(tq^3)^{n_2}\cdots\sum_{n_{j+1}=0}^{\infty}(tq^{2j+1})^{n_{j+1}}.$$

Thus, the series above generates partitions of n into odd parts, such that $2k+1$ equals the largest part plus twice the number of odd parts; call this number $B_k(n)$. The power of t is

$$k=j+1+n_1+n_2+\cdots+n_{j+1}=j+1+\nu(\pi)-1=j+\nu(\pi),$$

since $1+n_1+n_2+\cdots+n_{j+1}=\nu(\pi)$. Hence,

$$2k+1=2j+1+2\nu(\pi)=\text{ largest part }+2\nu(\pi).$$

Thus, we have shown that Theorem 4.1.2 and Theorem 4.1.3 are equivalent. □

Example 4.1.4. *Let* $A_k(n)$ *and* $B_k(n)$ *denote the coefficients of* t^kq^n *on the left and right sides of* (4.1.4)*, respectively. Let* $k = 8$ *and* $n = 14$*. We see that* $A_8(14) = 4 = B_8(14)$*, with the respective representations being*

$$8 + 6, 8 + 5 + 1, 8 + 4 + 2, 8 + 3 + 2 + 1;$$
$$13 + 1(17 = 13 + 2 \cdot 2), 9 + 3 + 1 + 1(17 = 9 + 2 \cdot 4),$$
$$5 + 5 + 1 + 1 + 1 + 1(17 = 5 + 2 \cdot 6), 5 + 3 + 3 + 1 + 1 + 1(17 = 5 + 2 \cdot 6).$$

Proof of Theorem 4.1.3. Using Corollary 3.1.3 twice below, we find that

$$\begin{aligned}
\sum_{j=0}^{\infty}(-q;q)_j(tq)^{j+1} &= \sum_{j=0}^{\infty}\frac{(q^2;q^2)_j(tq)^{j+1}}{(q;q)_j} \qquad (4.1.5)\\
&= tq(q^2;q^2)_\infty\sum_{j=0}^{\infty}\frac{(tq)^j}{(q;q)_j}\frac{1}{(q^{2j+2};q^2)_\infty}\\
&= tq(q^2;q^2)_\infty\sum_{j=0}^{\infty}\frac{(tq)^j}{(q;q)_j}\sum_{m=0}^{\infty}\frac{q^{(2j+2)m}}{(q^2;q^2)_m}\\
&= tq(q^2;q^2)_\infty\sum_{m=0}^{\infty}\frac{q^{2m}}{(q^2;q^2)_m}\sum_{j=0}^{\infty}\frac{t^jq^{j(2m+1)}}{(q;q)_j}\\
&= tq(q^2;q^2)_\infty\sum_{m=0}^{\infty}\frac{q^{2m}}{(q^2;q^2)_m(tq^{2m+1};q)_\infty}\\
&= \frac{tq(q^2;q^2)_\infty}{(tq;q)_\infty}\sum_{m=0}^{\infty}\frac{(tq;q^2)_m(tq^2;q^2)_mq^{2m}}{(q^2;q^2)_m}\\
&= \frac{tq(q^2;q^2)_\infty}{(tq;q)_\infty}\sum_{m=0}^{\infty}\frac{(tq;q^2)_m(tq^2;q^2)_mq^{2m}}{(q^2;q^2)_m},
\end{aligned}$$

where we have used the fact that $(tq;q)_{2m}(tq^{2m+1};q)_\infty = (tq;q)_\infty$. Applying Heine's transformation, Theorem 3.2.1, to the far right side of (4.1.5), with first q replaced by q^2, and then with $a = tq$, $b = tq^2$, $c = 0$, and $z = q^2$, we deduce that

$$\begin{aligned}
&\sum_{j=0}^{\infty}(-q;q)_j(tq)^{j+1}\\
&\quad= \frac{tq(q^2;q^2)_\infty}{(tq;q)_\infty}\frac{(tq^2;q^2)_\infty(tq^3;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(q^2;q^2)_mt^mq^{2m}}{(tq^3;q^2)_m(q^2;q^2)_m}\\
&\quad= \frac{tq(tq^2;q^2)_\infty(tq^3;q^2)_\infty}{(tq;q)_\infty}\sum_{m=0}^{\infty}\frac{t^mq^{2m}}{(tq^3;q^2)_m}\\
&\quad= \frac{tq}{1-tq}\sum_{m=0}^{\infty}\frac{t^mq^{2m}}{(tq^3;q^2)_m}
\end{aligned}$$

$$= \sum_{m=0}^{\infty} \frac{t^{m+1}q^{2m+1}}{(tq;q^2)_{m+1}},$$

and this completes the proof. □

Theorem 4.1.5. *The number of partitions of $a-c$ into exactly $b-1$ parts, none exceeding c equals the number of partitions of $a-b$ into exactly $c-1$ parts, none exceeding b, or*

$$P(c, b-1, a-c) = P(b, c-1, a-b).$$

We provide two proofs. The first is combinatorial; the second is analytic. We shall accompany our proof by an example.

First Proof of Theorem 4.1.5. Consider the Ferrers graph of a partition of the first type. For example, let $a = 21$, $c = 7$, and $b = 5$. Thus, below we have a Ferrers graph of a partition of $a - c = 14$ into exactly $5 - 1 = 4$ parts, with none exceeding the largest part $6 < 7$.

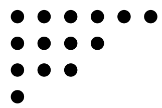

Figure 1. A Partition of $a - c$

Now add a row of c nodes to the top of the Ferrers graph above. So now we have a partition of $a - c + c$, in the Example 21.

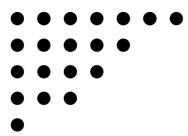

Figure 2. A Partition of $a - c + c$

We now delete the first column. Thus, we have a Ferrers graph of $a-b$. The top row has $c - 1$ nodes. And, there are less than or equal to b rows, in this example, ≤ 5 rows.

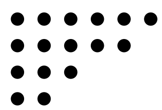

Figure 3. A Partition of $a - b$

Now take the conjugate of the partition above. Thus, we have a partition of $a - b$ into exactly $c - 1$ parts, with the biggest part less than or equal to b.

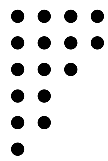

Figure 4. Conjugate Partition of $a - b$

All of these processes are reversible, and so we have shown simply with the use of Ferrers graphs that

$$Q(c, b - 1, a - c) = Q(b, c - 1, a - b).$$

□

Second Proof of Theorem 4.1.5. While our first proof of Theorem 4.1.5 was purely combinatorial, our second proof depends on the theory of q-series. We form a generating function

$$S(a, b, c) := \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} Q(a, b - 1, a - c) x^b y^c a^a. \tag{4.1.6}$$

We can write

$$S(a, b, c) = 1 + \sum_{n=1}^{\infty} x(yq)^n \prod_{j=1}^{n} (1 + xq^j + x^2 q^{2j} + x^3 q^{3j} + \cdots). \tag{4.1.7}$$

To see that (4.1.7) holds, first note that we have extracted x, because the exact number of parts is $b - 1$. Second, observe that $y^c q^a = (yq)^c q^{a-c}$, and so the power c of yq is the upper bound for parts. The running index $n = c$ then stands for the largest possible part in the partitions being generated. The product in (4.1.7) generates the remaining parts. Note that indeed all parts are less than or equal to n. Because we had previously extracted b, we acquire an x a total of $b - 1$ times, as required. Lastly, note that we are generating partitions of $a - c$ in the product. Summing each of the n geometric series on the right side of (4.1.7), we find that

$$S(a, b, c) = 1 + \sum_{n=1}^{\infty} \frac{x(yq)^n}{(xq; q)_n} =: f(x, y). \tag{4.1.8}$$

We see that the generating function for $Q(b, c - 1, a - b)$ is identical to that in (4.1.6), except that the roles of b and c have been switched. Thus, to complete the proof, it suffices to show that

$$f(x, y) = f(y, x). \tag{4.1.9}$$

After a reformulation of the definition of $f(x,y)$ in (4.1.8), we apply Heine's transformation (3.2.1) with $a=0$, $b=q$, $c=xq$, and $t=yq$. Hence,

$$f(x,y)-1+x=\sum_{n=0}^{\infty}\frac{x(yq)^n}{(xq;q)_n}=x\sum_{n=0}^{\infty}\frac{(0)_n(q)_n}{(xq)_n(q)_n}(yq)^n$$
$$=x\frac{(q)_\infty}{(xq)_\infty(yq)_\infty}\sum_{n=0}^{\infty}\frac{(x)_n(yq)_n}{(0)_n(q)_n}q^n. \tag{4.1.10}$$

Apply Heine's transformation (3.2.1) once again, now with $a=yq$, $b=x$, $c=0$, and $t=q$. Hence, from (4.1.10),

$$f(x,y)-1+x=x\frac{(q)_\infty}{(xq)_\infty(yq)_\infty}\frac{(yq^2)_\infty(x)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}\frac{(0)_n(q)_n}{(q)_n(yq^2)_n}x^n$$
$$=x(1-x)\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^n}{(yq)_{n+1}}$$
$$=(1-x)\sum_{n=1}^{\infty}\frac{x^n}{(yq)_n}, \tag{4.1.11}$$

where we replaced n by $n-1$. Therefore, rearranging (4.1.11), we deduce that

$$f(x,y)=(1-x)\sum_{n=0}^{\infty}\frac{x^n}{(yq)_n}$$
$$=\sum_{n=0}^{\infty}\frac{x^n}{(yq)_n}-\sum_{n=0}^{\infty}\frac{x^{n+1}}{(yq)_n}$$
$$=1+\sum_{n=1}^{\infty}\frac{x^n}{(yq)_n}-\sum_{n=1}^{\infty}\frac{x^n(1-yq^n)}{(yq)_n}$$
$$=1+\sum_{n=1}^{\infty}\frac{y(xq)^n}{(yq)_n}$$
$$=f(y,x). \tag{4.1.12}$$

Our last identity (4.1.12) is precisely (4.1.9), and so the proof is complete. □

We come now to one of the highlights of these lecture notes, our first proof of the epic Rogers–Ramanujan identities. All of the groundwork has been laid in Chapter 3. We first derive a corollary of Corollary 3.3.10.

Corollary 4.1.6. *We have*

$$1+\sum_{k=1}^{\infty}\frac{(-1)^k(1-aq^{2k})(aq)_{k-1}a^{2k}q^{k(5k-1)/2}}{(q)_k}=(aq)_\infty\sum_{k=0}^{\infty}\frac{a^kq^{k^2}}{(q)_k}. \tag{4.1.13}$$

Proof. Return to (3.3.43) and let $b,d,e\to\infty$. If we use

$$\lim_{\alpha\to\infty}\frac{(\alpha)_k}{\alpha^k}=(-1)^kq^{k(k-1)/2}$$

on both the left and right sides of (3.3.43), we easily deduce the truth of (4.1.13) to complete the proof. □

Theorem 4.1.7 (Rogers–Ramanujan Identities)**.** *We have*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{4.1.14}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{4.1.15}$$

Proof. In Corollary 4.1.6, set $a = 1$. Then

$$\begin{aligned}
(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1-q^{2n})(q)_{n-1} q^{n(5n-1)/2}}{(q)_n} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n (1+q^n) q^{n(5n-1)/2} \\
&= \sum_{n=0}^{\infty} (-1)^n q^{n(5n-1)/2} + \sum_{n=-\infty}^{-1} (-1)^n q^{-n} q^{n(5n+1)/2} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n-1)/2} \\
&= f(-q^2, -q^3) \\
&= (q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty,
\end{aligned}$$

by the Jacobi triple product identity, (1.1.7) or (3.1.18). Dividing both sides above by $(q)_\infty$, and simplifying, we complete the proof of (4.1.14).

Next, in Corollary 4.1.6, set $a = q$. Then

$$\begin{aligned}
(q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1-q^{2n+1})(q^2)_{n-1} q^{2n+n(5n-1)/2}}{(q)_n} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1-q^{2n+1}) q^{2n+n(5n-1)/2}}{1-q}.
\end{aligned}$$

Multiplying both sides above by $(1-q)$ and setting $n = -m-1$ in the second sum on the right side below, we arrive at

$$\begin{aligned}
(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} &= 1 - q + \sum_{n=1}^{\infty} (-1)^n q^{n(5n+3)/2} - \sum_{n=1}^{\infty} (-1)^n q^{n(5n+7)/2+1} \\
&= 1 - q + \sum_{n=1}^{\infty} (-1)^n q^{n(5n+3)/2} + \sum_{m=-\infty}^{-2} (-1)^m q^{m(5m+3)/2+1} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+3)/2}
\end{aligned}$$

$$= f(-q^4, -q)$$
$$= (q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty,$$

by the Jacobi triple product identity (3.1.18). Dividing both sides above by $(q)_\infty$, we finish the proof of (4.1.15). □

We next derive another corollary of Watson's transformation.

Corollary 4.1.8. *We have*

$$1+\sum_{k=1}^{\infty}\frac{(1-aq^{2k})(d)_k(aq)_{k-1}}{(q)_k(aq/d)_k}\left(\frac{a^2}{d}\right)^k q^{2k^2}=\frac{(aq)_\infty}{(aq/d)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(d)_k}{(q)_k}\left(\frac{a}{d}\right)^k q^{k(k+1)/2}. \tag{4.1.16}$$

Proof. Return to (3.3.42) and let b, c, and e tend to infinity. Hence, we find that

$$1+\sum_{k=1}^{\infty}\frac{(1-aq^{2k})(d)_k(a)_k}{(1-a)(q)_k(aq/d)_k}\left(\frac{a^2}{d}\right)^k q^{2k^2}=\frac{(aq)_\infty}{(aq/d)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(d)_k}{(q)_k}\left(\frac{a}{d}\right)^k q^{k(k+1)/2}. \tag{4.1.17}$$

Minor simplification completes the proof. □

If we replace q by q^2 and then set $d=-q$ in (4.1.16), we find that

$$1+\sum_{k=1}^{\infty}\frac{(1-aq^{4k})(-1)^k(-q;q^2)_k(aq^2;q^2)_{k-1}}{(q^2;q^2)_k(-aq;q^2)_k}a^{2k}q^{4k^2-k}=\frac{(aq^2;q^2)_\infty}{(-aq;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;q^2)_k}a^k q^{k^2}. \tag{4.1.18}$$

The next two theorems are the analytic versions of the two famous Göllnitz–Gordon identities [**60**]. However, the identities can be found in Ramanujan's lost notebook [**91**, p. 41], [**17**, pp. 36–37]. They can also be found in Slater's list [**97**, equations (36), (34)], but with q replaced by $-q$. The Göllnitz–Gordon identities have played a seminal role in the subsequent development in the theory of partition identities. They were first studied in this regard by H. Göllnitz [**59**] and by B. Gordon [**61**], [**62**]. A generalization by Andrews [**6**] led to a number of further discoveries culminating in [**8**]. After we prove both identities, we shall discuss their combinatorial implications.

Theorem 4.1.9 (First Göllnitz–Gordon Identity)**.** *We have*

$$\sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;q^2)_k}q^{k^2}=\frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}. \tag{4.1.19}$$

Proof. Letting $a = 1$ in (4.1.18), we deduce that

$$\begin{aligned}
\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;q^2)_k}q^{k^2} &= 1+\sum_{k=1}^{\infty}\frac{(1-q^{4k})(-1)^k(-q;q^2)_k(q^2;q^2)_{k-1}}{(q^2;q^2)_k(-q;q^2)_k}q^{4k^2-k}\\
&= 1+\sum_{k=1}^{\infty}(-1)^k(1+q^{2k})q^{4k^2-k}\\
&= \sum_{k=1}^{\infty}(-1)^k q^{4k^2-k}+\sum_{k=0}^{\infty}(-1)^k q^{4k^2+k}\\
&= \sum_{k=-1}^{-\infty}(-1)^k q^{4k^2+k}+\sum_{k=0}^{\infty}(-1)^k q^{4k^2+k}\\
&= \sum_{k=-\infty}^{\infty}(-1)^k q^{4k^2+k}\\
&= f(-q^5,-q^3)\\
&= (q^3;q^8)_\infty(q^5;q^8)_\infty(q^8;q^8)_\infty, \qquad (4.1.20)
\end{aligned}$$

by the Jacobi triple product identity (3.1.18). A slight rearrangement of (4.1.20) yields

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;q^2)_k}q^{k^2} &= \frac{(-q;q^2)_\infty(q^3;q^8)_\infty(q^5;q^8)_\infty(q^8;q^8)_\infty}{(q^2;q^2)_\infty}\\
&= \frac{(q^2;q^4)_\infty(q^3;q^8)_\infty(q^5;q^8)_\infty(q^8;q^8)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty}\\
&= \frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}.
\end{aligned}$$

Thus, the proof is complete. □

Theorem 4.1.10 (Second Göllnitz–Gordon Identity)**.** *We have*

$$\sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;q^2)_k}q^{k^2+2k} = \frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}. \qquad (4.1.21)$$

Proof. Set $a = q^2$ in (4.1.18) to find that

$$\begin{aligned}
\frac{(q^4;q^2)_\infty}{(-q^3;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;q^2)_k}q^{k^2+2k} &= 1+\sum_{k=1}^{\infty}\frac{(-1)^k(1-q^{4k+2})(-q;q^2)_k(q^4;q^2)_{k-1}}{(q^2;q^2)_k(-q^3;q^2)_k}q^{4k^2+3k}\\
&= 1+\sum_{k=1}^{\infty}\frac{(-1)^k(1-q^{2k+1})(1+q^{2k+1})(1+q)}{(1-q^2)(1+q^{2k+1})}q^{4k^2+3k}.
\end{aligned} \qquad (4.1.22)$$

Rearranging (4.1.22), and then setting $k = -n-1$ in the second sum below, we see that

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;q^2)_k}q^{k^2+2k} = 1-q+\sum_{k=1}^{\infty}(-1)^k(1-q^{2k+1})q^{4k^2+3k}$$

$$\begin{aligned}
&= 1 - q + \sum_{k=1}^{\infty}(-1)^k q^{4k^2+3k} - \sum_{k=1}^{\infty} q^{4k^2+5k+1}\\
&= 1 - q + \sum_{k=1}^{\infty}(-1)^k q^{4k^2+3k} + \sum_{n=-2}^{-\infty} (-1)^n q^{4n^2+3n}\\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2+3n}\\
&= f(-q^7, -q)\\
&= (q;q^8)_\infty (q^7;q^8)_\infty (q^8;q^8)_\infty,
\end{aligned}$$

by an appeal to the Jacobi triple product identity (3.1.18). Hence,

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(-q;q^2)_k}{(q^2;q^2)_k} q^{k^2+2k} &= \frac{(q^2;q^4)_\infty (q;q^8)_\infty (q^7;q^8)_\infty (q^8;q^8)_\infty}{(q;q^2)_\infty (q^2;q^2)_\infty}\\
&= \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty},
\end{aligned}$$

after cancellation. Thus, the proof is complete. □

Theorem 4.1.11 (Combinatorial Versions of the Göllnitz–Gordon Identities)**.** *(a) The first identity, Theorem 4.1.9, is equivalent to the statement: The number of partitions of* n *into distinct parts, with at least* 2 *between parts, and with no consecutive even parts, is equal to the number of partitions of* n *into parts congruent to* $1, 4,$ *or* 7 *modulo* 8*.*

(b) The second identity, Theorem 4.1.10, is equivalent to the statement: The number of partitions of n *into distinct parts, with at least* 2 *between parts, with no consecutive even parts, and with all parts* ≥ 3*, is equal to the number of partitions of* n *into parts congruent to* $3, 4,$ *or* 5 *modulo* 8*.*

Proof. On the left-hand side of (4.1.19), $1/(q^2;q^2)_k$ generates partitions into less than or equal to k even parts. Consider the Ferrers graph of such a partition. Next, $(-q;q^2)_k$ generates partitions into less than or equal to k distinct odd parts. Add such a partition with parts in decreasing order to the previous Ferrers graph. Note that if we examine two successive parts that are obtained by adjoining odd parts, then these parts differ by at least two. Now recall that $k^2 = 1+3+\cdots+(2k-1)$. Take the Ferrers graph above and add to it, in decreasing order, $(2k-1), (2k-3), \ldots, 1$. Thus, we now have a partition into exactly k parts. We see that in those instances where we had two successive odd parts before our last additions, that now we will have even parts, and that the even parts will differ by at least 4. In other words, there are no consecutive even parts. The partition-theoretic interpretation of the right-hand side of (4.1.19) is clear.

The proof of the second identity is very similar. The first two steps in constructing a Ferrers graph are the same. Now we note that, referring to the left-hand side of (4.1.21), $k^2 + 2k = (2k+1) + (2k-1) + \cdots + 3$. Thus, when we add these odd numbers to the

Ferrers graph obtained after the first two steps, the last part of the graph is at least 3. The partition-theoretic interpretation of the right-hand side of (4.1.21) is obvious. □

Example 4.1.12. *Let $n = 8$. The partitions of 8 into parts differing by at least 2 and with no consecutive even integers are:*

$$8, 7+1, 6+2, 5+3.$$

The partitions of 8 into parts congruent to 1, 4, 7 modulo 8 are

$$1+1+1+1+1+1+1+1, 4+1+1+1+1+, 7+1, 4+4.$$

The partitions of 8 into parts differing by at least 2, with no consecutive even integers are, and with all parts at least 3 are:

$$8, 5+3.$$

The partitions of 8 into parts congruent to 3, 4, 5 modulo 8 are

$$5+3, 4+4.$$

4.2. Exercises

1. Give a proof of Lebesgue's Identity, Theorem 4.1.1, by using Heine's transformation, Theorem 3.2.1.

2. Give a combinatorial proof of Lebesgue's Identity, Theorem 4.1.1.

Chapter 5

The Combinatorics of Theorems from Ramanujan's Lost Notebook

5.1. Simpler Theorems

In Ramanujan's last letter to Hardy, dated 12 January 1920, he stated the following identity without proof.

Theorem 5.1.1. *If* $|q| < 1$, *then*

$$\sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_n} = 1 + \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(q^{m+1})_{m+1}}. \tag{5.1.1}$$

Corollary 5.1.2 (Partition Theoretic Interpretation of Theorem 5.1.1)**.** *The number of partitions of* N *in which the smallest part does not repeat and the largest part is at most twice the smallest part equals the number of partitions of* N *where the largest part is odd and the smallest part is larger than half of the largest part.*

Example 5.1.3. *Let* $N = 7$. *Then the partitions of* 7 *satisfying the conditions on the left and right sides of Corollary 5.1.2 are, respectively,*

$$7 = 4 + 3 = 2 + 2 + 2 + 1; \qquad 7 = 3 + 2 + 2 = 1 + 1 + 1 + 1 + 1 + 1 + 1.$$

Let $N = 10$. *Then the desired partitions are, respectively,*

$$10 = 6 + 4 = 4 + 4 + 2;\ 5 + 5 = 3 + 3 + 2 + 2 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1.$$

Before commencing our proof, we offer some wide-spread notation in the theory of partitions.

Definition 5.1.4. *Let π be a partition. Then*

$$\lambda(\pi) = \text{the largest part of } \pi; \tag{5.1.2}$$

$$\nu(\pi) = \text{the number of parts of } \pi.. \tag{5.1.3}$$

Proof. Consider the Ferrers graph of a partition π such that $\lambda(\pi) = 2m + 1$; hence, the smallest part $\geq m + 1$. Suppose that $\nu(\pi) = n$. As an example, consider the partition $31 = 9+8+8+6$. Thus, $m = 4$ and $n = 6$. Isolate the Durfee $(m+1)\times n$ rectangle. Note that the number of such partitions is $q^{2m+1}/(q^{m+1})_{m+1}$, which is the general summand on the right side of (5.1.1). In our example, this rectangle has 5 columns and 4 rows. To the right of this rectangle is the Ferrers graph of another partition π_r. In our example, $\pi_r = 4 + 3 + 3 + 1$. Note that $\nu(\pi_r) \leq n$ and $\lambda(\pi_r) = m$. Now place this partition below the rectangle to obtain the Ferrers graph of a new partition, which we read from left to right, i.e., in columns. We note that the smallest part is n, and it does not repeat (since $m < m+1$). The largest part is $\leq 2n$. We now see that the generating function for such partitions is the series on the left in (5.1.1). □

5.2. Exercises

1. Give a q-series proof of Theorem 5.1.1.

Chapter 6

Analogues of Euler's Theorem; Partitions with Gaps

6.1. Motivation

Before proceeding further, let us introduce a standard, more economical notation for the partition of an integer. Instead of writing, for example, a partition with seven 5's in it by $\cdots 5+5+5+5+5+5+5+\cdots$, we shall write 5^7. We give further examples below.

Recall Euler's fundamental theorem: The number of partitions of a positive integer n into distinct parts is identical to the number of partitions of n into odd parts. If we arrange the parts of a partition of n in descending order, say, $a_1 > a_2 > \cdots > a_j$, then the distance between successive parts is at least 1. On the other hand, if we have a partition of n into odd parts, say, $b_1 > b_2 > \cdots > b_r$, then $b_i - b_j \equiv 0 \pmod 2$, or in other words, $b_i \equiv 1 \pmod 2$. Recall the definitions of the Rogers–Ramanujan functions,

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}.$$

The famous Rogers–Ramanujan identities are given by

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \qquad \text{and} \qquad H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{6.1.1}$$

The first identity is equivalent to: The set of partitions of n wherein the difference between any two parts is at least 2 is equinumerous with the set of partitions into parts congruent to either 1 or 4 modulo 5, i.e. the parts are $\equiv \pm 1 \pmod 5$.

The natural question that one should ask now is: Suppose the parts of a partition have a distance of at least 3 between successive parts. Is this set of partitions equinumerous with a set of partitions in which the parts belong to prescribed residue classes?

Suppose we have a class of partitions such that the gap between any two parts is at least 3. If we perform some calculations, it appears that partitions with parts congruent to $1, 5 \pmod 6$ are equinumerous with partitions with a distance of at least 3 between parts. In fact, the sizes of these sets of partitions are the same for $1 \leq n \leq 8$. However, when we reach 9, we find a discrepancy. The partitions of 9 with a gap of at least 3 between parts are: 9, 8+1, 7+2, 6+3. The partitions with parts congruent to 1 or 5 modulo 6 are: $71^2, 51^4, 1^9$. The first set has four representations, and the second set has 3 representations. For $n = 10, 11, 12, 13, 14$, the two aforementioned classes of partitions are equinumerous. Consider now $n = 15$. Those partitions with gaps of at least 3 between successive parts are: 15, 14+1, 13+2, 12+3, 11+4, 10+5, 10+4+1, 9+6, 9+5+1, 8+5+2. Those partitions of 15 with all parts congruent to 1 or 5 modulo 6 are: $1^2 13, 1^4 11, 17^2, 1^3 57, 1^8 7, 5^3, 1^5 5^2, 1^{10} 5, 1^{15}$. There are 10 elements in the first class, and 9 in the second class. Is there a requirement that would eliminate some of the partitions in the first class? Readers should refrain here from peeking at the answer, which will be given at the beginning of the next paragraph.

What we need to eliminate are those representations with consecutive multiples of 3, i.e., remove $6 + 3$ and $9 + 6$, respectively, from the two sets of partitions with gaps of at least 3 above. If we do so in the two examples above, or in any of the other cases, we will find that the sets are equinumerous.

We offer a remark here to indicate that, generally, if we prescribe the congruence conditions for the parts of partitions, then it is more difficult to find, if possible, the corresponding partitions with gaps. Normally, it is easier to begin with a description of the partitions with gaps. Depending on the starting conditions, we thus want to compute b_n or a_n in the generating function $F(q)$, which we write in the form

$$F(q) := \sum_{n=0}^{\infty} b_n q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-a_n}, \qquad b_0 = 1. \tag{6.1.2}$$

In our example, $a_n = 0$, when $n \equiv 0, 2, 3, 4 \pmod 6$; $a_n = 1$, when $n \equiv 1, 5 \pmod 6$.

Our goal is to compute a generating function for the sequence or sequences satisfying the aforementioned gap and congruence conditions. Taking the logarithmic derivative of (6.1.2), we find that

$$\frac{F'(q)}{F(q)} = -\frac{d}{dq} \sum_{n=1}^{\infty} a_n \log(1 - q^n) = \sum_{n=1}^{\infty} \frac{a_n n q^{n-1}}{1 - q^n},$$

from which it follows that

$$\begin{aligned} qF'(q) &= F(q) \sum_{n=1}^{\infty} \frac{a_n n q^n}{1 - q^n} = F(q) \sum_{n=1}^{\infty} a_n n \sum_{m=1}^{\infty} q^{mn} \\ &= F(q) \sum_{N=1}^{\infty} q^N \sum_{n \mid N} n a_n = F(q) \sum_{N=1}^{\infty} D_N q^N, \end{aligned} \tag{6.1.3}$$

where

$$D_N = \sum_{n|N} na_n.$$

Equating coefficients of q^M, $M \geq 1$, on both sides of (6.1.3) and using (6.1.2), we find that

$$Mb_M = \sum_{j=0}^{M} b_j D_{M-j}, \qquad \textit{with } D_0 = 0.$$

Thus, we have found a recurrence relation for $\{b_M\}$ in terms of $\{a_j\}$. Conversely, given $\{b_j\}$, we can calculate $\{a_j\}$. As an example, if $a_n \equiv 1$, then $b_n = p(n)$, and furthermore

$$Mp(M) = \sum_{j=0}^{M-1} p(j)\sigma(M-j).$$

which is in the spirit of several results that we had derived in the last portions of Chapter 1.

As with our example, $a_n = 0$, when $n \equiv 0, 2, 3, 4 \pmod 6$; $a_n = 1$, when $n \equiv 1, 5 \pmod 6$, it is usually more difficult to impose congruence conditions than it is to impose gap conditions. Let $\sigma(\pi)$ denote a partition, and let $\nu(\pi)$ denote the number of parts of π. Consider

$$f(z,q) := \sum_{\substack{\pi \\ \text{min. dif.}\geq 3 \\ \text{no consec. mult. of 3}}} z^{\nu(\pi)} q^{\sigma(\pi)} = \sum_{m,n-0}^{\infty} c(m,n) z^m q^n. \tag{6.1.4}$$

Let $f_i(z,q)$ denote the same function as in (6.1.4), but with the added condition that the smallest part of π is $> i$. Thus,

$$f_0(z,q) = f(z,q),$$

and

$$f_0(z,q) - f_1(z,q)$$

generates paritítions with minimal part equal to 1. Now delete 1 from all the partitions π. Subtract 3 from each of the remaining parts. Note that before subtracting 3 from each part, each part now is ≥ 4. Thus,

$$f_0(z,q) - f_1(z,q) = zqf_0(zq^3, q). \tag{6.1.5}$$

Now observe that $f_1(z,q) - f_2(z,q)$ is a general function of admissible partitions with 2 appearing. We repeat the argument above by subtracting 2 from each part of each partition. Hence, deleting 2 from each part of each partition π and subtracting 3 from each part, we find that

$$f_1(z,q) - f_2(z,q) = zq^2 f_1(zq^3, q). \tag{6.1.6}$$

Repeat the argument, but now recall that no consecutive multiples of 3 appear. So instead of all the parts being greater than 2, they really are all greater than 3. Hence,

$$f_2(z,q) - f_3(z,q) = zq^3 f_3(zq^3, q). \tag{6.1.7}$$

If we now subtract 3 from each part, we are back to our original setting, i.e.,

$$f_3(z,q) = f_0(zq^3,q). \tag{6.1.8}$$

Rewrite (6.1.5) in the form

$$f_1(z,q) = f_0(z,q) - zqf_0(zq^3,q). \tag{6.1.9}$$

Substitute (6.1.8) into (6.1.7) twice, once with z replaced by zq^3, to obtain

$$f_2(z,q) = f_0(zq^3,q) + zq^3 f_0(zq^6,q). \tag{6.1.10}$$

Now, substitute (6.1.9) and (6.1.10) into (6.1.6) to deduce that

$$\{f_0(z,q) - zqf_0(zq^3,q)\} - \{f_0(zq^3,q) + zq^3 f_0(zq^6,q)\} \\ = zq^2\{f_0(zq^3,q) - zq^4 f_0(zq^6,q)\},$$

which we solve for $f_0(z,q)$. We now delete the second argument, so that we have

$$f_0(z) = (1 + zq + zq^2)f_0(zq^3) + (zq^3 - z^2q^6)f_0(zq^6). \tag{6.1.11}$$

Now define

$$\varphi(z) = \frac{f_0(z)}{(z;q^3)_\infty}. \tag{6.1.12}$$

Thus, if we divide both sides by $(zq^3;q^3)_\infty$, (6.1.11) can be rewritten in the form

$$\begin{aligned}(1-z)\varphi(z) &= (1 + zq + zq^2)\varphi(zq^3) + zq^3(1 - zq^3)\frac{f_0(zq^6)}{(zq^3;q^3)_\infty} \\ &= (1 + zq + zq^2)\varphi(zq^3) + zq^3\varphi(zq^6).\end{aligned} \tag{6.1.13}$$

Let

$$\varphi(z) = \sum_{n=0}^{\infty} \alpha_n z^n, \qquad \varphi(0) = \alpha_0 = f_0(0) = 1. \tag{6.1.14}$$

Substitute (6.1.14) into (6.1.13) and equate coefficients of z^n to deduce that

$$\alpha_n - \alpha_{n-1} = \alpha_n q^{3n} + \alpha_{n-1}q^{3n-2} + \alpha_{n-1}q^{3n-1} + \alpha_{n-1}q^{6n-3},$$

which can be rewritten in the shape

$$\alpha_n = \frac{(1+q^{3n-2})(1+q^{3n-1})}{1-q^{3n}}\alpha_{n-1}. \tag{6.1.15}$$

Iterating (6.1.15) n times and employing the initial condition $\alpha_0 = 1$, we deduce that

$$\alpha_n = \frac{(-q;q^3)_n(-q^2;q^3)_n}{(q^3;q^3)_n}. \tag{6.1.16}$$

We now can conclude from (6.1.16) and (6.1.12) that

$$\begin{aligned}f(z,q) = f_0(z,q) = f_0(z) &= (z;q^3)_\infty \sum_{n=0}^{\infty} \frac{(-q;q^3)_n(-q^2;q^3)_n}{(q^3;q^3)_n} z^n \\ &= (z;q^3)_\infty {}_2\phi_1(-q,-q^2;0;q^3;z).\end{aligned} \tag{6.1.17}$$

We now apply Heine's transformation, Theorem 3.2.1, with q replaced by q^3, $a=-q$, $b=-q^2$, and $c=0$, on the right-hand side of (6.1.17) to deduce that

$$f(z,q)=(-q^2;q^3)_\infty(-qz;q^3)_\infty {}_2\phi_1(0,z;-qz;q^3;-q^2). \tag{6.1.18}$$

Let $z=1$ and note that $(1)_n=0$, $n\geq 1$. Thus,

$${}_2\phi_1(0,1;-q;q^3;-q^2)=1.$$

Thus, from (6.1.18), we can conclude that

$$\begin{aligned} f(1,q)&=(-q^2;q^3)_\infty(-q;q^3)_\infty\\ &=\frac{(-q;q^3)_\infty(-q^2;q^3)_\infty(q;q^3)_\infty(q^2;q^3)_\infty}{(q;q^3)_\infty(q^2;q^3)_\infty}\\ &=\frac{(q^2;q^6)_\infty(q^4;q^6)_\infty}{(q;q^3)_\infty(q^2;q^3)_\infty}\\ &=\frac{(q^2;q^6)_\infty(q^4;q^6)_\infty}{(q;q^6)_\infty(q^4;q^6)_\infty(q^2;q^6)_\infty(q^5;q^6)_\infty}\\ &=\frac{1}{(q;q^6)_\infty(q^5;q^6)_\infty}. \end{aligned} \tag{6.1.19}$$

We have just proved the following theorem, originally due to I. Schur in 1926 [**94**].

Theorem 6.1.1 (Schur)**.** *The set of partitions of n with minimal difference at least equal to 3, and with no consecutive multiples of 3 is equinumerous with the set of partitions of n into parts that are congruent to 1 or 5 modulo 6.*

Definition 6.1.2. *Let $S_1(n)$ denote the number of partitions of n into parts which differ by at least 3 and with no consecutive multiples of 3.*

Let $S_2(n)$ denote the number of partitions of n into distinct parts, with none of them divisible by 3.

Let $S_3(n)$ denote the number of partitions of n into parts congruent to either 1 or 5 modulo 6.

From the far right side of (6.1.19) and our construction, we have proved the following theorem.

Theorem 6.1.3. *We have*

$$\sum_{n=0}^{\infty} S_1(n)q^n=\frac{1}{(q;q^6)_\infty(q^5;q^6)_\infty}. \tag{6.1.20}$$

The next theorem is trivial.

Theorem 6.1.4. *We have*

$$\sum_{n=0}^{\infty} S_2(n)q^n=(-q;q^3)_\infty(-q^2;q^3)_\infty.$$

n	partitions, $R_2(n)$	partitions in S	conclusion
1	1	1	$1 \in S$
2	2	$1+1$	2 not $\in S$
3	3	$1+1+1$	3 not $\in S$
4	$4, 3+1$	$1^4, 4$	$4 \in S$
5	$5, 4+1$	$1^5, 4+1$	5 not $\in S$
6	$6, 5+1$	$1^6, 4+1+1$	6 not $\in S$
7	$7, 6+1, 5+2$	$1^7, 41^3, 7$	$7 \in S$
8	$8, 71, 62, 53$	$1^8, 4^2, 41^4, 1^8$	8 not $\in S$
9	$9, 81, 72, 63, 531$	$1^9, 4^21, 41^5, 71^2, 9$	$9 \in S$
10	$(10), 91, 82, 73, 631$	$1^{10}, 4^21^2, 41^6, 71^3, 91$	10 not $\in S$
11	$(11), (10)1, 92, 83, 74, 731$	$1^{11}, 4^21^3, 41^7, 71^4, 91^2, 74$	11 not $\in S$

Table 1. Partitions Enumerated by $R_2(n)$

From the far left side of (6.1.18) and the two previous theorems, we have established the following corollary.

Corollary 6.1.5. *We have*

$$S_1(n) = S_2(n) = S_3(n).$$

Definition 6.1.6. *Let $R_d(n)$ denote the number of partitions of n into parts that differ by at least d with no consecutive multiples of d allowed.*

Note that, by Theorem 6.1.3, for $d = 3$, we have proved that these partitions are equinumerous with the set of partitions into parts congruent to 1 or 5 modulo 6. For $d = 1$, we cannot have partitions into parts with no consecutive multiples of 1. However, if we drop the requirement that no consecutive multiples of 2 are allowed but require that the parts differ by at least 2, then these partitions, by the first Rogers–Ramanujan identity, are equinumerous with the set of partitions into parts that are congruent to 1 or 4 modulo 5. Let us see if we can find a generating function for $R_2(n)$, or find a set of congruences, say, $S = S(n)$, that is identical in number with the cardinality of $R_2(n)$.

Let us explain the reasoning in the right-hand column of Table 1. We see that 1 must belong to the targeted set S, for otherwise we would have no partitions in S, and the desired equal cardinalities could not happen. If $2 \in S$, then we would have two partitions in S, and so equal cardinalities would be impossible. The same reasoning holds for 3. There are two partitions enumerated by $R_2(4)$, and so to get two partitions in S, 4 must belong to S. Proceeding in this manner, up to $n = 11$, we find that $1, 4, 7, 9 \in S$. We see that S cannot be described by congruence classes modulo 2, 3, 5, 6, or 7. However, congruence classes modulo 8 might be possible from this limited table. Indeed, this conjecture is correct.

Theorem 6.1.7. *The cardinality of $R_2(n)$, $n \geq 1$, is equal to that of the subset S, where*

$$S = \{n : n \equiv 1, 4, 7 \pmod 8; n \geq 1\}.$$

Proof. Recall that $\sigma(\pi)$ denotes a partition and $\nu(\pi)$ denotes the number of parts of π. Define

$$g_i(z, q) := \sum_{\substack{\pi \\ \text{parts differ by } \geq 2 \\ \text{no consec. mult. of 2} \\ \text{each part} > i}} z^{\nu(\pi)} q^{\sigma(\pi)}. \tag{6.1.21}$$

Deleting the argument q from our notation, we proceed exactly in the same manner as we did in the proof of Theorem 6.1.3. The partitions in the difference $g_0(z) - g_1(z)$ have minimal part equal to 1. We delete 1 from each partition, and so

$$g_0(z) - g_1(z) = zqg_0(zq^2), \quad \text{or} \quad g_1(z) = g_0(z) - zqg_0(zq^2). \tag{6.1.22}$$

Deleting 2 from each partition and remembering that no consecutive multiples of 2 are allowed, we find that

$$g_1(z) - g_2(z) = zq^2 g_2(zq^2). \tag{6.1.23}$$

Furthermore,

$$g_2(z) = g_0(zq^2). \tag{6.1.24}$$

Put (6.1.22) and (6.1.24) into (6.1.23) to find that

$$\{g_0(z) - zqg_0(zq^2)\} - g_0(zq^2) = zq^2 g_0(zq^4),$$

which we rearrange in the form

$$g_0(z) = (1 + zq)g_0(zq^2) + zq^2 g_0(zq^4). \tag{6.1.25}$$

Now let

$$g_0(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g_0(0) = a_0 = 1. \tag{6.1.26}$$

Putting (6.1.26) into (6.1.25) and equating coefficients of z^n, $n \geq 1$, on both sides, we find that

$$a_n = a_n q^{2n} + a_{n-1} q^{2n-1} + a_{n-1} q^{4n-2},$$

which we reformulate in the recurrence relation

$$a_n \frac{q^{2n-1}(1 + q^{2n-1})}{1 - q^{2n}} a_{n-1}, \quad n \geq 1. \tag{6.1.27}$$

Iterating (6.1.27) n times and recalling that $a_0 = 1$, we deduce that

$$a_n = \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n}, \quad n \geq 0, \tag{6.1.28}$$

which, when put in (6.1.26), yields

$$g_0(z) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} z^n.$$

To complete the proof of the theorem that we sought, rewritten combinatorially below, we need to evaluate $g_0(1)$, which is not easy. One needs to use Watson's q-analogue of Whipple's theorem. In fact,

$$g_0(1) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty}. \tag{6.1.29}$$

□

We have just then "almost proved" the following theorem.

Theorem 6.1.8. *The number of partitions of* n *with parts differing by at least* 2 *and with no consecutive multiples of* 2 *is equal to the number of partitions of* n *into parts congruent to* $1, 4, 7$ *modulo* 8.

It is natural to ask if Theorem 6.1.8 has an analogue if we replace "no consecutive multiples of 2" by "no successive odd parts." The next result provides an affirmative answer.

Theorem 6.1.9. *The number of partitions of* n *into distinct parts, with a difference of at least* 2 *between parts, with no successive odd parts, and with each part at least* 2 *is equal to the number of partitions of* n *into parts congruent to* $2, 3, 7$ *modulo* 8.

Proof. Let

$$g_i(z,q) := g_i(z) := \sum_{\substack{\text{parts differ by at least 2} \\ \text{no consecutive odd parts} \\ \text{each part } > i}} z^{\nu(\pi)} q^{\sigma(\pi)}.$$

We want to determine $g_1(1)$. First, observe that

$$g_3(z) = g_1(zq^2). \tag{6.1.30}$$

Second, observe that if we want to determine those partitions with smallest part equal to 2, then

$$g_1(z) - g_2(z) = zq^2 g_1(zq^2). \tag{6.1.31}$$

If we want to determine those partitions with smallest part precisely equal to 3, then

$$g_2(z) - g_3(z) = zq^3 g_3(zq^2). \tag{6.1.32}$$

One might think that on the right side of (6.1.32), we should have $g_2(zq^2)$ instead of $g_3(zq^2)$. But remember that we are forbidden to have the successive pair of parts, $3, 5$, and for this reason g_3 appears, and not g_2. Using (6.1.30)–(6.1.32), we see that

$$\begin{aligned} g_1(z) &= \{g_3(z) + zq^3 g_3(zq^2)\} + zq^2 g_1(zq^2) \\ &= g_1(zq^2) + zq^3 g_1(zq^4) + zq^2 g_1(zq^2) \\ &= \left(1 + zq^2\right) g_1(zq^2) + zq^3 g_1(zq^4). \end{aligned} \tag{6.1.33}$$

If we set

$$g_1(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then equating coefficients of q^n on both sides of (6.1.33), we deduce that

$$a_n(1 - q^{2n}) = a_{n-1} q^{2n}(1 + q^{2n-1}), \qquad n \geq 1,$$

or

$$a_n = \frac{q^{2n}(1 + q^{2n-1})}{1 - q^{2n}} a_{n-1}.$$

If we iterate the recurrence relation above and note that $a_0 = 1$, we find that

$$a_n = \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n^2+n}, \qquad n \geq 1.$$

Hence,

$$g_1(z) = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n^2+n} z^n.$$

To evaluate $g_1(1)$, we apply Lebesque's theorem, Theorem 4.1.1, with q replaced by q^2 and with $b = 1/q$. Hence,

$$\begin{aligned} g_1(1) &= \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n^2+n} = \frac{(-q^3; q^4)_\infty}{(q^2; q^4)_\infty} \\ &= \frac{(q^6; q^8)_\infty}{(q^2; q^4)_\infty (q^3; q^4)_\infty} \qquad (6.1.34) \\ &= \frac{1}{(q^2; q^8)_\infty (q^3; q^8)_\infty (q^7; q^8)_\infty}, \qquad (6.1.35) \end{aligned}$$

which completes the proof of Theorem 6.1.9. □

Definition 6.1.10. *Let $q_d(n)$ denote the number of partitions of n with parts differing by at least d. Let $Q_d(n)$ be the number of partitions of n with parts congruent to $\pm 1 \pmod{d+3}$.*

In 1956, H. L. Alder [**4**] made the following conjecture.

Conjecture 6.1.11.

$$\Delta_d(n) = q_d(n) - Q_d(n) \geq 0, \qquad \textit{for all } d \textit{ and } n. \tag{6.1.36}$$

Let us examine some special cases. Suppose that $d = 1$. Then $q_1(n)$ denotes the number of partitions into distinct parts, while $Q_1(n)$ denotes the number of partitions into parts $\equiv \pm 1 \pmod 4$, i.e., odd parts. By Euler's Theorem, $\Delta_1(n) = 0$ Thus (6.1.36) is valid for $d = 1$.

Take $d = 2$. Thus, $q_2(n)$ stands for the number of partitions of n into parts differing by at least 2, while $Q_2(n)$ denotes the number of partitions of n into parts congruent to $\pm 1 \pmod 5$. By the first Rogers–Ramanujan identity, $q_2(n) = Q_2(n)$, and so $\Delta_2(n) = 0$. Therefore, Alder's conjecture is valid for $d = 2$.

Let $d = 3$. Then $q_3(n)$ denotes the number of partitions of n into parts differing by at least 3, and $Q_3(n)$ denotes the number of partitions of n into parts congruent to $\pm 1 \pmod 6$. Let $q_{c3}(n)$ denote the number of partitions of n into parts differing by at least 3 and with consecutive multiples of 3. Then Schur's Theorem can be cast in the form $q_3(n) - q_{c3}(n) = Q_3(n)$. Thus,

$$\Delta_3(n) = q_3(n) - Q_3(n) = q_3(n) - \{q_3(n) - q_{c3}(n)\} = q_{c3}(n) \geq 0.$$

In 1971, Andrews [**7**] proved Alder's Conjecture for $d = 2^r - 1$, $r \geq 4$. No further progress was made for over 30 years until A. J. Yee [**104**], [**105**] proved Alder's Conjecture for all $r \geq 31$ and for $r = 7$. Finally, C. Alfes, M. Jameson, and R. Lemke Oliver [**5**] used the *circle method* and computation to handle the remaining cases.

The next theorem, at first glance, does not appear to have an aesthetic appeal, and the proof does not seem to be enchanting either. It is due to J. J. Sylvester [**98**], and will serve as motivation to one of the most elementary proofs of the Rogers–Ramanujan identities, due to Andrews. The proof itself also has interesting consequences, which we will point out along the way. However, the method is deceptively elegant and powerful.

Theorem 6.1.12. *Let*

$$\begin{aligned} S(N;x,q) = 1 &+ \sum_{j=1}^{\infty} \begin{bmatrix} N+1-j \\ j \end{bmatrix} (-xq;q)_{j-1} q^{j(3j-1)/2} x^j \\ &+ \sum_{j=1}^{\infty} \begin{bmatrix} N-j \\ j \end{bmatrix} (-xq;q)_{j-1} q^{3j(j+1)/2} x^{j+1}. \end{aligned} \tag{6.1.37}$$

Then, for each positive integer N,

$$S(N;x,q) = (-xq;q)_N. \tag{6.1.38}$$

Observe that both sums in (6.1.37) are finite.

Proof. We first combine together the two sums in (6.1.37), while also using the definition of the Gaussian binomial coefficients. Then we add and subtract q^j within the expression in curly brackets. Next, we simplify the sums, and replace the summation index j by

$j+1$ in the first sum. Consequently, in order, we deduce that

$$\begin{aligned} S(N;x,q) &= 1+\sum_{j=1}^{\infty}\frac{(q)_{N-j}(-xq)_{j-1}q^{j(3j-1)/2}x^j}{(q)_j(q)_{N+1-2j}}\left\{(1-q^{N+1-j})+(1-q^{N+1-2j})xq^{2j}\right\} \\ &= 1+\sum_{j=1}^{\infty}\frac{(q)_{N-j}(-xq)_{j-1}q^{j(3j-1)/2}x^j}{(q)_j(q)_{N+1-2j}}\left\{(1-q^{j})+q^j(1+xq^j)(1-q^{N+1-2j})\right\} \\ &= 1+\sum_{j=0}^{\infty}\frac{(q)_{N-j-1}(-xq)_{j}q^{(j+1)(3j+2)/2}x^{j+1}}{(q)_j(q)_{N-1-2j}} \\ &\quad +\sum_{j=1}^{\infty}\frac{(q)_{N-j}(-xq)_{j}q^{j(3j+1)/2}x^j}{(q)_j(q)_{N-2j}}. \end{aligned} \tag{6.1.39}$$

If we now let $N\to\infty$ in (6.1.39) and assume, for the moment, that (6.1.38) holds, then we deduce Theorem 1.2.25. Thus, (6.1.37) can be considered to be a finite analogue of (1.2.20). We continue with the proof of Theorem 6.1.12.

Extract the term with $j=0$ from the first series on the right-hand side of (6.1.39). Then factor out $(1+xq)$ from both series that remain. Also, change the order of the two infinite series. Hence,

$$\begin{aligned} S(N;x,q) &= 1+xq+(1+xq)\left\{\sum_{j=1}^{\infty}\begin{bmatrix}(N-1)+1-j\\ j\end{bmatrix}(-xq^2)_{j-1}q^{j(3j-1)/2}(xq)^j\right. \\ &\quad \left.+\sum_{j=1}^{\infty}\begin{bmatrix}(N-1)-j\\ j\end{bmatrix}(-xq^2)_{j-1}q^{j(3j+3)/2}(xq)^{j+1}\right\} \\ &= (1+xq)S(N-1;xq;q) \\ &= (1+xq)(1+xq^2)S(N-2;xq^2;q) \\ &= \cdots \\ &= (-xq;q)_N S(0;xq^N,q). \end{aligned} \tag{6.1.40}$$

Return to the definition of $S(N;x,q)$ in (6.1.37), and set $N=0$. Recall that $\begin{bmatrix}n\\ m\end{bmatrix}=0$, for $m>n$. Thus, we see that in (6.1.37)

$$\begin{bmatrix}1-j\\ j\end{bmatrix},\quad \begin{bmatrix}-j\\ j\end{bmatrix}=0,\quad j\geq 1.$$

Thus, $S(0;xq^N,q)=1$. Hence, from (6.1.40), we see that we have proved (6.1.38). □

Let us return to (6.1.39) and let $N\to\infty$. We then obtain the following corollary, which can be thought of as a generalization of the pentagonal number theorem.

Corollary 6.1.13. *We have*

$$(-xq;q)_\infty = \sum_{j=0}^{\infty} \frac{(-xq)_j}{(q)_j} q^{(3j^2+j)/2} x^j \left\{q^{2j+1}x+1\right\}.$$

If we set $x=-1$ in Corollary 6.1.13, we obtain the pentagonal number theorem 1.2.26.

We next study, for nonnegative integers h, k,

Definition 6.1.14.

$$\begin{aligned} C_{k,h}(N;x,q) = &\sum_{j=0}^{\infty} \begin{bmatrix} N+h-kj \\ j \end{bmatrix} (xq)_j(-1)^j x^{kj} q^{(2k+1)j(j+1)/2-hj} \\ &- \sum_{j=0}^{\infty} \begin{bmatrix} N-kj \\ j \end{bmatrix} (xq)_j(-1)^j x^{kj+h} q^{(2k+1)j(j+1)/2+hj+h}. \end{aligned} \tag{6.1.41}$$

In particular, if $h=k=1$, then (6.1.41) reduces to

$$\begin{aligned} C_{1,1}(N;x,q) = &\sum_{j=0}^{\infty} \begin{bmatrix} N+1-j \\ j \end{bmatrix} (xq)_j(-1)^j x^{j} q^{3j(j+1)/2-j} \\ &- \sum_{j=0}^{\infty} \begin{bmatrix} N-j \\ j \end{bmatrix} (xq)_j(-1)^j x^{j+1} q^{3j(j+1)/2+j+1}. \end{aligned} \tag{6.1.42}$$

Extract the terms with $j=0$ in (6.1.42), namely, $1-xq$. In the remaining series (beginning with $j=1$), factor out $1-xq$. We then find that

$$C_{1,1}(N;x,q) = (1-xq)S(N;-xq,q) = S(N+1;-x,q), \tag{6.1.43}$$

by (6.1.37) and the second equality in (6.1.40).

We also note that

$$C_{k,0}(N;x,q) = 0, \tag{6.1.44}$$

since the terms cancel.

Theorem 6.1.15. *If* $k,h \geq 1$, *then*

$$\begin{aligned} &C_{k,h}(N;x,q) - C_{k,h-1}(N;x,q) \\ &\qquad = x^{h-1}q^{h-1}(1-xq)C_{k,k-h+1}(N-(k-h+1);xq,q). \end{aligned} \tag{6.1.45}$$

Proof. Combining the two series and invoking the second q-analogue of Pascal's formula (Chapter 2, Exercise 5), and noting that the terms with $j = 0$ cancel, we find that

$$
\begin{aligned}
&C_{k,h}(N;x,q) - C_{k,h-1}(N;x,q) \\
&= \sum_{j=0}^{\infty} \left\{ \begin{bmatrix} N+h-kj \\ j \end{bmatrix} - q^j \begin{bmatrix} N+h-1-kj \\ j \end{bmatrix} \right\} (xq)_j (-1)^j x^{kj} q^{(2k+1)j(j+1)/2-hj} \\
&\quad + \sum_{j=0}^{\infty} \begin{bmatrix} N-kj \\ j \end{bmatrix} (1-xq^{j+1})(xq)_j (-1)^j x^{kj+h-1} q^{(2k+1)j(j+1)/2+(h-1)j+h-1} \\
&= \sum_{j=1}^{\infty} \begin{bmatrix} N+h-1-kj \\ j-1 \end{bmatrix} (xq)_j (-1)^j x^{kj} q^{(2k+1)j(j+1)/2-hj} \\
&\quad + \sum_{j=0}^{\infty} \begin{bmatrix} N-kj \\ j \end{bmatrix} (xq)_{j+1} (-1)^j x^{kj+h-1} q^{(2k+1)j(j+1)/2+(h-1)j+h-1} \\
&= \sum_{j=0}^{\infty} \begin{bmatrix} N+h-1-kj-k \\ j \end{bmatrix} (xq)_{j+1} (-1)^{j+1} x^{k(j+1)} q^{(2k+1)(j+1)(j+2)/2-h(j+1)} \\
&\quad + \sum_{j=0}^{\infty} \begin{bmatrix} N-kj \\ j \end{bmatrix} (xq)_{j+1} (-1)^j x^{kj+h-1} q^{(2k+1)j(j+1)/2+(h-1)j+h-1}, \qquad (6.1.46)
\end{aligned}
$$

where we replaced j by $j+1$ in the first sum. Returning to (6.1.46), we switch the order of the two series above, extract common factors from both sums, and refer to (6.1.41) to conclude that

$$
\begin{aligned}
&C_{k,h}(N;x,q) - C_{k,h-1}(N;x,q) \\
&= x^{h-1}q^{h-1}(1-xq) \left\{ \sum_{j=0}^{\infty} \begin{bmatrix} N-kj \\ j \end{bmatrix} (xq^2)_j (-1)^j (xq)^{kj} q^{(2k+1)j(j+1)/2-(k-h+1)j} \right. \\
&\left. - \sum_{j=0}^{\infty} \begin{bmatrix} N-(k-h+1)-kj \\ j \end{bmatrix} (xq^2)_j (-1)^j (xq)^{kj+k-h+1} q^{(2k+1)j(j+1)/2+(k-h+1)j+k-h+1} \right\} \\
&= x^{h-1}q^{h-1}(1-xq) C_{k,k-h+1}(N-(k-h+1);xq,q). \qquad (6.1.47)
\end{aligned}
$$

Equation (6.1.47) is in agreement with (6.1.45), and so the proof is complete. □

As we shall see, the preceding theorems inspired Andrews to give a new proof of the Rogers–Ramanujan identities [**12**].

Theorem 6.1.16. *We have*

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \qquad (6.1.48)$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \qquad (6.1.49)$$

These identities were first proved by L. J. Rogers in 1894 [**93**], but his paper was hardly noticed. Ramanujan rediscovered them in India sometime probably between 1912 and early 1914 before he left for England. However, he did not possess a proof of the identities. He had communicated them in his first letter to G. H. Hardy, but unfortunately the page of the letter containing the identities has been lost. Published with Ramanujan's lost notebook [**91**] is a short manuscript that Ramanujan wrote containing four reasons why Ramanujan believed that the identities were true [**16**, Chapter 10]. After his arrival in Cambridge, the identities became famous, and their combinatorial interpretations were first found by P. A. MacMahon, who in his epic two volumes [**76**] presented them as open problems.

Corollary 6.1.17. *(a) The number of partitions of a positive integer n into parts differing by at least 2 is equal to the number of partitions of n into parts congruent to either 1 or 4 modulo 5.*

(b) The The number of partitions of a positive integer n into parts differing by at least 2 and with no 1's is equal to the number of partitions of n into parts congruent to either 2 or 3 modulo 5.

Proof. We first prove (a). It is clear that the right-hand side of (6.1.48) generates all partitions into parts congruent to either 1 or 4 modulo 5. On the left-hand side, consider the Ferrers graph of a partition π of an integer N arranged into less than or equal to n decreasing parts. We now recall that $1+3+\cdots+(2n-1)=n^2$, $1\leq n<\infty$. Now add to the Ferrers graph of π $2n-1$ nodes in the first row, $2n-3$ nodes in the second row, etc. and 1 node in the nth row, irrespective of whether the partition π actually has an nth part. We thus obtain a partition into exactly n distinct parts, and the difference between any two parts is at least equal to 2.

Consider next part (b). The combinatorial interpretation of the right-hand side of (6.1.49) should be clear. As in the proof of (a), consider the Ferrers graph of a partition π with less than or equal to n parts arranged in descending order. Recall that $2+4+\cdots+2n=n(n+1)$. Adjoin the parts $2n$, $2n-2$,. . . ,2 in descending order to the Ferrers graph of π. We then obtain a partition into exactly n parts, each differing from any other part by at least 2. However, the last part is at least equal to 2, because we added 2 to the last part of π. Hence, this establishes the combinatorial interpretation of the second Rogers–Ramanujan identity. □

Example 6.1.18. *To illustrate the first Rogers–Ramanujan identity, consider the partitions of 8 into the two types:*

$$8=7+1=6+2=5+3,$$
$$6+1+1=4+4=4+1+1+1+1=1+1+1+1+1+1+1+1.$$

To illustrate the second identity, again consider 8 with the following partitions:

$$8=6+2=5+3,$$
$$8=3+3+2=2+2+2+2.$$

One day during his stay at Cambridge, Ramanujan was perusing old issues of the *Proceedings of the London Mathematical Society*, and he saw that Rogers [**93**] had proved the identities that he had conjectured. Shortly thereafter, Ramanujan found his own proof. Rogers was contacted, and he himself found another proof, whereupon Hardy arranged for the two proofs to be published together [**85**]. It was first observed by R. A. Askey that, ironically, in India, Ramanujan had actually established a theorem from which the Rogers–Ramanujan identities could be deduced as corollaries. That identity is Entry 7 in Chapter 16 of Ramanujan's second notebook [**90**], [**26**, p. 16], and the aforementioned proof by Askey is found in [**26**, pp. 77–78]. One can also find a short history of the Rogers–Ramanujan identities in [**26**, pp. 77–79]. A much more complete and informative survey of proofs of the Rogers–Ramanujan identities has been written by Andrews [**13**].

It is now time to give Andrews's proof of the Rogers–Ramanujan identities [**12**] stimulated by the work of Sylvester.

Proof. Let $h = k = 2$ in (6.1.45) in Theorem 6.1.15 to deduce that

$$C_{2,2}(N;x,q) - C_{2,1}(N;x,q) = xq(1-xq)C_{2,1}(N-1;xq,q). \tag{6.1.50}$$

Next, set $h = 1$ and $k = 2$ in (6.1.45) and recall (6.1.44). Thus,

$$C_{2,1}(N;x,q) = (1-xq)C_{2,2}(N-2;xq,q). \tag{6.1.51}$$

Eliminate $C_{2,1}(N;x,q)$ from (6.1.50) and (6.1.51) to arrive at

$$C_{2,2}(N;x,q) = (1-xq)C_{2,2}(N-2;xq,q)+xq(1-xq)(1-xq^2)C_{2,2}(N-3;xq^2,q). \tag{6.1.52}$$

Define

$$D(N;x,q) := \sum_{0\le 2j\le N} \begin{bmatrix} N-j \\ j \end{bmatrix} x^j q^{j^2}. \tag{6.1.53}$$

Note that if we set $x = 1$ and $x = q$ in (6.1.53), and let $N \to \infty$, we obtain, respectively, the Rogers–Ramanujan functions $G(q)$ and $H(q)$ in (6.1.48) and (6.1.49), respectively.

Next, define

$$\Delta(N;x,q) = C_{2,2}(N;x,q) - (xq)_{[N/2]+1} D\left(\left[\tfrac{N}{2}\right]+2;x,q\right). \tag{6.1.54}$$

We now compute some polynomials related to $C_{2,2}(i;a,q)$, $D\left(\left[\frac{i}{2}\right]+2;a,q\right)$, and $\Delta(i;a,q)$.

On the basis of Table 2, we make the following conjecture, which we prove.

Theorem 6.1.19. *For each nonnegative integer N,*

$$x^{-2}q^{-2N-1}\Delta(2N-1;x,q) \quad \textit{and} \quad x^{-3}q^{-2N-4}\Delta(2N;x,q) \tag{6.1.55}$$

are polynomials in x and q.

i	$C_{2,2}(i;a,q)$	$(aq)_{[i/2]+1}D\left(\left[\frac{i}{2}\right]+2;a,q\right)$	$\Delta(i;a,q)$
0	$1-a^2q^2$	$(1-aq)(1+aq)$	0
1	$1-a^2q^2-a^2q^3+a^3q^4$	$(1-aq)(1+aq)$	$-a^2q^3+a^3q^4$
2	$1-a^2q^2-a^2q^3-a^2q^4$ $+a^3q^4+a^3q^5$	$(1-aq)(1-aq^2)$ $\times(1+aq+aq^2)$	0
3	$1-a^2q^2-a^2q^3-a^2q^4$ $-a^2q^5+a^3q^4+a^3q^5$ $+a^3q^6+a^4q^9-a^5q^{10}$	$(1-aq)(1-aq^2)$ $\times(1+aq+aq^2)$	$-a^2q^5+a^3q^6$ $+a^4q^9-a^5q^{10}$

Table 2. Table of Polynomials

Proof. It is clear from Table 2 that Theorem 6.1.19 is true for $N=0,1,2,3$. We use induction. Using the second q-analogue of Pascal's formula from Exercise 5 in Chapter 2, we write

$$D(N;x,q)=\sum_{j=0}^{\infty}\left(\begin{bmatrix}N-j-1\\ j-1\end{bmatrix}+q^j\begin{bmatrix}N-j-1\\ j\end{bmatrix}\right)x^jq^{j^2}.$$

We now switch the order of the two sums above and replace j by $j+1$ in what was formerly the first sum to find that

$$\begin{aligned}D(N;x,q)&=D(N-1;xq,q)+\sum_{j=0}^{\infty}\begin{bmatrix}N-j-2\\ j\end{bmatrix}x^{j+1}q^{(j+1)^2}\\ &=D(N-1;xq,q)+xqD(N-2;xq^2,q).\end{aligned}\tag{6.1.56}$$

Next, after using the distributive law, we observe that the terms with $j=0$ cancel. We appeal to the first q-analogue of Pascal's formula in Exercise 5 in Chapter 2. Thus,

$$\begin{aligned}&(1-xq^N)D(N+1;x,q)-D(N;x,q)\\ &=\sum_{j=0}^{\infty}\left(\begin{bmatrix}N+1-j\\ j\end{bmatrix}x^jq^{j^2}-\begin{bmatrix}N-j\\ j\end{bmatrix}x^jq^{j^2}\right)\\ &\quad-xq^N\sum_{j=0}^{\infty}\begin{bmatrix}N+1-j\\ j\end{bmatrix}x^jq^{j^2}\\ &=\sum_{j=1}^{\infty}\begin{bmatrix}N-j\\ j-1\end{bmatrix}x^jq^{N+(j-1)^2}-xq^N\sum_{j=0}^{\infty}\begin{bmatrix}N+1-j\\ j\end{bmatrix}x^jq^{j^2}\\ &=\sum_{j=0}^{\infty}\begin{bmatrix}N-j-1\\ j\end{bmatrix}x^{j+1}q^{N+j^2}-xq^N\sum_{j=0}^{\infty}\begin{bmatrix}N+1-j\\ j\end{bmatrix}x^jq^{j^2}\end{aligned}$$

$$= xq^N \sum_{j=0}^{\infty} \left(\begin{bmatrix} N-j-1 \\ j \end{bmatrix} - \begin{bmatrix} N+1-j \\ j \end{bmatrix} \right) x^j q^{j^2}. \tag{6.1.57}$$

We now apply the first of the two q-analogues of Pascal's formula from Exercise 5 of Chapter 2 to each of the q-binomial coefficients on the far right side of (6.1.57). Accordingly,

$$\begin{aligned} \begin{bmatrix} N-j-1 \\ j \end{bmatrix} - \begin{bmatrix} N+1-j \\ j \end{bmatrix} &= \begin{bmatrix} N-j \\ j \end{bmatrix} - q^{N-2j} \begin{bmatrix} N-j-1 \\ j-1 \end{bmatrix} \\ &\quad - \left(\begin{bmatrix} N-j \\ j \end{bmatrix} + q^{N-2j+1} \begin{bmatrix} N-j \\ j-1 \end{bmatrix} \right) \\ &= -q^{N-2j+1} \begin{bmatrix} N-j \\ j-1 \end{bmatrix} - q^{N-2j} \begin{bmatrix} N-j-1 \\ j-1 \end{bmatrix}. \end{aligned}$$

Substituting the foregoing calculation into (6.1.57), we deduce that

$$\begin{aligned} &(1-xq^N)D(N+1;x,q) - D(N;x,q) \\ &= -xq^N \sum_{j=0}^{\infty} \left(q^{N-2j+1} \begin{bmatrix} N-j \\ j-1 \end{bmatrix} + q^{N-2j} \begin{bmatrix} N-j-1 \\ j-1 \end{bmatrix} \right) x^j q^{j^2}. \end{aligned}$$

The term with $j = 0$ is equal to 0, and so we replace j by $j+1$ above and recall the definition of $D(N;x,q)$ in (6.1.53) to deduce that

$$\begin{aligned} &(1-xq^N)D(N+1;x,q) - D(N;x,q) \\ &= -xq^{2N} \sum_{j=0}^{\infty} \left(q^{-2j-1} \begin{bmatrix} N-1-j \\ j \end{bmatrix} + q^{-2j-2} \begin{bmatrix} N-2-j \\ j \end{bmatrix} \right) x^{j+1} q^{j^2+2j+1} \\ &= -x^2 q^{2N} D(N-1;x,q) - x^2 q^{2N-1} D(N-2;x,q). \end{aligned} \tag{6.1.58}$$

Assume now that Theorem 6.1.19 is valid for each subscript $< 2N$. Return to the definition of D in (6.1.54) and use (6.1.52) and (6.1.56) to deduce that

$$\begin{aligned} &\Delta(2N;x,q) = C_{2,2}(2N;x,q) - (xq)_{N+1} D(N+2;x,q) \\ &= (1-xq)C_{2,2}(2N-2;xq,q) + xq(1-xq)(1-xq^2)C_{2,2}(2N-3;xq^2,q) \\ &\quad - (xq)_{N+1} \left\{ D(N+1;xq,q) + xqD(N;xq^2,q) \right\} \\ &= (1-xq)\Delta(2(N-1);xq,q) + xq(1-xq)(1-xq^2)\Delta(2(N-1)-1;xq^2,q). \end{aligned} \tag{6.1.59}$$

Multiplying both sides of (6.1.59) by $x^{-3}q^{-2N-4}$, we arrive at

$$\begin{aligned} &x^{-3}q^{-2N-4}\Delta(2N;x,q) \\ &= q(1-xq)(xq)^{-3}q^{-2(N-1)-4}\Delta(2(N-1);xq,q) \\ &\quad + (1-xq)(1-xq^2)(xq^2)^{-2}q^{-2(N-1)-1}\Delta(2(N-1)-1;xq^2,q). \end{aligned} \tag{6.1.60}$$

Applying the induction hypothesis, we complete the proof in the case that the index is even.

We next consider the case when the index is odd and assume that the theorem is valid for all smaller indices. We once again use (6.1.52) and (6.1.56). However, in this case, we must add and subtract $xq(xq)_{N+2}D(N+1;xq^2,q)$ in order to bring us to $\Delta(2N-2;xq^2,q)$. Hence,

$$\begin{aligned}
\Delta(2N+1;x,q) &= C_{2,2}(2N+1;x,q) - (xq)_{N+1}D(N+2;x,q)\\
&= (1-xq)C_{2,2}(2N-1;xq,q) + xq(1-xq)(1-xq^2)C_{2,2}(2N-2;xq^2,q)\\
&\quad - (xq)_{N+1}\left\{D(N+1;xq,q) + xqD(N;xq^2,q)\right\}\\
&= (1-xq)\Delta(2N-1;xq,q) + xq(1-xq)(1-xq^2)\Delta(2N-2;xq^2,q)\\
&\quad + xq(xq)_{N+2}D(N+1;xq^2,q) - xq(xq)_{N+1}D(N;xq^2,q)\\
&= (1-xq)\Delta(2N-1;xq,q) + xq(1-xq)(1-xq^2)\Delta(2N-2;xq^2,q)\\
&\quad + xq(xq)_{N+1}\left\{-x^2q^4q^{2N}D(N-1;xq^2,q) - x^2q^4q^{2N-1}D(N-2;xq^2,q)\right\}, \qquad (6.1.61)
\end{aligned}$$

where we have employed (6.1.58). Multiply both sides of (6.1.61) by $x^{-2}q^{-2N-3}$ to see that

$$\begin{aligned}
&x^{-2}q^{-2N-3}\Delta(2N+1;x,q)\\
&= (1-xq)(xq)^{-2}q^{-2N-1}\Delta(2N-1;xq,q)\\
&\quad + x^2q^6(1-xq)(1-xq^2)(xq^2)^{-3}q^{-2N-2}\Delta(2N-2;xq^2,q)\\
&\quad - xq^2(xq)_{N+1}D(N-1;xq^2,q) - xq(xq)_{N+1}\Delta(N-2;xq^2,q). \qquad (6.1.62)
\end{aligned}$$

By induction and inspection, we see that each expression on the right-hand side of (6.1.62) is a polynomial in x and q. Thus, we have completed the proof for odd index. Hence, with our conclusions from (6.1.60) and (6.1.62), we have completed the proof of Theorem 6.1.19. □

We now complete Andrews's proof of the Rogers–Ramanujan identities.

Let us fix r and n and examine the coefficients of x^rq^n in $D(N;x,q)$ and $C_{2,2}(N;x,q)$. If N is sufficiently large, say $N \geq N_0(r,n)$, then these coefficients of x^rq^n are constant. To see this, we observe that as N increases, the new terms in q have powers larger than n.

Let us examine carefully $x^{-2}q^{-2N-1}\Delta(2N-1;x,q)$, and put $m = 2N-1$. From Theorem 6.1.19, we see that if $-2N-1+n < 0$, or equivalently, $m \geq n-1$, then this constant coefficient of q^n in $\Delta(2N-1;x,q)$ must be equal to 0. Suppose next that $m = 2N$. Then, by Theorem 6.1.19, if $-2N-4+n < 0$, or equivalently, $m \geq n-3$, this constant coefficient of q^n in $\Delta(2N;x,q)$ must also be equal to 0. In general, if $m \geq n-1$, the coefficient of q^n in

$$C_{2,2}(m;x,q) - (xq)_{[m/2]+1}D([\tfrac{m}{2}]+2;x,q)$$

must be equal to 0. In particular, if we let $m \to \infty$, we must conclude that the coefficient of q^n in

$$C_{2,2}(\infty;x,q) - (xq)_\infty D(\infty;x,q)$$

is equal to 0. In other words,

$$C_{2,2}(\infty; x, q) - (xq)_\infty D(\infty; x, q) \equiv 0. \tag{6.1.63}$$

Returning to the definition (6.1.53), we see that

$$\begin{aligned}\lim_{N\to\infty} D(N; 1, q) &= \lim_{N\to\infty} \sum_{0\le 2j\le N} \begin{bmatrix} N-j \\ j \end{bmatrix} q^{j^2} \\ &= \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_j},\end{aligned} \tag{6.1.64}$$

where we have used the notation from (6.1.48). Next, from the definition of $C_{2,2}$ in (6.1.41),

$$\begin{aligned}\lim_{N\to\infty} C_{2,2}(N; 1, q) &= \lim_{N\to\infty} \left(\sum_{j=0}^{\infty} \begin{bmatrix} N+2-2j \\ j \end{bmatrix} (q)_j (-1)^j q^{5j(j+1)/2-2j} \right. \\ &\qquad \left. - \sum_{j=0}^{\infty} \begin{bmatrix} N-2j \\ j \end{bmatrix} (q)_j (-1)^j q^{5j(j+1)/2+2j+2} \right) \\ &= \sum_{j=0}^{\infty} (-1)^j q^{j(5j+1)/2} - \sum_{j=0}^{\infty} (-1)^j q^{j(5j+9)/2+2} \\ &= \sum_{j=0}^{\infty} (-1)^j q^{j(5j+1)/2} + \sum_{n=-1}^{-\infty} (-1)^n q^{n(5n+1/2)} \\ &= \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \\ &= f(-q^3, -q^2) \\ &= (q^3; q^5)(q^2; q^5)(q^5; q^5) \\ &= \frac{(q;q)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty},\end{aligned} \tag{6.1.65}$$

where in the anti-penultimate line we set $j = -n-1$, and lastly invoked the Jacobi triple product identity (3.1.18).

Now put (6.1.64) and (6.1.65) in (6.1.63), with $x = 1$, to finally deduce that

$$\frac{(q;q)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} = (q;q)_\infty \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_j},$$

which proves the first Rogers–Ramanujan identity (6.1.48).

To prove the second identity, return to (6.1.51) and let $N \to \infty$, and then use (6.1.63). Hence,

$$\begin{aligned} C_{2,1}(\infty; x, q) &= (1 - xq)C_{2,2}(\infty; xq, q) \\ &= (1 - xq)(xq^2)_\infty D(\infty; xq, q) \\ &= (xq)_\infty D(\infty; xq, q). \end{aligned} \tag{6.1.66}$$

If $x = 1$, then (6.1.66) reduces to

$$C_{2,1}(\infty; 1, q) = (q)_\infty D(\infty; q, q) = (q)_\infty \sum_{j=0}^{\infty} \frac{q^{j(j+1)}}{(q;q)_j}. \tag{6.1.67}$$

Now, setting $k = 2$ and $h = 1$ in (6.1.41), and replacing j by $-j - 1$ in the second sum in the third equality below, we find that

$$\begin{aligned} \lim_{N\to\infty} C_{2,1}(N; 1, q) &= \lim_{N\to\infty} \left(\sum_{j=0}^{\infty} \begin{bmatrix} N+1-2j \\ j \end{bmatrix} (q)_j (-1)^j q^{5j(j+1)/2-j} \right. \\ &\qquad \left. - \sum_{j=0}^{\infty} \begin{bmatrix} N-2j \\ j \end{bmatrix} (q)_j (-1)^j q^{5j(j+1)/2+j+1} \right) \\ &= \sum_{j=0}^{\infty} (-1)^j q^{5j(j+1)/2-j} - \sum_{j=0}^{\infty} (-1)^j q^{5j(j+1)/2+j+1} \\ &= \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+3)/2} \\ &= f(-q^4, -q) \\ &= (q;q^5)_\infty (q^4;q^5)_\infty (q^5;q^5)_\infty = \frac{(q;q)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \end{aligned} \tag{6.1.68}$$

upon the invoking of the Jacobi triple product identity (3.1.18). The second Rogers–Ramanujan identity (6.1.48) follows immediately from (6.1.67) and (6.1.68). □

The ideas of Sylvester and Andrews have not been as fully exploited as they should be. In an email to the lecturer on February 9, 2014, Andrews wrote, "I do think that there is a lot left to do." Other papers by Andrews relevant to his proof and that of Sylvester are [**9**], [**14**], and [**15**].

6.2. The Rogers–Ramanujan Continued Fraction

A continued fraction is an expression of the sort

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots}}}}, \tag{6.2.1}$$

which is commonly written in the more compact form

$$b_0 + \frac{a_1}{b_1} \, {}_+ \, \frac{a_2}{b_2} \, {}_+ \, \frac{a_3}{b_3} \, {}_+ \, \frac{a_4}{b_4} \, {}_+ \cdots. \tag{6.2.2}$$

Suppose that we define the sequences P_n and Q_n, $n \geq -1$, by

$$\begin{gathered} P_n = b_n P_{n-1} + a_n P_{n-2}, \qquad n \geq 0, \\ Q_n = b_n Q_{n-1} + a_n Q_{n-2}, \qquad n \geq 0, \\ P_{-1} = 1, \quad Q_{-1} = 0, \quad P_0 = b_0, \quad Q_0 = 1. \end{gathered}$$

One can readily check that

$$\frac{P_n}{Q_n} := b_0 + \frac{a_1}{b_1} \, {}_+ \, \frac{a_2}{b_2} \, {}_+ \cdots {}_+ \, \frac{a_n}{b_n}. \tag{6.2.3}$$

Then if

$$\lim_{n\to\infty} \frac{P_n}{Q_n}$$

exists, we say that the continued fraction (6.2.2) converges; otherwise it diverges. For theorems providing criteria for the convergence or divergence of (6.2.2), we refer readers to the excellent text by L. Lorentzen and H. Waadeland [**75**, Chapter 1]. In particular, see [**75**, p. 35, Theorem 3].

Most mathematics students first encounter continued fractions in a course in elementary number theory. The first infinite continued fractions that students may be asked to evaluate are those in Exercise 6 below. These are, in fact, special cases of perhaps the most interesting continued fraction in mathematics, the Rogers–Ramanujan continued fraction, which first appeared in a paper by L. J. Rogers [**93**] in 1894.

Definition 6.2.1. *The Rogers–Ramanujan continued fraction $R(q)$ is defined by*

$$R(q) := \frac{q^{1/5}}{1} \, {}_+ \, \frac{q}{1} \, {}_+ \, \frac{q^2}{1} \, {}_+ \, \frac{q^3}{1} \, {}_+ \cdots, \tag{6.2.4}$$

provided that it converges. Furthermore, set

$$T(q) := 1 + \frac{q}{1} \, {}_+ \, \frac{q^2}{1} \, {}_+ \, \frac{q^3}{1} \, {}_+ \cdots. \tag{6.2.5}$$

By Exercise 6, we see that $R(\pm 1)$ converges. In general, $R(q)$ converges for $|q|<1$, but for $|q|=1$, the problem of convergence is open. However, at roots of unity on the unit circle, by a theorem of I. Schur [**95**, pp. 319–321] and Ramanujan [**90**], [**27**, p. 35], we do know the behavior of $R(q)$.

Theorem 6.2.2. *Recall that $T(q)$ is defined by* (6.2.5). *Let q be a primitive mth root of unity. If m is a multiple of* 5, *$T(q)$ diverges. Otherwise, $T(q)$ converges and*

$$T(q) = \alpha T(\alpha) q^{(1-\alpha\rho m)/5}, \tag{6.2.6}$$

where α denotes the Legendre symbol $\left(\frac{m}{5}\right)$ and ρ is the least positive residue of m modulo 5.

In our completion of this chapter, our primary aim is to establish the connection between the Rogers–Ramanujan functions $G(q)$ and $H(q)$ and the Rogers–Ramanujan continued fraction $R(q)$. More precisely, Rogers [**93**] and Ramanujan [**90**, Vol. II, Chapter 16, Sect. 15], [**26**, p. 30] proved that

$$R(q) = q^{1/5}\frac{H(q)}{G(q)}. \tag{6.2.7}$$

In fact, we first prove a *finite form* of (6.2.7), which can be found as Entry 16 in Chapter 16 of Ramanujan's second notebook [**90**], [**26**, p. 31], and from which (6.2.7) follows as an immediate corollary.

Theorem 6.2.3. *For each nonnegative integer n, let*

$$\mu := \mu_n(a,q) := \sum_{k=0}^{[(n+1)/2]} \frac{(q)_{n-k+1} a^k q^{k^2}}{(q)_k (q)_{n-2k+1}}, \tag{6.2.8}$$

$$\nu := \nu_n(a,q) := \sum_{k=0}^{[n/2]} \frac{(q)_{n-k} a^k q^{k(k+1)}}{(q)_k (q)_{n-2k}}, \tag{6.2.9}$$

where $[x]$ denotes the greatest integer less than or equal to x. Then, for $n \geq 1$,

$$\frac{\mu}{\nu} = 1 + \frac{aq}{1} \, + \, \frac{aq^2}{1} \, + \cdots + \, \frac{aq^n}{1}. \tag{6.2.10}$$

Proof. For each nonnegative integer r, define

$$F_r := F_r(a,q) := \sum_{k=0}^{[(n-r+1)/2]} \frac{(q)_{n-r-k+1} a^k q^{k(r+k)}}{(q)_k (q)_{n-r-2k+1}}.$$

Observe that

$$F_0 = \mu \qquad \text{and} \qquad F_1 = \nu. \tag{6.2.11}$$

Also note that

$$F_n = 1 \qquad \text{and} \qquad F_{n-1} = 1 + aq^n. \tag{6.2.12}$$

We now develop a recurrence relation for F_r. When we combine the two sums in the first step below, we conventionally set $1/(q)_{-1} = 0$. To that end,

$$\begin{aligned}
F_r - F_{r+1} &= \sum_{k=0}^{[(n-r+1)/2]} \frac{(q)_{n-r-k+1} a^k q^{k(r+k)}}{(q)_k (q)_{n-r-2k+1}} \\
&\quad - \sum_{k=0}^{[(n-r)/2]} \frac{(q)_{n-r-k} a^k q^{k(r+1+k)}}{(q)_k (q)_{n-r-2k}} \\
&= \sum_{k=1}^{[(n-r+1)/2]} \frac{(q)_{n-r-k} a^k q^{k(r+k)}}{(q)_k (q)_{n-r-2k}} \left(\frac{1 - q^{n-r-k+1}}{1 - q^{n-r-2k+1}} - q^k \right) \\
&= \sum_{k=1}^{[(n-r+1)/2]} \frac{(q)_{n-r-k} a^k q^{k(r+k)}}{(q)_k (q)_{n-r-2k}} \frac{(1 - q^k)}{(1 - q^{n-r-2k+1})} \\
&= \sum_{k=1}^{[(n-r+1)/2]} \frac{(q)_{n-r-k} a^k q^{k(r+k)}}{(q)_{k-1} (q)_{n-r-2k+1}} \\
&= \sum_{j=0}^{[(n-r-1)/2]} \frac{(q)_{n-r-j-1} a^{j+1} q^{(j+1)(r+j+1)}}{(q)_j (q)_{n-r-2j-1}} \\
&= aq^{r+1} \sum_{j=0}^{[(n-r-1)/2]} \frac{(q)_{n-(r+2)-j+1} a^j q^{j(r+2+j)}}{(q)_j (q)_{n-(r+2)-2j+1}} \\
&= aq^{r+1} F_{r+2}. \qquad (6.2.13)
\end{aligned}$$

Using (6.2.11), (6.2.13) repeatedly, and lastly (6.2.12), we conclude that

$$\begin{aligned}
\frac{\mu}{\nu} &= \frac{F_0}{F_1} = \frac{F_1 + aqF_2}{F_1} = 1 + \frac{aq}{F_1/F_2} \\
&= 1 + \frac{aq}{(F_2 + aq^2 F_3)/F_2} = 1 + \frac{aq}{1} + \frac{aq^2}{F_2/F_3} \\
&= 1 + \frac{aq}{1} + \frac{aq^2}{1} + \cdots + \frac{aq^{n-1}}{F_{n-1}/F_n} \\
&= 1 + \frac{aq}{1} + \frac{aq^2}{1} + \cdots + \frac{aq^{n-1}}{1} + \frac{aq^n}{1}.
\end{aligned}$$

□

Corollary 6.2.4. *For any complex number a and $|q| < 1$,*

$$\frac{\displaystyle\sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(q)_k}}{\displaystyle\sum_{k=0}^{\infty} \frac{a^k q^{k(k+1)}}{(q)_k}} = 1 + \frac{aq}{1} + \frac{aq^2}{1} + \cdots + \frac{aq^n}{1} + \cdots. \qquad (6.2.14)$$

Proof. Let $n \to \infty$ in (6.2.10). □

The continued fraction in (6.2.14) is called the *Generalized Rogers–Ramanujan Continued Fraction*.

Recall that the Rogers–Ramanujan identities are given by

$$G(q)=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \qquad \text{and} \qquad H(q)=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{6.2.15}$$

Using (6.2.15), we immediately deduce the elegant representation for $R(q)$ in the next theorem.

Theorem 6.2.5. *We have*

$$R(q)=q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{6.2.16}$$

Proof. Set $a=1$ in Corollary 6.2.4, take the reciprocal of both sides, use the definitions of $G(q)$ and $H(q)$ in (6.2.15), and lastly use (6.1.1). □

We now establish a very important formula for the Rogers–Ramanujan continued fraction that has had many applications, including the explicit evaluation of the Rogers–Ramanujan continued fraction at certain arguments.

Theorem 6.2.6. *If* $T(q)$ *is defined in* (6.2.5). *Then*

$$T(q^5)-q-\frac{q^2}{T(q^5)}=\frac{(q;q)_\infty}{(q^{25};q^{25})_\infty}. \tag{6.2.17}$$

We now prove a more general theorem from which Theorem 6.2.6 follows by specialization. We employ the notation (1.1.3).

Theorem 6.2.7. *For any complex number* a,

$$\begin{aligned}(a,a^2,q/a,q/a^2,q;q)_\infty \\ =(q^5;q^5)_\infty\Bigg(&\frac{(a^5q;q^5)_\infty(a^{-5}q^4;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}-a\frac{(a^5q^2;q^5)_\infty(a^{-5}q^3;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\\ &-a^2\frac{(a^5q^3;q^5)_\infty(a^{-5}q^2;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}+a^3\frac{(a^5q^4;q^5)_\infty(a^{-5}q;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}\Bigg).\end{aligned} \tag{6.2.18}$$

Before proving Theorem 6.2.7, we show that Theorem 6.2.6 follows immediately from Theorem 6.2.7.

Proof of Theorem 6.2.6. If we replace q by q^5 in (6.2.18), then set $a=q$, realize that $(1;q^{25})_\infty=0$, and recall (6.2.16), we deduce (6.2.17) forthwith. □

The decompositions in Theorems 6.2.6 and 6.2.7 are called 5-*dissections*, because in the former theorem, the series terms of $(q;q)_\infty$ are separated out in powers of q according to their residue classes modulo 5, and in the latter theorem, the terms are separated out in powers of a according to their residue classes modulo 5.

Proof of Theorem 6.2.7. Using the Jacobi triple product identity (1.1.7) twice, we find that

$$\begin{aligned}(a, a^2, q/a, q/a^2, q; q)_\infty &= \frac{(a, q/a, q; q)_\infty (a^2, q/a^2, q; q)_\infty}{(q; q)_\infty} \\ &= \frac{1}{(q; q)_\infty} \sum_{r=-\infty}^{\infty} (-1)^r a^r q^{(r^2-r)/2} \sum_{s=-\infty}^{\infty} (-1)^s a^{2s} q^{(s^2-s)/2} \\ &= \frac{1}{(q; q)_\infty} \sum_{r,s=-\infty}^{\infty} (-1)^{r+s} a^{r+2s} q^{(r^2-r+s^2-s)/2} \\ &= \sum_{n=-\infty}^{\infty} a^n c_n(q), \end{aligned} \tag{6.2.19}$$

where, for $-\infty < n < \infty$,

$$c_n(q) := \frac{1}{(q; q)_\infty} \sum_{\substack{r,s=-\infty \\ r+2s=n}}^{\infty} (-1)^{r+s} q^{(r^2-r+s^2-s)/2}.$$

We now determine $c_n(q)$ according to the residue class of n modulo 5.

First, consider the residue class $0 \pmod 5$. Replace n by $5n$ and make the change of variables $r = n - 2t$ and $s = 2n + t$. Note that $r + 2s = 5n$. Then, simplifying and applying the Jacobi triple product identity (1.1.7), we find that

$$\begin{aligned} c_{5n}(q) &= \frac{1}{(q; q)_\infty} \sum_{t=-\infty}^{\infty} (-1)^{n+t} q^{((n-2t)^2-(n-2t)+(2n+t)^2-(2n+t))/2} \\ &= \frac{(-1)^n q^{(5n^2-3n)/2}}{(q; q)_\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{(5t^2+t)/2} \\ &= \frac{(-1)^n q^{(5n^2-3n)/2}}{(q; q)_\infty} f(-q^3, -q^2) \\ &= \frac{(-1)^n q^{(5n^2-3n)/2}}{(q; q^5)_\infty (q^4; q^5)_\infty}. \end{aligned} \tag{6.2.20}$$

Second, consider the residue class $1 \pmod 5$. Set $r = n - 2t + 1$ and $s = 2n + t$, so that $r + 2s = 5n + 1$. Then, upon simplification and the use of the Jacobi triple product identity (1.1.7), we see that

$$\begin{aligned} c_{5n+1}(q) &= \frac{1}{(q; q)_\infty} \sum_{t=-\infty}^{\infty} (-1)^{n+t+1} q^{((n-2t+1)^2-(n-2t+1)+(2n+t)^2-(2n+t))/2} \\ &= \frac{(-1)^{n+1} q^{(5n^2-n)/2}}{(q; q)_\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{(5t^2-3t)/2} \\ &= \frac{(-1)^{n+1} q^{(5n^2-n)/2}}{(q; q)_\infty} f(-q, -q^4) \end{aligned}$$

$$= \frac{(-1)^{n+1}q^{(5n^2-n)/2}}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{6.2.21}$$

It should now be clear how to calculate the three remaining cases,

$$c_{5n+2}(q) = \frac{(-1)^{n+1}q^{(5n^2+n)/2}}{(q^2;q^5)_\infty(q^3;q^5)_\infty}, \tag{6.2.22}$$

$$c_{5n+3}(q) = \frac{(-1)^{n}q^{(5n^2+3n)/2}}{(q;q^5)_\infty(q^4;q^5)_\infty}, \tag{6.2.23}$$

$$c_{5n+4}(q) = 0. \tag{6.2.24}$$

Substitute (6.2.20)–(6.2.24) in (6.2.19) and use the Jacobi triple product identity (1.1.7) four times to conclude that

$$\begin{aligned}
&(a, a^2, q/a, q/a^2, q; q)_\infty \\
&= \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n a^{5n}q^{(5n^2-3n)/2} \\
&\quad - \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n a^{5n+1}q^{(5n^2-n)/2} \\
&\quad - \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n a^{5n+2}q^{(5n^2+n)/2} \\
&\quad + \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n a^{5n+3}q^{(5n^2+3n)/2} \\
&= (q^5;q^5)_\infty\left(\frac{(a^5q;q^5)_\infty(a^{-5}q^4;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} - a\frac{(a^5q^2;q^5)_\infty(a^{-5}q^3;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\right. \\
&\quad \left. -a^2\frac{(a^5q^3;q^5)_\infty(a^{-5}q^2;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} + a^3\frac{(a^5q^4;q^5)_\infty(a^{-5}q;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}\right).
\end{aligned}$$

Thus, the proof of Theorem 6.2.7 is complete. □

The results in this section can be greatly generalized in that many more general continued fractions can be represented as quotients of two q-series. See, in particular, [**16**, Chapter 6] and [**26**, pp. 30–31].

6.3. Exercises

1. Prove combinatorially that $S_2(n) = S_3(n)$.
2. Use Lebesque's Identity to evaluate $g_0(q)$.
3. Find an evaluation of (6.1.29) that is simpler than using Watson's q-analogue of Whipple's theorem.
4. Give a bijective proof of Corollary 6.1.13.

5. Prove (6.2.3).
6. Prove that

$$1+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\cdots=\frac{\sqrt{5}+1}{2} \tag{6.3.1}$$

and

$$1-\frac{1}{1}+\frac{1}{1}-\frac{1}{1}+\cdots=\frac{\sqrt{5}-1}{2}. \tag{6.3.2}$$

7. Prove (6.2.22)–(6.2.24).

Chapter 7

The Frobenius and Generalized Frobenius Symbols

7.1. First Generalization of Frobenius Partitions

Recall that in Chapter 3 we defined the Frobenius symbol in Definition 3.1.10.

Definition 7.1.1. *Consider the Ferrers graph of a positive integer n. Let r be the size of a Durfee square. Form the diagonal of the Durfee square, which will have r nodes. To the right of the diagonal is a graphical representation of a partition of no more than r parts, reading from top to bottom, say $a_1, a_2, \ldots, a_r$. To the left of the diagonal is a graphical representation of another partition of no more than r parts, reading from left to right, namely $b_1, b_2, \ldots, b_r$, say. Thus, $n = r + \sum_{j=1}^{r}(a_j + b_j)$. A matrix representation corresponding to these two partitions can be given by the Frobenius symbol*

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}.$$

From the constructive definition that we gave, it is clear that $a_1 > a_2 > \cdots > a_r$ and that $b_1 > b_2 > \cdots > b_r$. Note that it possible that a_r or b_r could be equal to 0. Recall that the number of such Frobenius symbols is $p(n)$. We will examine a couple generalizations.

Suppose that we allow repetition in any row at most k times. We keep the requirement that $n = r + \sum_{j=1}^{r}(a_j + b_j)$.

Example 7.1.2. *Let $k = 2$ and $n = 3$. Then the possible generalized Frobenius symbols are*

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with the respective values of $r = 1, 1, 1, 2, 2$.

Definition 7.1.3. *If we allow k repetitions in the Frobenius symbol, then we let $\varphi_k(n)$ denote the number of these* Generalized Frobenious Symbols.

Clearly, $\varphi_1(n) = p(n)$. Let

$$\Phi_k(q) := \sum_{n=0}^{\infty} \varphi_k(n) q^n. \tag{7.1.1}$$

At the beginning of Chapter 1, we showed that $\Phi_1(q) = 1/(q;q)_\infty$. We also showed in Chapter 3 that $\Phi_1(q)$ was the constant term in the product

$$\prod_{n=1}^{\infty}(1+zq^n)(1+q^{n-1}/z) = (-zq;q)_\infty(-1/z;q)_\infty.$$

By the same reasoning, we see that $\Phi_k(q)$ is the constant term

$$\begin{aligned} c_0 &:= [z^0]\prod_{n=1}^{\infty}(1+zq^n+\cdots+z^kq^{kn})(1+z^{-1}q^{n-1}+\cdots+z^{-k}q^{k(n-1)}) \\ &= [z^0]\prod_{n=1}^{\infty}\frac{(1-z^{k+1}q^{(k+1)n})}{(1-zq^n)}\frac{(1-z^{-(k+1)}q^{(k+1)(n-1)})}{(1-z^{-1}q^{n-1})}. \end{aligned} \tag{7.1.2}$$

In the next theorem, we prove that c_0 can be written as a product of theta functions. However, even with this representation, it remains very difficult to determine $\phi_k(n)$. With a little effort, we shall explicitly determine $\phi_2(n)$.

Theorem 7.1.4. *Recall that $f(a,b)$ is defined in* (1.1.6). *Let $\zeta = \exp(2\pi i/(k+1)$. Then*

$$c_0 = [z^0]\frac{1}{(q;q)_\infty^k}\prod_{j=1}^{k} f(\zeta^j zq, \zeta^{-j}z^{-1}). \tag{7.1.3}$$

Proof. First observe that

$$\prod_{j=0}^{k}(1-\zeta^j x) = 1 - x^{k+1}. \tag{7.1.4}$$

Using (7.1.4) twice in (7.1.2), replacing j by $k+1-j$ in the second product in the third equality below, and using the Jacobi triple product identity (3.1.17), we find that

$$\begin{aligned} c_0 &= [z^0]\prod_{n=1}^{\infty}\frac{1}{(1-zq^n)(1-z^{-1}q^{n-1})}\prod_{j=0}^{k}(1-\zeta^j zq^n)(1-\zeta^j z^{-1}q^{n-1}) \\ &= [z^0]\prod_{n=1}^{\infty}\prod_{j=1}^{k}(1-\zeta^j zq^n)(1-\zeta^j z^{-1}q^{n-1}) \\ &= [z^0]\prod_{n=1}^{\infty}\prod_{j=1}^{k}(1-\zeta^j zq^n)(1-\zeta^{-j} z^{-1}q^{n-1}) \\ &= [z^0]\frac{1}{(q;q)_\infty^k}\prod_{j=1}^{k}\prod_{n=1}^{\infty}(1-\zeta^j zq^n)(1-\zeta^{-j}z^{-1}q^{n-1})(1-q^n) \end{aligned}$$

$$= [z^0]\frac{1}{(q;q)_\infty^k}\prod_{j=1}^{k} f(\zeta^j zq, \zeta^{-j}z^{-1}),$$

and so the proof of (7.1.3) is complete. □

Let $k = 1$ in Theorem 7.1.4. Then $\Phi_1(q)$ is the constant term in

$$\begin{aligned}\Phi_1(q) &= [z^0]\frac{1}{(q;q)_\infty}\sum_{n_1=0}^{\infty}(-1)^{n_1}(-1)^{n_1}z^{n_1}q^{n_1(n_1+1)/2}\\ &= [z^0]\frac{1}{(q;q)_\infty}\sum_{n_1=0}^{\infty}z^{n_1}q^{n_1(n_1+1)/2}\\ &= \frac{1}{(q;q)_\infty}\cdot 1 = \frac{1}{(q;q)_\infty},\end{aligned}$$

in agreement with our previous observation in Chapters 1 and 3.

Theorem 7.1.5. *We have*

$$\Phi_2(q) = \frac{(q^6;q^{12})_\infty}{(q;q)_\infty(q^2;q^4)_\infty(q^3;q^6)_\infty}. \tag{7.1.5}$$

Proof. Let $\zeta = e^{2\pi i/3}$. If n_1 and n_2 are the two indices of summation for the theta functions in (7.1.3), we see that for the constant term, $n_1 = -n_2$. Hence, using the Jacobi product identity below (3.1.17), we find that

$$\begin{aligned}\Phi_2(q) &= \frac{1}{(q;q)_\infty^2}\sum_{n_2=-\infty}^{\infty}\zeta^{-1\cdot n_2+2n_2}q^{-n_2(-n_2+1)/2+n_2(n_2+1)/2}\\ &= \frac{1}{(q;q)_\infty^2}\sum_{n=-\infty}^{\infty}\zeta^n q^{n^2}\\ &= \frac{1}{(q;q)_\infty^2}f(\zeta q, \zeta^{-1}q)\\ &= \frac{1}{(q;q)_\infty^2}(-\zeta q;q^2)_\infty(-\zeta^{-1}q;q^2)_\infty(q^2;q^2)_\infty\\ &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\prod_{n=1}^{\infty}\left(1+(\zeta+\zeta^{-1})q^{2n-1}+q^{4n-2}\right)\\ &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\prod_{n=1}^{\infty}\left(1-q^{2n-1}+q^{4n-2}\right)\\ &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\prod_{n=1}^{\infty}\frac{1+q^{6n-3}}{1+q^{2n-1}}\\ &= \frac{(q^2;q^2)_\infty(-q^3;q^6)_\infty}{(q;q)_\infty^2(-q;q^2)_\infty}\\ &= \frac{(q^2;q^2)_\infty(q^6;q^{12})_\infty}{(q;q)_\infty^2(-q;q^2)_\infty(q^3;q^6)_\infty}\end{aligned}$$

$$= \frac{(q^6;q^{12})_\infty}{(q;q)_\infty(q;q^2)_\infty(-q;q^2)_\infty(q^3;q^6)_\infty}$$
$$= \frac{(q^6;q^{12})_\infty}{(q;q)_\infty(q^2;q^4)_\infty(q^3;q^6)_\infty}.$$

Thus, obtaining (7.1.5), we complete the proof. □

7.2. Second Generalization of Frobenius Partitions

We now consider k copies of the nonnegative integers with a total ordering as follows:

$$0_1 < 0_2 < \cdots < 0_k < 1_1 < 1_2 < \cdots < 1_k < 2_1 < 2_2 < \cdots. \tag{7.2.1}$$

Definition 7.2.1. *The generalized Frobenius symbol, associated with the natural number n, is given by*

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

where the elements are arranged in strictly decreasing order according to the convention (7.2.1). *Moreover,*

$$n = r + \sum_{j=1}^{r}(a_j + b_j).$$

Let $c\varphi_k(n)$ denote the number of such partitions of n, and let

$$c\Phi_k(q) = \sum_{n=0}^{\infty} c\varphi_k(n)q^n \tag{7.2.2}$$

denote the generating function of $c\varphi_k(n)$. (The appendage c is an indication that we can regard these partitions of n as partitions in k colors.)

Example 7.2.2. *Let $k = 2$ and $n = 2$. The possible generalized partitions of n are:*

$$\begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_1 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}.$$

Therefore, $c\varphi_2(2) = 9$.

Analogously to how we argued in (3.1.41) and Definition 3.1.10, $c\Phi_k(q)$ is the constant term in

$$\prod_{n=1}^{\infty}(1 + zq^n)^k(1 + z^{-1}q^{n-1})^k. \tag{7.2.3}$$

Using the same kind of argument that we used above to deduce (7.1.3), except that now $\zeta = 1$ and z is replaced by $-z$, and recalling the notation (2.2.3), we find that

$$\begin{aligned} c\Phi_k(q) &= [z^0]\frac{1}{(q;q)_\infty^k} \sum_{n_1,n_2,\ldots,n_k=0}^{\infty} q^{n_1(n_1+1)/2+n_2(n_2+1)/2+\cdots+n_k(n_k+1)/2} z_1^{n_1} z_1^{n_2} \cdots z_1^{n_k} \\ &= \frac{1}{(q;q)_\infty^k} \sum_{\substack{n_1,n_2,\ldots,n_k=0 \\ n_1+n_2+\cdots n_k=0}}^{\infty} q^{n_1(n_1+1)/2+n_2(n_2+1)/2+\cdots+n_k(n_k+1)/2}. \end{aligned} \tag{7.2.4}$$

In particular,

$$c\Phi_1(q) = \frac{1}{(q;q)_\infty}$$

and, by the Jacobi triple product identity (3.1.17),

$$c\Phi_2(q) = \frac{1}{(q;q)_\infty^2} \sum_{n_1=-\infty}^{\infty} q^{n_1^2} = \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty}{(q;q)_\infty^2}. \tag{7.2.5}$$

We close our discussion of $c\varphi_k(n)$ by deriving a simple congruence for $c\varphi_k(n)$ modulo any prime $k = p$. From (7.2.4) and (7.2.3),

$$\begin{aligned} \sum_{n=0}^{\infty} c\varphi_k(n) q^n &= [z^0](-zq;q)_\infty^k (-z^{-1};q)_\infty^k \\ &\equiv [z^0](-z^k q^k;q^k)_\infty (-z^{-k};q^k)_\infty \pmod{k} \\ &\equiv [z^0]\frac{1}{(q^k;q^k)_\infty} \sum_{n=0}^{\infty} z^{kn} q^{kn(n+1)/2} \pmod{k} \\ &= \frac{1}{(q^k;q^k)_\infty} \pmod{k} \\ &= \sum_{m=0}^{\infty} p(m) q^{km} \pmod{k}. \end{aligned}$$

Equating coefficients of q^n above, we deduce the following theorem.

Theorem 7.2.3. *We have for every pair of natural numbers k, n,*

$$c\varphi_k(n) \equiv \begin{cases} 0 \pmod{k}, & \text{if } k \nmid n, \\ p(n/k) \pmod{k}, & \text{if } k | n. \end{cases}$$

Chapter 8

Congruences for $p(n)$

8.1. Introduction

In about 1916, P. A. MacMahon used the most common recurrence formula for $p(n)$ to calculate $p(n)$ for the first 200 values of n. He conveniently arranged the values in segmented columns with five values in each grouping. Ramanujan noticed that every fifth value was divisible by 5, i.e., $p(5n+4) \equiv 0 \pmod{5}$, $0 \leq n \leq 39$. He also noticed further congruences modulo 7 and 11. More precisely, in 1919, Ramanujan [**86**], [**89**, pp. 210–213] announced that he had found three simple congruences satisfied by $p(n)$, namely,

$$p(5n+4) \equiv 0 \pmod{5}, \tag{8.1.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{8.1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{8.1.3}$$

He gave proofs of (8.1.1) and (8.1.2) in [**86**] and later in a short one page note [**87**], [**89**, p. 230] announced that he had also found a proof of (8.1.3). He also remarks in [**87**] that "It appears that there are no equally simple properties for any moduli involving primes other than these three." It was not until 2003 that this speculative observation was proved by S. Ahlgren and M. Boylan [**3**]. In a posthumously published paper [**88**], [**89**, pp. 232–238], G. H. Hardy extracted different proofs of (8.1.1)–(8.1.3) from an unpublished manuscript of Ramanujan on $p(n)$ and $\tau(n)$ [**91**, pp. 133–177], [**35**], [**18**, Chapter 5].

In [**86**], Ramanujan offered a more general conjecture. Let $\delta = 5^a 7^b 11^c$ and let λ be an integer such that $24\lambda \equiv 1 \pmod{\delta}$. Then

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta}. \tag{8.1.4}$$

In more detail, Ramanujan conjectured:

$$\text{If } 24N \equiv 1 \pmod{5^n}, \quad \text{then } p(N) \equiv 0 \pmod{5^n}; \tag{8.1.5}$$

$$\text{if } 24N \equiv 1 \pmod{7^n}, \quad \text{then } p(N) \equiv 0 \pmod{7^n}; \tag{8.1.6}$$

$$\text{if } 24N \equiv 1 \pmod{11^n}, \quad \text{then } p(N) \equiv 0 \pmod{11^n}. \tag{8.1.7}$$

In his unpublished manuscript [**91**, pp. 133–177], [**35**], Ramanujan gave a proof of (8.1.4) for arbitrary a and $b = c = 0$. He also began a proof of his conjecture for arbitrary b and $a = c = 0$, but he did not complete it. If he had completed his proof, he would have noticed that his conjecture in this case needed to be modified. Recall that Ramanujan had formulated his conjectures after studying MacMahon's table of values of $p(n), 0 \leq n \leq 200$. After Ramanujan died, H. Gupta [**63**], [**64**, pp. 47–53] extended MacMahon's table up to $n = 300$. Upon examining Gupta's table in 1934, S. Chowla [**46**] found that $p(243) = 133978259344888$ is not divisible by 7^3, despite the fact that $24 \cdot 243 \equiv 1 \pmod{7^3}$. To correct Ramanujan's conjecture, define $\delta' = 5^a 7^{b'} 11^c$, where $b' = b$, if $b = 0, 1, 2$, and $b' = [(b+2)/2]$, if $b > 2$. Then

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta'}. \tag{8.1.8}$$

In particular,

$$\text{If } 24N \equiv 1 \pmod{7^n}, \quad \text{then } p(N) \equiv 0 \pmod{7^{[(n+2)/2]}}.$$

In 1938, G. N. Watson [**99**] published a proof of (8.1.8) for $a = c = 0$ and gave a more detailed version of Ramanujan's proof of (8.1.8) in the case $b = c = 0$. It was not until 1967 that A. O. L. Atkin [**23**] proved (8.1.8) for arbitrary c and $a = b = 0$. M. D. Hirschhorn and D. C. Hunt [**70**] constructed a proof of (8.1.8) for arbitrary powers of 5, while F. Garvan [**54**] devised a proof of (8.1.8) for general powers of 7, both in the spirit of Ramanujan's proof. An account of the two general congruences for powers of 5 and 7 in the spirit of modular forms can be found in M. I. Knopp's excellent book [**73**].

8.2. Ramanujan's Congruence $p(5n + 4) \equiv 0 \pmod 5$

We shall give several proofs of Ramanujan's congruence for $p(n)$ modulo 5. We offer three proofs in this section. The first and the second are more elementary than the third, but the third gives more information.

Theorem 8.2.1. *For each nonnegative integer n,*

$$p(5n + 4) \equiv 0 \pmod 5. \tag{8.2.1}$$

First Proof of Theorem 8.2.1. Our first proof is taken from Ramanujan's paper [**86**], [**89**, pp. 210–213] and is reproduced in Hardy's book [**66**, pp. 87–88].

We begin by writing

$$q(q;q)_\infty^4 \frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} = q\frac{(q^5;q^5)_\infty}{(q;q)_\infty} = (q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)q^{m+1}. \tag{8.2.2}$$

By the binomial theorem,

$$(q;q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod 5 \qquad \text{or} \qquad \frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} \equiv 1 \pmod 5. \tag{8.2.3}$$

Hence, by (8.2.2) and (8.2.3),

$$q(q;q)_\infty^4 \equiv (q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)q^{m+1} \pmod 5. \tag{8.2.4}$$

We now see from (8.2.4) that in order to show that $p(5n+4) \equiv 0 \pmod 5$ we must show that the coefficients of q^{5n+5} on the left side of (8.2.4) are multiples of 5.

By the pentagonal number theorem, Corollary 1.2.26, and Jacobi's identity (3.1.30),

$$\begin{aligned} q(q;q)_\infty^4 =& q(q;q)_\infty (q;q)_\infty^3 \\ =& q \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j+1)/2} \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)/2} \\ =& \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k}(2k+1) q^{1+j(3j+1)/2+k(k+1)/2}. \end{aligned} \tag{8.2.5}$$

Our objective is to determine when the exponents on the right side are multiples of 5. Observe that

$$2(j+1)^2 + (2k+1)^2 = 8\left\{1 + \tfrac{1}{2}j(3j+1) + \tfrac{1}{2}k(k+1)\right\} - 10j^2 - 5.$$

Thus, $1 + \frac{1}{2}j(3j+1) + \frac{1}{2}k(k+1)$ is a multiple of 5 if and only if

$$2(j+1)^2 + (2k+1)^2 \equiv 0 \pmod 5. \tag{8.2.6}$$

It is easily checked that $2(j+1)^2 \equiv 0, 2,$ or 3 modulo 5 and that $(2k+1)^2 \equiv 0, 1,$ or 4 modulo 5. We therefore see that (8.2.6) is true if and only if

$$2(j+1)^2 \equiv 0 \pmod 5 \qquad \text{and} \qquad (2k+1)^2 \equiv 0 \pmod 5.$$

In particular, $2k+1 \equiv 0 \pmod 5$, which, by (8.2.5), implies that the coefficient of q^{5n+5}, $n \geq 0$, in $q(q;q)_\infty^4$ is a multiple of 5. The coefficient of q^{5n+5} on the right side of (8.2.4) is therefore also a multiple of 5, i.e., $p(5n+4)$ is a multiple of 5. □

We next give a variant of Ramanujan's proof due to Mike Hirschhorn [**69**].

Second Proof of Theorem 8.2.1. Recall Jacobi's Identity (3.1.30)

$$J := \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)/2} = (q;q)_\infty^3. \tag{8.2.7}$$

If on the left side of (8.2.7) we collect together terms according to the residue classes of the powers of q modulo 5, we find that

$$J \equiv J_0 + J_1 \pmod 5, \tag{8.2.8}$$

where J_j contains all of the terms in which the power of q is congruent to j modulo 5. On the other hand, by the generating function for $p(n)$, the binomial theorem, and (8.2.8),

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= \frac{1}{(q;q)_\infty} = \frac{(q;q)_\infty^9}{(q;q)_\infty^{10}} \\ &= \frac{J^3}{((q;q)_\infty^5)^2} \equiv \frac{(J_0+J_1)^3}{(q^5;q^5)_\infty^2} \\ &= \frac{J_0^3 + 3J_0^2J_1 + 3J_0J_1^2 + J_1^3}{(q^5;q^5)_\infty^2} \pmod 5.\end{aligned}$$

A careful inspection of the terms on the right-hand side above shows that none of the powers are congruent to 4 modulo 5. Thus, $p(5n+4) \equiv 0 \pmod 5$. ☐

We now give a third simple proof due to G. E. Andrews [**11**] and based on the simple lemma given below. See also an extensive generalization of this lemma by Andrews and R. Roy [**22**]. In particular, taking a special case of their general theorem, Andrews and Roy establish the congruence $p(7n+5) \equiv 0 \pmod 7$.

Lemma 8.2.2. *Let* $\{a_n\}$, $n \geq 0$, *be any sequence of integers. Then the coefficient of* q^{5n+3}, $n \geq 0$, *in*

$$L(q) := \frac{1}{(q;q)_\infty^2} \sum_{n=0}^{\infty} a_n q^{n^2} \tag{8.2.9}$$

is divisible by 5.

Proof. Write (8.2.9) in the form

$$L(q) = (q;q)_\infty^3 \frac{1}{(q;q)_\infty^5} \sum_{m=0}^{\infty} a_m q^{m^2} \equiv (q;q)_\infty^3 \frac{1}{(q^5;q^5)_\infty} \sum_{m=0}^{\infty} a_m q^{m^2} \pmod 5,$$

by the binomial theorem. Using Jacobi's identity (3.1.30), we thus see that it suffices to examine the coefficient of q^{5n+3} in

$$(q;q)_\infty^3 \sum_{m=0}^{\infty} a_m q^{m^2} = \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2} \sum_{m=0}^{\infty} a_m q^{m^2}. \tag{8.2.10}$$

We want those terms above for which $j(j+1)/2 + m^2 = 5n+3$, where $n \geq 0$. It is easy to see that this condition is equivalent to the congruence

$$(2j+1)^2 + 3m^2 \equiv 0 \pmod 5. \tag{8.2.11}$$

Since $(2j+1)^2 \equiv 0, \pm 1 \pmod 5$ and $3m^2 \equiv 0, 2, 3 \pmod 5$, we see that (8.2.11) holds only when

$$m \equiv 2j+1 \equiv 0 \pmod 5. \tag{8.2.12}$$

The coefficients of q^{5n+3} in (8.2.10) are then composed of terms of the sort $(-1)^j(2j+1)a_m$, which, by (8.2.12), are all multiples of 5. ☐

Third Proof of Theorem 8.2.1. Using (8.6.1), we find that

$$\sum_{k=0}^{\infty} p(k)q^{2k} = \frac{1}{(q^2;q^2)_\infty} = \frac{1}{(q;q)_\infty(-q;q)_\infty} = \frac{1}{(q;q)_\infty^2}\frac{(q;q)_\infty}{(-q;q)_\infty}$$
$$= \frac{1}{(q;q)_\infty^2}\left(1+2\sum_{m=1}^{\infty}(-1)^m q^{m^2}\right).$$

By Lemma 8.2.2, the coefficients $p(k)$ on the left side above are multiples of 5 whenever $2k \equiv 5j+3 \pmod 5$, i.e., whenever $k = 5n+4$. This then completes our third proof. □

Recall that $\varphi_2(n)$ is defined in Definition 7.1.3.

Corollary 8.2.3. *We have*

$$\varphi_2(5n+3) \equiv 0 \pmod 5. \tag{8.2.13}$$

Proof. Recall from (7.1.1) and the third line of the proof of Theorem 7.1.5 that

$$\Phi_2(q) = \sum_{n=0}^{\infty}\varphi_2(n)q^n = \frac{1}{(q;q)_\infty^2}\sum_{n=-\infty}^{\infty}\zeta^n q^{n^2}, \quad \zeta = e^{2\pi i/3}. \tag{8.2.14}$$

Now,

$$\sum_{n=-\infty}^{\infty}\zeta^n q^{n^2} = \sum_{m=-\infty}^{\infty}\zeta^{3m}q^{9m^2} + \sum_{\substack{n=1\\ n\not\equiv 0 \pmod 3}}^{\infty}(\zeta^n+\zeta^{-n})q^{n^2}. \tag{8.2.15}$$

But,

$$\zeta^n+\zeta^{-n} = \begin{cases} 2\cos(2\pi/3) = -1, & n \equiv 1 \pmod 3, \\ 2\cos(4\pi/3) = -1, & n \equiv 2 \pmod 3. \end{cases} \tag{8.2.16}$$

If we use (8.2.16) in (8.2.15) and then (8.2.15) in (8.2.14), we see that the hypotheses of Lemma 8.2.2 are satisfied. Hence, by the aforementioned lemma, the proof of (8.2.13) is complete. □

Corollary 8.2.4. *We have*

$$c\varphi_2(5n+3) \equiv 0 \pmod 5. \tag{8.2.17}$$

Proof. Recall from (7.2.2) and (7.2.5) that

$$c\Phi_2(q) = \sum_{n=0}^{\infty}c\varphi_2(n)q^n = \frac{1}{(q;q)_\infty^2}\sum_{n=-\infty}^{\infty}q^{n^2} = \frac{1}{(q;q)_\infty^2}\left(1+2\sum_{n=0}^{\infty}q^{n^2}\right). \tag{8.2.18}$$

Thus, from (8.2.18), we see that (8.2.17) follows immediately to complete the proof. □

Although the proofs of the congruence $p(5n+4) \equiv 0 \pmod 5$ that we have given so far are attractive, they must take second place to the proof arising from the following beautiful identity.

Theorem 8.2.5. *We have*

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{8.2.19}$$

It is obvious that the congruence $p(5n+4) \equiv 0 \pmod 5$ follows directly from (8.2.19).

In singling out an elegant formula of Ramanujan that characterizes Ramanujan's mathematics, Hardy remarked [**89**, p. xxxv], "and, if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting a formula from [**86**], viz.

$$p(4) + p(9)x + p(14)x^2 + \cdots = 5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^6},$$

where $p(n)$ is the number of partitions of n."

The proof of Theorem 8.2.5 that we shall give is based on another beautiful formula of Ramanujan.

Theorem 8.2.6. *If* $\left(\frac{n}{5}\right)$ *denotes the Legendre symbol, then*

$$\sum_{n=0}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2} = q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}. \tag{8.2.20}$$

We first demonstrate that Theorem 8.2.5 follows from Theorem 8.2.6.

Proof. From (8.2.20),

$$q(q^5;q^5)_\infty^5 \sum_{n=0}^{\infty} p(n)q^n = \sum_{n=0}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2} = \sum_{n=0}^{\infty} \left(\frac{n}{5}\right) \sum_{k=1}^{\infty} kq^{nk}. \tag{8.2.21}$$

We now equate the terms in (8.2.21) in which the powers of q are multiples of 5. We note that these powers arise only when k is a multiple of 5, since $\left(\frac{n}{5}\right) = 0$ when n is a multiple of 5. Hence,

$$q(q^5;q^5)_\infty^5 \sum_{n=0}^{\infty} p(5n+4)q^{5n+4} = 5\sum_{n=0}^{\infty} \left(\frac{n}{5}\right) \sum_{k=1}^{\infty} kq^{5nk} = 5q^5\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty}, \tag{8.2.22}$$

where we applied Theorem 8.2.6, or (8.2.21), once again, but now with q replaced by q^5. Rewriting (8.2.22) slightly, we see that

$$\sum_{n=0}^{\infty} p(5n+4)q^{5n+4} = 5q^4\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}.$$

Cancelling q^4 on both sides above and replacing q^5 by q, we complete the proof of Theorem 8.2.6. □

We next show that Theorem 8.2.6 is a special instance of the next theorem.

Theorem 8.2.7. *For any complex numbers x and y,*

$$\sum_{n=-\infty}^{\infty}\left\{\frac{xq^n}{(1-xq^n)^2}-\frac{yq^n}{(1-yq^n)^2}\right\}$$
$$=\frac{(x-y)(1-xy)}{(1-x)^2(1-y)^2}\frac{(xyq)_\infty(q/(xy))_\infty(xq/y)_\infty(yq/x)_\infty(q;q)_\infty^4}{(xq)_\infty^2(q/x)_\infty^2(yq)_\infty^2(q/y)_\infty^2}. \quad (8.2.23)$$

Proof. As promised, we show that Theorem 8.2.6 follows from Theorem 8.2.7. Replace q by q^5 and set $x=q$ and $y=q^2$ in (8.2.23). Hence, the left-hand side of (8.2.23) becomes

$$\sum_{n=-\infty}^{\infty}\left\{\frac{q^{5n+1}}{(1-q^{5n+1})^2}-\frac{q^{5n+2}}{(1-q^{5n+2})^2}\right\}$$
$$=\sum_{n=0}^{\infty}\left\{\frac{q^{5n+1}}{(1-q^{5n+1})^2}-\frac{q^{5n+2}}{(1-q^{5n+2})^2}\right\}+\sum_{j=0}^{\infty}\left\{\frac{q^{-5j-4}}{(1-q^{-5j-4})^2}-\frac{q^{-5j-3}}{(1-q^{-5j-3})^2}\right\}$$
$$=\sum_{n=0}^{\infty}\left\{\frac{q^{5n+1}}{(1-q^{5n+1})^2}-\frac{q^{5n+2}}{(1-q^{5n+2})^2}\right\}+\sum_{j=0}^{\infty}\left\{\frac{q^{5j+4}}{(1-q^{5j+4})^2}-\frac{q^{5j+3}}{(1-q^{-5j-3})^2}\right\}$$
$$=\sum_{n=0}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}, \quad (8.2.24)$$

where we set $n=-j-1$ in the sums over the negative indices.

On the other hand, the right-hand side of (8.2.23) becomes, under the above-mentioned substitutions,

$$\frac{(q-q^2)(1-q^3)(q^8;q^5)_\infty(q^2;q^5)_\infty(q^4;q^5)_\infty(q^6;q^5)_\infty(q^5;q^5)_\infty^4}{(1-q)^2(1-q^2)^2(q^6;q^5)_\infty^2(q^4;q^5)_\infty^2(q^7;q^5)_\infty^2(q^3;q^5)_\infty^2}$$
$$=q\frac{(q^3;q^5)_\infty(q^2;q^5)_\infty(q^5;q^5)_\infty^4}{(q;q^5)_\infty(q^4;q^5)_\infty(q^2;q^5)_\infty^2(q^3;q^5)_\infty^2}=q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}. \quad (8.2.25)$$

If we now put together (8.2.24) and (8.2.25), we deduce Theorem 8.2.6. □

Proof of Theorem 8.2.7. Let

$$F(x,y):=\frac{(x-y)(1-xy)}{(1-x)^2(1-y)^2}\frac{(xyq)_\infty(q/(xy))_\infty(xq/y)_\infty(yq/x)_\infty(q;q)_\infty^4}{(xq)_\infty^2(q/x)_\infty^2(yq)_\infty^2(q/y)_\infty^2}. \quad (8.2.26)$$

and

$$G(x,y):=\frac{(xyq)_\infty(q/(xy))_\infty(xq/y)_\infty(yq/x)_\infty(q;q)_\infty^4}{(xq)_\infty^2(q/x)_\infty^2(yq)_\infty^2(q/y)_\infty^2}. \quad (8.2.27)$$

Suppose that $y\neq q^n$, $-\infty<n<\infty$. Regard $F(x,y)$ as a function of x, with y constant for the time being. Observe that $F(x,y)$ has double poles at $x=q^n$, $-\infty<n<\infty$. Our first goal is to find the principal parts of $F(x,y)$ at these poles. It will help us to first derive some functional equations. By (8.2.26),

$$\begin{aligned}F(qx,y)&=\frac{(qx-y)(1-qxy)}{(1-qx)^2(1-y)^2}\frac{(xyq^2)_\infty(1/(xy))_\infty(xq^2/y)_\infty(y/x)_\infty(q;q)_\infty^4}{(xq^2)_\infty^2(1/x)_\infty^2(yq)_\infty^2(q/y)_\infty^2}\\&=\frac{(xq-y)(1-1/(xy))(1-y/x)}{(1-qx/y)(1-y)^2(1-1/x)^2}G(x,y)\\&=\frac{(x-y)(1-xy)}{(1-x)^2(1-y)^2}G(x,y)\\&=F(x,y).\end{aligned}\tag{8.2.28}$$

Also, by (8.2.28),

$$F(q^{-1}x,y)=F(q(q^{-1}x),y)=F(x,y).\tag{8.2.29}$$

Observe that, from the definition (8.2.26),

$$\lim_{x\to 1}(1-x)^2F(x,y)=1.\tag{8.2.30}$$

Thus, using the definition (8.2.27), we so far know that

$$F(x,y)-\frac{1}{(x-1)^2}=\frac{1}{(x-1)^2}\left(\frac{(x-y)(1-xy)}{(1-y)^2}G(x,y)-1\right).\tag{8.2.31}$$

We now need to calculate

$$\begin{aligned}\lim_{x\to 1}(x-1)\left(F(x,y)-\frac{1}{(x-1)^2}\right)&=\lim_{x\to 1}\frac{1}{x-1}\left(\frac{(x-y)(1-xy)}{(1-y)^2}G(x,y)-1\right)\\&=\lim_{x\to 1}\left(\left\{\frac{1-xy}{(1-y)^2}-\frac{y(x-y)}{(1-y)^2}\right\}G(x,y)+\frac{(x-y)(1-xy)}{(1-y)^2}\frac{\partial}{\partial x}G(x,y)\right)\\&=1+\lim_{x\to 1}\frac{(x-y)(1-xy)}{(1-y)^2}G(x,y)\frac{\partial}{\partial x}\log G(x,y)\\&=1+\lim_{x\to 1}\frac{\partial}{\partial x}\log G(x,y),\end{aligned}\tag{8.2.32}$$

because $G(1,y)=1$. Now

$$\begin{aligned}\frac{\partial}{\partial x}\log G(x,y)=\sum_{n=1}^{\infty}\Bigg(&-\frac{yq^n}{1-xyq^n}+\frac{q^n/(x^2y)}{1-q^n/(xy)}-\frac{q^n/y}{1-xq^n/y}\\&+\frac{yq^n/x^2}{1-yq^n/x}+2\frac{q^n}{1-xq^n}-2\frac{q^n/x^2}{1-q^n/x}\Bigg).\end{aligned}$$

It follows from the calculation above that

$$\lim_{x\to 1}\frac{\partial}{\partial x}\log G(x,y)=0.\tag{8.2.33}$$

We provide another argument suggested by Dan Schultz. From (8.2.27), it is easily seen that $G(x,y)=G(1/x,y)$. Thus, by the chain rule, if $u=1/x$,

$$\frac{\partial}{\partial x}G(x,y)=-\frac{1}{x^2}\frac{\partial}{\partial u}G(u,y).$$

If we now set $x = 1$ (so that $u = 1$) and replace x by u on the right-hand side, we see that

$$\frac{\partial}{\partial x}G(1,y) = -\frac{\partial}{\partial x}G(1,y).$$

Hence,

$$\frac{\partial}{\partial x}G(1,y) = 0. \tag{8.2.34}$$

So, using either (8.2.33) or (8.2.34) in (8.2.32), we conclude that

$$\lim_{x\to 1}(x-1)\left(F(x,y) - \frac{1}{(x-1)^2}\right) = 1. \tag{8.2.35}$$

Hence, by (8.2.30) and (8.2.32), the principal part of $F(x,y)$ about $x = 1$ equals

$$\frac{1}{(x-1)^2} + \frac{1}{x-1} = \frac{x}{(x-1)^2}.$$

Since, by (8.2.28) and (8.2.29), $F(x,y) = F(qx,y) = F(q/x,y)$, it follows that the principal part about the pole q^{-n}, $-\infty < n < \infty$, equals

$$\frac{xq^n}{(1-xq^n)^2}.$$

Hence, we have shown so far that

$$\begin{aligned} F(x,y) &= \frac{x}{(1-x)^2} + \sum_{n=1}^{\infty}\left(\frac{xq^n}{(1-xq^n)^2} + \frac{x^{-1}q^n}{(1-x^{-1}q^n)^2}\right) + \cdots \\ &=: G(x) + H(x,y), \end{aligned} \tag{8.2.36}$$

say. Now write

$$H(x,y) = \sum_{n=-\infty}^{\infty} a_n(y)x^n. \tag{8.2.37}$$

Since we have subtracted all of the principal parts of $F(x,y)$ (except around $x = 0$) in (8.2.36) in defining $H(x,y)$, it follows that (8.2.37) is valid for $0 < |x| < \infty$. Now, $F(qx,y) = F(x,y)$, and clearly from above $G(x) = G(qx)$. It follows from (8.2.36) that $H(x,y) = H(qx,y)$. Hence,

$$H(qx,y) = \sum_{n=-\infty}^{\infty} a_n(y)(qx)^n = H(x,y) = \sum_{n=-\infty}^{\infty} a_n(y)x^n.$$

It follows that

$$a_n(y)q^n = a_n(y), \qquad -\infty < n < \infty.$$

Hence, $a_n(y) = 0$, $n \neq 0$, and so

$$H(x,y) = a_0(y).$$

It then follows from (8.2.36) that

$$F(x,y) = G(x) + a_0(y).$$

Putting $x = y$ above, we conclude that

$$0 = F(y,y) = G(y) + a_0(y), \qquad \text{or} \qquad a_0(y) = -G(y).$$

Thus,

$$F(x,y) = G(x) - G(y),$$

which is what we wanted to prove. □

Theorem 8.2.5 can be utilized to provide a proof of Ramanujan's congruence for $p(n)$ modulo 25.

Theorem 8.2.8. *For every nonnegative integer n,*

$$p(25n+24) \equiv 0 \pmod{25}. \tag{8.2.38}$$

Proof. Applying the binomial theorem on the right side of (8.2.19), we find that

$$\sum_{n=0}^{\infty} p(5n+4)q^n \equiv 5\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty} = 5(q^5;q^5)_\infty^4 \sum_{n=0}^{\infty} p(n)q^n \pmod{25}. \tag{8.2.39}$$

From Theorem 8.2.1 we know that the coefficients of $q^4, q^9, q^{14}, \dots, q^{5n+4}, \dots$ on the far right side of (8.2.39) are all multiples of 25. It follows that the coefficients of $q^{5n+4}, n \geq 0$, on the far left side of (8.2.39) are also multiples of 25, i.e.,

$$p(25n+24) \equiv 0 \pmod{25}.$$

This completes the proof. □

8.3. Ramanujan's Congruence $p(7n+5) \equiv 0 \,(\textbf{mod}\, 7)$

Theorem 8.3.1. *For each nonnegative integer n,*

$$p(7n+5) \equiv 0 \pmod{7}. \tag{8.3.1}$$

Proof. Our proof is again taken from Ramanujan's paper [**86**] and was sketched by Hardy [**66**, p. 88].

First, by the binomial theorem,

$$\begin{aligned} q^2(q^7;q^7)_\infty \sum_{n=0}^{\infty} p(n)q^n = q^2\frac{(q^7;q^7)_\infty}{(q;q)_\infty} &= q^2(q;q)_\infty^6 \frac{(q^7;q^7)_\infty}{(q;q)_\infty^7} \\ &\equiv q^2(q;q)_\infty^6 \pmod{7}. \end{aligned} \tag{8.3.2}$$

Hence, if we can show that the coefficient of $q^{7n+7}, n \geq 0$, in $q^2(q;q)_\infty^6$ is a multiple of 7, it will follow from (8.3.2) that the coefficient of q^{7n+7} on the far left side is a multiple of 7, i.e., $p(7n+5) \equiv 0 \pmod{7}$.

Applying Jacobi's identity, Theorem 8.2.7, we find that

$$\begin{aligned} q^2(q;q)_\infty^6 &= q^2\{(q;q)_\infty^3\}^2 \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-1)^{j+k}(2j+1)(2k+1)q^{2+j(j+1)/2+k(k+1)/2}. \end{aligned} \tag{8.3.3}$$

As we saw in the previous paragraph, we want to know when the exponents above are multiples of 7. Now observe that

$$(2j+1)^2 + (2k+1)^2 = 8\{2 + \tfrac{1}{2}j(j+1) + \tfrac{1}{2}k(k+1)\} - 14,$$

and so $2 + \frac{1}{2}j(j+1) + \frac{1}{2}k(k+1)$ is a multiple of 7 if and only if

$$(2j+1)^2 + (2k+1)^2 \equiv 0 \pmod{7}. \tag{8.3.4}$$

We easily see that $(2j+1)^2, (2k+1)^2 \equiv 0, 1, 2, 4 \pmod{7}$, and so the only way (8.3.4) can hold is if both $(2j+1)^2, (2k+1)^2 \equiv 0 \pmod{7}$. In such cases, we trivially see that the coefficients on the right side of (8.3.3) are multiples of 7. Hence, the coefficient of $q^{7n+7}, n \geq 1$, on the left side of (8.3.3) is a multiple of 7. As we demonstrated in the foregoing paragraph, this implies that $p(7n+5) \equiv 0 \pmod{7}$. □

The following variant of Ramanujan's proof is due to Mike Hirschhorn [**69**].

First Proof of Theorem 8.2.1. Return to Jacobi's Identity (8.2.7) and gather together terms according to the residue classes of the powers of q modulo 7. We thus find that

$$J \equiv J_0 + J_1 + J_3 \pmod{7}, \tag{8.3.5}$$

where J_j contains all of the terms in which the power of q is congruent to j modulo 7. On the other hand, by the generating function for $p(n)$, the binomial theorem, and (8.3.5),

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= \frac{1}{(q;q)_\infty} = \frac{(q;q)_\infty^6}{(q;q)_\infty^7} \\ &= \frac{J^2}{(q;q)_\infty^7} \equiv \frac{(J_0+J_1+J_3)^2}{(q^7;q^7)_\infty} \\ &= \frac{J_0^2 + J_1^2 + J_3^2 + 2J_0J_1 + 2J_0J_3 + 2J_1J_3}{(q^7;q^7)_\infty^2} \pmod{7}.\end{aligned}$$

A careful examination of the terms on the right-hand side above shows that none of the powers are congruent to 5 modulo 7. Thus, $p(7n+5) \equiv 0 \pmod{7}$. □

Theorem 8.3.2. *We have*

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}. \tag{8.3.6}$$

Theorem 8.3.2 is clearly an analogue of Theorem 8.2.6. Furthermore, it is clear that Theorem 8.3.1 is an immediate corollary of Theorem 8.3.2. Ramanujan's proof of Theorem 8.3.2 can be found in his unpublished manuscript on $p(n)$ and $\tau(n)$, published for the first time with his lost notebook [**91**]. Ramanujan, however, provided almost no details, which were worked out for the first time by the present author, Ae Ja Yee, and Jinhee Yi [**37**]. The account of Ramanujan's proof of Theorem 8.3.2 that we give below is taken from [**37**].

Proof of Theorem 8.3.2. Using the pentagonal number theorem (1.2.23) in both the numerator and denominator and then separating the indices of summation in the numerator into residue classes modulo 7, we readily find that

$$\frac{(q^{1/7};q^{1/7})_\infty}{(q^7;q^7)_\infty} = J_1 + q^{1/7}J_2 - q^{2/7} + q^{5/7}J_3, \tag{8.3.7}$$

where J_1, J_2, and J_3 are power series in q with integral coefficients, and where the pentagonal number theorem was used to calculate the coefficient of $q^{2/7}$. Cubing both sides of (8.3.7), we find that

$$\begin{aligned}
&\frac{(q^{1/7};q^{1/7})^3_\infty}{(q^7;q^7)^3_\infty}\\
&= (J_1^3 + 3J_2^2J_3q - 6J_1J_3q) + q^{1/7}(3J_1^2J_2 - 6J_2J_3q + J_3^2q^2)\\
&\quad + 3q^{2/7}(J_1J_2^2 - J_1^2 + J_3q) + q^{3/7}(J_2^3 - 6J_1J_2 + 3J_1J_3^2q)\\
&\quad + 3q^{4/7}(J_1 - J_2^2 + J_2J_3^2q) + 3q^{5/7}(J_2 + J_1^2J_3 - J_3^2q)\\
&\quad + q^{6/7}(6J_1J_2J_3 - 1).
\end{aligned} \tag{8.3.8}$$

On the other hand, using Jacobi's identity, Corollary 3.1.9, and separating the indices of summation in the numerator on the left side of (8.3.8) into residue classes modulo 7, we easily find that

$$\frac{(q^{1/7};q^{1/7})^3_\infty}{(q^7;q^7)^3_\infty} = G_1 + q^{1/7}G_2 + q^{3/7}G_3 - 7q^{6/7}, \tag{8.3.9}$$

where G_1, G_2, and G_3 are power series in q with integral coefficients, and where Jacobi's identity, Corollary 3.1.9, was used to determine the coefficient of $q^{6/7}$. Comparing coefficients in (8.3.8) and (8.3.9), we conclude that

$$\begin{cases}
J_1J_2^2 - J_1^2 + J_3q &= 0,\\
J_1 - J_2^2 + J_2J_3^2q &= 0,\\
J_2 + J_1^2J_3 - J_3^2q &= 0,\\
6J_1J_2J_3 - 1 &= -7.
\end{cases} \tag{8.3.10}$$

Replace $q^{1/7}$ by $\omega q^{1/7}$ in (8.3.7), where ω is any seventh root of unity. Therefore,

$$\frac{(\omega q^{1/7};\omega q^{1/7})_\infty}{(q^7;q^7)_\infty} = J_1 + \omega q^{1/7}J_2 - \omega^2q^{2/7} + \omega^5q^{5/7}J_3. \tag{8.3.11}$$

Taking the products of both sides of (8.3.11) over all seven seventh roots of unity, we find that

$$\frac{(q;q)^8_\infty}{(q^7;q^7)^8_\infty} = \prod_\omega (J_1 + \omega q^{1/7}J_2 - \omega^2q^{2/7} + \omega^5q^{5/7}J_3). \tag{8.3.12}$$

Using the generating function for $p(n)$, (8.3.7), and (8.3.12), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= \frac{1}{(q;q)_\infty} = \frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}\frac{(q^7;q^7)_\infty^8}{(q^{49};q^{49})_\infty^8}\frac{(q^{49};q^{49})_\infty}{(q;q)_\infty}\\ &= \frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}\frac{\prod_\omega (J_1+\omega qJ_2-\omega^2q^2+\omega^5q^5J_3)}{J_1+qJ_2-q^2+q^5J_3}\\ &= \frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}\left\{\prod_{\omega\neq 1}(J_1+\omega qJ_2-\omega^2q^2+\omega^5q^5J_3)\right\}. \qquad (8.3.13)\end{aligned}$$

We only need to compute the terms in $\prod_{\omega\neq 1}(J_1+\omega qJ_2-\omega q^2+\omega q^5J_3)$ where the powers of q are of the form $7n+5$ to complete the proof. In order to do this, we need to prove several identities using the identities of (8.3.10). More precisely, we need to prove

$$J_1^7+J_2^7q+J_3^7q^5 = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8}+14q\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4}+57q^2, \qquad (8.3.14)$$

$$J_1^3J_2+J_2^3J_3q+J_3^3J_1q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4}-8q, \qquad (8.3.15)$$

$$J_1^2J_2^3+J_3^2J_1^3q+J_2^2J_3^3q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4}-5q. \qquad (8.3.16)$$

Since $J_2^2=J_1+J_2J_3^2q$, $J_1^2=J_1J_2^2+J_3q$, $J_3^2q=J_2+J_1^2J_3$, and $J_1J_2J_3=-1$ by (3.1.36), we find that

$$\begin{aligned}J_1^2J_2^3+J_3^2J_1^3q+J_2^2J_3^3q^2 &= J_1^3J_2+J_1^2J_2^2J_3^2q+J_1^2J_2^2J_3^2q+J_3^3J_1q^2+J_2^3J_3q+J_1^2J_2^2J_3^2q\\ &= J_1^3J_2+J_2^3J_3q+J_3^3J_1q^2+3q, \qquad (8.3.17)\\ J_1J_2^5+J_3J_1^5+J_2J_3^5q^3 &= J_1J_2(J_1+J_2J_3^2q)^2+J_3J_1(J_1J_2^2+J_3q)^2+J_2J_3(J_2+J_1^2J_3)^2q\\ &= J_1^3J_2+2J_1^2J_2^2J_3^2q+J_1J_2^3J_3^4q^2+J_3J_1^3J_2^4+2J_1^2J_2^2J_3^2q+J_3^3J_1q^2\\ &\quad +J_2^3J_3q+2J_1^2J_2^2J_3^2q+J_2J_3^3J_1^4q\\ &= J_1^3J_2+J_2^3J_3q+J_3^3J_1q^2-(J_1^2J_2^3+J_3^2J_1^3q+J_2^2J_3^3q^2)+6q\\ &= 3q, \qquad (8.3.18)\end{aligned}$$

where (8.3.18) is obtained from (8.3.17). (Observe from (8.3.17) that it suffices to prove only (8.3.15) or (8.3.16).) By squaring the left side of (8.3.15) and using (3.1.36), (8.3.18), and (8.3.17), we find that

$$\begin{aligned}(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2)^2 =& J_1^6J_2^2 + J_2^6J_3^2q^2 + J_3^6J_1^2q^4\\ &+ 2(J_1^3J_2^4J_3q + J_1J_2^3J_3^4q^3 + J_1^4J_2J_3^3q^2)\\ =& J_1^7 + J_1^6J_2J_3^2q + J_2^7q + J_1^2J_2^6J_3q + J_3^7q^5 + J_1J_2^2J_3^6q^4\\ &- 2(J_1^2J_2^3q + J_3^2J_1^3q^2 + J_2^2J_3^3q^3)\\ =& J_1^7 + J_2^7q + J_3^7q^5 - (J_1J_2^5q + J_3J_1^5q + J_2J_3^5q^4)\\ &- 2(J_1^2J_2^3q + J_2^2J_3^3q^3 + J_1^3J_3^2q^2)\\ =& J_1^7 + J_2^7q + J_3^7q^5 - 2q(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2) - 9q^2.\end{aligned}$$

Thus,

$$(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2 + q)^2 = (J_1^7 + J_2^7q + J_3^7q^5) - 8q^2. \tag{8.3.19}$$

Expanding the right side of (3.3.26) and using (3.1.36), (8.3.18), and (8.3.17), we obtain

$$\begin{aligned}\frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8} =& J_1^7 + J_2^7q + J_3^7q^5 + 7(J_1J_2^5q + J_3J_1^5q + J_2J_3^5q^4) + 7(J_1^4J_2^2J_3q + J_1J_2^4J_3^2q^2\\ &+ J_2J_3^4J_1^2q^3) + 7(J_1^3J_2q + J_2^3J_3q^2 + J_3^3J_1q^3)\\ &+ 14(J_1^2J_2^3q + J_3^2J_1^3q^2 + J_2^2J_3^3q^3) + 7J_1^2J_2^2J_3^2q^2 + 14J_1J_2J_3q^2 - q^2\\ =& J_1^7 + J_2^7q + J_3^7q^5 + 21q^2 - 7q(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2) + 7q(J_1^3J_2 + J_2^3J_3q\\ &+ J_3^3J_1q^2) + 14q(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2 + 3q) + 7q^2 - 14q^2 - q^2\\ =& J_1^7 + J_2^7q + J_3^7q^5 + 14q(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2) + 55q^2.\end{aligned} \tag{8.3.20}$$

Combining (8.3.19) and (8.3.20), we find that

$$\begin{aligned}\frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8} &= (J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2 + q)^2 + 8q^2 + 14(J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2)q + 55q^2\\ &= (J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2 + 8q)^2.\end{aligned}$$

By (3.1.33), we see that for q sufficiently small and positive, $J_2 < 0$. Thus, taking the square root of both sides above, we find that

$$J_1^3J_2 + J_2^3J_3q + J_3^3J_1q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 8q, \tag{8.3.21}$$

which proves (8.3.15). We now see that (8.3.14) follows from (8.3.20) and (8.3.21), and (8.3.16) follows from (8.3.17) and (8.3.21).

Returning to (3.3.27), we are now ready to compute the terms in $\prod_{i=1}^6(J_1 + \omega^i qJ_2 - \omega^{2i}q^2 + \omega^{5i}q^5J_3)$ where the powers of q are of the form $7n+5$. Using the computer algebra system MAPLE, (8.3.15), (8.3.16), and (8.3.18), we find that the desired terms with powers of the form q^{7n+5} are equal to

$$\begin{aligned}
&-(J_1J_2^5+J_3J_1^5+3J_1^3J_2+4J_1^2J_2^3)q^5-(3J_2^3J_3+4J_3^2J_1^3-8)q^{12}\\
&-(4J_2^2J_3^3+3J_3^3J_1)q^{19}-J_2J_3^5q^{26}\\
=&-3(J_1^3J_2+J_2^3J_3q^7+J_3^3J_1q^{14})q^5-4(J_1^2J_2^3+J_3^2J_1^3q^7+J_2^2J_3^3q^{14})q^5\\
&-(J_1J_2^5+J_3J_1^5+J_2J_3^5q^{21})q^5+8q^{12}\\
=&7\frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4}q^5+49q^{12}. \qquad (8.3.22)
\end{aligned}$$

Choosing only those terms on each side of (8.3.13) where the powers of q are of the form $7n+5$ and using the omitted calculations, we find that

$$\sum_{\substack{n=0\\ n\equiv 5\,(\mathrm{mod}\,7)}}^{\infty} p(n)q^n = q^5\frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}\left(7\frac{(q^7;q^7)_\infty^4}{(q^{49};q^{49})_\infty^4}+49q^7\right),$$

or

$$\sum_{n=0}^{\infty} p(7n+5)q^{7n} = 7\frac{(q^{49};q^{49})_\infty^3}{(q^7;q^7)_\infty^4}+49q^7\frac{(q^{49};q^{49})_\infty^7}{(q^7;q^7)_\infty^8}. \qquad (8.3.23)$$

Replacing q^7 by q in (8.3.23), we complete the proof of (8.3.6). □

By comparing (3.1.33) with Entry 17(v) in Chapter 19 of Ramanujan's second notebook [**90**], [**26**, p. 303], we see that

$$J_1=\frac{f(-q^2,-q^5)}{f(-q,-q^6)},\qquad J_2=-\frac{f(-q^3,-q^4)}{f(-q^2,-q^5)},\qquad \text{and}\qquad J_3=\frac{f(-q,-q^6)}{f(-q^3,-q^4)},$$

where

$$f(a,b):=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2},\qquad |ab|<1.$$

In the notation of Section 18 of Chapter 19 in [**90**], [**26**, p. 306],

$$\alpha=u^{1/7}=q^{-2/7}J_1,\qquad \beta=-v^{1/7}=q^{-1/7}J_2,\qquad \text{and}\qquad \gamma=w^{1/7}=q^{3/7}J_3. \qquad (8.3.24)$$

Thus, the identity (3.1.38) is equivalent to an identity in Entry 18 in Chapter 19 of Ramanujan's second notebook [**90**], [**26**, p. 305, eq. (18.2)]. The proof of (3.1.38) given here is much simpler than that given in [**26**, pp. 306–312].

Recall that the identity of Theorem 8.2.5 yielded in Theorem 8.2.8 a congruence for $p(n)$ modulo 5^2. Similarly, the identity in Theorem 8.3.2 yields a congruence for $p(n)$ modulo 7^2, as we now demonstrate.

Theorem 8.3.3. *For each nonnegative integer n,*

$$p(49n+47)\equiv 0\,(\mathrm{mod}\,49). \qquad (8.3.25)$$

Proof. Write (8.3.6) in the form

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3 (q;q)_\infty^3}{(q;q)_\infty^7} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}$$
$$\equiv 7(q^7;q^7)_\infty^2 \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{m(m+1)/2} \pmod{49}, \qquad (8.3.26)$$

by the binomial theorem and Jacobi's identity, Corollary 3.1.9. We now examine the terms on the right side of (8.3.26) where the powers of q are of the form $7n+6$. Separating the summands into residue classes modulo 7, we see that the only terms yielding such exponents are when $m \equiv 3 \pmod 7$. But then $2m+1 \equiv 0 \pmod 7$. Thus, the coefficient of the power $q^{7n+6}, n \geq 1$, on the right side of (8.3.26) is a multiple of 49. The same must be true, of course, on the left side of (8.3.26), i.e., the coefficient $p(49n+47)$ must be a multiple of 49, i.e., (8.3.25) has been established. □

8.4. Ramanujan's Congruence $p(11n+6) \equiv 0 \pmod{11}$

The following lemma is a special case of Ramanujan's ${}_1\psi_1$-summation. It is also a special case of a considerably more general theorem on partial fraction decompositions of q-products [**45**]. Observe the symmetry in x and y in the lemma.

Lemma 8.4.1. *For* $|q| < |x| < 1$,

$$\frac{[xy]_\infty (q)_\infty^2}{[x,y]_\infty} = \sum_{n=-\infty}^{\infty} \frac{x^n}{1-yq^n}. \qquad (8.4.1)$$

Proof. We regard

$$F(y) := \frac{[xy]_\infty (q)_\infty^2}{[x,y]_\infty}, \quad |x|, |q/x| < 1, \qquad (8.4.2)$$

as a function of the complex variable y. We calculate the partial fraction expansion of $F(y)$. Observe that $F(y)$ has simple poles at $y = q^n$, $-\infty < n < \infty$. Set

$$\begin{aligned} L_n &:= \lim_{y\to q^{-n}} (y-q^{-n})F(y) \\ &= \lim_{y\to q^{-n}} -q^{-n} \frac{(xy;q)_\infty (q/(xy);q)_\infty (q;q)_\infty^2}{(x;q)_\infty (q/x;q)_\infty (1-y)\cdots(1-yq^{n-1})(1-yq^{n+1})\cdots(q/y;q)_\infty} \\ &= -q^{-n} \frac{(x/q^n;q)_\infty (q^{n+1}/x;q)_\infty (-1)^n q^{n(n+1)/2} (q;q)_\infty^2}{(x;q)_\infty (q/x;q)_\infty (q;q)_n (q;q)_\infty (q^{n+1};q)_\infty} \\ &= -q^{-n} \frac{(-1)^n x^n (q/x;q)_\infty (x;q)_\infty (-1)^n q^{n(n+1)/2} (q;q)_\infty^2}{q^{n(n+1)/2} (x;q)_\infty (q/x;q)_\infty (q;q)_\infty^2} \\ &= -q^{-n} x^n. \end{aligned}$$

Thus, the associated term in the partial fraction decomposition of $F(y)$ is equal to

$$-\frac{q^{-n}x^n}{y-q^{-n}} = \frac{x^n}{1-yq^n}, \quad -\infty < n < \infty.$$

Hence, for some entire function $G(y)$,

$$F(y) = \sum_{n=-\infty}^{\infty} \frac{x^n}{1 - yq^n} + G(y) =: H(y) + G(y).$$

We want to show that $G(y) \equiv 0$. Observe that

$$F(qy) = \frac{[xqy]_\infty (q)_\infty^2}{[x, qy]_\infty} = \frac{(1 - 1/(xy))(1 - y)}{(1 - xy)(1 - 1/y)} F(y) = \frac{F(y)}{x}.$$

It is easy to see that

$$H(qy) = \frac{H(y)}{x}.$$

It follows that

$$F(y/q) = xF(q \cdot y/q) = xF(y) \qquad \text{and} \qquad H(y/q) = xH(y). \tag{8.4.3}$$

On $0 \leq |y| \leq 1$, $G(y) = F(y) - H(y)$ is a bounded analytic function. By (8.4.3), $|G(y/q)| = |xG(y)|$, and by induction,

$$|G(y/q^n)| = |x^n G(y)|. \tag{8.4.4}$$

Hence, as $n \to \infty$, we see that $G(y/q^n)$ tends to 0, since $|x| < 1$. In conclusion, $G(y)$ is an entire function that tends to 0 as y tends to infinity. It is therefore a bounded entire function, and so, by Liouville's theorem, $G(y)$ is a constant. But since $G(y) \to 0$, as $y \to \infty$, this constant must be 0. Hence, (8.4.1) follows, and the proof is finished. □

Lemma 8.4.2. *[Halphen's Identity] For arbitrary complex numbers a, b, c,*

$$\begin{aligned} H(a, b, c, q) &:= \frac{[ab, bc, ca]_\infty (q)_\infty^2}{[a, b, c, abc]_\infty} \\ &= 1 + F(a, q) + F(b, q) + F(c, q) - F(abc, q), \end{aligned} \tag{8.4.5}$$

where

$$F(x, q) := \sum_{k=0}^{\infty} \frac{xq^k}{1 - xq^k} - \sum_{k=1}^{\infty} \frac{q^k/x}{1 - q^k/x}, \qquad (|q| < 1). \tag{8.4.6}$$

In this particular form, Halphen's identity was first established by Andrews, R. Lewis, and Z.–G. Liu [**21**]. However, equivalent versions of (8.4.5) in terms of Weierstrass elliptic functions appeared in the literature much earlier, in particular, as exercises in the classical text by Whittaker and Watson [**100**, Examples 19 and 20, p. 458], and also in G. H. Halphen's paper [**65**, p. 187]. We will follow the lead of Whittaker and Watson and leave Halphen's identity as an exercise.

Lemma 8.4.3. *For $|q| < 1$,*

$$1 + F(a, q) = -F(1/a, q). \tag{8.4.7}$$

Proof. Extracting the term with $n=0$ in the first sum below and then adding and subtracting the term with $n=0$ in the second sum, we find that

$$\begin{aligned}F(a,q) &= \sum_{n=0}^{\infty}\frac{aq^n}{1-aq^n}-\sum_{n=1}^{\infty}\frac{q^n/a}{1-q^n/a}\\ &= \frac{a}{1-a}+\frac{\frac{1}{a}}{1-\frac{1}{a}}+\sum_{n=1}^{\infty}\frac{q^n/\frac{1}{a}}{1-q^n/\frac{1}{a}}-\sum_{n=0}^{\infty}\frac{\frac{1}{a}q^n}{1-\frac{1}{a}q^n}\\ &= -1-F(1/a,q).\end{aligned}$$

□

Theorem 8.4.4 (Winquist's Identity). *We have*

$$\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}q^{(3m^2+3n^2+3m+n)/2}\left(a^{-3m}b^{-3n}-a^{-3m}b^{3n+1}\right.$$
$$\left.-b^{-3m-1}a^{-3n+1}+b^{-3m-1}a^{3n+2}\right)=(q;q)_\infty^2[a,b,ab,a/b;q]_\infty \quad (8.4.8)$$

Proof. We first rewrite (8.4.8) in the equivalent form

$$\begin{aligned}&[a,b,ab,a/b;q]_\infty(q;q)_\infty^2\\ &=[a^3;q^3]_\infty(q^3;q^3)_\infty^2\left([b^3q;q^3]_\infty-b[b^3q^2;q^3]_\infty\right)\\ &\quad-\frac{a}{b}[b^3;q^3]_\infty(q^3;q^3)_\infty^2\left([a^3q;q^3]_\infty-a[a^3q^2;q^3]_\infty\right).\end{aligned} \quad (8.4.9)$$

To derive this representation of Winquist's identity, we need to identify each of the eight bilateral series on the left side of (8.4.8) in terms of Ramanujan's theta function $f(\alpha,\beta)$ and then apply the Jacobi triple product identity (1.1.7) to each of the eight series. We leave this task as an exercise. Our goal is thus to prove (8.4.9).

Let ω be a cubic root of unity. For brevity, set $F(a,q)=F(a)$. In Halphen's identity (8.4.5), we replace a, b, and c by $a\omega$, $b\omega^2$, and ω^2/b, respectively, to deduce that

$$H(a\omega,b\omega^2,\omega^2/b,q)=1+F(a\omega)+F(b\omega^2)+F(\omega^2/b)-F(a\omega^2). \quad (8.4.10)$$

Simplifying the left side of (8.4.10), we deduce that

$$\begin{aligned}H(a\omega,b\omega^2,\omega^2/b,q)&=\frac{[ab,a/b,\omega;q]_\infty(q;q)_\infty^2}{[a\omega,b\omega^2,\omega^2/b,a\omega^2;q]_\infty}\\ &=-b\omega(1-\omega)\frac{[a,b,ab,a/b;q]_\infty(q;q)_\infty^2}{[a^3,b^3,q;q^3]_\infty}.\end{aligned} \quad (8.4.11)$$

Note that

$$\begin{aligned}&F(a\omega)-F(a\omega^2)\\ &=\sum_{k=0}^{\infty}\frac{a\omega q^k}{1-a\omega q^k}-\sum_{k=1}^{\infty}\frac{\omega^2q^k/a}{1-\omega^2q^k/a}-\sum_{k=0}^{\infty}\frac{a\omega^2q^k}{1-a\omega^2q^k}+\sum_{k=1}^{\infty}\frac{\omega q^k/a}{1-\omega q^k/a}\end{aligned}$$

$$= (\omega-\omega^2)\sum_{k=0}^{\infty}\frac{(1-aq^k)aq^k}{1-a^3q^{3k}} + (\omega-\omega^2)\sum_{k=1}^{\infty}\frac{(1-q^k/a)q^k/a}{1-q^{3k}/a^3}$$
$$= (\omega-\omega^2)\sum_{k=0}^{\infty}\frac{(1-aq^k)aq^k}{1-a^3q^{3k}} + (\omega-\omega^2)\sum_{k=1}^{\infty}\frac{(1-aq^{-k})aq^{-k}}{1-a^3q^{-3k}}$$
$$= a(\omega-\omega^2)\sum_{k=-\infty}^{\infty}\frac{q^k}{1-a^3q^{3k}} - a^2(\omega-\omega^2)\sum_{k=-\infty}^{\infty}\frac{q^{2k}}{1-a^3q^{3k}}$$
$$= a(\omega-\omega^2)\frac{[a^3q;q^3]_\infty(q^3;q^3)_\infty^2}{[a^3,q;q^3]_\infty} - a^2(\omega-\omega^2)\frac{[a^3q^2;q^3]_\infty(q^3;q^3)_\infty^2}{[a^3,q^2;q^3]_\infty}, \quad (8.4.12)$$

where we applied Lemma 8.4.1 twice in the last equality. By invoking (8.4.7), we similarly deduce that

$$1+F(b\omega^2)+F(\omega^2/b)$$
$$= F(b\omega^2)-F(b\omega)$$
$$= b^2(\omega-\omega^2)\frac{[b^3q^2;q^3]_\infty(q^3;q^3)_\infty^2}{[b^3,q^2;q^3]_\infty} - b(\omega-\omega^2)\frac{[b^3q;q^3]_\infty(q^3;q^3)_\infty^2}{[b^3,q;q^3]_\infty}. \quad (8.4.13)$$

Substituting (8.4.11) on the left side of (8.4.10), and (8.4.12) and (8.4.13) on the right side of (8.4.10), and dividing both sides by $-b\omega(1-\omega)$, we arrive at

$$\frac{[a,b,ab,a/b;q]_\infty(q;q)_\infty^2}{[a^3,b^3,q;q^3]_\infty} = \frac{[b^3q;q^3]_\infty(q^3;q^3)_\infty^2}{[b^3,q;q^3]_\infty} - b\frac{[b^3q^2;q^3]_\infty(q^3;q^3)_\infty^2}{[b^3,q^2;q^3]_\infty}$$
$$- \frac{a}{b}\frac{[a^3q;q^3]_\infty(q^3;q^3)_\infty^2}{[a^3,q;q^3]_\infty} + \frac{a^2}{b}\frac{[a^3q^2;q^3]_\infty(q^3;q^3)_\infty^2}{[a^3,q^2;q^3]_\infty}.$$

Upon multiplying both sides by $[a^3,b^3,q;q^3]_\infty$, we obtain Winquist's identity in the form (8.4.9). □

Theorem 8.4.5. *For every nonnegative integer n,*

$$p(11n+6)\equiv 0 \pmod{11}. \quad (8.4.14)$$

Proof. In Winquist's identity (8.4.8), replace a and b by a^2 and b^2, respectively, and then multiply both sides by $a^{-1}b^{-1}$. Noting that

$$a^{-1}b^{-1}(1-a^2)(1-b^2) = (a-a^{-1})(b-b^{-1}),$$

we find that

$$\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}q^{(3m^2+3n^2+3m+n)/2}\big(a^{-6m-1}b^{-6n-1}$$
$$-a^{-6m-1}b^{6n+1} - b^{-6m-3}a^{-6n+1} + b^{-6m-3}a^{6n+3}\big)$$
$$= (a-a^{-1})(b-b^{-1})(q;q)_\infty^2(a^2q;q)_\infty(a^{-2}q;q)_\infty(b^2q;q)_\infty(b^{-2}q;q)_\infty$$
$$\times (a^2b^2;q)_\infty(qa^{-2}b^{-2};q)_\infty(a^2b^{-2};q)_\infty(qa^{-2}b^2;q)_\infty. \quad (8.4.15)$$

Differentiate both sides of (8.4.15) with respect to b and then set $b = 1$. Then multiply both sides by a^{-2}. Because $b - b^{-1} = 0$ when $b = 1$, we see that only one expression on the right-hand side survives, namely, that from differentiating $b - b^{-1}$. We note that

$$\frac{d}{db}(b - b^{-1})\Big|_{b=1} = 1 + b^{-2}\big|_{b=1} = 2.$$

Also,

$$a^{-2}(1 - a^2b^2)(1 - a^2b^{-2})\big|_{b=1} = (a - a^{-1})^2.$$

With all of these remarks in mind, we find that the differentiation of (8.4.15) with respect to b and the setting of $b = 1$ yields

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty} &(-1)^{m+n} q^{(3m^2+3n^2+3m+n)/2} \\ &\times \left(-2(6n+1)a^{-6m-3} + (6m+3)a^{-6n-1} - (6m+3)a^{6n+1}\right) \\ &= 2(a - a^{-1})^3 (q;q)_\infty^4 (a^2q;q)_\infty^3 (a^{-2}q;q)_\infty^3. \end{aligned} \tag{8.4.16}$$

We now apply the operator $a\frac{d}{da}$ three times to (8.4.16). Then we set $a = 1$, and lastly divide both sides by 2. On the right-hand side, all of the expressions will equal 0 when we set $a = 1$, except for the contribution obtained by three applications of the aforementioned operator to $(1 - a^{-2})^3$. To help readers with these calculations, we write $2(a - a^{-1})^3 = 2a^3(1 - a^{-2})^3$. The contributions after three applications of the given operator are:

$$\begin{aligned} 6a^3(1 - a^{-2})^2 \cdot 2a^{-3} \cdot a &= 12a(1 - a^{-2})^2, \\ 24a(1 - a^{-2}) \cdot 2a^{-3} \cdot a &= 48a^{-1}(1 - a^{-2}), \\ 48a^{-1} \cdot 2a^{-3} \cdot a &= 96a^{-3}. \end{aligned}$$

With the aid of the calculations above, we find that

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{(3m^2+3n^2+3m+n)/2} \left\{(6m+3)^3(6n+1) - (6m+3)(6n+1)^3\right\} \\ = 48(q;q)_\infty^{10}. \end{aligned} \tag{8.4.17}$$

Now set $\alpha = 6m + 3,\ \beta = 6n + 1$. A straightforward calculation shows that

$$3m^2 + 3n^2 + 3m + n = \frac{1}{12}\left(\alpha^2 + \beta^2 - 10\right). \tag{8.4.18}$$

Hence, we may write (8.4.17) in the shape

$$\sum_{\substack{\alpha,\beta=-\infty \\ \alpha\equiv 3 \pmod 6 \\ \beta\equiv 1 \pmod 6}}^{\infty} (-1)^{(\alpha+\beta-4)/6}(\alpha^3\beta - \alpha\beta^3) q^{(\alpha^2+\beta^2-10)/24} = 48(q;q)_\infty^{10}. \tag{8.4.19}$$

If we set

$$(q;q)_\infty^{10} =: \sum_{n=0}^{\infty} a(n) q^n, \tag{8.4.20}$$

then, equating coefficients of q^n, $n \geq 0$, in (8.4.19), we find that

$$a(n) = \frac{1}{48} \sum_{\substack{\alpha,\beta=-\infty \\ \alpha\equiv 3 \pmod 6 \\ \beta\equiv 1 \pmod 6 \\ (\alpha^2+\beta^2-10)/24=n}}^{\infty} (-1)^{(\alpha+\beta-4)/6}(\alpha^3\beta - \alpha\beta^3).$$

Now,

$$\begin{aligned} &\frac{1}{24}(\alpha^2 + \beta^2 - 10) \equiv 6 \pmod{11}, \\ \iff\ &\alpha^2 + \beta^2 - 10 \equiv 1 \pmod{11}, \\ \iff\ &\alpha^2 + \beta^2 \equiv 0 \pmod{11}, \\ \iff\ &\alpha, \beta \equiv 0 \pmod{11}. \end{aligned}$$

Hence, if $n \equiv 6 \pmod{11}$, then $a(n) \equiv 0 \pmod{11^4}$.

From (8.4.20),

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{(q;q)_\infty^{10}}{(q;q)_\infty^{11}} \equiv \frac{(q;q)_\infty^{10}}{(q^{11};q^{11})_\infty} = \frac{1}{(q^{11};q^{11})_\infty} \sum_{n=0}^{\infty} a(n)q^n \pmod{11}.$$

Extracting those terms with powers of q congruent to 6 modulo 11 and using the subsequent congruence $a(n) \equiv 0 \pmod{11^4}$, we see that

$$\sum_{n=0}^{\infty} p(11n+6)q^{11n+6} = \frac{1}{(q^{11};q^{11})_\infty} \sum_{n=0}^{\infty} a(11n+6)q^{11n+6}.$$

Thus,

$$p(11n+6) \equiv 0 \pmod{11},$$

which is what we wanted to prove. □

Another proof of the identity (8.4.17) for $(q;q)_\infty^{10}$ has been given by the author, S. H. Chan, Z.–G. Liu, and H. Yesilyurt [**33**]. We remark that (8.4.17) shows that $(q;q)_\infty^{10}$ is lacunary. By the pentagonal number theorem (1.2.23) and Jacobi's identity (3.1.30), respectively, $(q;q)_\infty$ and $(q;q)_\infty^3$ are also lacunary. It is natural to ask what powers $(q;q)_\infty^n$ are lacunary, and this was answered by J.–P. Serre [**96**]. Briefly, there are very few powers of the eta-function that are lacunary.

8.5. A More General Partition Function

In a letter to Hardy written from Fitzroy House late in 1918 [**36**, pp. 192–193], Ramanujan writes, "I have considered more or less exhaustively about the congruency of $p(n)$ and in general that of $p_r(n)$ where

$$\sum p_r(n)x^n = \frac{1}{(x;x)_\infty^r},$$

by four different methods." This declaration appears to imply that he had established several results about $p_r(n)$. However, the only work that remains of Ramanujan on this more general partition function is page 182 in the volume containing Ramanujan's lost notebook. The page has "5" written in the upper right-hand corner clearly indicating that this page belonged to a much longer manuscript that has evidently been lost. On this page, Ramanujan's general partition function is defined slightly differently, and we adopt this definition here. Define the more general partition function $p_r(n)$ by

$$\frac{1}{(q;q)_\infty^r} = \sum_{n=0}^{\infty} p_r(n)q^n, \qquad |q| < 1. \tag{8.5.1}$$

This definition is actually not provided on page 182, but it is clear that it must have been given somewhere in the missing pages 1–4 of the manuscript. Of course, $p_1(n) = p(n)$. The elementary methods developed by Ramanujan to prove congruences for $p(n)$ modulo 5 and 7 are sufficient to establish all the results on this page. We follow the exposition of the writer, C. Gugg, and S. Kim in their paper [**34**]. Formerly, a brief account of page 182 was given by K. G. Ramanathan [**83**].

Entry 8.5.1 (p. 182)**.** *Let δ denote any integer, and let n denote a nonnegative integer. Suppose that ϖ is a prime of the form $6\lambda - 1$. Then*

$$p_{\delta\varpi-4}\left(n\varpi - \frac{\varpi+1}{6}\right) \equiv 0 \pmod{\varpi}. \tag{8.5.2}$$

Proof. Recalling that $\lambda = (\varpi+1)/6$, consider

$$\sum_{n=0}^{\infty} p_{\delta\varpi-4}(n)q^{n+\lambda} = (q;q)_\infty^{-\delta\varpi}(q;q)_\infty^3(q;q)_\infty q^\lambda \tag{8.5.3}$$
$$\equiv (q^\varpi;q^\varpi)_\infty^{-\delta} \sum_{\mu=0}^{\infty}\sum_{\nu=-\infty}^{\infty} (-1)^{\mu+\nu}(2\mu+1)q^{\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(3\nu+1)+\lambda} \pmod{\varpi},$$

upon the use of Euler's pentagonal number theorem (1.2.23) and Jacobi's identity (3.1.30). We want to examine those terms for which

$$\frac{1}{2}\mu(\mu+1) + \frac{1}{2}\nu(3\nu+1) + \frac{\varpi+1}{6} \equiv 0 \pmod{\varpi}. \tag{8.5.4}$$

Our goal is to prove that

$$\varpi \mid (2\mu+1). \tag{8.5.5}$$

Multiply (8.5.4) by 24 to obtain the equivalent congruence

$$12\mu(\mu+1) + 12\nu(3\nu+1) + 4\varpi + 4 \equiv 0 \pmod{\varpi},$$

or

$$3(2\mu+1)^2 + (6\nu+1)^2 \equiv 0 \pmod{\varpi}. \tag{8.5.6}$$

Using the fact that, for each prime p, the Legendre symbol $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$, and the law of quadratic reciprocity, we find that

$$\left(\frac{-3}{\varpi}\right) = \left(\frac{\varpi}{3}\right) = \left(\frac{-1}{3}\right) = -1.$$

Thus, the only way that (8.5.6) can hold is for (8.5.5) to happen. But then, from the right-hand side of (8.5.3), we can conclude that

$$p_{\delta\varpi-4}\left(n\varpi - \frac{\varpi+1}{6}\right) \equiv 0 \pmod{\varpi}.$$

Thus, the proof is complete. ☐

Corollary 8.5.2 (p. 182). *For each positive integer n,*

$$p_6(5n-1) \equiv 0 \pmod{5},$$
$$p_7(11n-2) \equiv 0 \pmod{11}.$$

Proof. The first congruence arises from the case $\varpi = 5$ and $\delta = 2$, while the second arises from the case $\varpi = 11$ and $\delta = 1$ in Entry 8.5.1. ☐

Next, Ramanujan gives an elementary proof of the congruence $p(7n-2) \equiv 0 \pmod{7}$. He begins with the same first three lines of [**86**, eq. (13)], [**89**, p. 212], and then argues in a somewhat more abbreviated fashion than he does in [**86**] to deduce the congruence

$$p_{-6}(7n-2) \equiv 0 \pmod{49}, \tag{8.5.7}$$

from which it follows that

$$p(7n-2) \equiv 0 \pmod{7}. \tag{8.5.8}$$

It should be remarked that the stronger congruence (8.5.7) is not mentioned by Ramanujan in [**86**], although it is implicit in his argument.

Unfortunately, the one-page manuscript ends with (8.5.8). It would seem that Ramanujan would have next offered a theorem analogous to Entry 8.5.1, and so we shall state and prove such a theorem here, but, of course, Ramanujan probably would have had lots more to say to us, if his manuscript had survived.

Theorem 8.5.3. *For a prime ϖ with $4 \mid (\varpi+1)$, any integer δ, and any positive integer n,*

$$p_{\delta\varpi-6}\left(n\varpi - \frac{\varpi+1}{4}\right) \equiv 0 \pmod{\varpi}. \tag{8.5.9}$$

In the case $\delta = 0$ above, we can strengthen (8.5.9).

Entry 8.5.4 (p. 182). *We have*

$$p_{-6}\left(n\varpi - \frac{\varpi+1}{4}\right) \equiv 0 \pmod{\varpi^2}. \tag{8.5.10}$$

Observe that (8.5.7) is the special case $\varpi = 7$ of (8.5.10), and so, with slight exaggeration, we affix "p. 182" to the entry above.

Corollary 8.5.5. *For each positive integer n,*

$$p_{3\delta-6}(3n-1) \equiv 0 \pmod{3}. \tag{8.5.11}$$

Proof. Set $\varpi = 3$ in Theorem 8.5.3. ☐

For the case $\delta = 3$ in (8.5.11), N.D. Baruah and K.K. Ojah [**25**], using more sophisticated means, obtained the stronger result

$$p_3(3n-1) \equiv 0 \pmod{3^2}.$$

Proof of Theorem 8.5.3. Consider, for $\lambda = (\varpi+1)/4$,

$$\begin{aligned}\sum_{n=0}^{\infty} p_{\delta\varpi-6}(n)q^{n+\lambda} &= (q;q)_\infty^{-\delta\varpi}(q;q)_\infty^6 q^\lambda \qquad (8.5.12)\\ &\equiv (q^\varpi;q^\varpi)_\infty^{-\delta}\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}(-1)^{\mu+\nu}(2\mu+1)(2\nu+1)q^{\frac12\mu(\mu+1)+\frac12\nu(\nu+1)+\lambda} \pmod{\varpi},\end{aligned}$$

upon the use of Jacobi's identity (3.1.30). We need to show that if

$$\frac12\mu(\mu+1)+\frac12\nu(\nu+1)+\frac{\varpi+1}{4} \equiv 0 \pmod{\varpi}, \qquad (8.5.13)$$

then

$$\varpi^2 \mid (2\mu+1)(2\nu+1). \qquad (8.5.14)$$

The congruence (8.5.9) will then follow from (8.5.14) and (8.5.12). Multiply (8.5.13) by 8 to obtain

$$4\mu(\mu+1)+4\nu(\nu+1)+2\varpi+2 \equiv 0 \pmod{\varpi},$$

or

$$(2\mu+1)^2+(2\nu+1)^2 \equiv 0 \pmod{\varpi}.$$

Since

$$\left(\frac{-1}{\varpi}\right) = -1,$$

we conclude that

$$\varpi \mid (2\mu+1) \quad \text{and} \quad \varpi \mid (2\nu+1),$$

which completes the proof of (8.5.14). □

Observe that if $\delta = 0$, then the congruence in (8.5.12) can be replaced by an equality. Hence, in (8.5.9), the congruence modulo ϖ can be replaced by a congruence modulo ϖ^2 in view of (8.5.14). Entry 8.5.4 therefore follows.

Although Entry 8.5.1 and Theorem 8.5.3 are not special cases of the general theorem of Andrews and Roy [**22**], they would be instances of the general theorem envisioned by the authors in Section 5 of their paper [**22**].

Recall next that a corollary of Winquist's identity is given by [**102**]

$$48(q;q)_\infty^{10} = \sum_{m,n=-\infty}^{\infty}(-1)^{m+n}((6m+3)^3(6n+1)-(6m+3)(6n+1)^3)q^{\frac12(3m^2+3m+3n^2+n)}. \qquad (8.5.15)$$

Theorem 8.5.6. *For a prime ϖ with $12 \mid (\varpi+1)$, and any integer δ, we have*

$$p_{\delta\varpi-10}\Big(n\varpi - \frac{5(\varpi+1)}{12}\Big) \equiv 0 \pmod{\varpi}.$$

Proof. Let $\lambda = 5(\varpi+1)/12$, and from (8.5.15) consider

$$\sum_{n=0}^{\infty} p_{\delta\varpi-10}(n)q^{n+\lambda} = (q;q)_\infty^{-\delta\varpi}(q;q)_\infty^{10}q^\lambda \tag{8.5.16}$$

$$\equiv (q^\varpi;q^\varpi)_\infty^{-\delta}\frac{1}{48}\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}((6m+3)^3(6n+1) - (6m+3)(6n+1)^3)q^{\frac{1}{2}(3m^2+3m+3n^2+n)+\lambda} \pmod{\varpi}.$$

If

$$\frac{1}{2}(3m^2+3m+3n^2+n)+\lambda \equiv 0 \pmod{\varpi},$$

then upon multiplying both sides above by 24, we find that

$$12(3m^2+3m+3n^2+n)+10(\varpi+1) \equiv 0 \pmod{\varpi},$$

or

$$(6m+3)^2+(6n+1)^2 \equiv 0 \pmod{\varpi}.$$

Since

$$\Big(\frac{-1}{\varpi}\Big) = -1,$$

we see that

$$\varpi \mid (6m+3) \qquad \text{and} \qquad \varpi \mid (6n+1).$$

Using these observations in (8.5.16), we complete the proof. □

We observe that in the special case $\delta = 0$, our proof yields a stronger result.

Corollary 8.5.7. *For a prime ϖ with $12 \mid (\varpi+1)$, we have*

$$p_{-10}\Big(n\varpi - \frac{5(\varpi+1)}{12}\Big) \equiv 0 \pmod{\varpi^4}.$$

8.6. Exercises

1. Use the Jacobi triple product identity to show that

$$\varphi(-q) = \frac{(q;q)_\infty}{(-q;q)_\infty}. \tag{8.6.1}$$

2. Find a proof of Theorem 8.4.5 that is in the spirit of Ramanujan's elementary proofs of the congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$, in particular, that depends on the pentagonal number theorem and Jacobi's identity.

3. Give a proof of Halphen's Identity.

4. Prove (8.4.11).

Define the colored partition function $p_m(n)$ by

$$(q;q)_\infty^m =: \sum_{n=0}^{\infty} p_m(n)q^n. \tag{8.6.2}$$

5. If $p_5(n), n \geq 0$, is defined by (8.6.2), then

$$\sum_{n=0}^{\infty} p_5(5n)q^n = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty}.$$

Hint: Write

$$\sum_{n=0}^{\infty} p_5(n)q^n = (q;q)_\infty^5 = \frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty} \frac{(q;q)_\infty^5}{(q^{25};q^{25})_\infty^5} \frac{(q^{25};q^{25})_\infty^6}{(q^5;q^5)_\infty^6}.$$

6. If $p_7(n), n \geq 0$, is defined by (8.6.2), then

$$\sum_{n=0}^{\infty} p_7(7n)q^n = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty} + 49q(q;q)_\infty^4(q^7;q^7)_\infty^3.$$

Hint: Write

$$\sum_{n=0}^{\infty} p_7(n)q^n = (q;q)_\infty^7 = \frac{(q^7;q^7)_\infty^8}{(q^{49};q^{49})_\infty} \frac{(q;q)_\infty^7}{(q^{49};q^{49})_\infty^7} \frac{(q^{49};q^{49})_\infty^8}{(q^7;q^7)_\infty^8}.$$

7. Recall that

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2} \qquad \text{and} \qquad \psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}. \tag{8.6.3}$$

In each case, determine for which positive integers n, $\varphi^n(q)$ and $\psi^n(q)$ are lacunary.

8. Prove (8.3.7), (8.3.9), and (8.3.12).

9. Prove that Winquist's identity (8.4.8) can be put in the form (8.4.9).

Chapter 9

Ranks and Cranks of Partitions

9.1. Definitions and Generating Functions for the Rank and Crank

In attempting to find a combinatorial interpretation for Ramanujan's famous congruences for the partition function $p(n)$, the number of ways of representing the positive integer n as a sum of positive integers, in 1944, F. J. Dyson [**48**] defined the *rank of a partition.*

Definition 9.1.1. *The rank of a partition is the largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of n with rank m.*

We find a generating function for $N(m,n)$. Consider the Ferrers diagram of a partition. Suppose that the Durfee square has side n.

Theorem 9.1.2. *If $N(m,n)$ denotes the number of partitions of n with rank m, then its generating function is given by*

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(q/z)_n}. \tag{9.1.1}$$

Proof. Consider the Ferrers graph of any partition π. Suppose that the largest Durfee square has side n. To the right of this Durfee square, there lies the graph of a partition into parts that are no larger than n, where we read from left to right, i.e., we are reading the conjugate of a portion of our original partition π. The partitions to the right of our Durfee square are generated by $1/(zq)_n$, where the power of z denotes the number of parts. If s_r denotes this number of parts, then note that the largest part of our original partition is equal to $n + s_r$.

Next, consider the partition below the Durfee square of side n. Reading from top to bottom, we see that the parts are no greater than n; let us say that there are s_ℓ

partition of 4	rank	partition of 5	rank	partition of 6	rank
4	3	5	4	6	5
$3+1$	1	4+1	2	5+1	3
$2+2$	0	3+2	1	4+2	2
$2+1+1$	-1	3+1+1	0	4+1+1	1
$1+1+1+1$	-3	2+2+1	-1	3+3	1
		2+1+1+1	-2	3+2+1	0
		1+1+1+1+1	-4	3+1+1+1	-1
				2+2+2	-1
				2+2+1+1	-2
				2+1+1+1+1	-3
				1+1+1+1+1+1	-5

Table 1

parts. The generating function for these partitions is $1/(z^{-1}q)_n$, where the power of z^{-1} denotes the number of parts.

Hence, the partitions with Durfee square of side n are generated by $q^{n^2}/(zq)_n(q/z)_n$. If we sum over all Durfee squares of side n, then we find that the generating function for all partitions is equal to

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(q/z)_n}.$$

Note that the original partition π has $n+s_\ell$ parts. Hence, the rank of our original partition π is equal to

$$(n+s_r)-(n+s_\ell)=s_r-s_\ell.$$

The power of z is equal to s_r-s_ℓ, which we have shown to be the rank of π. Hence, (9.1.1) follows. □

Dyson offered several conjectures, including combinatorial interpretations of Ramanujan's famous congruences $p(5n+4)\equiv 0 \pmod 5$ and $p(7n+5)\equiv 0 \pmod 7$. To see how Dyson formulated his conjectures, let us calculate some ranks of partitions.

We see that each residue class modulo 5 contains the rank of one of the five partitions of 4. In general, Dyson found that the ranks of the partitions of $5n+4$ fall into equinumerous residue classes modulo 5. Likewise, we see that each residue class modulo

7 contains exactly one of the ranks of the seven partitions of 5. Further calculations of Dyson seemed to indicate that the ranks of partitions of $7n+5$ are distributed into equinumerous residue classes modulo 7. These conjectures, as well as further conjectures of Dyson, were first proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer [**24**] in 1954.

The corresponding analogue does not hold for $p(11n+6) \equiv 0 \pmod{11}$. In particular, we see that the residue classes 1 and 10 are represented twice, and the residue classes 4 and 7 are not represented. Dyson conjectured the existence of another statistic, which he could not identify and which he whimsically called the *crank*, that would combinatorially explain the congruences for numbers of the form $11n+6$ and hopefully also for those of the kinds $5n+4$ and $7n+5$.

In his doctoral dissertation [**55**], motivated by material in Ramanujan's lost notebook [**91**], Garvan defined the crank of certain vector partitions. This is not quite the crank envisioned by Dyson, because it is associated with vector partitions. The goal is to find a crank for ordinary partitions. But, as we shall see, Garvan's crank for vector partitions led to the crank for ordinary partitions. We now describe Garvan's crank.

Definition 9.1.3. *Let π denote a partition; let $\#(\pi)$ denote the number of parts of π; and let $\sigma(\pi)$ denote the sum of the parts of π, i.e., usually also denoted by n. Let V denote the set of vector partitions defined by*

$$V = \{(\pi_1, \pi_2, \pi_3) : \pi_1 = \textit{ partition into distinct parts };\pi_2, \pi_3 = \textit{ unrestricted partitions}\}.$$

For brevity, put $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$. Furthermore, set

$$\begin{aligned} s(\vec{\pi}) &= \sigma_1(\pi) + \sigma_2(\pi) + \sigma_3(\pi), \\ w(\vec{\pi}) &= (-1)^{\#(\pi_1)} \qquad (\textit{weight of } \vec{\pi}), \\ r(\vec{\pi}) &= \#(\pi_2) - \#(\pi_3) \qquad (\textit{crank of } \vec{\pi}). \end{aligned}$$

As an example, let $\vec{\pi} = (5+3+2, 2+2+1, 2+1+1)$. Then

$$\begin{aligned} s(\vec{\pi}) &= 10+5+4 = 19, \\ w(\vec{\pi}) &= (-1)^3 = -1, \\ r(\vec{\pi}) &= 3-3 = 0. \end{aligned}$$

Definition 9.1.4. *The number of vector partitions of n with crank m is defined by*

$$N_V(m,n) := \sum_{\substack{\vec{\pi}\in V \\ s(\vec{\pi})=n \\ r(\vec{\pi})=m}} w(\vec{\pi}).$$

Furthermore, let

$$N_V(k,t,n) = \sum_{\substack{\vec{\pi}\in V \\ s(\vec{\pi})=n \\ r(\vec{\pi})\equiv k \pmod{t}}} w(\vec{\pi}) = \sum_{m=-\infty}^{\infty} N_V(mt+k,n).$$

Since we may interchange the roles of π_2 and π_3, we easily see that

$$N_V(m,n) = N_V(-m,n),$$

which implies that

$$N_V(t-m,t,n) = N_V(m,t,n).$$

If we recall the definition of the crank of a partition $\vec{\pi}$ from Definition 9.1.3, we can readily verify the truth of the next theorem.

Theorem 9.1.5. *If $N_V(m,n)$ is defined by Definition 9.1.4, then*

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_V(m,n)z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty (q/z;q)_\infty}. \tag{9.1.2}$$

Observe that if $z=1$, (9.1.2) reduces to the ordinary generating function for $p(n)$.

The *true crank* was discovered by Andrews and Garvan [**20**] on June 6, 1987. The instructor unwittingly played a role in the discovery by Andrews and Garvan. On Monday–Friday, June 1–5, 1987, a major international conference was held at the University of Illinois to commemorate the centenary of Ramanujan's birth and to highlight the influence of Ramanujan's mathematics in many areas of contemporary research in a series of several diverse lectures. At that time, air fares were considerably cheaper if one stayed over on a Saturday night. Thus, if this lecturer had any wits about him, he would have arranged for the meeting to be held on June 2–6. Unfortunately, because he failed this test of elementary reasoning, many of the conference participants, including Andrews and Garvan stayed an extra day. Having nothing better to do at the Illinois Street Residence Hall where they were staying, Andrews and Garvan began to discuss Garvan's definition of a vector crank on the afternoon of June 6, and before the afternoon had come to a close, they had discovered the true crank envisioned by Dyson, which we now define.

Definition 9.1.6. *For a partition π, let*

$$\begin{aligned}\ell(n) &= \textit{ the largest part of } \pi,\\ \mu(\pi) &= \textit{ the number of ones in } \pi,\\ \nu(\pi) &= \textit{ the number of parts of } \pi \textit{ larger than } \mu(\pi).\end{aligned}$$

The crank $c(\pi)$ is then defined to be

$$c(\pi) = \begin{cases}\ell(\pi), & \textit{if } \mu(\pi)=0,\\ \nu(\pi)-\mu(\pi), & \textit{if } \mu(\pi)>0.\end{cases} \tag{9.1.3}$$

The crank not only leads to a combinatorial interpretation of $p(11n+6) \equiv 0 \pmod{11}$, as predicted by Dyson, but also to similar interpretations for $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$. Let us perform some calculations in the following table.

partition of 4	crank	partition of 5	crank	partition of 6	crank
4	4	5	5	6	6
$3+1$	0	4+1	0	5+1	0
$2+2$	2	3+2	3	4+2	4
$2+1+1$	-2	3+1+1	-1	4+1+1	-1
$1+1+1+1$	-4	2+2+1	1	3+3	3
		2+1+1+1	-3	3+2+1	1
		1+1+1+1+1	-5	3+1+1+1	-3
				2+2+2	2
				2+2+1+1	-2
				2+1+1+1+1	-4
				1+1+1+1+1+1	-6

Table 2

We note that for each set of congruences, each residue class, modulo 5, 7, and 11, respectively, is represented once. Indeed, the work of Garvan [**55**] and Andrews and Garvan [**20**] shows that the cranks of partitions of numbers of the forms $5n+4$, $7n+5$, and $11n+6$ are equinumerous in the residue classes modulo 5, 7, and 11, respectively. Thus, the dream of Dyson has been realized.

For $n > 1$, let $M(m, n)$ denote the number of partitions of n with crank m. We now demonstrate that for $n > 1$, $N_V(m, n) = M(m, n)$. As we shall demonstrate later, for $n = 1$ the proposed equality is not satisfied.

Theorem 9.1.7. *The number of partitions of π with $c(\pi) = m$ is equal to $N_V(m, n)$ for $n > 1$, i.e., $N_V(m, n) = M(m, n)$ for $n > 1$.*

Proof. Using Thoerem 9.1.5 and the q-binomial theorem, Theorem 3.1.2, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_V(m,n)z^m q^n &= \frac{(1-q)(q^2;q)_\infty}{(zq;q)_\infty (q/z;q)_\infty} \\
&= \frac{1-q}{(zq;q)_\infty}\sum_{j=0}^{\infty}\frac{(zq;q)_j (q/z)^j}{(q;q)_j} \\
&= \frac{1-q}{(zq;q)_\infty} + \sum_{j=1}^{\infty}\frac{z^{-j}q^j}{(q^2;q)_j (zq^{j+1};q)_\infty}. \qquad (9.1.4)
\end{aligned}$$

If we write the summands on the far right side of (9.1.4) in the form

$$\frac{z^{-j}q^{1+1+\cdots+1}}{(q^2;q)_j(zq^{j+1};q)_\infty},$$

we see that the sum generates partitions with $w(\pi)=j$ and with contributions to the exponent of z arising from the 1's and only those parts that are larger than $w(\pi)=j$, i.e., the exponent on z is equal to

$$\mu(\pi)-w(\pi)=c(\pi).$$

There remains the examination of $(1-q)/(zq;q)_\infty$ in (9.1.4). First, $1/(zq;q)_\infty$ generates partitions with the power of z keeping track of the number of parts of the partitions. We want to consider the conjugate partitions, so that the exponent of z keeps track of the largest part of partitions. Second, examine $q/(zq;q)_\infty$, which, when considering the conjugate partitions, generates partitions in which the power of z keeps track of the largest part, but now with at least one 1 in the partitions. (In the Ferrers graph, we put the 1 at the right end of the top row of the Ferrers graph.) Thus, in

$$\frac{1}{(zq;q)_\infty}-\frac{q}{(zq;q)_\infty},$$

all of the partitions of the first quotient are cancelled by those in the second quotient, except those that do not have any 1's in them, because the second quotient generates partitions with at least one 1. Note that our argument, however, fails if $n=1$, because then the exponent on z would be equal to 0, which is not equal to the largest part, namely 1. In conclusion,

$$\frac{1-q}{(zq;q)_\infty}$$

counts partitions with no 1's and with the exponent on z, i.e., $c(\pi)=m$, equaling the largest part of the partition, i.e., $c(\pi)=\ell(\pi)$. □

Let us now examine the cases when $n=0,1$. Observe first that $c(1)=0-1=-1$. Thus, we should (but we won't) define

$$M(-1,1)=1 \qquad \text{and} \qquad M(0,1)=M(1,1)=0.$$

We are not defining $M(m,1)$ as above, because these values do not agree with the coefficients in the generating function (9.1.2). We observe that

$$\begin{aligned} N_V(0,1)&=-1, \qquad \text{for } \vec{\pi}=(1,0,0),\\ N_V(1,1)&=1, \qquad \text{for } \vec{\pi}=(0,1,0),\\ N_V(-1,1)&=1, \qquad \text{for } \vec{\pi}=(0,0,1). \end{aligned}$$

Hence, the values of $N_V(m,1)$ do not agree with the crank, except when $m=-1$. In the sequel, we shall thus define $M(m,1)=N_V(m,1)$, for the possible values $m=0,\pm1$. By convention, we also define $M(0,0)=1$ and $M(-j,0)=0$, for $j\geq 1$.

9.2. Dissections of Cranks

After Ramanujan, let

$$F(q) := F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} = \sum_{n=0}^{\infty} \lambda_n q^n. \tag{9.2.1}$$

Thus, from Theorem 9.1.5, Theorem 9.1.7, and the discussion following Theorem 9.1.7,

$$\lambda_n = \sum_{m=-\infty}^{\infty} M(m,n) a^m.$$

On page 179 in his lost notebook, Ramanujan states the following claim.

Entry 9.2.1. *If $F(q)$ is defined by* (9.2.1) *and $f(a,b)$ is defined by* (1.1.6)*, then*

$$F(\sqrt{q}) \equiv \frac{f(-q^3,-q^5)}{(-q^2;q^2)_\infty} + \left(a - 1 + \frac{1}{a}\right)\sqrt{q}\frac{f(-q,-q^7)}{(-q^2;q^2)_\infty} \mod \left(a^2 + \frac{1}{a^2}\right). \tag{9.2.2}$$

The congruence in Entry 9.2.1 is in the ring of power series in the variables a, q. The congruence (9.2.2) may be expressed in terms of an equality of the form

$$f(a,q) = g(a,q) + h(a,q)\left(a^2 + \frac{1}{a^2}\right).$$

If we replace q by q^2 in (9.2.2), we see that Ramanujan, modulo (a^2+1/a^2), is dissecting $F_a(q)$ into even powers and odd powers. On the same page, Ramanujan also states a congruence for $F_a(q^{1/3})$ modulo (a^3+1+1/a^3).

Set $\lambda_2 = a^2 + 1/a^2$. Then, in succession,

$$a^2\lambda_2 = a^4 + 1 \Rightarrow a^4 \equiv -1 \pmod{\lambda_2} \Rightarrow a^8 \equiv 1 \pmod{\lambda_2}.$$

Thus, a behaves like a primitive 8th root of unity modulo λ_2.

Definition 9.2.2. *Let $P(q)$ be a power series in q. Then the t-dissection of $P(q)$ is given by*

$$P(q) = \sum_{k=0}^{t-1} q^k P_k(q^t).$$

A similar definition can be made for a congruence t-dissection of $P(q)$. Thus, in Entry 9.2.1, Ramanujan is offering a congruence 2-dissection of $F(q)$.

Now let $a = e^{2\pi i/8}$. Then $a - 1 + 1/a = 2\cos(\pi/4) - 1 = \sqrt{2} - 1$ and $a^2 + 1/a^2 = 2\cos(\pi/2) = 0$. Hence, with $a = e^{2\pi i/8}$ and q replaced by q^2, we may state Entry 9.2.1 as an equality, namely,

$$F(q) = \frac{f(-q^6,-q^{10})}{(-q^4;q^4)_\infty} + \left(\sqrt{2}-1\right) q \frac{f(-q^2,-q^{14})}{(-q^4;q^4)_\infty}. \tag{9.2.3}$$

In his lost notebook [**91**], as mentioned above, Ramanujan offers the 3-dissection of $F_a(q)$. On page 20 in the lost notebook, he states the 5-dissection of $F_a(q)$ in terms of an equality. Ramanujan also found the 7- and 11-dissections of $F_a(q)$, although he did not state them explicitly. On page 71, he offers (upside down) the quotients of theta functions that appear in the 7-dissection, and on page 70, he records the quotients of theta functions that appear in the 11-dissection of $F_a(q)$.

Proof of Entry 9.2.1. We prove Entry 9.2.1 with q replaced by q^2. First,

$$\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} = (q;q)_\infty \prod_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}(aq^n)^k\right)\left(\sum_{k=0}^{\infty}(q^n/a)^k\right). \tag{9.2.4}$$

We subdivide the sums in the product on the right-hand side above into residue classes modulo 8 while using congruences for powers of a modulo 8. Hence, modulo λ_2,

$$\begin{aligned}
\sum_{k=0}^{\infty}(aq^n)^{8k} &\equiv \frac{1}{1-q^{8n}}, & \sum_{k=0}^{\infty}(aq^n)^{8k+1} &\equiv \frac{aq^n}{1-q^{8n}},\\
\sum_{k=0}^{\infty}(aq^n)^{8k+2} &\equiv \frac{a^2q^{2n}}{1-q^{8n}}, & \sum_{k=0}^{\infty}(aq^n)^{8k+3} &\equiv \frac{a^3q^{3n}}{1-q^{8n}},\\
\sum_{k=0}^{\infty}(aq^n)^{8k+4} &\equiv -\frac{q^{4n}}{1-q^{8n}}, & \sum_{k=0}^{\infty}(aq^n)^{8k+5} &\equiv -\frac{aq^{5n}}{1-q^{8n}},\\
\sum_{k=0}^{\infty}(aq^n)^{8k+6} &\equiv -\frac{a^2q^{6n}}{1-q^{8n}}, & \sum_{k=0}^{\infty}(aq^n)^{8k+7} &\equiv -\frac{a^3q^{7n}}{1-q^{8n}}.
\end{aligned}$$

We now observe that

$$\frac{1-q^{4n}}{1-q^{8n}} + aq^n\frac{1-q^{4n}}{1-q^{8n}} + a^2q^{2n}\frac{1-q^{4n}}{1-q^{8n}} + a^3q^{3n}\frac{1-q^{4n}}{1-q^{8n}} = \frac{1+aq^n+a^2q^{2n}+a^3q^{3n}}{1+q^{4n}}.$$

Using these calculations in (9.2.4), we find that

$$\begin{aligned}
&\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty}\\
&\quad\equiv (q;q)_\infty \prod_{n=1}^{\infty}\frac{(1+aq^n+a^2q^{2n}+a^3q^{3n})(1+a^{-1}q^n+a^{-2}q^{2n}+a^{-3}q^{3n})}{(1+q^{4n})^2} \pmod{\lambda_2}.
\end{aligned} \tag{9.2.5}$$

Using the facts that

$$a^2q^{2n}+a^{-2}q^{2n} \equiv 0 \pmod{\lambda_2}, \quad \text{and} \quad a^3q^{3n}+a^{-3}q^{3n} \equiv -\left(\frac{1}{a}+a\right)q^{3n} \pmod{\lambda_2},$$

we find that

$$\begin{aligned}
&(1+aq^n+a^2q^{2n}+a^3q^{3n})(1+a^{-1}q^n+a^{-2}q^{2n}+a^{-3}q^{3n})\\
&\equiv 1+(a+a^{-1})q^n+q^{2n}+0q^{3n}+q^{4n}+(a+a^{-1})q^{5n}+q^{6n} \pmod{\lambda_2}\\
&= (1+(a+a^{-1})q^n+q^{2n})+q^{4n}(1+(a+a^{-1})q^n+q^{2n}) \pmod{\lambda_2}\\
&= (1+q^{4n})(1+aq^n)(1+a^{-1}q^n) \pmod{\lambda_2}.
\end{aligned} \tag{9.2.6}$$

Putting (9.2.6) into (9.2.5) and invoking the Jacobi triple product identity (1.1.7), we arrive at

$$\begin{aligned}\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} &= \frac{(q;q)_\infty}{(-q^4;q^4)_\infty}\prod_{n=1}^{\infty}(1+aq^n)(1+a^{-1}q^n)\\ &= \frac{(q;q)_\infty(-aq;q)_\infty(-q/a;q)_\infty}{(-q^4;q^4)_\infty}\\ &= \frac{f(a,q/a)}{(1+a)(-q^4;q^4)_\infty}. \qquad (9.2.7)\end{aligned}$$

We now apply Lemma 1.1.2 with a, $b=q/a$, and $N=4$. Recall also that $a^4 \equiv -1 \pmod{\lambda_2}$. Hence, we deduce that

$$\begin{aligned}f(a,q/a) &= f(a^4q^6,q^{10}/a^4)+af(a^4q^{10},q^6/a^4)+a^2qf(a^4q^{14},q^2/a^4)\\ &\quad + a^3q^3f(a^4q^{18},1/(a^4q^2))\\ &= f(a^4q^6,q^{10}/a^4)+af(a^4q^{10},q^6/a^4)+a^2qf(a^4q^{14},q^2/a^4)\\ &\quad + (q/a)f(q^{14}/a^4,a^4q^2)\\ &\equiv (1+a)f(-q^6,-q^{10})+(a^2+1/a)qf(-q^2,-q^{10}) \pmod{\lambda_2}, \qquad (9.2.8)\end{aligned}$$

where in the penultimate line we applied (1.1.11) from Lemma 1.1.1. Now

$$\frac{a^2+1/a}{1+a} \equiv a-1+1/a \pmod{\lambda_2}. \qquad (9.2.9)$$

Hence, using (9.2.9) in (9.2.8), we find that

$$\frac{f(a,q/a)}{1+a} \equiv f(-q^6,-q^{10})+\left(a-1+\frac{1}{a}\right)qf(-q^2,-q^{14}) \pmod{\lambda_2}. \qquad (9.2.10)$$

Inserting (9.2.10) in (9.2.7), we deduce (9.2.2) with q replaced by q^2, namely,

$$F_a(q) \equiv \frac{f(-q^6,-q^{10})}{(-q^4;q^4)_\infty}+(A_1-1)q\frac{f(-q^2,-q^{14})}{(-q^4;q^4)_\infty} \pmod{A_2}, \qquad (9.2.11)$$

where

$$A_n := a^n + a^{-n}. \qquad (9.2.12)$$

This completes the proof of Entry 9.2.1. □

Our next goal is to prove the 5-dissection of $F_a(q)$, which in fact yields Ramanujan's congruence $p(5n+4) \equiv 0 \pmod 5$ as a corollary. It is natural to ask if Ramanujan had any application in mind for (9.2.11). In view of the aforementioned application of the 5-dissection of $F_a(q)$, was he intending to use (9.2.11) to study the parity of $p(n)$? We will discuss the parity of $p(n)$ in Chapter 10. Unfortunately, to the best of our knowledge, no one has been able to use (9.2.11) to study this important problem.

Set

$$S_n := \sum_{k=-n}^{n} a^k. \qquad (9.2.13)$$

Theorem 9.2.3. *If A_n is defined by* (9.2.12) *and S_n is defined by* (9.2.13), *then*

$$\begin{aligned}F_a(q) \equiv & \frac{f(-q^{10},-q^{15})}{f^2(-q^5,-q^{20})}f^2(-q^{25}) + (A_1-1)q\frac{f^2(-q^{25})}{f(-q^5,-q^{20})} \\ & + A_2q^2\frac{f^2(-q^{25})}{f(-q^{10},-q^{15})} - A_1q^3\frac{f(-q^5,-q^{20})}{f^2(-q^{10},-q^{15})}f^2(-q^{25}) \pmod{S_2}.\end{aligned} \tag{9.2.14}$$

Observe that (9.2.14) provides the 5-dissection of $F_a(q)$. Also, note that there are no powers congruent to 4 modulo 5 in this dissection. Furthermore, by (9.2.13),

$$a^4 + a^3 + a^2 + a + 1 = a^2S_2 \equiv 0 \pmod{S_2}. \tag{9.2.15}$$

Hence, a acts like a fifth root of unity.

Proof of Theorem 9.2.3. For each positive integer n, we can see from (9.2.15) that

$$(1-q^n)(1-aq^n)(1-a^2q^n)(1-a^3q^n)(1-a^4q^n) \equiv 1 - q^{5n} \pmod{S_2}. \tag{9.2.16}$$

Hence, by (9.2.16),

$$\begin{aligned}F_a(q) = \frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} &= \frac{(q;q)^2_\infty(a^2q;q)_\infty(a^3q;q)_\infty}{(q;q)_\infty(aq;q)_\infty(q/a;q)_\infty(a^2q;q)_\infty(a^3q;q)_\infty} \\ &\equiv \frac{(q;q)^2_\infty(a^2q;q)_\infty(a^3q;q)_\infty}{(q^5;q^5)_\infty}.\end{aligned} \tag{9.2.17}$$

To proceed further, we need two facts about the Rogers–Ramanujan continued fraction, defined in (6.2.4) by

$$R(q) := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots, \qquad |q| < 1. \tag{9.2.18}$$

First, recall that the Rogers–Ramanujan identities enabled us to represent $R(q)$ in Theorem 6.2.5 as a quotient of theta functions

$$R(q) = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = q^{1/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)}. \tag{9.2.19}$$

Second, we need a famous result about $R(q)$ that was proved in Theorem 6.2.6 in a slightly different notation, namely,

$$\frac{1}{R(q)} - R(q) - 1 = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}. \tag{9.2.20}$$

If we replace q by q^5 in (9.2.20) and employ (9.2.19), we find that

$$\frac{f(-q)}{qf(-q^{25})} = \frac{f(-q^{10},-q^{15})}{qf(-q^5,-q^{20})} - \frac{qf(-q^5,-q^{20})}{f(-q^{10},-q^{15})} - 1. \tag{9.2.21}$$

Next, by (9.2.15) and the Jacobi triple product identity (1.1.7),

$$\begin{aligned}(q;q)_\infty(a^2q;q)_\infty(a^3q;q)_\infty &= \frac{(q;q)_\infty(a^2;q)_\infty(a^3q;q)_\infty}{1-a^2} \\ &\equiv \frac{(q;q)_\infty(a^2;q)_\infty(q/a^2;q)_\infty}{1-a^2} = \frac{f(-a^2,-q/a^2)}{1-a^2} \pmod{S_2}.\end{aligned} \tag{9.2.22}$$

To obtain a 5-dissection, we need to apply Lemma 1.1.2 with a replaced by $-a^2$, b replaced by $-q/a^2$, and $n = 5$. Elementary calculations yield, for each positive integer n,

$$U_n = (-1)^n a^{2n} q^{n(n-1)/2} \quad \text{and} \quad V_n = (-1)^n a^{-2n} q^{n(n+1)/2}.$$

Hence,

$$\begin{aligned}
&f(-a^2, -q/a^2)\\
&= f(-a^{10}q^{10}, -a^{-10}q^{15}) - a^2 f(-a^{10}q^{15}, -a^{-10}q^{10}) + a^4 q f(-a^{10}q^{20}, -a^{-10}q^5)\\
&\quad - a^6 q^3 f(-a^{10}q^{25}, -a^{-10}) + a^8 q^6 f(-a^{10}q^{30}, -a^{-10}q^{-5})\\
&\equiv (1-a^2) f(-q^{10}, -q^{15}) + a^4 q f(-q^5, -q^{20}) - a^3 q^6 \cdot q^{-5} f(-q^5, -q^{20}) \pmod{S_2}\\
&\equiv (1-a^2) f(-q^{10}, -q^{15}) + q(a^4 - a^3) f(-q^5, -q^{20}) \pmod{S_2}, \qquad (9.2.23)
\end{aligned}$$

where in the penultimate line we applied (1.1.10) and (1.1.11). Observe that, by (9.2.15) and (9.2.12),

$$\frac{a^4 - a^3}{1 - a^2} \equiv A_1 \pmod{S_2}. \qquad (9.2.24)$$

Using (9.2.23) and (9.2.24) in (9.2.22), we conclude that

$$\begin{aligned}
(q;q)_\infty (a^2 q;q)_\infty (a^3 q;q)_\infty &\equiv f(-q^{10}, -q^{15}) + q \frac{a^4 - a^3}{1 - a^2} f(-q^5, -q^{20})\\
&\equiv f(-q^{10}, -q^{15}) + q A_1 f(-q^5, -q^{20}) \pmod{S_2}. \qquad (9.2.25)
\end{aligned}$$

We now return to (9.2.17). For brevity, let us set

$$x := f(-q^{10}, -q^{15}) \qquad \text{and} \qquad y := f(-q^5, -q^{20}).$$

Using (9.2.21) and (9.2.25) in (9.2.17), we finally conclude that

$$F_a(q) \equiv \frac{f(-q^{25})}{f(-q^5)} (x + q A_1 y) \left(\frac{x}{y} - q^2 \frac{y}{x} - q \right) \pmod{S_2}. \qquad (9.2.26)$$

Using the Jacobi triple product identity (1.1.7), we readily find that

$$\begin{aligned}
xy &= (q^{10}; q^{25})_\infty (q^{15}; q^{25})_\infty (q^{25}; q^{25})_\infty (q^5; q^{25})_\infty (q^{20}; q^{25})_\infty (q^{25}; q^{25})_\infty\\
&= f(-q^5) f(-q^{25}). \qquad (9.2.27)
\end{aligned}$$

We are now ready to determine the 5-dissection of $F_a(q)$ from (9.2.26). First, by (9.2.26) and (9.2.27),

$$[q^0] F_a(q) = \frac{f(-q^{25})}{f(-q^5)} \frac{x^2}{y} = \frac{f(-q^{25})}{f(-q^5)} xy \frac{x}{y^2} = \frac{f(-q^{25})}{f(-q^5)} f(-q^5) f(-q^{25}) \frac{x}{y^2} = f^2(-q^{25}) \frac{x}{y^2}. \qquad (9.2.28)$$

Second, by (9.2.26) and (9.2.27),

$$[q^1] F_a(q) = \frac{f(-q^{25})}{f(-q^5)} (A_1 x - x) = \frac{f(-q^{25})}{f(-q^5)} xy \frac{A_1 - 1}{y} = f^2(-q^{25}) \frac{A_1 - 1}{y}. \qquad (9.2.29)$$

Third, by (9.2.26) and (9.2.27),

$$[q^2]F_a(q) = -\frac{f(-q^{25})}{f(-q^5)}(y + A_1 y) = -\frac{f(-q^{25})}{f(-q^5)}xy\frac{A_1+1}{x}$$
$$= -f^2(-q^{25})\frac{A_1+1}{x} \equiv -f^2(-q^{25})\frac{A_2}{x} \pmod{S_2}. \qquad (9.2.30)$$

Fourth, by (9.2.26) and (9.2.27),

$$[q^3]F_a(q) = -\frac{f(-q^{25})}{f(-q^5)}\frac{A_1 y^2}{x} = -\frac{f(-q^{25})}{f(-q^5)}xy\frac{A_1 y}{x^2} = -f^2(-q^{25})\frac{A_1 y}{x^2}. \qquad (9.2.31)$$

Fifth, observe that

$$[q^4]F_a(q) = 0. \qquad (9.2.32)$$

Hence, employing (9.2.28)–(9.2.32) in (9.2.26), we complete the proof of (9.2.14). □

Corollary 9.2.4. *For each nonnegative integer n,*

$$p(5n+4) \equiv 0 \pmod 5. \qquad (9.2.33)$$

Proof. Let $a = 1$ in Theorem 9.2.3 to deduce that

$$\frac{1}{(q;q)_\infty} \equiv \frac{f(-q^{10},-q^{15})}{f^2(-q^5,-q^{20})}f^2(-q^{25}) + q\frac{f^2(-q^{25})}{f(-q^5,-q^{20})}$$
$$+ 2q^2\frac{f^2(-q^{25})}{f(-q^{10},-q^{15})} - 2q^3\frac{f(-q^5,-q^{20})}{f^2(-q^{10},-q^{15})}f^2(-q^{25}) \pmod 5. \qquad (9.2.34)$$

The coefficient of q^{5n+4} on the right-hand side of (9.2.34) is clearly equal to 0. Thus, the coefficient of q^{5n+4} on the left-hand side modulo 5 must also be equal to 0, i.e., $p(5n+4) \equiv 0 \pmod 5$. □

We close this section by establishing the 7-dissection for the crank generating function. In some ways, the proof is easier than that for the 5-dissection. We shall ask the reader in an exercise to supply the missing details.

Theorem 9.2.5. *Recall that A_n and S_n are defined in* (9.2.12) *and* (9.2.13), *respectively. Let*

$$A := f(-q^{21},-q^{28}), \quad B := f(-q^{14},-q^{35}), \quad \text{and} \quad C := f(-q^7,-q^{42}).$$

Then

$$F_a(q) \equiv \frac{1}{f(-q^7)}\{A^2 + (A_1-1)qAB + A_2q^2B^2 + (A_3+1)q^3AC$$
$$-A_1q^4BC - (A_2+1)q^6C^2\} \pmod{S_3}. \qquad (9.2.35)$$

Proof. We observe that a now acts like a seventh root of unity. Proceeding as we did at the beginning of the proof of Theorem 9.2.3, we find that

$$F_a(q) = \frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} \equiv \frac{(q;q)_\infty^2(a^2q;q)_\infty(a^{-2}q;q)_\infty(a^3q;q)_\infty(a^{-3}q;q)_\infty}{(q^7;q^7)_\infty}$$
$$\equiv \frac{1}{f(-q^7)}\frac{f(-a^2,-q/a^2)}{1-a^2}\frac{f(-a^3,-q/a^3)}{1-a^3}\,(\mathrm{mod}\, S_3). \quad (9.2.36)$$

Next, apply Lemma 1.1.2 twice with $n = 7$, first with a replaced by $-a^2$ and b replaced by $-q/a^2$, and second with a replaced by $-a^3$ and b replaced by $-q/a^3$. Leaving the details to readers, we arrive at

$$\frac{f(-a^2,-q/a^2)}{1-a^2} \equiv A - q\frac{a^5-a^4}{1-a^2}B + q^3\frac{a^3-a^6}{1-a^2}C\,(\mathrm{mod}\, S_3), \quad (9.2.37)$$
$$\frac{f(-a^3,-q/a^3)}{1-a^3} \equiv A - q\frac{a^4-a^6}{1-a^3}B + q^3\frac{a-a^2}{1-a^3}C\,(\mathrm{mod}\, S_3). \quad (9.2.38)$$

Substitute the congruences (9.2.37) and (9.2.38) into (9.2.36). Separate out the terms for each coefficient of q^n, $0 \le n \le 6$. We therefore readily deduce (9.2.35) to complete the proof of Theorem 9.2.5. □

Corollary 9.2.6. *For each nonnegative integer n,*

$$p(7n+5) \equiv 0\,(\mathrm{mod}\, 7). \quad (9.2.39)$$

Proof. If we set $a = 1$ in (9.2.35), we find that

$$F_1(q) \equiv \frac{1}{f(-q^7)}\left(A^2 + qBC + 2q^2B^2 + 3q^3AC - 2q^4BC - 3q^6C^2\right). \quad (9.2.40)$$

On the right side of the congruence (9.2.40), there are no powers of q congruent to 5 modulo 7. Hence, the same must be true on the left side, i.e., the congruence (9.2.39) follows. □

Proofs of the 5-dissection have been given by Garvan [**56**] and A. B. Ekin [**51**], and Berndt, H. H. Chan, S. H. Chan, and W.–C. Liaw [**30**]. The first explicit statement and proof of the 7-dissection of $F(q)$ was given by Garvan [**56**, Thm. 5.1]. As mentioned earlier, although Ramanujan did not state the 7-dissection of $F(q)$, he clearly knew it, because the six quotients of theta functions that appear in the 7-dissection are found on the bottom of page 71 (written upside down) in his lost notebook. The first appearance of the 11-dissection of $F(q)$ in the literature also can be found in Garvan's paper [**56**, Thm. 6.7]. Further proofs have been given by M. D. Hirschhorn [**68**] and Ekin [**50**], [**51**], who also gave a different proof of the 7-dissection. However, again, it is very likely that Ramanujan knew the 11-dissection, since he offers the quotients of theta functions which appear in the 11-dissection on page 70 of his lost notebook [**91**]. Garvan [**57**] has also found 8-, 9-, and 10-dissections of the crank generating function.

We have emphasized that there are two approaches to establishing dissections for $F_a(q)$ – through congruences or through roots of unity. Ramanujan stated the 2- and 3-dissections as congruences, but the 5-dissection in terms of roots of unity. We do not know how Ramanujan derived his 7- and 11-dissections, since he did not completely state his results. One might ask if one approach is more general than the other. In fact, in terms of generality, they are equal. This was first observed by Liaw (unofficially) and Garvan, who provided a proof to the authors of [**30**]; see [**30**, pp. 118–119]. The four authors of [**30**] gave two different proofs of each of the 2-, 3-, 5-, 7-, and 11-dissections in terms of congruences. Their first proof uses a method of "rationalization," which was used in the proofs of the 5- and 7-dissections given above. Their second method uses a result of Ramanujan given on page 59 in his lost notebook [**91**], which we now state.

Theorem 9.2.7. *If*

$$A_n := a^n + a^{-n}, \tag{9.2.41}$$

then

$$\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} = 1 - \sum_{m=1,n=0}^{\infty} (-1)^m q^{m(m+1)/2+mn}(A_{n+1} - A_n). \tag{9.2.42}$$

9.3. Another Crank

The original crank of Ramanujan was hidden behind an opaque curtain for several decades, its existence was conjectured by Dyson during this period, and its discovery on June 6, 1987 by Andrews and Garvan heralded a new era in the theory of partitions. In recent years, cranks have been discovered for a few additional partition functions. How can we recognize a crank? First, its generating function should have two variables. (It may be possible to construct crank generating functions in more than two variables, but we are unaware of such instances.) Second, its generating function should reduce to the generating function for the given partition function when the second variable is appropriately specialized. The extra variable should enable us to gain further information about the partition function, usually combinatorial in nature.

We briefly discuss one further partition function, studied by Z. Reti [**92**], H.–C. Chan [**41**]–[**43**], and B. Kim [**71**]. As the unrestricted partition function $p(n)$ is associated, as we have seen, with the Rogers–Ramanujan continued fraction, the partition function $a(n)$ defined below is associated with Ramanujan's cubic continued fraction [**26**, p. 345].

Definition 9.3.1. *Define the partition function* $a(n)$ *by*

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty(q^2;q^2)_\infty}. \tag{9.3.1}$$

Thus, $a(n)$ *counts the number of partitions of* n *into two colors, orange and blue, with the odd parts appearing only in orange.*

Theorem 9.3.2. *We have*

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4}. \tag{9.3.2}$$

Corollary 9.3.3. *For each nonnegative integer n,*

$$a(3n+2) \equiv 0 \pmod 3. \tag{9.3.3}$$

Clearly, Theorem 9.3.2 is an analogue of Ramanujan's "beautiful" identity in Theorem 8.2.5.

Recall from Entry 1 of Chapter 20 in Ramanujan's second notebook [**90**], [**26**, p. 345] the definition of Ramanujan's cubic continued fraction

$$v := v(q) := \frac{q^{1/3}}{1} \,{+}\, \frac{q+q^2}{1} \,{+}\, \frac{q^2+q^4}{1} \,{+}\, \cdots. \tag{9.3.4}$$

Many of the properties of v arise from the theory of modular equations of degree 3, which are discussed in Chapter 18 of Ramanujan's second notebook [**90**] and the author's two books [**26**, pp. 230–238, Entry 5], [**28**, pp. 144–150].

Lemma 9.3.4. [**26**, p. 345] *Recall that* $\psi(q)$ *and* $\varphi(q)$ *denote Ramanujan's theta functions are defined in* (8.6.3). *Then*

$$1+\frac{1}{v} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}, \tag{9.3.5}$$

$$1-2v = \frac{\varphi(-q^{1/3})}{\varphi(-q^3)}. \tag{9.3.6}$$

Lemma 9.3.5. *Let* $x := x(q) := q^{-1/3}v(q)$. *Then*

$$\frac{1}{x} - q^{1/3} - 2q^{2/3}x = \frac{(q^{1/3};q^{1/3})_\infty (q^{2/3};q^{2/3})_\infty}{(q^3;q^3)_\infty (q^6;q^6)_\infty}. \tag{9.3.7}$$

Proof. Multiply (9.3.5) by (9.3.6). On the left side, we arrive at

$$q^{1/3}\left(1+\frac{1}{v}\right)(1-2v) = q^{1/3} + \frac{q^{1/3}}{v} - 2q^{1/3}v - 2q^{1/3} = -q^{1/3} + \frac{1}{x} - 2q^{2/3}x. \tag{9.3.8}$$

On the right side, by the product representations for $\varphi(q)$ from (3.1.28) and $\psi(q)$ from (3.1.27), and Euler's identity, we find that

$$\begin{aligned}\frac{q^{1/3}\psi(q^{1/3})}{q^{1/3}\psi(q^3)}\frac{\varphi(-q^{1/3})}{\varphi(-q^3)} &= \frac{(q^{2/3};q^{2/3})_\infty (q^3;q^6)_\infty (q^{1/3};q^{1/3})_\infty (-q^3;q^3)_\infty}{(q^{1/3};q^{2/3})_\infty (q^6;q^6)_\infty (-q^{1/3};q^{1/3})_\infty (q^3;q^3)_\infty}\\ &= \frac{(q^{1/3};q^{1/3})_\infty (q^{2/3};q^{2/3})_\infty}{(q^3;q^3)_\infty (q^6;q^6)_\infty}. \end{aligned} \tag{9.3.9}$$

Combining (9.3.8) and (9.3.9), we complete the proof of Lemma 9.3.5. □

Lemma 9.3.6. *If x is defined in Lemma 9.3.5, then*

$$\frac{1}{x^3(q^3)} - 7q^3 - 8q^6x^3(q^3) = \left(\frac{(q^3;q^3)_\infty (q^6;q^6)_\infty}{(q^9;q^9)_\infty (q^{18};q^{18})_\infty}\right)^4. \tag{9.3.10}$$

Proof. Replace q by q^3 in Lemma 9.3.5 to deduce that

$$\frac{1}{x(q^3)} - q - 2q^2 x(q^3) = \frac{(q;q)_\infty (q^2;q^2)_\infty}{(q^9;q^9)_\infty (q^{18};q^{18})_\infty}. \tag{9.3.11}$$

Next, replace q by ωq in (9.3.11), where ω is any cube root of unity, and then multiply the three equalities together. On the right-hand side, we obtain the quotient

$$\begin{aligned} R(q) &:= \frac{\prod_\omega (\omega q;\omega q)_\infty (\omega^2 q^2;\omega^2 q^2)_\infty}{(q^9;q^9)_\infty^3 (q^{18};q^{18})_\infty^3} \\ &= \frac{(q^3;q^3)_\infty^4}{(q^9;q^9)_\infty} \frac{(q^6;q^6)_\infty^4}{(q^{18};q^{18})_\infty} \frac{1}{(q^9;q^9)_\infty^3 (q^{18};q^{18})_\infty^3} \\ &= \left(\frac{(q^3;q^3)_\infty (q^6;q^6)_\infty}{(q^9;q^9)_\infty (q^{18};q^{18})_\infty} \right)^4. \end{aligned} \tag{9.3.12}$$

On the other hand, from the left-hand side of (9.3.11), with $x = x(q^3)$, we deduce that

$$\begin{aligned} L(q) &:= \left(\frac{1}{x} - q - 2q^2 x\right)\left(\frac{1}{x} - \omega q - 2\omega^2 q^2 x\right)\left(\frac{1}{x} - \omega^2 q - 2\omega q^2 x\right) \\ &= \frac{1}{x^3(q^3)} - 7q^3 - 8q^6 x^3(q^3), \end{aligned} \tag{9.3.13}$$

by a straightforward multiplication with the repeated use of the fact $\omega + \omega^2 = -1$. Combining (9.3.12) and (9.3.13), we see that the equality $R(q) = L(q)$ implies (9.3.10), which was our aim. □

Proof of Theorem 9.3.2. Set $x = x(q^3)$. Using Lemmas 9.3.5 and 9.3.6 and long division, we find that

$$\begin{aligned} \frac{1}{(q;q)_\infty (q^2;q^2)_\infty} &= \frac{1}{(q^9;q^9)_\infty (q^{18};q^{18})_\infty} \left(\frac{(q;q)_\infty (q^2;q^2)_\infty}{(q^9;q^9)_\infty (q^{18};q^{18})_\infty} \right)^{-1} \\ &= \frac{1}{(q^9;q^9)_\infty (q^{18};q^{18})_\infty} \frac{1}{1/x - q - 2q^2 x} \\ &\quad \times \left(\frac{(q^3;q^3)_\infty (q^6;q^6)_\infty}{(q^9;q^9)_\infty (q^{18};q^{18})_\infty} \right)^{-4} \left(\frac{1}{x^3} - 7q^3 - 8q^6 x^3 \right) \\ &= \frac{(q^9;q^9)_\infty^3 (q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4 (q^6;q^6)_\infty^4} \frac{x^{-3} - 7q^3 - 8q^6 x^3}{x^{-1} - q - 2q^2 x} \\ &= \frac{(q^9;q^9)_\infty^3 (q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4 (q^6;q^6)_\infty^4} \left(4q^2x^2 - 2q^3 x + 3q^2 + qx^{-1} + x^{-2}\right). \end{aligned} \tag{9.3.14}$$

Extract the terms with powers q^{3n+2} on the extremal sides of (9.3.14) to deduce that

$$\sum_{n=0}^\infty a(3n+2) q^{3n+2} = 3q^2 \frac{(q^9;q^9)_\infty^3 (q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4 (q^6;q^6)_\infty^4}. \tag{9.3.15}$$

Cancelling q^2 on both sides of (9.3.15) and then replacing q^3 by q, we immediately deduce (9.3.2) to complete the proof. □

We now relate Byungchan Kim's crank for the partition function $a(n)$ [**71**]. In view of Ramanujan's crank for $p(n)$, it is natural to define the crank generating function for $a(n)$ by

$$F(x,q) := \frac{(q;q)_\infty (q^2;q^2)_\infty}{(xq;q)_\infty (q/x;q)_\infty (xq^2;q^2)_\infty (q^2/x;q^2)_\infty}. \tag{9.3.16}$$

We shall demonstrate that indeed (9.3.16) is a natural crank generating function after reviewing Garvan's vector crank.

Definition 9.3.7. *Let λ_1, λ_2, and λ_3, denote arbitrary partition functions, and let λ denote an arbitrary vector partition, defined below. Let $\mathbb{P}$ and $\mathbb{D}$ denote the sets of unrestricted partitions and partitions into distinct parts, respectively. Set*

$$\begin{aligned}
V &:= \{\lambda = (\lambda_1, \lambda_2, \lambda_3) : \lambda_1 \in \mathbb{D}, \lambda_2, \lambda_3 \in \mathbb{P}\},\\
\#(\lambda) &:= \textit{the number of parts of } \lambda,\\
\sigma(\lambda) &:= \textit{the sum of the parts of } \lambda (\textit{often denoted by } n),\\
s(\lambda) &:= \sigma(\lambda_1) + \sigma(\lambda_2) + \sigma(\lambda_3),\\
w(\lambda) &:= (-1)^{\#(\lambda_1)} \quad (\textit{weight of the partition } \lambda),\\
t(\lambda) &:= \#(\lambda_2) - \#(\lambda_3) \quad (\textit{vector crank of the partition } \lambda),\\
N_V(m,n) &:= \textit{the number of weighted vector partitions of } n \textit{ with crank } m, \textit{i.e.},\\
&\qquad N_V(m,n) = \sum_{\substack{\lambda \in V\\ s(\lambda)=n\\ t(\lambda)=m}} w(\lambda).
\end{aligned}$$

Furthermore, let λ_r denote a vector partition with all parts of color red, and λ_b denote a partition, arising from blue parts only, formed by dividing each of the blue parts by 2. Define the number of weighted vector partitions of n with crank m associated with the partition function $a(n)$ by

$$N_V^a(m,n) := \sum_{\substack{\lambda_r, \lambda_b \in V\\ s(\lambda_r)+2s(\lambda_b)=n\\ t(\lambda_r)+t(\lambda_b)=m}} \mathrm{wt}(\lambda_r)\mathrm{wt}(\lambda_b). \tag{9.3.17}$$

Our motivation for defining the crank function $N_V(m,n)$ arises precisely from Garvan's original defintion of a vector crank. From the definition (9.3.16), we indeed see that

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_V^a(m,n) x^m q^n = F(x,q)$$

and

$$\sum_{m=-\infty}^{\infty} N_V^a(m,n) = a(n), \quad n \geq 0.$$

Recall that the crank discovered by Andrews and Garvan does not agree with the vector crank for $n = 1$. Likewise, the crank defined by Byungchan Kim does not agree with the natural vector crank. To make a proper definition of the *true* crank, we need to define two further copies of the vector 1. Set

$$1 = ((1), \emptyset, \emptyset), \quad 1^* = (\emptyset, (1), \emptyset), \quad 1^{**} = (\emptyset, \emptyset, (1)). \tag{9.3.18}$$

We then define an extended set of partitions, $\mathbb{P}^*$ by

$$\mathbb{P}^* := \{(\emptyset), (1), (1^*), (1^{**}), (1,1), (2), \dots\}. \tag{9.3.19}$$

We emphasize that all partitions in $\mathbb{P}^*$ are ordinary partitions, except for the two enhanced partitions $(1^*), (1^{**})$.

We now define the weights, cranks, and sums $\sigma*$ for partitions $\lambda \in \mathbb{P}^*$. First, the weight of λ is defined by

$$\mathrm{wt}(\lambda) := \begin{cases} 1, & \text{if } \lambda \in \mathbb{P}, \\ w(\lambda), & \text{otherwise.} \end{cases} \tag{9.3.20}$$

We note that

$$\begin{aligned} \mathrm{wt}(1) &= \mathrm{wt}((1), \emptyset, \emptyset) = -1, \\ \mathrm{wt}(1^*) &= \mathrm{wt}(\emptyset, (1), \emptyset) = 1, \\ \mathrm{wt}(1) &= \mathrm{wt}((1), \emptyset, \emptyset, (1)) = 1. \end{aligned}$$

Next, define the crank by

$$c^*(\lambda) = \begin{cases} c(\lambda), & \text{if } \lambda \in \mathbb{P}, \\ t(\lambda), & \text{otherwise.} \end{cases} \tag{9.3.21}$$

We observe that

$$\begin{aligned} c^*(1) &= t((1), \emptyset, \emptyset) = 0, \\ c^*(1^*) &= t(\emptyset, (1), \emptyset) = 1, \\ c^*(1^{**}) &= t(\emptyset, \emptyset, (1)) = -1. \end{aligned}$$

Lastly, define "the sum of parts" function by

$$\sigma^*(\lambda) = \begin{cases} \sigma(\lambda), & \text{if } \lambda \in \mathbb{P}, \\ s(\lambda), & \text{otherwise.} \end{cases} \tag{9.3.22}$$

We note that

$$\sigma^*(1) = \sigma^*(1^*) = \sigma^*(1^{**}) = 1.$$

In summary, we see that

$$\begin{aligned} \frac{(q;q)_\infty}{(xq;q)_\infty (q/x;q)_\infty} &= 1 + (-1 + x + x^{-1})q + (x^2 + x^{-2})q^2 + (x^3 + 1 + x^{-3})q^3 + \cdots \\ &= \sum_{\lambda \in \mathbb{P}^*} \mathrm{wt}(\lambda) x^{c^*(\lambda)} q^{\sigma*(\lambda)}. \end{aligned}$$

Having carefully set out the crank function $c^*(\lambda)$ for vector partitions, we want to extend our definitions to cubic partitions, i.e., to elements $\lambda \in \mathbb{P}^* \times \mathbb{P}^*$. Define

$$\sigma_a(\lambda) = \sigma^*(\lambda_r) + 2\sigma^*(\lambda_b),$$
$$\mathrm{wt}_a(\lambda) = \mathrm{wt}(\lambda_r) \cdot \mathrm{wt}_b(\lambda_b).$$

Hence,

$$\begin{aligned}
\sum_{\lambda\in\mathbb{P}^*\times\mathbb{P}^*} \mathrm{wt}(\lambda) x^{c_a(\lambda)} q^{\sigma_a(\lambda)} &= \sum_{\lambda\in\mathbb{P}^*} \mathrm{wt}(\lambda_r) x^{c^*(\lambda_r)} q^{\sigma^*(\lambda_r)} \cdot \sum_{\lambda\in\mathbb{P}^*} \mathrm{wt}(\lambda_b) x^{c^*(\lambda_b)} q^{2\sigma^*(\lambda_b)} \\
&= \left(1 + (-1 + x + x^{-1})q + \sum_{\lambda_r\in\mathbb{P}} x^{c(\lambda_r)} q^{\sigma^*(\lambda_r)}\right) \\
&\quad \cdot \left(1 + (-1 + x + x^{-1})q^2 + \sum_{\lambda_b\in\mathbb{P}} x^{c(\lambda_b)} q^{2\sigma^*(\lambda_b)}\right) \\
&= \left(1 + (-1 + x + x^{-1})q + \sum_{n=2}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) x^m q^n\right) \\
&\quad \cdot \left(1 + (-1 + x + x^{-1})q^2 + \sum_{n=2}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) x^m q^{2n}\right) \\
&= \frac{(q;q)_\infty}{(xq;q)_\infty (q/x;q)_\infty} \cdot \frac{(q^2;q^2)_\infty}{(xq^2;q^2)_\infty (q^2/x;q^2)_\infty}.
\end{aligned}$$

Definition 9.3.8. *Let $M'(m,n)$ denote the number of extended cubic partitions of n with crank m counted according to weight wt_a.*

Then

$$M'(m,n) = \sum_{\substack{\lambda\in\mathbb{P}^*\times\mathbb{P}^* \\ c_a(\lambda)=m \\ \sigma_a(\lambda)=n}} \mathrm{wt}_a(\lambda)$$

and

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M'(m,n) x^m q^n = F(x,q).$$

In summary, we have demonstrated the following theorem.

Theorem 9.3.9. *For $n \geq 1$,*

$$M'(m,n) = N_V^a(m,n).$$

Definition 9.3.10. *Let $M'(m,N,n)$ denote the number of extended cubic partitions of n with crank congruent to m modulo N according to the weight wt_a.*

Thus,

$$M'(m,N,n) = \sum_{r\equiv m \pmod N} M'(r,n) = \sum_{\substack{\lambda\in\mathbb{P}^*\times\mathbb{P}^* \\ c_a(\lambda)\equiv m \pmod N \\ \sigma_a(\lambda)=n}} \mathrm{wt}_a(\lambda).$$

Theorem 9.3.11. *Let $M'(m,N,n)$ denote the number of extended cubic partitions of n with crank congruent to m modulo N according to the weight* wt_a. *Then*

$$M'(0,3,3n+2) \equiv M'(1,3,3n+2) \equiv M'(2,3,3n+2) \pmod 3. \tag{9.3.23}$$

Corollary 9.3.12. *For each nonnegative integer n,*

$$a(3n+2) \equiv 0 \pmod 3. \tag{9.3.24}$$

Proof. Since

$$a(3n+2) = \sum_{j=0}^{2} M'(j,3,3n+2),$$

the congruence (9.3.24) follows immediately from Theorem 9.3.11. □

Proof of Theorem 9.3.11. Let ω denote a primitive cube root of unity. Then,

$$\begin{aligned} F(\omega,q) &= \frac{(q;q)_\infty}{(\omega q;q)_\infty(\omega^{-1}q;q)_\infty} \cdot \frac{(q^2;q^2)_\infty}{(\omega q^2;q^2)_\infty(\omega^{-1}q^2;q^2)_\infty} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{2} M'(k,3,3n+2)\omega^k q^n. \end{aligned} \tag{9.3.25}$$

Furthermore, using (9.3.25), and then using below the product representation of $\varphi(-q)$ in (3.1.28) and Jacobi's identity, Corollary 3.1.9, we find that

$$\begin{aligned} &\frac{(q;q)_\infty(\omega q^2;q^2)_\infty}{(\omega q;q)_\infty(\omega^{-1}q;q)_\infty(\omega q^2;q^2)_\infty(\omega^{-1}q^2;q^2)_\infty} = \frac{(q;q)_\infty^2(q^2;q^2)_\infty^2}{(q^3;q^3)_\infty(q^6;q^6)_\infty} \\ &= \frac{(q;q^2)_\infty^2(q^2;q^2)_\infty^4}{(q^3;q^3)_\infty(q^6;q^6)_\infty} = \frac{\varphi(-q)(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty} \\ &= \frac{1}{(q^3;q^3)_\infty(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{m(m+1)}. \end{aligned} \tag{9.3.26}$$

We now extract the terms of the type q^{3k+2} from the left and right sides of (9.3.26). We note that $n^2 \equiv 0,1 \pmod 3$ and that $m(m+1) \equiv 0,2 \pmod 3$. Thus, to obtain the desired powers, we need to take $n \equiv 0 \pmod 3$ and $m \equiv 1 \pmod 3$. Thus, replacing n by $3n$ and m by $3m+1$, we find that

$$\sum_{n=0}^{\infty} a(3n+2)q^{3n+2} = \frac{1}{(q^3;q^3)_\infty(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2}\sum_{m=0}^{\infty}(-1)^m(6m+3)q^{9m^2+9m+2}. \tag{9.3.27}$$

Observe that the coefficients on the right-hand side of (9.3.27) are multiples of 3. Thus, from (9.3.25),

$$\sum_{k=0}^{2} M'(k,3,3n+2)\omega^k = 3N,$$

for some integer N. Since $1+\omega+\omega^2$ is a minimal polynomial in $\mathbb{Z}[\omega]$, we must conclude that

$$M'(0,3,3n+2) \equiv M'(1,3,3n+2) \equiv M'(2,3,3n+2) \pmod 3,$$

(λ_r, λ_b)	$\mathrm{wt}_a(\lambda)$	$c_a(\lambda) \pmod 3$
$((1,1), \emptyset)$	$+1$	$-2 \equiv 1$
$((2), \emptyset)$	$+1$	$2 \equiv 2$
$(\emptyset, (1))$	-1	$0 \equiv 0$
$(\emptyset, (1^*))$	$+1$	$1 \equiv 1$
$(\emptyset, (1^{**}))$	$+1$	$-1 \equiv 2$

i.e., (9.3.23) holds. □

We offer an example to illustrate Theorem 9.3.11.

Example 9.3.13. *Let $n = 2$. There are 5 extended cubic partitions of 2. We represent these as elements (λ_r, λ_b) of $\mathcal{P}^* \times \mathcal{P}^*$ in the following table. We see that*

$$M'(0,3,2) = -1, \quad M'(1,3,2) = M'(2,3,2) = 2$$

and

$$M'(0,3,2) \equiv M'(1,3,2) \equiv M'(2,3,2) \pmod 3.$$

Hence, we furthermore find that

$$M'(0,3,2) = -1, \quad M'(1,3,2) = M'(2,3,2) = 2, \tag{9.3.28}$$

and so

$$M'(0,3,2) \equiv M'(1,3,2) \equiv M'(2,3,2) \pmod 3,$$

which is in agreement with Theorem 9.3.11.

By (9.3.25) and (9.3.26), which demonstrates that $F(\omega, q)$ is independent of ω, we see that, for $n \geq 1$,

$$M'(1,3,n) = M'(2,3,n). \tag{9.3.29}$$

Thus, the second equality in (9.3.28) is not an anomaly. Moreover, by (9.3.29),

$$M'(1,3,n)\omega + M'(2,3,n)\omega^2 = -M'(1,3,n).$$

Hence, from (9.3.26),

$$\sum_{n=0}^{\infty} \left(M'(0,3,n) - M'(1,3,n)\right) q^n = \frac{(q;q)_\infty^2 (q^2;q^2)_\infty^2}{(q^3;q^3)_\infty (q^6;q^6)_\infty}.$$

9.4. Concluding Remarks

In his lost notebook, Ramanujan devoted considerable space to the study of $F_a(q)$. In particular, he calculated by hand over 300 values of λ_n. There is considerable evidence that Ramanujan's last creative endeavors were devoted to cranks before he died on April 26, 1920 at the age of 32 [**31**].

On pages 179 and 180 in his lost notebook [**91**], Ramanujan offers ten tables of indices of coefficients λ_n satisfying certain congruences. On page 61 in [**91**], he offers rougher drafts of nine of the ten tables; Table 6 is missing on page 61. Unlike the tables on pages 179 and 180, no explanations are given on page 61. To verify Ramanujan's claims, the aforementioned four authors calculated λ_n up to $n = 500$ with the use of Maple V. Ramanujan evidently thought that each table is complete in that there are no further values of n for which the prescribed divisibility property holds [**31**]. Indeed, Ramanujan's assumptions were correct, as O.–Yeat Chan [**44**] used the circle method to verify that all of Ramanujan's claims were correct.

Several pages of the lost notebook are devoted to the crank. In particular, Ramanujan extensively factored λ_n. There are several pages of scratch work, with apparently most of them devoted to calculations involving the crank. Discovering the meanings of these calculations seems to be an almost impossible task. Ramanujan did not record in his lost notebook a combinatorial interpretation of $F_a(q)$, but he clearly was aware of its importance, and he obviously had reasons for studying it that we have been unable to discern. Perhaps he was looking for more congruences or more general congruences. As mentioned above, there is strong evidence that Ramanujan devoted the last days of his life to cranks, and we refer readers to a paper by the lecturer, Chan, Chan, and Liaw for more details [**32**].

9.5. Exercises

1. Complete the proof of Theorem 9.2.5 by providing the missing details.

Chapter 10

The Parity of $p(n)$

10.1. The Parity of $p(n)$: History

Ask anyone who has ever heard of the partition function the following question: How often is $p(n)$ even? Alternatively, how often is $p(n)$ odd? Almost everyone would say that $p(n)$ should be even about half of the time and odd about half of the time. Indeed, it has long been conjectured that $p(n)$ is even approximately half of the time, or, more precisely,

$$\#\{n \leq N : p(n) \, \text{is even}\} \sim \tfrac{1}{2}N, \tag{10.1.1}$$

as $N \to \infty$. Despite the venerability of the problem, it was not even known that $p(n)$ assumes either even or odd values infinitely often until 1959, when O. Kolberg [**74**] established these facts. Other proofs of Kolberg's theorem were later found by J. Fabrykowski and M. V. Subbarao [**52**] and by M. Newman [**78**]. In 1967, T. R. Parkin and D. Shanks [**82**] undertook the first extensive computations, providing strong evidence that indeed (10.1.1) is most likely true. In 1983, L. Mirsky [**77**] established the first quantitative result by showing that

$$\#\,\{n \leq N : p(n) \;\text{ is even (odd)}\} > \frac{\log\log N}{2\log 2}. \tag{10.1.2}$$

An improvement was made by J.–L. Nicolas and A. Sárközy [**79**], who proved that

$$\#\,\{n \leq N : p(n) \;\text{ is even (odd)}\} > (\log N)^c, \tag{10.1.3}$$

for some positive constant c.

In the most recent investigations, the methods for finding lower bounds for the number of occurrences of even values of $p(n)$ have been somewhat different from those for odd values of $p(n)$. Greatly improving on previous results, Nicolas, I. Z. Ruzsa, and Sárközy [**80**] in 1998 proved that

$$\#\,\{n \leq N : p(n) \;\text{ is even }\} \gg \sqrt{N} \tag{10.1.4}$$

and, for each $\epsilon > 0$,

$$\#\{n \leq N : p(n) \text{ is odd }\} \gg \sqrt{N} e^{-(\log 2+\epsilon)\frac{\log N}{\log\log N}}. \tag{10.1.5}$$

In an appendix to the paper by Nicolas, Ruzsa, and Sárközy [**80**], J.–P. Serre used modular forms to prove that

$$\lim_{N\to\infty} \frac{\#\{n \leq N : p(n) \text{ is even }\}}{\sqrt{N}} = \infty. \tag{10.1.6}$$

At present, this is the best known result for even values of $p(n)$. The lower bound (10.1.5) has been improved first by S. Ahlgren [**2**], who utilized modular forms, and second by Nicolas [**81**], who used more elementary methods, to prove that

$$\#\{n \leq N : p(n) \text{ is odd }\} \gg \frac{\sqrt{N}(\log\log N)^K}{\log N}, \tag{10.1.7}$$

for some positive number K. Ahlgren proved (10.1.7) with $K = 0$. An elegant, elementary proof of (10.1.7) when $K = 0$ was established by D. Eichhorn [**49**]. The lower bound (10.1.7) is currently the best known result for odd values of $p(n)$.

10.2. Proofs of (10.1.5) and (10.1.4)

Our goal in this section is to prove (10.1.4) and (10.1.5) of Nicolas, I. Z. Ruzsa, and Sárközy [**80**] by relatively simple means. Our proof is a special instance of an argument devised by Berndt, Yee, and A. Zaharescu [**38**], who proved considerably more general theorems that are applicable to a wide variety of partition functions. Except for one step, when we must appeal to a theorem of S. Wigert [**101**] and Ramanujan [**84**], our proof is elementary and self-contained.

Theorem 10.2.1. *For each fixed c with $c > 2\log 2$ and N sufficiently large,*

$$\#\{n \leq N : p(n) \textit{ is odd}\} \geq N^{\frac{1}{2}-\frac{c}{\log\log N}}. \tag{10.2.1}$$

Theorem 10.2.2. *For each fixed constant c with $c < 1/\sqrt{6}$, and for N sufficiently large,*

$$\#\{n \leq N : p(n) \textit{ is even}\} \geq c\sqrt{N}. \tag{10.2.2}$$

Before we begin our proofs of Theorems 10.2.1 and 10.2.2, we need to establish some terminology. Although we shall use language from modern algebra, readers need not know any theorems from the subject. In fact, some of the information conveyed in the next two paragraphs will not be used in the sequel, but we think these facts are interesting in themselves.

Let $A := \mathbf{F}_2[[X]]$ be the ring of formal power series in one variable X over the field with two elements $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$, i.e.,

$$A = \left\{ f(X) = \sum_{n=0}^{\infty} a_n X^n : a_n \in \mathbf{F}_2, \quad 0 \leq n < \infty \right\}. \tag{10.2.3}$$

The ring A is an integral domain; note that an element $f(X) = \sum_{n=0}^{\infty} a_n X^n \in A$ is invertible if and only if $a_0 = 1$. Since 0 and 1 are the only elements of $\mathbf{F}_2$, we may write any element $f(X) \in A$ in the form

$$f(X) = X^{n_1} + X^{n_2} + \cdots , \tag{10.2.4}$$

where the sum may be finite or infinite and $0 \le n_1 < n_2 < \cdots$. For any $f(X) \in A$, observe that

$$f^2(X) = f(X^2). \tag{10.2.5}$$

In other words, if $f(X)$ is given by (10.2.4), then

$$f^2(X) = X^{2n_1} + X^{2n_2} + \cdots . \tag{10.2.6}$$

On A there exists a natural derivation which sends $f(X) \in A$ to $f'(X) = \frac{df}{dX} \in A$, i.e., if

$$f(X) = \sum_{n=0}^{\infty} a_n X^n, \quad \text{then} \quad f'(X) = \sum_{n=1}^{\infty} n a_n X^{n-1}. \tag{10.2.7}$$

Note that for any $f(X) \in A$,

$$f''(X) = 0. \tag{10.2.8}$$

We also remark that for any $f(X)$ given in the form (10.2.4), the condition

$$f'(X) = 0 \tag{10.2.9}$$

is equivalent to the condition that all the exponents n_j are even numbers.

In our proof of Theorem 10.2.2, we need to know the shape (10.2.4) of the series $f(X)/(1-X)$. For any integers $0 \le a < b$, we see that in A

$$\frac{X^a + X^b}{1-X} = \frac{X^a(1-X^{b-a})}{1-X} = X^a + X^{a+1} + \cdots + X^{b-1}. \tag{10.2.10}$$

We put together pairs of consecutive terms $X^{n_{2k+1}} + X^{n_{2k+2}}$ to obtain the equality

$$\begin{aligned} \frac{f(X)}{1-X} &= \frac{X^{n_1} + X^{n_2}}{1-X} + \frac{X^{n_3} + X^{n_4}}{1-X} + \cdots + \frac{X^{n_{2k+1}} + X^{n_{2k+2}}}{1-X} + \cdots \\ &= \left(X^{n_1} + X^{n_1+1} + \cdots + X^{n_2-1}\right) + \left(X^{n_3} + \cdots + X^{n_4-1}\right) + \cdots \\ &\quad + \left(X^{n_{2k+1}} + \cdots + X^{n_{2k+2}-1}\right) + \cdots . \end{aligned} \tag{10.2.11}$$

If the sum on the right side of (10.2.4) defining $f(X)$ is finite, say $f(X) = X^{n_1} + X^{n_2} + \cdots + X^{n_s}$, then

$$\begin{aligned} \frac{f(X)}{1-X} &= \left(X^{n_1} + X^{n_1+1} + \cdots + X^{n_2-1}\right) + \cdots \\ &\quad + \left(X^{n_{s-1}} + X^{n_{s-1}+1} + \cdots + X^{n_s-1}\right), \end{aligned} \tag{10.2.12}$$

if s is even, and

$$\frac{f(X)}{1-X} = \left(X^{n_1} + \cdots + X^{n_2-1}\right) + \cdots \\ + \left(X^{n_{s-2}} + \cdots + X^{n_{s-1}-1}\right) + \sum_{n=n_s}^{\infty} X^n, \qquad (10.2.13)$$

if s is odd.

Before commencing our proofs of Theorems 10.2.1 and 10.2.2, we introduce some standard notation in analytic number theory. We say that $f(N) = O(g(N))$, as N tends to ∞, if there exists a positive constant $A > 0$ and a number $N_0 > 0$ such that $|f(N)| \leq A|g(N)|$ for all $N \geq N_0$. To emphasize that this positive constant A above may depend upon another parameter c, we write $f(N) = O_c(g(N))$, as N tends to ∞.

Proof of Theorem 10.2.1. We begin with the pentagonal number theorem (1.2.23)

$$\sum_{n=0}^{\infty}(-1)^n q^{n(3n-1)/2} + \sum_{n=1}^{\infty}(-1)^n q^{n(3n+1)/2} = (q;q)_\infty. \qquad (10.2.14)$$

By reducing the coefficients modulo 2 and replacing q by X in (10.2.14), we find that, if $1/F(X)$ is the image of the infinite series of (10.2.14) in A, then

$$1 = F(X)\left(1 + \sum_{n=1}^{\infty}\left(X^{n(3n-1)/2} + X^{n(3n+1)/2}\right)\right). \qquad (10.2.15)$$

We write $F(X)$ in the form

$$F(X) = 1 + X^{n_1} + X^{n_2} + \cdots + X^{n_j} + \cdots, \qquad (10.2.16)$$

where, of course, $n_1, n_2, \ldots$ are positive integers. Clearly, from the generating function of the partition function $p(n)$ and (10.2.16),

$$\#\{1 \leq n \leq N : p(n) \text{ is odd}\} = \#\{n_j \leq N\} \qquad (10.2.17)$$

and

$$\#\{1 \leq n \leq N : p(n) \text{ is even}\} = N - \#\{n_j \leq N\}. \qquad (10.2.18)$$

We first establish a lower bound for $\#\{n_j \leq N\}$. Using (10.2.16), write (10.2.15) in the form

$$\left(\sum_{j=1}^{\infty} X^{n_j}\right)\left(1 + \sum_{n=1}^{\infty}\left(X^{n(3n-1)/2} + X^{n(3n+1)/2}\right)\right) \\ = \sum_{m=1}^{\infty}\left(X^{m(3m-1)/2} + X^{m(3m+1)/2}\right). \qquad (10.2.19)$$

Asymptotically, there are $\sqrt{2N/3}$ terms of the form $X^{m(3m-1)/2}$ less than X^N on the right side of (10.2.19). For a fixed positive integer n_j, we determine how many of these terms appear in a series of the form

$$X^{n_j}\left(1+\sum_{n=1}^{\infty}\left(X^{n(3n-1)/2}+X^{n(3n+1)/2}\right)\right), \tag{10.2.20}$$

arising from the left side of (10.2.19). Thus, for fixed $n_j < N$, we estimate the number of integral pairs (m,n) of solutions of the equation

$$n_j + \tfrac{1}{2}n(3n-1) = \tfrac{1}{2}m(3m-1), \tag{10.2.21}$$

which we put in the form

$$2n_j = (m-n)(3m+3n-1). \tag{10.2.22}$$

By a result of S. Wigert [**101**] and Ramanujan [**84**], [**89**, p. 80], the number of divisors of $2n_j$ is no more than $O_c\left(N^{\frac{c}{\log\log N}}\right)$ for any fixed $c > \log 2$. Thus, each of the numbers $m-n$ and $3m+3n-1$ can assume at most $O_c\left(N^{\frac{c}{\log\log N}}\right)$ values. Since the pair $(m-n, 3m+3n-1)$ uniquely determines the pair (m,n), it follows that the number of solutions to (10.2.22) is $O_c\left(N^{\frac{c}{\log\log N}}\right)$, where c is any constant such that $c > 2\log 2$. A similar argument can be made for the terms in (10.2.19) of the form $X^{m(3m+1)/2}$.

Returning to (10.2.19) and (10.2.20), we see that each series of the form (10.2.20) has at most $O_c\left(N^{\frac{c}{\log\log N}}\right)$ terms $X^{m(3m-1)/2}$ up to X^N that appear on the right side of (10.2.19). It follows that there are at least $O_c\left(N^{\frac{1}{2}-\frac{c}{\log\log N}}\right)$ numbers $n_j \le N$ that are needed to match all the (asymptotically $\sqrt{2N/3}$) terms $X^{m(3m-1)/2}$ up to X^N on the right side of (10.2.19). Again, an analogous argument holds for terms of the form $X^{m(3m+1)/2}$. We have therefore completed the proof of Theorem 10.2.1. $\square$

Proof of Theorem 10.2.2. Next, a lower bound for $\#\{n \le N : p(n) \text{ is even}\}$ is provided. Let $\{m_1, m_2, \dots\}$ be the complement of the set $\{0, n_1, n_2, \dots\}$ in the set of natural numbers $\{0,1,2,\dots\}$, and define

$$G(X) := X^{m_1} + X^{m_2} + \cdots \in A. \tag{10.2.23}$$

Then

$$G(X) + F(X) = 1 + X + X^2 + \cdots + X^k + \cdots = \frac{1}{1-X}. \tag{10.2.24}$$

Since, by (10.2.18),

$$\#\{m_j \le N\} = N - \#\{n_j \le N\} = \{n \le N : p(n) \text{ is even}\}, \tag{10.2.25}$$

we need a lower bound for $\#\{m_j \le N\}$. Using (10.2.24) in (10.2.15), we find that

$$\begin{aligned}1 + G(X)&\left(1+\sum_{n=1}^{\infty}\left(X^{n(3n-1)/2}+X^{n(3n+1)/2}\right)\right)\\ &= \frac{1}{1-X}\left(1+\sum_{n=1}^{\infty}\left(X^{n(3n-1)/2}+X^{n(3n+1)/2}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-X}\left(1+X+X^2+X^5+X^7+\cdots\right)\\
&= \frac{1}{1-X}\Bigg((1+X)+(X^2+X^5)+\cdots\\
&\quad + (X^{(n-1)(3(n-1)+1)/2}+X^{n(3n-1)/2})\\
&\quad + (X^{n(3n+1)/2}+X^{(n+1)(3(n+1)-1)/2})+\cdots\Bigg).
\end{aligned}
\tag{10.2.26}
$$

By (10.2.11), we see that the right side of (10.2.26) equals

$$
\begin{aligned}
1+(X^2+X^3+X^4)+\cdots+(X^{(n-1)(3(n-1)+1)/2}+\cdots+X^{n(3n-1)/2-1})&\\
+(X^{n(3n+1)/2}+\cdots+X^{(n+1)(3(n+1)-1)/2-1})+\cdots .&
\end{aligned}
\tag{10.2.27}
$$

Observe that the gap between $X^{n(3n-1)/2-1}$ and $X^{n(3n+1)/2}$ contains n terms that are missing from the series (10.2.27). This gap comes after a segment of

$$\tfrac{1}{2}n(3n-1)-1-\tfrac{1}{2}(n-1)(3(n-1)+1)+1 = 2n-1$$

terms that do appear in (10.2.27). So we see that (10.2.27) contains asymptotically

$$\frac{2n-1}{n+(2n-1)}N = \frac{2n-1}{3n-1}N \sim \frac{2}{3}N$$

terms up to X^N. Now the sum in parentheses on the left side of (10.2.26) has asymptotically $2\sqrt{2N/3}$ nonzero terms up to X^N. Thus $G(X)$ must have at least $\sqrt{N/6}$ nonzero terms up to X^N in order for the left side of (10.2.26) to have at least $2N/3$ terms up to X^N to match those on the right side of (10.2.26). We have therefore completed the proof of Theorem 10.2.2. ☐

The general theorems proved by the author, Yee, and Zaharescu [**38**], [**39**] apply to a wide variety of partition functions as special cases. We close this chapter of our notes with three examples.

Theorem 10.2.3. *Let S be a set of positive integers that contains all of the odd integers, and let $p_S(n)$ denote the number of partitions of n in elements of S. Then, for N sufficiently large,*

$$\#\{n \leq N : p_S(n) \text{is even}\} \gg \sqrt{N}.$$

Let $p_C(n)$ denote the number of partitions of n with $C(m)$ colors, or, more precisely, we define $p_C(n)$ by

$$\sum_{n=0}^{\infty} p_C(n)q^n := \prod_{m=1}^{\infty}(1-q^m)^{-C(m)}.$$

Thus, each integer m is allowed to have $C(m)$ colors. We have already studied the special case when $C(m)=r$, where r is a constant positive integer. More precisely, in (8.5.1) we defined $p_r(n)$ by

$$\frac{1}{(q;q)_\infty^r} = \sum_{n=0}^{\infty} p_r(n)q^n, \qquad |q|<1.$$

Theorem 10.2.4. *Let $C(m)$ be odd for all but finitely many odd numbers m. Then, for each fixed $c > 2\log 2$, and for N sufficiently large,*

$$\#\{n \leq N : p_C(n) \text{ is odd}\} \gg N^{\frac{1}{2} - \frac{c}{\log\log N}}.$$

Theorem 10.2.5. *Let $C(m)$ be odd for all but an even number of odd numbers m. Then, for N sufficiently large,*

$$\#\{n \leq N : p_C(n) \text{ is even}\} \gg \sqrt{N}.$$

Chapter 11

Plane Partitions

11.1. Introduction

An ordinary partition is a non-increasing one-dimensional array of positive integers.

Definition 11.1.1. *A plane partition is a two-dimensional array of positive integers, non-increasing in both directions. Let $p_2(n)$ denote the number of plane partitions of n.*

Thus, $p_2(0) = 1$, $p_2(1) = 1$, $p_2(2) = 3$, $p_2(3) = 6$, $p_2(4) = 13$, etc. In the next example, we list the 13 plane partitions for 4.

Example 11.1.2. *Let $n = 4$. Then the plane partitions of 4 are:*

$$4,\quad 3\,1,\quad \begin{matrix}3\\1\end{matrix},\quad 2\,2,\quad \begin{matrix}2\\2\end{matrix},\quad 2\,1\,1\quad \begin{matrix}2\,1\\1\end{matrix},\quad \begin{matrix}2\\1\\1\end{matrix},\quad 1\,1\,1\,1,\quad \begin{matrix}1\,1\,1\\1\end{matrix},\quad \begin{matrix}1\,1\\1\,1\end{matrix},\quad \begin{matrix}1\,1\\1\\1\end{matrix},\quad \begin{matrix}1\\1\\1\\1\end{matrix}.$$

Thus, $p_2(4) = 13$.

The primary theorem in this chapter provides a generating function for $p_2(n)$.

Theorem 11.1.3. *If $p_2(n)$ is defined in Definition 11.1.1, then*

$$\sum_{n=0}^{\infty} p_2(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}. \tag{11.1.1}$$

Definition 11.1.4. *An r-dimensional partition is an r-dimensional array of positive integers, non-decreasing in each of the r directions. Let $p_r(n)$ denote the number of r-dimensional partitions.*

In his book [**76**], P. A. MacMahon conjectured that

$$\sum_{n=0}^{\infty} p_r(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\binom{n+r-2}{r-1}}}. \tag{11.1.2}$$

By (1.2.1) and (11.1.1), MacMahon's conjecture is valid for $r=1$ and $r=2$, respectively. However, for $r>2$, (11.1.2) is false, as we now demonstrate.

Define the coefficients $\mu_r(n)$, $n\geq 0$, $k\geq 1$, by

$$\begin{aligned}\sum_{n=0}^{\infty}\mu_r(n)q^n &= \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{\binom{n+r-2}{r-1}}} \qquad (11.1.3)\\
&= \frac{1}{1-q}\frac{1}{(1-q^2)^r}\frac{1}{(1-q^3)^{\binom{r+1}{2}}}\frac{1}{(1-q^4)^{\binom{r+2}{3}}}\frac{1}{(1-q^5)^{\binom{r+3}{4}}}\frac{1}{(1-q^6)^{\binom{r+4}{5}}}\cdots\\
&= (1+q+q^2+q^3+q^4+q^5+q^6+\cdots)\left(1+rq^2+\binom{r+1}{2}q^4+\binom{r+2}{3}q^6+\cdots\right)\\
&\quad\times\left(1+\binom{r+1}{2}q^3+\frac{1}{2}\binom{r+1}{2}\left(\binom{r+1}{2}+1\right)q^6+\cdots\right)\\
&\quad\times\left(1+\binom{r+2}{3}q^4+\cdots\right)\left(1+\binom{r+3}{4}q^5+\cdots\right)\left(1+\binom{r+4}{5}q^6+\cdots\right)\times\cdots.\end{aligned}$$

Comparing these coefficients with the r-dimensional partition function $p_r(n)$, we find that

$$\begin{aligned}\mu_r(0)&=p_r(0)=1,\\
\mu_r(1)&=p_r(1)=1,\\
\mu_r(2)&=p_r(2)=r+1,\\
\mu_r(3)&=p_r(3)=1+2r+\binom{r}{2},\\
\mu_r(4)&=p_r(4)=1+4r+4\binom{r}{2}+\binom{r}{3},\\
\mu_r(5)&=p_r(5)=1+6r+11\binom{r}{2}+7\binom{r}{3}+\binom{r}{4},\\
\mu_r(6)&=1+10r+27\binom{r}{2}+29\binom{r}{3}+12\binom{r}{4}+\binom{r}{5}\\
&=p_r(6)+\binom{r}{3}+\binom{r}{4}.\end{aligned}$$

We leave the calculation of $\mu_r(n)$ from (11.1.3) to the reader. The first two values of $\mu_r(n)$ and $p_r(n)$ are obvious. For $n=2$, we observe that there are r ways to choose the vector $1,1$ but only one way to record 2. For $n=3$, we see that there is one way to record the partition 3, r ways to write each of the partitions 2,1 and 1,1,1, and finally $\binom{r}{2}$ ways to choose the two vectors 1,1 and 1,1 arising from $\begin{smallmatrix}1&1\\1&\end{smallmatrix}$.

For $n=4$, there is one way to write 4. There are r ways to display the partitions 3,1; 2,2; 2,1,1; and 1,1,1,1. For the arrangements of $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$ and $\begin{smallmatrix}2&1\\1&\end{smallmatrix}$, there are $\binom{r}{2}$ possibilities for each. Next, for $\begin{smallmatrix}1&1&1\\1&&\end{smallmatrix}$, we have $2\binom{r}{2}$ possible representations. Finally, there are $\binom{r}{3}$ ways to represent the four 1's in 3-space.

We leave the calculation of $p_r(5)$ and $p_r(6)$ as an exercise.

In summary, we see that $\mu_r(n) = p_r(n)$ for $0 \le n \le 5$, but that $\mu_r(6) \ne p_r(6)$, for $r \ge 3$. Thus, MacMahon's conjecture is quite reasonable, assuming that he had calculated both arithmetical functions up to $n = 5$.

11.2. Proof of Theorem 11.1.3

Definition 11.2.1. *Let $\pi_r(n_1, n_2, \ldots, n_k; q)$ denote the generating function for the number of plane partitions with at most r columns and exactly k rows, where n_j, $1 \le j \le k$, is the first entry in the jth row.*

Observe that

$$\pi_1(n_1, n_2, \ldots, n_k; q) = q^{n_1+n_2+\cdots+n_k}. \tag{11.2.1}$$

Lemma 11.2.2. *For each pair of nonnegative integers m, n,*

$$\sum_{j=0}^{n} \begin{bmatrix} m+j \\ j \end{bmatrix} q^j = \begin{bmatrix} m+n+1 \\ m+1 \end{bmatrix}. \tag{11.2.2}$$

Proof. We induct on n. It is clear that (11.2.2) is valid for $n = 0$. Using the induction hypothesis and the second q-analogue of Pascal's formula (2.3.2), we find that

$$\begin{aligned}
\sum_{j=0}^{n+1} \begin{bmatrix} m+j \\ j \end{bmatrix} q^j &= \begin{bmatrix} m+n+1 \\ m+1 \end{bmatrix} + q^{n+1} \begin{bmatrix} m+n+1 \\ n+1 \end{bmatrix} \\
&= \begin{bmatrix} m+n+1 \\ n \end{bmatrix} + q^{n+1} \begin{bmatrix} m+n+1 \\ n+1 \end{bmatrix} \\
&= \begin{bmatrix} m+n+2 \\ n+1 \end{bmatrix},
\end{aligned}$$

which completes the proof of (11.2.2). □

Theorem 11.2.3. *For each nonnegative integer r,*

$$\pi_{r+1}(n_1, n_2, \ldots, n_k; q) = q^{n_1+n_2+\cdots+n_k} \sum_{\substack{m_1,\ldots,m_k \\ m_2 \le m_1 \le n_1 \\ m_3 \le m_2 \le n_2 \\ \vdots \\ 0 \le m_k \le n_k}} \pi_r(n_1, n_2, \ldots, n_k; q). \tag{11.2.3}$$

Proof. With the help of (11.2.1), we see that (11.2.3) follows from a little cogitation. □

We use Theorem 11.2.3 to explicitly calculate $\pi_2(n_1, n_2; q)$.

Theorem 11.2.4. *We have*

$$\pi_2(n_1, n_2; q) = q^{n_1+n_2} \begin{vmatrix} \begin{bmatrix} n_1+1 \\ 1 \end{bmatrix} & q \begin{bmatrix} n_2+1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} n_1+1 \\ 0 \end{bmatrix} & \begin{bmatrix} n_2+1 \\ 1 \end{bmatrix} \end{vmatrix}. \tag{11.2.4}$$

Proof. We employ (11.2.3) and (11.2.1). Furthermore, in the algebraic manipulation that follows, we add and subtract q^{n_2+1} and then add and subtract q. Hence,

$$\begin{aligned}
\pi_2(n_1,n_2;q) &= q^{n_1+n_2}\sum_{m_2=0}^{n_2}\sum_{m_1=m_2}^{n_1} q^{m_1+m_2}\\
&= q^{n_1+n_2}\sum_{m_2=0}^{n_2} q^{m_2}\frac{q^{m_2}-q^{n_1+1}}{1-q}\\
&= \frac{q^{n_1+n_2}}{1-q}\left(\frac{1-q^{2(n_2+1)}}{1-q^2}-\frac{q^{n_1+1}(1-q^{n_2+1})}{1-q}\right)\\
&= \frac{q^{n_1+n_2}}{(1-q)(1-q^2)}\left(1-q^{2(n_2+1)}-q^{n_1+1}+q^{n_1+n_2+2}-q^{n_1+2}+q^{n_1+n_2+3}\right)\\
&= \frac{q^{n_1+n_2}}{(1-q)(1-q^2)}(1-q^{n_2+1})(1+q^{n_2+1}-(1+q)q^{n_1+1})\\
&= \frac{q^{n_1+n_2}}{(1-q)(1-q^2)}(1-q^{n_2+1})\left((1-q^{n_1+1})(1+q)-q(1-q^{n_2})\right)\\
&= q^{n_1+n_2}\left(\begin{bmatrix} n_2+1\\ 1\end{bmatrix}\begin{bmatrix} n_1+1\\ 1\end{bmatrix}-q\begin{bmatrix} n_2+1\\ 2\end{bmatrix}\right)\\
&= q^{n_1+n_2}\begin{vmatrix} \begin{bmatrix} n_1+1\\ 1\end{bmatrix} & q\begin{bmatrix} n_2+1\\ 2\end{bmatrix}\\ \begin{bmatrix} n_1+1\\ 0\end{bmatrix} & \begin{bmatrix} n_2+1\\ 1\end{bmatrix}\end{vmatrix}.
\end{aligned}$$

Thus, we have completed the proof of (11.2.4). □

A close examination of Theorem 11.2.4 leads us to the following conjecture, which, not surprisingly, is a theorem.

Theorem 11.2.5. *For each nonnegative integer* r*,*

$$\pi_r(n_1,n_2;q) = q^{n_1+n_2}\begin{vmatrix} \begin{bmatrix} n_1+r-1\\ r-1\end{bmatrix} & q\begin{bmatrix} n_2+r-1\\ r\end{bmatrix}\\ \begin{bmatrix} n_1+r-1\\ r-2\end{bmatrix} & \begin{bmatrix} n_2+r-1\\ r-1\end{bmatrix}\end{vmatrix}. \tag{11.2.5}$$

Observe that, by (11.2.1) and Theorem 11.2.4, Theorem 11.2.5 is valid for $r=1$ and $r=2$, respectively.

We shall apply Lemma 11.2.2 several times in what follows. For some of these applications, it may be helpful to note the trivial identity

$$q\sum_{m_1=m_2}^{n_1}\begin{bmatrix} m_1+r-1\\ r-2\end{bmatrix} q^{m_1} = \sum_{j=m_2+1}^{n_1+1}\begin{bmatrix} j+r-2\\ r-2\end{bmatrix} q^{j}. \tag{11.2.6}$$

Proof of Theorem 11.2.5 In the sequel, we apply Lemma 11.2.2 several times, some times with the aid of (11.2.6). Accordingly, by induction on r,

$$\begin{aligned}\pi_{r+1}(n_1,n_2;q) &= q^{n_1+n_2}\sum_{m_2=0}^{n_2}\sum_{m_1=m_2}^{n_1}\left(\begin{bmatrix}m_1+r-1\\ r-1\end{bmatrix}\begin{bmatrix}m_2+r-1\\ r-1\end{bmatrix}\right.\\ &\quad \left. -q\begin{bmatrix}m_2+r-1\\ r\end{bmatrix}\begin{bmatrix}m_1+r-1\\ r-2\end{bmatrix}\right)q^{m_1+m_2}\\ &= q^{n_1+n_2}\sum_{m_2=0}^{n_2}\left(\left\{\begin{bmatrix}n_1+1+r-1\\ r\end{bmatrix}-\begin{bmatrix}m_2+r-1\\ r\end{bmatrix}\right\}\begin{bmatrix}m_2+r-1\\ r-1\end{bmatrix}\right.\\ &\quad \left. -\left\{\begin{bmatrix}n_1+2+r-2\\ r-1\end{bmatrix}-\begin{bmatrix}m_2+1+r-2\\ r-1\end{bmatrix}\right\}\begin{bmatrix}m_2+r-1\\ r\end{bmatrix}\right)q^{m_2}\\ &= q^{n_1+n_2}\sum_{m_2=0}^{n_2}\left(\begin{bmatrix}n_1+r\\ r\end{bmatrix}\begin{bmatrix}m_2+r-1\\ r-1\end{bmatrix}-\begin{bmatrix}n_1+r\\ r-1\end{bmatrix}\begin{bmatrix}m_2+r-1\\ r\end{bmatrix}\right)q^{m_2}\\ &= q^{n_1+n_2}\left(\begin{bmatrix}n_1+r\\ r\end{bmatrix}\begin{bmatrix}n_2+r\\ r\end{bmatrix}-q\begin{bmatrix}n_1+r\\ r-1\end{bmatrix}\begin{bmatrix}n_2+r\\ r+1\end{bmatrix}\right)\\ &= q^{n_1+n_2}\begin{vmatrix}\begin{bmatrix}n_1+r\\ r\end{bmatrix} & q\begin{bmatrix}n_2+r\\ r+1\end{bmatrix}\\ \begin{bmatrix}n_1+r\\ r-1\end{bmatrix} & \begin{bmatrix}n_2+r\\ r\end{bmatrix}\end{vmatrix}.\end{aligned}$$

Thus, our conjecture has been shown to be correct. □

Our next task is to derive a general formula for $\pi_r(n_1,n_2,\ldots,n_k;q)$ that generalizes our formula for $\pi_r(n_1,n_2;q)$ in Theorem 11.2.5.

Theorem 11.2.6. *For arbitrary positive integers r and k,*

$$\pi_r(n_1,n_2,\ldots,n_k;q) = q^{n_1+\cdots+n_k}\det\left(q^{\binom{i-j}{2}}\begin{bmatrix}n_j+r-1\\ r-i+j-1\end{bmatrix}\right)_{k\times k}. \tag{11.2.7}$$

Suppose that $r=1$ in Theorem 11.2.6. Then

$$\pi_1(n_1,n_2,\ldots,n_k;q) = q^{n_1+\cdots+n_k}\det\left(q^{\binom{i-j}{2}}\begin{bmatrix}n_j+r-1\\ -i+j\end{bmatrix}\right)_{k\times k}. \tag{11.2.8}$$

Observe that the matrix on the right side above is upper-triangular and that the diagonal entries are all equal to 1. Hence, the determinant is equal to 1, and (11.2.8) reduces to (11.2.1), as we would expect.

Proof of Theorem 11.2.6. It will be convenient to derive a variant of Lemma 11.2.2, which will be used several times in the proof that follows. Using Lemma 11.2.2 and setting $j = C - A + k$, we find that

$$\begin{aligned}\sum_{j=s}^{B} \begin{bmatrix} A+j \\ C \end{bmatrix} q^j &= \sum_{k=s-C+A}^{B-C+A} \begin{bmatrix} C+k \\ C \end{bmatrix} q^{C-A+k} \\ &= q^{C-A} \left(\sum_{k=0}^{B-C+A} \begin{bmatrix} C+k \\ C \end{bmatrix} q^k - \sum_{k=0}^{s-C+A-1} \begin{bmatrix} C+k \\ C \end{bmatrix} q^k \right) \\ &= q^{C-A} \left(\begin{bmatrix} C+B-C+A+1 \\ C+1 \end{bmatrix} - \begin{bmatrix} C+s-C+A-1+1 \\ C+1 \end{bmatrix} \right) \\ &= q^{C-A} \left(\begin{bmatrix} A+B+1 \\ C+1 \end{bmatrix} - \begin{bmatrix} A+s \\ C+1 \end{bmatrix} \right). \end{aligned} \tag{11.2.9}$$

We have seen from (11.2.8) that Theorem 11.2.6 holds for $r = 1$. Thus, we shall proceed by induction on r. Hence,

$$\begin{aligned} &\pi_{r+1}(n_1, n_2, \ldots, n_k; q) \qquad (11.2.10) \\ &= q^{n_1+\cdots+n_k} \sum_{m_k=0}^{n_k} \cdots \sum_{m_2=m_3}^{n_2} \sum_{m_1=m_2}^{n_1} q^{m_1+\cdots+m_k} \det \left(q^{\binom{i-j}{2}} \begin{bmatrix} m_j+r-1 \\ r-i+j-1 \end{bmatrix} \right)_{k\times k}. \end{aligned}$$

First observe that the innermost sum applies only to the first column of the matrix. For $1 \le i \le k$, by (11.2.9),

$$\begin{aligned} q^{\binom{i-1}{2}} \sum_{m_1=m_2}^{n_1} \begin{bmatrix} m_1+r-1 \\ r-i \end{bmatrix} q^{m_1} &= q^{\binom{i-1}{2}+r-i-(r-1)} \left(\begin{bmatrix} n_1+1+r-1 \\ r-i+1 \end{bmatrix} - \begin{bmatrix} m_2+r-1 \\ r-i+1 \end{bmatrix} \right) \\ &= q^{\binom{i-1}{2}+1-i} \begin{bmatrix} n_1+r \\ r+1-i \end{bmatrix} - q^{\binom{i-2}{2}-1} - \begin{bmatrix} m_2+r-1 \\ r+1-i \end{bmatrix}. \end{aligned} \tag{11.2.11}$$

We note that the second column of our matrix is given by

$$q^{\binom{i-2}{2}} \begin{bmatrix} m_2+r-1 \\ r+1-i \end{bmatrix}. \tag{11.2.12}$$

Multiply the second column, given by (11.2.12), by q^{-1} and add it to the new first column given by (11.2.11). The new first column then becomes

$$q^{\binom{i-1}{2}+1-i} \begin{bmatrix} n_1+r \\ r+1-i \end{bmatrix}. \tag{11.2.13}$$

We now examine the sum of the second column

$$q^{\binom{i-2}{2}} \sum_{m_2=m_3}^{n_2} \begin{bmatrix} m_2+r-1 \\ r-(i-1) \end{bmatrix} q^{m_2}. \tag{11.2.14}$$

We note that the argument is the same as above, but with i replaced by $i-1$, n_1 replaced by n_2, and m_2 replaced by m_3. Thus,

$$q^{\binom{i-2}{2}} \sum_{m_2=m_3}^{n_2} \begin{bmatrix} m_2+r-1 \\ r-(i-1) \end{bmatrix} = q^{\binom{i-2}{2}+2-i} \begin{bmatrix} n_2+r \\ r+2-i \end{bmatrix} - q^{\binom{i-3}{2}-1} - \begin{bmatrix} m_3+r-1 \\ r+2-i \end{bmatrix}. \tag{11.2.15}$$

The third column is given by

$$q^{\binom{i-3}{2}} \begin{bmatrix} m_3 + r - 1 \\ r + 2 - i \end{bmatrix}. \tag{11.2.16}$$

Multiply this third column (11.2.16) by q^{-1} and add it to the second column to arrive at

$$q^{\binom{i-2}{2}+2-i} \begin{bmatrix} n_2 + r \\ r + 2 - i \end{bmatrix}. \tag{11.2.17}$$

In general, after summing the jth column, we arrive at

$$q^{\binom{i-j}{2}+j-i} \left(\begin{bmatrix} n_j + r \\ r + j - i \end{bmatrix} - \begin{bmatrix} m_{j+1} + r - 1 \\ r - i + j \end{bmatrix} \right). \tag{11.2.18}$$

The $(j+1)st$ column is

$$q^{\binom{i-j-1}{2}} \begin{bmatrix} m_{j+1} + r - 1 \\ r - i + j \end{bmatrix}. \tag{11.2.19}$$

Multiply (11.2.19) by q^{-1} and add the result to (11.2.18). Noting that

$$\binom{i-j}{2} + j - i = \binom{i-j-1}{2} - 1,$$

we obtain, for $1 \leq j \leq k$,

$$q^{\binom{i-j}{2}+j-i} \begin{bmatrix} n_j + r \\ r + j - i \end{bmatrix}. \tag{11.2.20}$$

With (11.2.20) in mind, we conclude that, after summing on m_1, m_2, $\ldots$,m_k, we obtain

$$q^{n_1+\cdots+n_k} \det\left(q^{\binom{i-j}{2}+j-i} \begin{bmatrix} n_j + r \\ r + j - i \end{bmatrix} \right)_{k\times k}. \tag{11.2.21}$$

Now factor q^j, $1 \leq j \leq k$, out of the jth column and q^{-i}, $1 \leq i \leq k$, out of the ith row to arrive at the determinant

$$q^{n_1+\cdots+n_k+1+2+\cdots+k-1-2-\cdots-k} \det\left(q^{\binom{i-j}{2}} \begin{bmatrix} n_j + r \\ r + j - i \end{bmatrix} \right)_{k\times k}$$
$$= q^{n_1+\cdots+n_k} \det\left(q^{\binom{i-j}{2}} \begin{bmatrix} n_j + r \\ r + j - i \end{bmatrix} \right)_{k\times k},$$

which is precisely equal to the right-hand side of (11.2.7) with r replaced by $r+1$. Hence, the induction is complete, and the proof of Theorem 11.2.6 is finished. ☐

Definition 11.2.7. *We now define $\pi_{k,r}(n;q)$ to be the generating function for plane partitions with $\leq k$ rows, $\leq r$ columns, and each part $\leq n$.*

Now consider a typical partition counted by $\pi_{k,r}(n;q)$, say,

$$\begin{matrix} n_{11} & n_{12} & \cdots & n_{1r} \\ n_{21} & n_{22} & \cdots & n_{2r} \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ n_{k1} & n_{k2} & \cdots & n_{kr} \end{matrix} \quad . \tag{11.2.22}$$

We now adjoin a column of n's to the left of the matrix (11.2.22) to arrive at

$$\begin{matrix} n & n_{11} & n_{12} & \cdots & n_{1r} \\ n & n_{21} & n_{22} & \cdots & n_{2r} \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ n & n_{k1} & n_{k2} & \cdots & n_{kr} \end{matrix} \quad . \tag{11.2.23}$$

The cardinality of the set of partitions of the form (11.2.22) counted by the generating function $\pi_{k,r}(n;q)$ is identical to the cardinality of the set of partitions (11.2.23) counted by $\pi_{r+1}(n,\ldots,n;q)$, except that the two generating functions differ by q^{nk}. More precisely,

$$\begin{aligned} \pi_{k,r}(n;q) &= q^{-kn}\pi_{r+1}(n,\ldots,n;q) \\ &= q^{-kn}q^{kn}\det\left(q^{\binom{i-j}{2}}\begin{bmatrix} n_j+r \\ r-i+j \end{bmatrix}\right)_{k\times k}. \end{aligned} \tag{11.2.24}$$

Let us work a couple examples.

Example 11.2.8. *Let $k=1$. Then* (11.2.24) *reduces to*

$$\pi_{1,r}(n;q) = \begin{bmatrix} n+r \\ r \end{bmatrix}, \tag{11.2.25}$$

a result that we have previously proved in (2.2.2) *of Theorem 2.2.4.*

Example 11.2.9. *Let $k=2$. By* (11.2.24)*,*

$$\begin{aligned} \pi_{2,r}(n;q) &= \begin{vmatrix} \begin{bmatrix} n+r \\ r \end{bmatrix} & q\begin{bmatrix} n+r \\ r+1 \end{bmatrix} \\ \begin{bmatrix} n+r \\ r-1 \end{bmatrix} & \begin{bmatrix} n+r \\ r \end{bmatrix} \end{vmatrix} \\ &= \begin{bmatrix} n+r \\ r \end{bmatrix}^2 - q\begin{bmatrix} n+r \\ r+1 \end{bmatrix}\begin{bmatrix} n+r \\ r-1 \end{bmatrix} \\ &= \frac{(q)_{n+r}^2}{(q)_r^2(q)_n^2}\left\{1 - \frac{q(1-q^n)(1-q^r)}{(1-q^{r+1})(1-q^{n+1})}\right\} \\ &= \frac{(q)_{n+r}^2}{(q)_r(q)_{r+1}(q)_n(q)_{n+1}}\left\{(1-q^{n+1})(1-q^{r+1}) - q(1-q^n)(1-q^r)\right\} \end{aligned}$$

$$= \frac{(q)_{n+r}^2}{(q)_r(q)_{r+1}(q)_n(q)_{n+1}} \left\{(1-q)(1-q^{n+r+1})\right\}$$
$$= \frac{(1-q)(q)_{n+r}(q)_{n+r+1}}{(q)_r(q)_{r+1}(q)_n(q)_{n+1}}. \tag{11.2.26}$$

Suppose that $k = 2$, $r = 3$, *and* $n = 2$. *Then* (11.2.26) *yields*

$$\pi_{2,3}(n;q) = \frac{(1-q)(q)_5(q)_6}{(q)_3(q)_4(q)_2(q)_3}$$
$$= 1 + q + 3q^2 + 4q^3 + 6q^4 + 6q^5 + 8q^6 + 6q^7 + 6q^8 + 4q^9 + 3q^{10} + q^{11} + q^{12}.$$

Theorem 11.2.10. *For* $k, r \geq 1$,

$$\pi_{k,r}(n;q) = \frac{(q)_1(q)_2\cdots(q)_{k-1}}{(q)_r(q)_{r+1}\cdots(q)_{r+k-1}} \frac{(q)_{n+r}(q)_{n+r+1}\cdots(q)_{n+r+k-1}}{(q)_n(q)_{n+1}\cdots(q)_{n+k-1}}. \tag{11.2.27}$$

Observe that (11.2.26) is the special case $k = 2$ of (11.2.27).

Proof. The proof given below is due to L. Carlitz [**40**]. Let

$$W(k,r) := \det\left(q^{ri+i(i-1)/2}\begin{bmatrix} j \\ i \end{bmatrix}\right)_{0\leq i,j\leq k-1}. \tag{11.2.28}$$

Define the matrix $(c_{i,j})_{0\leq i,j\leq k-1}$ by

$$\pi_{k,r}(n;q)W(k,r) = \det(c_{i,j})_{0\leq i,j\leq k-1}. \tag{11.2.29}$$

Recall from (11.2.24) that

$$\pi_{k,r}(n;q) = \det\left(q^{\binom{i-j}{2}}\begin{bmatrix} n+r \\ r-i+j \end{bmatrix}\right)_{1\leq i,j\leq k}.$$

If we replace i by $i+1$ and j by $j+1$, $i-j$ is invariant, and so we can write

$$\pi_{k,r}(n;q) = \det\left(q^{\binom{i-j}{2}}\begin{bmatrix} n_j+r \\ r-i+j \end{bmatrix}\right)_{0\leq i,j\leq k-1}.$$

Multiplying the two matrices on the left-hand side of (11.2.29), we find that

$$c_{i,j} = \sum_{s=0}^{k-1} q^{(i-s)(i-s-1)/2}\begin{bmatrix} n+r \\ r-i+s \end{bmatrix} q^{rs+s(s-1)/2}\begin{bmatrix} j \\ s \end{bmatrix}$$
$$= q^{i(i-1)/2}\sum_{s=0}^{k-1} q^{s^2-is+rs}\begin{bmatrix} n+r \\ r-i+s \end{bmatrix}\begin{bmatrix} j \\ s \end{bmatrix}$$
$$= q^{i(i-1)/2}\sum_{s=0}^{j} q^{s^2-is+rs}\begin{bmatrix} n+r \\ r-i+s \end{bmatrix}\begin{bmatrix} j \\ s \end{bmatrix}$$
$$= q^{i(i-1)/2}\sum_{t=0}^{j} q^{(j-t)^2-i(j-t)+r(j-t)}\begin{bmatrix} n+r \\ r-i+j-t \end{bmatrix}\begin{bmatrix} j \\ t \end{bmatrix} \tag{11.2.30}$$

where in the penultimate line we used the fact $\begin{bmatrix} j \\ s \end{bmatrix} = 0$, if $s > j$, and in the last line replaced j by $j-t$. We now replace the upper index j by $r-i+j$. If $r-i+j > j$, then we will be adding terms, but because of the presence of $\begin{bmatrix} j \\ t \end{bmatrix}$, all of these added terms are equal to 0. If $r-i+j < j$, then we will be subtracting terms from the sum, but because of the presence of $\begin{bmatrix} n+r \\ r-i+j-t \end{bmatrix}$, these subtracted terms will all be equal to 0. Hence, we can write (11.2.30) in the form

$$c_{i,j} = q^{i(i-1)/2} \sum_{t=0}^{r-i+j} q^{(j-t)(j-t-i+r)} \begin{bmatrix} n+r \\ r-i+j-t \end{bmatrix} \begin{bmatrix} j \\ t \end{bmatrix}. \tag{11.2.31}$$

We now apply Exercise 1 below with n replaced by j, m replaced by $n+r$, and h replaced by $r-i+j$. Accordingly,

$$\begin{aligned} c_{i,j} = q^{i(i-1)/2} \begin{bmatrix} n+r+j \\ r-i+j \end{bmatrix} &= q^{i(i-1)/2} \begin{bmatrix} n+r+j \\ n+i \end{bmatrix} \\ &= q^{i(i-1)/2} \frac{(q)_{n+r+j}}{(q)_{r-i+j}(q)_{n+i}}. \end{aligned} \tag{11.2.32}$$

In the ith row, $1/(q)_{n+i}$ is a constant, $0 \le i \le k-1$. Factor each from the determinant. In the jth column, $(q)_{n+r+j}$ is a constant for $0 \le j \le k-1$, and so we shall factor each of these q-products from the determinant. Thus, from (11.2.32) and (11.2.29), we have shown that

$$\pi_{k,r}(n;q)W(k,r) = \frac{(q)_{n+r}(q)_{n+r+1}\cdots(q)_{n+r+k-1}}{(q)_n(q)_{n+1}\cdots(q)_{n+k-1}} \det\left(\frac{q^{i(i-1)/2}}{(q)_{r-i+j}}\right). \tag{11.2.33}$$

Let

$$\det\left(\frac{q^{i(i-1)/2}}{(q)_{r-i+j}}\right) =: C(k,r).$$

Observe that $W(k,r)$ and $C(k,r)$ are independent of n. Thus, we can determine the quotient $C(k,r)/W(k,r)$ by setting $n=0$ in (11.2.33). Noting that $\pi(0;q)=1$, we see that (11.2.33) yields

$$\frac{C(k,r)}{W(k,r)} = \frac{(q)_0(q)_1(q)_2\cdots(q)_{k-1}}{(q)_r(q)_{r+1}\cdots(q)_{r+k-1}}. \tag{11.2.34}$$

Using (11.2.34) in (11.2.33), we immediately deduce (11.2.27). □

Definition 11.2.11. *Let $p_{k,r}(m,n)$ denote the number of plane partitions of m with at most k rows, at most r columns, and each part no larger than n. Hence,*

$$\pi_{k,r}(n;q) = \sum_{m=0}^{\infty} p_{k,r}(m,n)q^m.$$

We note that

$$p_2(m) = p_{\infty,\infty}(m,\infty). \tag{11.2.35}$$

Corollary 11.2.12. *We have*

$$\sum_{m=0}^{\infty} p_{k,\infty}(m,\infty)q^m = \prod_{j=1}^{\infty}(1-q^j)^{-\min(k,j)}. \tag{11.2.36}$$

Proof. Let $r, n \to \infty$ in Theorem 11.2.10 to immediately deduce that

$$\begin{aligned}
\pi_{k,\infty}(\infty;q) &= \sum_{m=0}^{\infty} p_{k,\infty}(m,\infty)q^m \\
&= \frac{(q)_\infty^k (q)_1 (q)_2 \cdots (q)_{k-1}}{(q)_\infty^k (q)_\infty^k} \\
&= \frac{1}{(1-q)^1(1-q^2)^2(1-q^3)^3\cdots(1-q^{k-1})^{k-1}(1-q^k)^k(1-q^{k+1})^k\cdots} \\
&= \prod_{j=1}^{\infty}(1-q^j)^{-\min(k,j)}.
\end{aligned}$$

□

Theorem 11.2.13. *For each nonnegative integer n,*

$$p_2(n) = \prod_{j=1}^{\infty}(1-q^j)^{-j}.$$

Proof. Let k tend to ∞ in Corollary 11.2.12 and use the fact $p_2(n) = p_{\infty,\infty}(n,\infty)$. □

11.3. Exercises

1. Prove the following q-analogue of VanderMonde's Theorem, i.e.,

$$\sum_{k=0}^{h} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ h-k \end{bmatrix} q^{(n-k)(h-k)} = \begin{bmatrix} m+n \\ h \end{bmatrix}.$$

2. Calculate $p_r(5)$ and $p_r(6)$.

Chapter 12

Glossary of Notation

(1) $p(n)$, Definition 1.2.1, the number of unrestricted partitions of n.

(2) $p(S, m, n)$, Definition 1.2.4, the number of partitions of n into exactly m parts taken from S.

(3) $p(m, n)$, Definition 1.2.4, the number of partitions of n into precisely m parts.

(4) $p_m(n)$, Definition 1.2.2, the number of partitions of n with largest part less than or equal to m.

(5) $p_m(j, n)$, Definition 2.1.13, the number of partitions of n into at most m parts, with the largest part being j.

(6) $p(N, M, n)$, Definition 2.2.1, the number of partitions of n into at most M parts, each $\leq N$.

(7) $Q(N, M, n)$, Definition 2.2.1, the number of partitions of n into exactly M distinct parts, each $\leq N$.

(8) $Q_m(j, n)$, Definition 2.1.13, the number of partitions of n into at most m distinct parts, with the largest part being j.

(9) $Q_m^{(k,\ell)}(n)$, Definition 2.1.11, the number of partitions of n into exactly m distinct parts, where each part (in descending order) differs from the next by at least k, and where the smallest part is $\geq \ell$.

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$$1+\frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})}\cdot x+\frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^{\gamma})(1-q^{\gamma+1})}\cdot x^2+\cdots,$$

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