Mat. Sbornik
Tom 92 (134) (1973), No. 3

Math. USSR Sbornik
Vol. 21 (1973), No. 3

PERIODS OF PARABOLIC FORMS* AND p-ADIC HECKE SERIES

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To Igor Rostislavovič Šafarevič
on his fiftieth birthday

Abstract. The author proves an algebraicity theorem for the periods of parabolic forms of any weight for the full modular group, gives explicit formulas for the coefficients of the forms, and constructs p-adic analogs of their Mellin transforms in the manner of Iwasawa and Mazur.

Bibliography: 13 items.

§1. Introduction. Basic results

The concentrated attention of mathematicians to number theory, while never lessening, has in recent years taken on new forms. Elementary questions about congruences and equations have found themselves becoming interwoven in an intricate and rich complex of constructions drawn from abstract harmonic analysis, topology, highly technical ramifications of homological algebra, algebraic geometry, measure theory, logic, and so on—corresponding to the spirit of Gödel's theorem on the incompleteness of the techniques of elementary arithmetic and on our capabilities of recognizing even those truths which we are in a position to "prove" (see for example [2], [9]).

A new "synthetic" number theory, taking in the legacy of the "analytic" theory, is possibly taking shape under our very eyes. (And here, perhaps, all the connotations of the word "synthetic" are appropriate.)

In our canvas, scarcely encompassable at a glance, any points of contact with concrete number-theoretical facts, whether old or new, take on especial significance. They discipline the imagination, and they provide a breathing space and the opportunity to evaluate the stunning beauty of past discoveries. Ramanujan's peculiar congruence modulo 691 serves as a fresh example. This congruence (and its analogs; see [7]) is so far, our only clue to understanding the 11-dimensional étale cohomology of the so-called Sato variety (see [9], [10], [11], [12]).

The present article is bssed on an analysis of several recent works on modular forms and has the modest aim of making explicit a part of the number-theoretic informa-

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**Editors's note.* In English one usually speaks of cusp forms.

AMS (MOS) subject classifications (1970). Primary 10D15, 12B30; Secondary 12B40.

tion which they contain implicitly. A subsequent translation of this information back into, say the language of representation theory is certainly possible and nontrivial, but remains a matter for the future.

Let us now describe in more detail the contents of the article.

1.1. Let $\Gamma = SL(2, Z)/(\pm 1)$, let w be an even integer, and let S_{w+2} be the space of parabolic forms with respect to Γ on the upper half-plane of one complex variable (see [8] for the basic notions).

To each $\Phi(z) \in S_{w+2}$ is associated the following collection of invariants:

a) The periods of Φ: the numbers $r_k(\Phi) = \int_0^{i\infty} \Phi(z) z^k\, dz$, $0 \le k \le w$.

b) The coefficients of Φ: the numbers λ_n such that $\Phi(z) = \sum_{n=1}^{\infty} \lambda_n e^{2\pi i n z}$.

c) The Hecke series $L_\Phi(s) = \sum_{n=1}^{\infty} \lambda_n n^{-s}$.

In §§2–7 of this article we construct the system of equations which connects the periods and the coefficients of Φ, and show that solving this system of equations allows us to obtain extremely precise information on both the periods and the coefficients. In particular, Ramanujan's congruence is included in a new series of formulas.

In §§8–10 we construct p-adic analytic functions which are related to the $L_\Phi(s)$ just as the Leopoldt-Kubota-Iwasawa functions [1] are related to ordinary Dirichlet L-functions.

The first group of results is obtained by the application to forms of higher weight of the techniques developed in the author's articles [3]–[5] for forms of weight 2 using Shimura's theory [13]. The second part generalizes to forms of arbitrary weight the methods of Mazur and Swinnerton-Dyer, introduced in [6] for forms of weight 2.

We restrict ourselves here to the consideration of the full modular group along the lines explained above.

We now formulate some of the basic results of the article.

1.2. **The Periods Theorem.** *Let*

$$\Phi(z) = \sum_{n=1}^{\infty} \lambda_n e^{2\pi i n z} \in S_{w+2}$$

be a nonzero form which is an eigenform for all the Hecke operators, so that $\Phi | T_n = \lambda_n \Phi$. *Then the ratios*

$$(r_0(\Phi) : r_2(\Phi) : \ldots : r_w(\Phi)), \quad (r_1(\Phi) : \ldots : r_{w-1}(\Phi))$$

are rational over the algebraic number field $Q(\lambda_1, \cdots, \lambda_n, \cdots)$.

Example. Let $w = 10$. Then $S_{12} = C\Delta(z)$, where

$$\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}.$$

We have

$$(r_0 : r_2 : r_4) = \left(1 : -\frac{691}{2^2 \cdot 3^4 \cdot 5} : \frac{691}{2^3 \cdot 3^2 \cdot 5 \cdot 7}\right), \quad (r_1 : r_3 : r_5) = \left(1 : -\frac{5^2}{2^4 \cdot 3} : \frac{5}{2^2 \cdot 3}\right).$$

The values of the remining periods are determined by the general relation $r_k(\Phi) = (-1)^{k+1} r_{w-k}(\Phi)$ (see §2 below).

As we have said, Theorem 1.2 is proved by constructing an infinite system of homogeneous linear equations for the periods of Φ. This system naturally breaks up into equations for the even ($k \equiv 0\ (2)$) and odd ($k \equiv 1\ (2)$) periods. We check that each of these subsystems has a 1-dimensional space of solutions; furthermore, the coefficients of the equations lie in $Q(\lambda_1, \cdots, \lambda_n, \cdots)$.

A part of these equations (which all $\Phi \in S_{w+2}$ have in common) has been pointed out by Shimura [13]. Another part, which is different for each Φ, is written out here for the first time in terms of the Hecke operators. For $w = 10$ Shimura's equations provide all the values of the periods with the exception of r_0. As w increases, more and more of the information is contained in the remaining equations.

Let us introduce another interpretation of the periods $r_k(\Phi)$ in terms of the Hecke series. Since

$$L_\Phi(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty \Phi(iy)\, y^{s-1}\, dy,$$

it is easy to see that

$$r_k(\Phi) = \frac{k!\, i^{k+1}}{(2\pi)^{k+1}} L_\Phi(k+1), \quad 1 \leqslant k+1 \leqslant w+1.$$

Thus the Periods Theorem, roughly speaking, allows us to compute the values of $L_\Phi(s)$ *at the integer points within the critical strip*, which stretches from $\sigma = 0$ to $\sigma = w+2$.

It would be very interesting to extend these computations to the remaining integer points. We also remark that our methods do not have anything to say on the arithmetic nature of the "common transcendencental multiplier" for the $r_{2k}(\Phi)$ and the $r_{2k+1}(\Phi)$ respectively.

1.3. The Coefficients Theorem. *Under the hypotheses of Theorem 1.2, for all $n \geq 2$ we have*

$$(\sigma_{w+1}(n) - \lambda_n)\, r_0 = \sum_{n=\Delta\Delta'+\delta\delta'}' \sum_{l=1}^{\left[\frac{w-1}{4}\right]}{}' 2 \binom{w}{2l} r_{2l} (\Delta^{2l}\delta^{w-2l} - \Delta^{w-2l}\delta^{2l}).^* \tag{1}$$

Here $\sigma_{w+1}(n) = \sum_{d|n} d^{w+1}$, $r_i = r_i(\Phi)$, and in the outer summation on the right the sum is taken over all integer solutions of the equation $n = \Delta\Delta' + \delta\delta'$ which satisfy

$$\Delta > \delta > 0, \textit{ and}$$

$$\textit{either } \Delta' > \delta' > 0 \textit{ or } \Delta | n,\ \Delta' = n/\Delta,\ \delta' = 0 \textit{ and } 0 < \delta/\Delta \leq 1/2. \tag{2}$$

**Translator's note.* Here and below the convention is observed that Σ' denotes a summation in which certain "boundary" terms are to be taken with the coefficient ½.

Furthermore, the terms in which $\delta/\Delta = ½$ are to be taken with the coefficient ½.

Example. Again for $w = 10$ we find

$$\sigma_{11}(n) - \tau(n) = \sum_{n=\Delta\Delta'+\delta\delta'}' \frac{691}{18}(\Delta^8\delta^2 - \Delta^2\delta^8) + \frac{691}{6}(\Delta^6\delta^4 - \Delta^4\delta^6).$$

In particular, this formula gives a new proof of Ramanujan's congruence

$$\tau(n) \equiv \sigma_{11}(n) \mod 691.$$

The table given in §7 shows that formula (1) also gives other well-known congruences.

The Coefficients Theorem simply displays the explicit form of a certain part of the above-mentioned equations for the periods. To prove the Periods Theorem we need to use, among other things, information on the coefficient of $r_0(\Phi)$ in (1). On the other hand, if $r_0(\Phi) \neq 0$, then (1) can be considered as an explicit formula for the coefficients λ_n of Φ, and hence the name of the theorem. It is interesting to note that to compute λ_n we must sum some polynomial of degree w in Δ and δ over the solutions of the universal equation $n = \Delta\Delta' + \delta\delta'$; only this polynomial depends on Φ. Note further that $\sigma_{w+1}(n) - \lambda_n$ is the nth coefficient of the modular form $E_{w+2}(z) - \Phi(z)$, E_{w+2} being the Eisenstein series of weight $w + 2$.

Thus the Coefficients Theorem provides a series of relations which can be written

$$E_{w+2}(z) - \Phi(z) = a_0 + \sum_{\substack{\Delta,\delta \\ \Delta',\delta'}}' F(\Delta, \delta)\, e^{2\pi i(\Delta\Delta'+\delta\delta')z}, \tag{3}$$

where $F(\Delta, \delta)$ is a homogeneous polynomial of degree w, and the summation is over the integer points of the region defined by (2). The series on the right of (3) can be considered to be a sort of theta function "with spherical polynomials" constructed for the indefinite quadratic form $\Delta\Delta' + \delta\delta'$. The role of this series still needs to be made explicit.

1.4. To formulate the theorem on p-adic Hecke series let us introduce some more notation.

First of all, fix a form $\Phi(z) \in S_{w+2}$ by the conditions that $\Phi | T_n = \lambda_n \Phi$ for all n, $\Phi(z) = \Sigma_1^\infty \lambda_n e^{2\pi i n z}$. Next, choose a prime p and an integer Δ_0 with $(\Delta_0, p) = 1$, and put $\Delta = \Delta_0 p$ (for $p \geq 3$) or $\Delta = 4\Delta_0$ (if $p = 2$). Furthermore, we fix an imbedding $O \hookrightarrow \overline{Q}_p$ of the field of all algebraic numbers into the algebraic closure of the p-adic field. In the sequel, all identifications of algebraic numbers with p-adic ones will be with respect to this imbedding.

Consider a primitive Dirichlet character χ modulo Δp^m, $m \geq 0$. χ can be uniquely written as a product $\chi_0\chi_1$, where χ_0 is a primitive character modulo Δ, and χ_1 is a primitive character of period p^m. χ_0 is called the *tame* component of χ, and χ_1 the *wild* component. Set

$$G(\chi) = \sum_{b \bmod \Delta p^m} \chi(b) e^{2\pi i \frac{b}{\Delta p^m}}$$

(the Gauss sum).

Consider the Hecke-Dirichlet series

$$L_\Phi(s, \chi) = \sum_{n=1}^{\infty} \chi(n) \lambda_n n^{-s} = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty \Phi_\chi(iy) y^{s-1} dy.$$

From the Periods Theorem we will later deduce that the values of $L_\Phi(k, \chi)$ for $k = 1, \cdots, w+1$ and any χ are "almost algebraic". Here we only need the following fact: *there exist two numbers* $\omega^\pm \in C$ *such that the values of* $(1/2\pi i\omega^\pm)L_\Phi(1, \chi)$ *are integers of the field* $Q(\lambda_1, \cdots, \lambda_n, \cdots, G(\chi), \chi(z))$ (where we divide by ω^+ or ω^- according as $\chi(-1) = +1$ or -1). Fix such ω^+ and ω^-.

We are now in a position to state the final result. All the notation which we have introduced in this subsection is used explicitly or implicitly in our formulation.

1.5. **Theorem on p-adic Hecke series.** *Suppose that* λ_p *is a p-adic unit, and denote by* ρ *that p-adic root of the equation* $X^2 - \lambda_p X + p^{w+1} = 0$ *which is also a p-adic unit.*

Then there exists a unique power series $g_{\chi_0}(T)$ *with coefficients in the p-adic completion of the ring of integers of the field* $Q(\lambda_1, \cdots, \lambda_n, \cdots)$ *such that for any character* $\chi = \chi_0\chi_1$ *modulo* Δp^m, $m \geq 0$, *which has tame component* χ_0 *we have*

$$\frac{\Delta p^m}{G(\chi)} \frac{1}{2\pi i \cdot \omega^\pm} L_\Phi(1, \chi) = \rho^m g_{\chi_0}(\chi_1(1+q) - 1), \tag{4}$$

where either $q = p$ *or* $q = 4$ (*for* $p = 2$).

We will prove this theorem by means of Mazur's "p-adic Mellin transform" of modular forms. This beautiful and important construction allows us to hope for adelic analogs of p-adic Hecke series.

Our result is clearly incomplete: the domain of values of the argument of L_Φ in (4) is too narrow. By analogy with the Leopoldt-Kubota-Iwasawa theory [1] one should expect a formula of type (4) for all the $L_\Phi(k, \chi)$, $k \in Z$, or at any rate for $k = 1, \cdots, w+1$.

It is a pleasure for me to be able to thank I. I. Pjateckiĭ-Šapiro and A. Nasybullin for extremely useful discussions, and B. Mazur and H. P. F. Swinnerton-Dyer for their kindness in sending me their article [6] before publication.

§2. The Shimura-Eichler relations

2.1. **Proposition.** *Let* $\Phi \in S_{w+2}$, *and let* $r_k(\Phi)$ *be the periods of* Φ. *Then for any* $k = 0, 1, \cdots, w$ *we have*

$$r_k(\Phi) + (-1)^k r_{w-k}(\Phi) = 0, \tag*{$(5)_k$}$$

$$r_k(\Phi)+(-1)^k\sum_{\substack{0\leqslant i\leqslant k\\ i\equiv 0(2)}}\binom{k}{i}r_{w-k+i}(\Phi)+(-1)^k\sum_{\substack{0\leqslant i\leqslant w-k\\ i\equiv k(2)}}\binom{w-k}{i}r_i(\Phi)=0, \qquad (6)_k$$

$$\sum_{\substack{1\leqslant i\leqslant k\\ i\equiv 1(2)}}\binom{k}{i}r_{w-k+i}(\Phi)+\sum_{\substack{0\leqslant i\leqslant w-k\\ i\not\equiv k(2)}}\binom{w-k}{i}r_i(\Phi)=0. \qquad (7)_k$$

Proof. Let $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, with $\det g\neq 0$, and let $F(z)$ be a polynomial of weight $\leq w$. Let

$$F^g(z)=(cz+d)^w F\left(\frac{az+b}{cz+d}\right).$$

Since Φ is a parabolic form of weight $w+2$ for Γ, for any $g\in\Gamma$ we have

$$\int_{g(0)}^{g(i\infty)}\Phi(z)F(z)\,dz=\int_0^{i\infty}\Phi(gz)F(gz)\,d(gz)=\int_0^{i\infty}\Phi(z)F^g(z)\,dz. \qquad (8)$$

(These and subsequent integrals are to be taken along geodesic arcs.) Put $s=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}$ and $t=\begin{pmatrix} 1 & -1\\ 1 & 0\end{pmatrix}$. These are the standard generators of Γ, and $s^2=t^3=1$. It is easy to verify that

$$\left(\int_0^{i\infty}+\int_{s(0)}^{s(i\infty)}\right)\Phi(z)F(z)\,dz=\left(\int_0^{i\infty}+\int_{t(0)}^{t(i\infty)}+\int_{t^2(0)}^{t^2(i\infty)}\right)\Phi(z)F(z)\,dz=0. \qquad (9)$$

Indeed, $s(i\infty)=0$ and $s(0)=i\infty$, so that the first sum vanishes. In the second sum the integral is taken around the perimeter of a triangle with vertices $(0, i\infty, 1)$; replacing this by a nearby triangle lying entirely within the upper half-plane, and estimating the error terms using standard inequalities, we obtain the desired result.

Now to obtain $(5)_k$, $(6)_k$ and $(7)_k$ we must substitute $F(z)=z^k$ in (9), replace all the paths of integration by the path $(0, i\infty)$ using (8), and use the fact that by definition

$$(z^k)^s=(-1)^k z^{w-k},\qquad (z^k)^t=z^{w-k}(z-1)^k,\qquad (z^k)^{t^2}=(-1)^k(z-1)^{w-k}.$$

The first part of (9) then gives us $(5)_k$, whereas the second, in view of the fact that

$$z^k+(z^k)^t+(z^k)^{t^2}=z^k+\sum_{i=0}^{k}\binom{k}{i}(-1)^{k+i}z^{w-k+i}+\sum_{i=0}^{w-k}\binom{w-k}{i}(-1)^i z^i, \qquad (10)$$

gives us an equation which only differs from $(6)_k$ in that the summation runs over all i rather than just over those having the same parity as 0 or k. To break up this equation into two, note that S_{w+2} has a basis made up of *real* forms (that is, forms with real coefficients). But if Φ is real, then the periods $r_{2k+1}(\Phi)$ are real, and the $r_{2k}(\Phi)$ pure imaginary. Hence, in the linear relation in the $r_k(\Phi)$ which arises out of (10), we can take real and imaginary parts; this then leads to $(6)_k$ and $(7)_k$, which proves the proposition.

2.2. **Definition.** Let S^+ be the space of real solutions of the system of equations $(5)_k$, $(6)_k$ for even k, and $(7)_k$ for odd k, and let S^- be the space of real solutions of the system $(5)_k$, $(6)_k$ for odd k, and $(7)_k$ for even k.

We clearly have $S^+ \subset R^{(w+2)/2}$ and $S^- \subset R^{w/2}$, where $R^{(w+2)/2}$ and $R^{w/2}$ are the real vector spaces with basis indexed by the even and odd integers respectively in the interval $[0, w]$.

Note that $(1, 0, \cdots, 0, -1) \in S^+$; indeed, a straightforward check shows that $r_0(\Phi)$ and $r_w(\Phi)$ have the same coefficient (either 1 or 2) in all the equations (5)–(7).

2.3. **Proposition (Eichler, Shimura).** *Let S^0_{w+2} be the space of all real parabolic forms of weight $w+2$. Then the following assertions are true:*

a) *The map*

$$S^0_{w+2} \to S^- : \Phi \longmapsto (r_1(\Phi), \quad r_3(\Phi), \ \ldots, \ r_{w-1}(\Phi))$$

is an isomorphism.

b) *The map*

$$S^0_{w+2} \to S^+ : \Phi \longmapsto (r_0(\Phi), r_2(\Phi), \ \ldots, \ r_w(\Phi))$$

is an imbedding of S^0_{w+2} as a subspace of codimension 1 in S^+ which does not contain the vector $(1, 0, \cdots, 0, -1)$.

This result can be considered an immediate reformulation of a particular case of a theorem of Shimura's ([13], Theorem 1, §5; compare also §9 of the same paper). We omit the proof, since it does not have any direct relation to what will follow. Apart from that, the proof which Shimura gives is not the most natural one here, since it relies on an independent computation of the dimensions of S^+ and S^- carried out by Eichler. In fact Proposition 2.3 follows from the theorems of de Rham and Dolbeault, applied to the Sato variety—the w-fold fiber product of the "general" elliptic curve. The periods of the forms Φ are the periods of the differentials of the first kind of the highest order on such a variety.

§3. The action of the Hecke operators on the periods

3.1. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\Phi \in S_{w+2}$. Recall that if we set

$$(\Phi | g)(z) = (\det g)^{\frac{w+2}{2}} (cz+d)^{-w-2} \Phi(gz)$$

then the Hecke operator T_n on S_{w+2} is defined by the formula

$$T_n = n^{\frac{w}{2}} \sum_{d|n} \sum_{b \bmod d} \begin{pmatrix} nd^{-1} & b \\ 0 & d \end{pmatrix}. \tag{11}$$

It is well known that S^0_{w+2} is invariant with respect to all the Hecke operators T_n and has a unique decomposition as a direct sum of 1-dimensional invariant subspaces.

3.2. **Proposition.** *For all $i, k = 0, \cdots, w$ and $n \geq 0$ there exist coefficients $A_{ik}(n) \in Z$, zero if $i \not\equiv k$ (2), such that*

$$\int_0^{i\infty} (\Phi | T_n)(z) z^k dz = \sum_{l=0}^{w} A_{lk}(n) r_l(\Phi)$$

for all $\Phi \in S^0_{w+2}$. *In other words, there exist endomorphisms* $T^{\pm}_n$ *of the spaces* $R^{(w+2)/2}$ *and* $R^{w/2}$ *of* §2.2,

$$T_n^+ : R^{\frac{w+2}{2}} \to R^{\frac{w+2}{2}},\ T_n^- : R^{\frac{w}{2}} \to R^{\frac{w}{2}},$$

satisfying the following conditions:

a) *The matrix of* $T^{\pm}_n$ *in the standard basis has integer entries.*

b) *The images of* S^0_{w+2} *in* $R^{(w+2)/2}$ *and* $R^{w/2}$ *(see Proposition* 2.3) *are invariant under the* T^+_n *and* T^-_n *respectively.*

c) *The restriction of* $T^{\pm}_n$ *to these images coincides with the action of* T_n *on* S^0_{w+2}.

Proof. First of all let $F(z)$ be an arbitrary polynomial of degree $\le w$. In the expression $\int_0^{i\infty} (\Phi | T_n)(z) F(z)\, dz$ substitute the explicit form (11) for T_n. We will carry out the summation over b mod d over the interval $-d/2 \le b \le d/2$; the symbol Σ' will indicate that in the summation in question the terms with $b = \pm d/2$ are to be taken with the coefficient ½. We have

$$\int_0^{i\infty} (\Phi | T_n)(z) F(z)\, dz = \int_0^{i\infty} \sum_{d|n} \sum_{-\frac{d}{2} \le b \le \frac{d}{2}}{}' \Phi\left(\frac{n}{d^2} z + \frac{b}{d}\right) \frac{n^{w+1}}{d^{w+2}} F(z)\, dz.$$

Replacing $nz/d^2 + b/d$ by z in each term, we bring this expression to the form

$$\int_0^{i\infty} (\Phi | T_n)(z) F(z)\, dz = \sum_{d|n} \frac{n^w}{d^w} \sum_{-\frac{d}{2} \le b \le \frac{d}{2}}{}' \left(\int_0^{i\infty} - \int_0^{b/d} \right) \Phi(z) F\left(\frac{d^2}{n} z - \frac{bd}{n}\right) dz. \quad (12)$$

To deduce Proposition 3.2. from (12), set $F(z) = z^k$ and expand $(d^2 z/n - bd/n)^k$ in powers of z. Clearly we obtain on the right an integral linear combination of the periods $r_l(\Phi)$ and of integrals of the form $\int_0^{b/d} \Phi(z) z^l\, dz$ with $l \le k$. Hence it suffices to check that all such integrals are integer linear combinations of the periods of Φ.

In fact, let $b > 0$, $(b, d) = 1$, and let $b/d = b_m/d_m, b_{m-1}/d_{m-1}, \cdots, b_0/d_0 = 0/1$ be the successive convergents to b/d in irreducible form. It is well known that

$$g_k = \begin{pmatrix} b_k & (-1)^{k-1} b_{k-1} \\ d_k & (-1)^{k-1} d_{k-1} \end{pmatrix} \in SL(2, Z),$$

and hence by (8)

$$\int_0^{b/d} \Phi(z) z^l\, dz = \sum_{k=1}^{m} \int_{b_{k-1}/d_{k-1}}^{b_k/d_k} \Phi(z) z^l\, dz = \sum_{k=1}^{m} \int_{g_k(0)}^{g_k(i\infty)} \Phi(z) z^l\, dz$$

$$= \sum_{k=1}^{m} \int_0^{i\infty} \Phi(z) (b_k z + (-1)^{k-1} b_{k-1})^l (d_k z + (-1)^{k-1} d_{k-1})^{w-l}\, dz. \quad (13)$$

This completes the proof. We note that the matrices $T_n^{\pm}$ can be computed explicitly without especial difficulty.

§4. The proof of Theorem 1.2 for odd periods

4.1. Let $\Phi \in S_{w+2}^0$, and suppose that $\Phi|T_n = \lambda_n \Phi$ for all $n \geq 1$. Then the vector $(r_1(\Phi), \cdots, r_{w-1}(\Phi)) \in S^-$ is an eigenvector for all the operators T_n^-, with eigenvalues λ_n (Proposition 3.2 c)). But the subspace $S^- \subset R^{w/2}$ is defined by the Eichler-Shimura equations, which have integer coefficients, and the T_n^- also have integer coefficients. Finally, S^-, together with S_{w+2}^0, has a uniquely defined decomposition into one-dimensional subspaces invariant under all the T_n^-. Hence it follows that the coordinates of the eigenvector $(r_1(\Phi), \cdots, r_{w-1}(\Phi))$ are, up to a common multiple, rational over $Q(\lambda_1, \cdots, \lambda_n, \cdots)$, and this proves one half of the Periods Theorem.

4.2. The only obstacle to proving the second half of the Periods Theorem in precisely the same way is the fact that S^+ has dimension one greater than S_{w+2}^0 (Proposition 2.3 b)), and hence it could a priori turn out that all the T_n^+ have an invariant plane in S^+, on which T_n^+ acts by multiplication by λ_n. Then for one exceptional form Φ we could not guarantee the rationality of $(r_0(\Phi): \cdots : r_w(\Phi))$ over $Q(\lambda_1, \cdots, \lambda_n, \cdots)$.

In fact there is no such exceptional form, but to prove this we need more precise information on the form of the T_n^+, so that it will be convenient to first prove Theorem 1.3.

§5. Proof of Theorem 1.3

5.1. Formula (1) will be obtained as the result of a series of transformations of (12). First substitute $F = 1$ in (12), transfer all the $\int_0^{i\infty}$ from right to left and change signs. If we use the fact that $\Phi|T_n = \lambda_n \Phi$, this gives

$$(n\sigma_{w-1}(n) - \lambda_n)\int_0^{i\infty} \Phi(z)\,dz = \sum_{d|n} \frac{n^w}{d^w} \sideset{}{'}\sum_{-\frac{d}{2} \leqslant b \leqslant \frac{d}{2}} \int_0^{b/d} \Phi(z)\,dz.$$

Fix b and d with $(b, d) = 1$ and collect together the terms $\int_0^{b\delta/d\delta}$ with all $\delta|(n/d)$. The coefficient of such an integral will then be

$$\frac{n^w}{d^w} \sum_{\delta \left| \left(\frac{n}{d}\right)\right.} \delta^{-w} = \sigma_w\left(\frac{n}{d}\right),$$

so that

$$(n\sigma_{w-1}(n) - \lambda_n)\int_0^{i\infty} \Phi(z)\,dz = \sum_{d|n} \sigma_w\left(\frac{n}{d}\right) \sideset{}{'}\sum_{\substack{1 \leqslant b \leqslant \frac{d}{2} \\ (b,d)=1}} \left(\int_0^{b/d} + \int_0^{-b/d}\right) \Phi(z)\,dz. \qquad (14)$$

The term on the right with $d = 1$ vanishes. The term with $d = 2$ is

$$\sigma_w\left(\frac{n}{2}\right)\int_0^{1/2}\Phi(z)\,dz = \sigma_w\left(\frac{n}{2}\right)\int_{\binom{10}{21}(0)}^{\binom{10}{21}(i\infty)}\Phi(z)\,dz = \sigma_w\left(\frac{n}{2}\right)\int_0^{i\infty}\Phi(z)(2z+1)^w\,dz$$

$$= \sigma_w\left(\frac{n}{2}\right)(1-2^w)\,r_0(\Phi) + \sigma_w\left(\frac{n}{2}\right)\sum_{k=1}^{w-1}2^k\binom{w}{k}r_k(\Phi). \tag{15}$$

We transform the terms with $d \geq 3$ using (13) with $l = 0$, and the completely analogous formula with $-b/d$ in place of b/d. In the notation of (13) we get

$$\left(\int_0^{b/d} + \int_0^{-b/d}\right)\Phi(z)\,dz = \sum_{k=1}^{m}\int_0^{i\infty}\Phi(z)\left[(d_k z + (-1)^{k-1}d_{k-1})^w - (d_k + (-1)^{k-1}d_{k-1}z)^w\right]dz$$

$$= \sum_{k=1}^{m}\sum_{i=0}^{w}\binom{w}{i}d_k^i(-1)^{(k-1)(w-i)}d_{k-1}^{w-i}\left(r_i(\Phi) - r_{w-i}(\Phi)\right).$$

On account of $(5)_i$ we may leave on the right-hand side only the terms with even i, and then sum only over $i = 2l \leq (w-2)/2$. Taking out the $r_0(\Phi)$ term again, we get

$$\left(\int_0^{b/d} + \int_0^{-b/d}\right)\Phi(z)\,dz = \sum_{k=1}^{m}2\left(d_{k-1}^w - d_k^w\right)r_0(\Phi)$$

$$+ \sum_{k=1}^{m}\sum_{l=1}^{\left[\frac{w-2}{4}\right]}2\binom{w}{2l}\left(d_k^{2l}d_{k-1}^{w-2l} - d_k^{w-2l}d_{k-1}^{2l}\right)r_{2l}(\Phi). \tag{16}_{b,d}$$

The first sum on the left is obviously equal to $2(1 - d^w)r_0(\Phi)$.

Now substitute (15) and $(16)_{b,d}$ into the right-hand side of (14) and take all the terms in $r_0(\Phi)$ over to the left. The coefficient of $r_0(\Phi)$ on the left is then of the form

$$n\sigma_{w-1}(n) - \lambda_n - \sum_{d|n}\sigma_w\left(\frac{n}{d}\right)\cdot 2\,(1 - d^w)\cdot\frac{1}{2}\varphi(d). \tag{17}$$

(Here $\phi(d)/2$ for $d \geq 3$ is the number of b with $1 \leq b \leq d/2$ and $(b, d) = 1$; for $d = 2$ the same formula automatically holds by (15).) An elementary calculation shows that (17) is equal to $\sigma_{w+1}(n) - \lambda_n$; that is, the coefficient of $r_0(\Phi)$ in (1).

It remains to check that we can also reduce the right-hand side of the formula obtained from (14) to the same form as the right-hand side of (1).

Up to this point in the computation the right-hand side of (14) has the following shape (after taking the terms in $r_0(\Phi)$ over to the left and using the fact that in (15) we need take only the terms with even k, the others being real):

$$\sigma_w\left(\frac{n}{2}\right)\sum_{l=1}^{\left[\frac{w-2}{4}\right]}\binom{w}{2l}(2^{2l}-2^{w-2l})\,r_{2l}(\Phi)$$

$$+\sum_{\substack{d|n\\ d\geqslant 3}}\sigma_w\left(\frac{n}{d}\right)\sum_{\substack{1\leqslant b<\frac{d}{2}\\ (b,d)=1}}\ \sum_{k=1}^{m\left(\frac{b}{d}\right)}\ \sum_{l=1}^{\left[\frac{w-2}{4}\right]} 2\binom{w}{2l}(d_k^{2l}d_{k-1}^{w-2l}-d_k^{w-2l}d_{k-1}^{2l})\,r_{2l}(\Phi). \tag{18}$$

Here the first sum has arisen from (15) by the use of $(5)_k$; and the second from $(16)_{b,d}$; in summing over b and d in (18) one should bear in mind that the length $m(b/d)$ of the expression for b/d as a continued fraction and the denominators d_k are functions of (b/d).

To compare (18) with (1), we use the following lemma of Heilbronn ([4], Lemma 7.7):

5.2. Lemma. *The family of pairs (d_k, d_{k-1}) of successive denominators of convergents to all rationals b/d with given $d\geq 3$ and $1\leq b<d/2$, $(b,d)=1$, is precisely the family of pairs (Δ,δ) taken from the solutions of the equation $d=\Delta\Delta'+\delta\delta'$, subject to the conditions $\Delta>\delta>0$, $(\Delta,\delta)=1$ and either $\Delta'>\delta'>0$, $(\Delta',\delta')=1$ or $\Delta=d$, $\Delta'=1$, $1\leq\delta'<d/2$, $\delta=0$, $(\Delta',\delta')=1$.*

5.3. The conditions imposed on the solutions of the equation $d=\Delta\Delta'+\delta\delta'$ in Heilbronn's Lemma only differ from the conditions (2) in the addition of the requirement $(\Delta,\delta)=(\Delta',\delta')=1$; that is, in the requirement that the solutions be primitive. That is compensated for by the factors $\sigma_w(n/d)$ in (18).

More precisely, write $\sigma_w(n/d)=\Sigma_{D|n/d}D^w$ and take D^w inside the internal summation:

$$D^w(d_k^{2l}d_{k-1}^{w-2l}-d_k^{w-2l}d_{k-1}^{2l})=(Dd_k)^{2l}(Dd_{k-1})^{w-2l}-(Dd_k)^{w-2l}(Dd_{k-1})^{2l}.$$

After this, to any primitive solution of $d=\Delta\Delta'+\delta\delta'$ and to any $D|n/d$ we make correspond the (not necessarily primitive) solution of $n=\overline{\Delta\Delta}'+\overline{\delta\delta}'$:

$$\bar\Delta=D\Delta,\quad \bar\delta=D\delta,\quad \bar\Delta'=\frac{n}{Dd}\Delta',\quad \bar\delta'=\frac{n}{Dd}\delta'.$$

The whole sum (18) then turns into a sum of the same form as the right-hand side of (1): the individual terms clearly have the same form, and the summation will be over the same region by Heilbronn's Lemma. This completes the proof of Theorem 1.3.

§6. Proof of Theorem 1.2 for even periods

6.1. Lemma. *Let $\Phi\in S^0_{w+2}$, and let $\Phi|T_n=\lambda_n\Phi$ for all $n\geq 1$. Furthermore, let $r'\in S^+$ (Definition 2.2) be a vector such that $r'|T_n^+=\lambda_n r'$ for all $n\geq 1$. Then r' is also the vector of periods for some form $\Phi'\in S^0_{w+2}$ (obviously proportional to Φ).*

Proof. According to Theorem 1.3, for any $\Phi\in S^0_{w+2}$ we have

$$\int_0^{i\infty} (\sigma_{w+1}(n)\Phi' - \Phi'|T_n)(z)\,dz = \sum_{l=1}^{\frac{w-2}{2}} A_{0,2l}(n)\, r_{2l}(\Phi'), \quad A_{0,2l}(n) \in Z. \tag{19}$$

Let $r' = (r_0', r_2', \cdots, r_w')$ and $r'|T_n^+ = \lambda_n r'$ for all $n \geq 1$. Since (19) describes the action of T_n^+ on the r_0' coordinate, we have

$$\sigma_{w+1}(n)\, r_0' - \lambda_n r_0' = \sum_{l=1}^{\frac{w-2}{2}} A_{0,2l}(n)\, r_{2l}'. \tag{20}$$

On the other hand, by Proposition 2.3 there exists a form $\Phi' \in S_{w+2}^0$ such that $ir_2' = r_2(\Phi'), \cdots, ir_{w-2}' = r_{w-2}(\Phi')$. We will show that it then follows that $ir_0' = r_0(\Phi')$. It will then follow from $(5)_0$ that $ir_w' = r_w(\Phi')$, and the lemma will be proved.

Substitute $ir_{2l}' = r_{2l}(\Phi')$ for $1 \leq l \leq (w-2)/2$ in (19). This gives

$$\int_0^{i\infty} (\sigma_{w+1}(n)\Phi' - \Phi'|T_n)(z)\,dz = i\sum_{l=1}^{\frac{w-2}{2}} A_{0,2l}(n)\, r_{2l}'. \tag{21}$$

Comparing (20) and (21) gives us

$$\int_0^{i\infty} (\sigma_{w+1}(n)\Phi' - \Phi'|T_n)(z)\,dz = \sigma_{w+1}(n)\, ir_0' - \lambda_n ir_0'. \tag{22}$$

Now let us divide (22) by $\sigma_{w+1}(n)$ and take the limit as $n \to \infty$. It is well known that $\lim_{n\to\infty} \sigma_{w+1}(n)^{-1}\lambda_n = 0$ for any Φ. It then follows that

$$\lim_{n\to\infty} \sigma_{w+1}(n)^{-1} \int_0^{i\infty} \Phi'|T_n\,dz = 0$$

for any Φ'. Finally we obtain

$$\int_0^{i\infty} \Phi'\,dz = ir_0',$$

which completes the proof of the lemma.

6.2. **Proof of Theorem 1.2. Conclusion.** Now we can repeat the arguments of §4.1 without alteration: the exceptional form referred to in §4.2 does not exist, since a collection $(\lambda_1, \cdots, \lambda_n, \cdots)$ of eigenvalues of the T_n^+ on $R^{(w+2)/2}$ corresponds by 6.1 to a 1-dimensional invariant subspace of S^+ (provided that $\Phi|T_n = \lambda_n\Phi$ for all n and some $\Phi \in S_{w+2}^0$).

§7. Examples

7.1. We give here a table of the even periods for those values of $w+2$ for which $\dim S_{w+2} = 1$; that is, for $w = 16, 18, 20, 22, 26$. The case $w+2 = 12$ was used as an illustration in the Introduction.

$w+2$	r_0	r_2	r_4	r_6	r_8	r_{10}
16	1	$-\frac{3617}{2^4\cdot 3^2\cdot 5\cdot 7\cdot 13}$	$\frac{3617}{2^3\cdot 3^2\cdot 5\cdot 11\cdot 13}$	$-\frac{3617}{2^4\cdot 3^3\cdot 5\cdot 7\cdot 13}$		
18	1	$-\frac{43867}{2^4\cdot 3^3\cdot 5^4}$	$\frac{43867}{2^6\cdot 3^2\cdot 5^2\cdot 7\cdot 13}$	$-\frac{43867}{2^6\cdot 3^2\cdot 5^3\cdot 7\cdot 11}$		
20	1	$-\frac{283\cdot 617}{2^2\cdot 3^4\cdot 5\cdot 7^2\cdot 17}$	$\frac{283\cdot 617}{2^3\cdot 3^2\cdot 5\cdot 7^2\cdot 17}$	$-\frac{283\cdot 617}{2^4\cdot 3^4\cdot 5\cdot 7^2\cdot 13\cdot 17}$	$\frac{283\cdot 617}{2^3\cdot 3^5\cdot 5\cdot 7^2\cdot 11\cdot 17}$	
22	1	$-\frac{131\cdot 593}{3^2\cdot 5^4\cdot 7\cdot 19}$	$\frac{131\cdot 593}{2^4\cdot 3^2\cdot 5^3\cdot 17\cdot 19}$	$-\frac{131\cdot 593}{2^4\cdot 3^1\cdot 5^4\cdot 7\cdot 19}$	$\frac{131\cdot 593}{2^5\cdot 3^1\cdot 5^3\cdot 7\cdot 13\cdot 19}$	
26	1	$-\frac{43\cdot 657931}{2^3\cdot 3^3\cdot 5^3\cdot 7^2\cdot 13\cdot 23}$	$\frac{97\cdot 657931}{2^5\cdot 3^4\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 23}$	$-\frac{29\cdot 657931}{2^5 3^3\cdot 5^3\cdot 7^2\cdot 13\cdot 19\cdot 23}$	$\frac{657931}{2^4\cdot 3^4\cdot 5^2\cdot 7^2\cdot 13\cdot 17\cdot 23}$	$-\frac{657931}{2^5\cdot 3^4\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 23}$

These values have been obtained by solving the system of the Eichler-Shimura equations for the even periods, to which was added one more equation of the Hecke type (for $n = 2$) to compute r_0.

An interesting feature of these solutions is the fact that from them congruences of the Ramanujan type modulo large primes automatically follow. More precisely, let

$$S^0_{w+2} = R\left(1 + \sum_{n=2}^{\infty}{}' \lambda_n^{(w+2)} e^{2\pi i n z}\right).$$

Then from our table and the Coefficients Theorem (formula (1)) we find immediately that

$$\lambda_n^{(16)} \equiv \sigma_{15}(n) \mod 3617, \quad \lambda_n^{(18)} \equiv \sigma_{17}(n) \mod 43867,$$
$$\lambda_n^{(20)} \equiv \sigma_{19}(n) \mod 283 \cdot 617,$$
$$\lambda_n^{(22)} \equiv \sigma_{21}(n) \mod 131 \cdot 593, \quad \lambda_n^{(26)} \equiv \sigma_{25}(n) \mod 657931.$$

For a systematic discussion of congruences of this type from the point of view of l-adic representations see the articles of Serre [9] and Swinnerton-Dyer [12].

§8. Mazur's p-adic integrals

8.1. In [6] Mazur constructs a p-adic analog of the Hecke series of a parabolic form Φ of weight 2, starting out from the integral representation

$$\frac{\Gamma(s)}{(2\pi)^s} L_\Phi(s) = \int_0^\infty \Phi(iy)\, y^{s-1}\, dy. \tag{23}$$

In this section we describe this construction. Rewrite the integral (23) in the form $\int_{R^*_+} \chi^{(s)}(y)\mu_\Phi^{(\infty)}$, where R^*_+ is the multiplicative group of positive real numbers, $\chi^{(s)}: R^*_+ \to C$, given by $\chi^{(s)}(y) = y^s$, is a quasi-character of R^*_+, and $\mu_\Phi^{(\infty)} = \Phi(iy)y^{-1}\,dy$ is a measure on R^*_+ which is associated with Φ.

We start with the description of the p-adic objects corresponding to R^*_+; after which we will introduce a class of p-adic measures with respect to which we can integrate, define the Mazur integral and prove that the p-adic L-function is analytic. The only novelty of our exposition is the consideration of measures which are not necessarily bounded; this is essential for the applications.

In the following sections we will construct p-adic measures $\mu_{\Phi}^{(p)}$ associated to parabolic forms $\Phi \in S_{w+2}$; we will then compute the corresponding archimedean and nonarchimedean integrals and compare them.

8.2. The analog of R_{+}^{*}. The following notation and conventions will be fixed from now on:

p will be a prime number, $\Delta_0 > 0$ an integer with $(\Delta_0, p) = 1$, and $q = p$ if $p \geq 3$, but $q = 4$ if $p = 2$. Let $\Delta = \Delta_0 q$, and let $Z_\Delta = \varprojlim Z/\Delta p^n$. Z_Δ is a ring with p-adic topology.

The analog of R_{+}^{*} will be *the multiplicative group* Z_Δ^*. Note that

$$Z_\Delta^* \cong \mathrm{Gal}\ Q(1^{p^{-\infty}}, 1^{\Delta_0^{-1}})/Q.$$

There is a canonical group isomorphism $Z_\Delta^* \simeq (Z/(\Delta))^* \times (1 + qZ_p)^*$. The projection onto the first factor is given by $a \mapsto a \bmod \Delta$. The inclusion $(Z/(\Delta))^* \hookrightarrow Z_\Delta^*$ (which of course cannot be extended to an imbedding of rings) defines the subgroups of "Teichmüller representatives". The subgroup $(1 + qZ_p)^* = \Gamma$ (in Iwasawa's notation) is isomorphic to Z_p; $1 + q$ is a canonical generator for it. We will frequently write elements of Γ in the form $(1 + q)^\alpha$, with $\alpha \in Z_p$. If $\alpha = \lim \alpha_n$ $(\alpha_n \in Z)$ in the p-adic topology, then $(1 + q)^\alpha = \lim (1 + q)^{\alpha_n}$ in the topology of the ring Z_Δ.

8.3. *The analog of* $\chi^{(s)}$. Choose a finite extension K of the field Q_p. Let O be ring of integers of K, and $m \subset O$ the maximal ideal. Consider the group of p-adically continuous homomorphisms

$$X = \mathrm{Hom}\,(Z_\Delta^*, O^*).$$

The elements of X will be called p-adic K-characters of Z_Δ^*. It is they that will be the analogs of $\chi^{(s)}$ (rather than the more restricted class of objects introduced below).

Any character $\chi \in X$ can be uniquely represented in the form $\chi = \chi_0 \cdot \chi_1$, where χ_0 is trivial on $(1 + qZ_p)^*$ and χ_1 is trivial on $(Z/(\Delta))^*$. The character χ_0 is the tame component of χ, and χ_1 the wild one. The group of tame characters is isomorphic to $\mathrm{Hom}\,((Z/(\Delta))^*, O^*)$ and is obviously finite.

The group of wild characters can naturally be given the structure of p-adic analytic group, as $(1 + m)^* \subset O^*$. Indeed, let us associate to the wild character χ the element $t = \chi(1 + q) - 1$. Then $t \in m$, since $\lim_{n\to\infty} (1 + q)^{p^n} = 1$ in Z_Δ^*; hence $\lim_{n\to\infty}(1 + t)^{p^n} = 1$ in O^*. Furthermore, χ is uniquely determined by t, since $1 + q$ is a topological generator of $(1 + qZ_p)^*$, and t can be taken to be any element of m.

We have thus introduced the *local coordinate* t in a neighborhood of the identity character of X:

$$m \ni t \longmapsto \chi_{(t)} : \chi_{(t)}\,(\varepsilon\,(1 + q)^\alpha) = (1 + t)^\alpha$$

for all $\alpha \in Z_p$.

It is sometimes convenient to use the *local coordinate* s, analogous to the classical argument s of Dirichlet series:

$$O \ni s \longmapsto \chi^{(s)} : \chi^{(s)}\,(\varepsilon\,(1 + q)^\alpha) = (1 + q)^{\alpha s} = \exp\,(\alpha s \log\,(1 + q)).$$

The character $\chi^{(s)}$ is defined for those s for which the series exp is p-adically convergent. In this domain $t = (1+q)^s - 1$. However, for instance, in the case $(1+t)^{p^n} = 1$ the required value of s clearly does not exist, so that the s-coordinate parametrizes a smaller neighborhood of unity than t (which covers all wild characters).

8.4. *The analog of* $\mu^{(\infty)}$. As in §8.3, let K be a finite extension of Q_p. The symbol $|\ |$ will denote the p-adic valuation of K for which $|p| = p^{-1}$; ord will denote the p-adic denominator, so that $\operatorname{ord}(p) = 1$.

We will call any finitely additive function of the open-and-closed subsets of Z_Δ^* with values in K a *K-measure* μ on Z_Δ^*.

Set $I_{a,m} = a + p^m \Delta Z_\Delta$ for $a \in Z_\Delta^*$ and $m \geq 0$. For any measure μ we have

$$\forall a, m \quad \mu(I_{a,m}) = \sum_{b \equiv a \bmod p^{m+1}\Delta} \mu(I_{b,\, m+1}). \tag{24}$$

Conversely, if we fix a measure on the $I_{a,m}$, subject to the relations (24), then it has a unique extension to a measure μ on the whole of Z_Δ^*, because any open-and-closed set is a disjoint finite union of "intervals" $I_{a,m}$. We will make frequent use of this remark.

Example 1. $\mu(I_{a,m}) = \text{const} \cdot p^{-m}$. This is the invariant measure. Note that as $m \to \infty$, while $I_{a,m}$ is contracting to a point, the measure $\mu(I_{a,m})$ grows infinitely in the p-adic topology of the field K in which μ is taking its values: $|p^{-m}| = p^m \to \infty$.

This is the phenomenon which is vital for our theory of integration. It is clear that the K-measure $\mu(I_{a,m})$ cannot tend to zero as $m \to \infty$ as it does for measures with archimedean values; but it could be bounded. We will see in what follows that the class of functions which are important for our purposes can be integrated with respect to any measure which grows slower than the invariant measure. This remark is essential for the consideration of values of p which are "supersingular" for a given form Φ, as was first shown by Nasybullin.

Example 2. Let $\Delta_0 = 1$, let c be an integer with $c > 0$, and let $0 < a < p^n$, with $(a, p) = 1$. Set

$$\mu_c(a + (p^n)) = \left[\frac{ac}{p^n}\right] - \frac{c-1}{2} \in \frac{1}{2} Z \subset Q_p = K, \tag{25}$$

where $[\ ]$ is the usual integer part of a number, and let us check that conditions (24) are fulfilled. (Mazur introduced this measure and showed that the Kubota-Leopoldt p-adic L-functions are defined by an integration with respect to it. It would be exceedingly interesting to determine whether analogous measures exist for the ζ-functions of any totally real fields, which have recently been constructed by Serre [11].)

We must prove that

$$\mu_c(a + (p^n)) = \sum_{b=0}^{p-1} \mu_c(a + bp^n + (p^{n+1})).$$

It follows from (25) that if $i/c \leq a/p^n < (i+1)/c$, then $\mu_c(a + (p^n)) = i - (c-1)/2$ for $i = 0, \cdots, c-1$. Fix i and a satisfying these conditions, and an integer b with $0 \leq b \leq p-1$.

One can easily see that

$$\frac{k}{c} \leqslant \frac{a+bp^n}{p^{n+1}} < \frac{k+1}{c}, \text{ where } k = \left[\frac{i+bc}{p}\right].$$

Hence the additivity relation is equivalent to the following:

$$i - \frac{c-1}{2} = \sum_{b=0}^{p-1} \left(\left[\frac{i+bc}{p}\right] - \frac{c-1}{2}\right).$$

We will check this by induction on i. For $i = 0$ we get the classical identity

$$\sum_{b=0}^{p-1} \left[\frac{bc}{p}\right] = \frac{(p-1)(c-1)}{2},$$

which is proved by counting the integer points under the diagonal of the rectangle having vertices $(1,1)$, $(p-1,1)$, $(p-1,c)$ and $(1,c)$. The inductive step from i to $i+1$ is based on the remark that on the right-hand side the summands corresponding to b with $i + bc \equiv -1 \bmod p$ increase by 1, and the others do not change.

8.5. *The analog of integration.* Let μ be a K-measure on Z_Δ^*, let R_m be a system of representatives for $(Z_\Delta/(p^m\Delta))^*$ in Z_Δ^*, and let $f: Z_\Delta^* \to K$ be a function. We will denote by

$$S(f; R_m) = \sum_{b \in R_m} f(b)\, \mu(I_{b,m})$$

the corresponding "Riemann sum".

8.6. **Theorem-Definition.** *There exists a unique limit*

$$\lim S(f; R_m) \stackrel{\text{def}}{=} \int_{Z_\Delta^*} f\mu$$

taken over all R_m as $m \to \infty$, provided that the following conditions hold:

a) *The measure μ is of moderate growth; that is, by definition,*

$$\varepsilon_m = \max_b |\mu(I_{b,m})|\, p^{-m} \to 0 \quad as \quad m \to \infty.$$

b) *The function f satisfies the "Lipschitz condition": there exists a constant C such that*

$$b \equiv b' \bmod \Delta p^m \Rightarrow |f(b) - f(b')| < Cp^{-m} \quad as \quad m \to \infty.$$

In the applications, the measure μ will be bounded, and hence of moderate growth. The function f will usually be (piecewise) analytic, or even locally constant.

Proof. Let R_m and R_m' be two different systems of representatives mod Δp^m. Then

$$|S(f; R_m) - S(f; R_m')| \leqslant \max_{b \equiv b'(\Delta p^m)} |f(b) - f(b')| \max_b |\mu(I_{b,m})| \leqslant C\varepsilon_m. \qquad (26)$$

Similarly,

$$|S(f;\, R_{m+1}) - S(f;\, R_m)| \leqslant \max_{b \equiv b'(\Delta p^m)} |f(b) - f(b')| \max_b |\mu(I_{b,m+1})| \leqslant Cp\varepsilon_{m+1} \tag{27}$$

(using (24)). Since $\epsilon_m \to 0$, the set of all sums $S(f, R_m)$ is bounded. It follows from the fact that K is locally compact that it has an accumulation point, and it can then be seen from (26) and (27) that there can only be one limit point for the sums as $m \to \infty$.

Remark. Archimedean Riemann sums have a limit because although the individual summands are small, there are a correspondingly large number of them to compensate. Nonarchimedean Riemann sums have a limit because although the individual terms are large, there are a correspondingly large number of the special form const $\cdot\, p^n$ of them to compensate. If the summands are not too large (moderate growth of μ), and are not too far apart (Lipschitz condition), summing them is "practically the same" as multiplying any one of them by const $\cdot\, p^n$; but $|p^n| \to 0$.

Now let χ be some p-adic K-character of Z_Δ^*, and let μ be a K-measure on Z_Δ^*. We will show that the *p-adic Mellin-Mazur transform*

$$L(\mu, \chi) = \int_{Z_\Delta^*} \mu\chi$$

is an analytic function of χ. The integral exists in any case, if μ is of moderate growth. This is clear from 8.3: $|(1+t)^{ap^n} - 1| < \text{const} \cdot p^{-n}$ as $n \to \infty$, $a \in Z_p$.

8.7. **Theorem.** *Let μ be of moderate growth, and let $\chi = \chi_0\chi_{(t)}$, where χ_0 is the tame and $\chi_{(t)}$ the wild component, as in §8.3. Then there exists a unique power series $g(\chi_0, T) \in K[[T]]$, depending only on χ_0, convergent for all $t \in m$, and such that*

$$L(\mu, \chi_0\chi_{(t)}) = g(\chi_0, t) \quad \textit{for all} \quad t \in m.$$

Proof. For any $m \geq 1$ we define the following system of representatives $R_m \subset Z_\Delta^*$ for the classes mod Δp^m:

$$R_m = \{\varepsilon(1+q)^k \mid \varepsilon \in (Z/(\Delta))^*,\ 0 \leqslant k \leqslant p^m - 1\}.$$

By definition, $L(\mu, \chi_0\chi_{(t)}) = \lim_{m\to\infty} S(R_m)$, where the Riemann sums have the form

$$S(R_m) = S(\chi_0\chi_{(t)}, R_m) = \sum_{k=0}^{p^m-1} a_k^{(m)}(1+t)^k, \tag{28}$$

$$a_k^{(m)} = \sum_{\varepsilon \in (Z/\Delta)^*} \chi_0(\varepsilon)\,\mu(\varepsilon(1+q)^k + (p^m\Delta)). \tag{29}$$

Set

$$S_m(T) = \sum_{k=0}^{p^m-1} a_k^{(m)}(1+T)^k \in K[T] \tag{30}$$

and let us prove the following facts:

a) *In the coefficient-by-coefficient convergence topology, the limit* $\lim_{m\to\infty} S_m(t) \in K[[T]]$ *exists.*

Denote this limit by $g(\chi_0, T)$.

b) *If* $g(\chi_0, T) = \Sigma_0^\infty b_n T^n$, *then* $|b_n| \lesssim \text{const}\cdot p^k$ *for* $p^k \le n \le p^{k+1}-1$, *so that* $g(\chi_0, t)$ *is convergent for all* $t \in m$.

c) *For fixed* $t \in m$

$$\lim_{m\to\infty}\left(S_m(t) - \sum_{n=0}^{p^m-1} b_n t^n\right) = 0,$$

so that $g(\chi_0, t)$ *converges to* $\lim S_m(t) = L(\mu, \chi_0\chi_{(t)})$.

We begin with the following lemma.

Lemma. *For all* $0 \le k \le p^m - 1$,

$$\sum_{i=0}^{p-1} a_{k+ip^m}^{(m+1)} = a_k^{(m)}. \tag{31}$$

Proof of the Lemma. Substituting definition (29) on both sides of (31) and equating the coefficients of $\chi_0(\epsilon)$ on either side, we see that it suffices to check the identity

$$\sum_{i=0}^{p-1} \mu\left(\varepsilon(1+q)^{k+ip^m} + (p^{m+1}\Delta)\right) = \mu\left(\varepsilon(1+q)^k + (p^m\Delta)\right),$$

which follows from the fact that μ is finitely additive.

We now return to the proof of the theorem. By using (30) and the lemma, we find that

$$S_{m+1}(T) = \sum_{k=0}^{p^{m+1}-1} a_k^{(m+1)}(1+T)^k = \sum_{k=0}^{p^m-1}\sum_{i=0}^{p-1} a_{k+ip^m}^{(m+1)}(1+T)^{k+ip^m},$$

$$S_m(T) = \sum_{k=0}^{p^m-1} a_k^{(m)}(1+T)^k = \sum_{k=0}^{p^m-1}\sum_{i=0}^{p-1} a_{k+ip^m}^{(m+1)}(1+T)^k,$$

and hence

$$S_{m+1}(T) - S_m(T) = \sum_{k=0}^{p^m-1}\sum_{i=0}^{p-1} a_{k+ip^m}^{(m+1)}(1+T)^k\left[(1+T)^{ip^m}-1\right] \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} A_n^{(m)}T^n. \tag{32}$$

To estimate the coefficients of the series (32) we use the following estimate for binomial coefficients: $|\binom{r}{s}| \le |r/s|$ (in the p-adic norm). Indeed, this inequality is equivalent to $\operatorname{ord}\binom{r}{s} \ge \operatorname{ord}(r) - \operatorname{ord}(s)$. If the right side is nonpositive, the assertion is trivial. In any case

$$\operatorname{ord}\binom{r}{s} = \sum_{i=1}^{\infty}\left(\left[\frac{r}{p^i}\right] - \left[\frac{s}{p^i}\right] - \left[\frac{r-s}{p^i}\right]\right).$$

But for any $\alpha, \beta \ge 0$ we obviously have $[\alpha+\beta] - [\alpha] - [\beta] \ge 0$, and even $[\alpha+\beta] -$

$[\alpha]-[\beta]\geq 1$ if $\alpha+\beta$ is an integer but α and β are not. This last possibility occurs for those terms of the sum for which $\operatorname{ord}(s)+1\leq i\leq \operatorname{ord}(r)$.

Now turn to (32). Taking account of the fact that $|a^{(m+1)}_{k+ip^m}|\leq \epsilon_{m+1}p^{m+1}$, and using (29) and 8.6 a), we get

$$|A_n^{(m)}|\leqslant \varepsilon_{m+1}p^{m+1}\max_{\substack{s\leqslant n\\ 1\leqslant i\leqslant p-1}}\left|\frac{ip^m}{s}\right|.$$

Suppose $p^{m_0-1}\leq n\leq p^{m_0}-1$. Then the maximum on the right is equal to p^{-m+m_0-1}, and hence

$$|A_n^{(m)}|\leqslant \varepsilon_{m+1}p^{m_0}\quad\text{for all}\quad p^{m_0-1}\leqslant n\leqslant p^{m_0}-1,\quad m\geqslant m_0. \tag{33}^{m_0}$$

We then have

$$b_n=\lim_{m\to\infty}a_n^{(m)}=a_n^{(m_0)}+(a_n^{(m_0+1)}-a_n^{(m_0)})+\ \dots\ =a_n^{(m_0)}+\sum_{M\geqslant m_0}A_n^{(M)},$$

and the series on the right is convergent by (33) and the fact that $\epsilon_{M+1}\to 0$ as $M\to\infty$. Hence the series $\Sigma_0^\infty b_nT^n=\lim S_m(T)$ is defined.

Furthermore, if $p^{m_0-1}\leq n\leq p^{m_0}-1$, then

$$|b_n|\leqslant\max(|a_n^{(m_0)}|,\ |A_n^{(m_0)}|,|A_n^{(m_0+1)}|,\dots)\leqslant p^{m_0}\max(\varepsilon_{m_0},p\varepsilon_{m_0+1},p\varepsilon_{m_0+2},\dots)$$

by (33). Hence $\Sigma_0^\infty b_nt^n$ is convergent for all $|t|<1$.

Finally, by $(33)_k$,

$$\left|\sum_{n=0}^{p^m-1}b_nt^n-S_m(t)\right|=\left|\sum_{n=0}^{p^m-1}(b_n-a_n^{(m)})t^n\right|=\left|\sum_{n=0}^{p^m-1}\Big(\sum_{M\geqslant m}A_n^{(m)}\Big)t^n\right|$$
$$\leqslant\max_{k\leqslant m}(\varepsilon_{m+1}p^k|t|^{p^{k-1}})\to 0\quad\text{as}\quad m\to\infty$$

for $k\leq m$. This completes the proof of Theorem 8.7.

§9. Measures associated to modular forms

9.1. In this section we will fix a form $\Phi\in S^0_{w+2}$ and two nonzero numbers $\omega^+\in Ri$ and $\omega^-\in R$, which will be chosen later. Consider the two functions on the rational numbers $P^\pm\colon Q\to C$:

$$P^+(x)=\frac{i}{\omega^+}\operatorname{Im}\int_x^{i\infty}\Phi(z)\,dz,\quad P^-(x)=\frac{1}{\omega^-}\operatorname{Re}\int_x^{i\infty}\Phi(z)\,dz. \tag{34}$$

We list the properties of these functions which we will need.

9.2. **Proposition.** a) $P^\pm(x+1)=P^\pm(x)$.

b) *P^+ is even, and P^- is odd.*

c) *Suppose that $\Phi|T_n=\lambda_n\Phi$ for all $n\geq 1$, and let the numbers $r_{2i+1}(\Phi)/\omega^+$ and $r_{2i}(\Phi)/\omega^-$ be algebraic integers for all integers $2i$, $2i+1\in[0,w]$ (see Theorem 1.2). Then $P^\pm(x)$ (for all $x\in Q$) takes values which are algebraic integers in some finite extension of Q.*

d) *Let p be a prime and $\Phi|T_p = \lambda_p\Phi$. Then*

$$\lambda_p P^{\pm}(x) = p^w P^{\pm}(px) + \sum_{k=0}^{p-1} P^{\pm}\left(\frac{x+k}{p}\right).$$

Proof. a) follows from the fact that $\Phi(z+1) = \Phi(z)$. b) follows since the coefficients of Φ are real:

$$\overline{\int_x^{i\infty} \Phi(z)\,dz} = \int_x^{i\infty} \Phi(-\bar{z})\,d\bar{z} = -\int_x^{i\infty} \Phi(z)\,dz. \tag{35}$$

To prove c) it will be sufficient to establish that

$$\forall x \in Q, \quad \int_x^{i\infty} \Phi(z)\,dz \in \sum_{l=0}^{w} Zr_l(\Phi), \tag{36}$$

and then to separate out the real and imaginary parts, using (35) and the fact that the odd periods are real, whereas the even ones are pure imaginary. But (36) has been established on the way to the proof of Proposition 3.2.

Finally, the equation d) just describes the action of the Hecke operator T_p (see (11) in §3):

$$\lambda_p \int_x^{i\infty} \Phi(z)\,dz = \int_x^{i\infty} \Phi|T_p\,dz = \int_x^{i\infty} p^{\frac{w}{2}}\left(p^{\frac{w+2}{2}}\Phi(pz) + \sum_{k=0}^{p-1} p^{-\frac{w+2}{2}}\Phi\left(\frac{z+k}{p}\right)\right)dz$$
$$= p^w \int_{px}^{i\infty} \Phi(z)\,dz + \sum_{k=0}^{p-1} \int_{\frac{x+k}{p}}^{i\infty} \Phi(z)\,dz.$$

Separating out real and imaginary parts completes the proof.

9.3. We will now fix Φ, ω^+ and ω^- satisfying 9.2 c). Choose a prime number p and denote by K one of the p-adic completions of the field in which the functions $P^{\pm}$ take their values. We will identify the $P^{\pm}(x)$ with elements of K without especial mention.

The following lemma was pointed out to me by Nasybullin. It is an improvement and a generalization of the original construction of Mazur and the author. In the notation of §8, let Q_Δ be the set of rational numbers whose denominators divide Δp^n for all $n \geq 1$.

9.4. Lemma. *Let $R: Q_\Delta \to K$ be a function with the following properties: for some A and $B \in K$ and all $x \in Q_\Delta$*

$$R(x+1) = R(x) \quad \textit{and} \quad \sum_{k=0}^{p-1} R\left(\frac{x+k}{p}\right) = AR(x) + BR(px). \tag{37}$$

Furthermore, let ρ denote any root of the equation $\rho^2 = A\rho + Bp$, with $\rho \neq 0$. Then *there exists a $K(\rho)$-measure μ on Z_Δ^* such that for all $m \geq 0$ and all $a \in Z$*

$$\mu(a+(p^m\Delta)) = \rho^{-m} R\left(\frac{a}{p^m\Delta}\right) + B\rho^{-(m+1)} R\left(\frac{a}{p^{m-1}\Delta}\right). \tag{38}$$

Proof. We must check formula (24). We have

$$\sum_{k=0}^{p-1} \mu(a+p^m\Delta k + (p^{m+1}\Delta)) = \sum_{k=0}^{p-1} \rho^{-(m+1)} R\left(\frac{a+p^m\Delta k}{p^{m+1}\Delta}\right) + \sum_{k=0}^{p-1} B\rho^{-(m+2)} R\left(\frac{a+p^m\Delta k}{p^m\Delta}\right).$$

We transform the first term by means of (37), and the second using the periodicity of R. This gives

$$\rho^{-(m+1)}\left(AR\left(\frac{a}{p^m\Delta}\right)+BR\left(\frac{a}{p^{m-1}\Delta}\right)\right) + B\rho^{-(m+2)} pR\left(\frac{a}{p^m\Delta}\right)$$
$$= \rho^{-(m+2)}(A\rho + Bp) R\left(\frac{a}{p^m\Delta}\right) + B\rho^{-(m+1)} R\left(\frac{a}{p^{m-1}\Delta}\right)$$
$$= \rho^{-m} R\left(\frac{a}{p^m\Delta}\right) + B\rho^{-(m+1)} R\left(\frac{a}{p^{m-1}\Delta}\right) = \mu(a+(p^m\Delta)),$$

and this proves that μ is finitely additive.

9.5. We now apply Lemma 9.4 to the functions $P^{\pm}(x)$ to obtain the basic result of this section:

Starting from given Φ, ω^+, ω^- and K as above, we can construct $K(\rho)$-measures $\mu_\Phi^\pm$ on Z_Δ^* such that

$$\mu_\Phi^\pm(a+(p^m\Delta)) = \rho^{-m} P^\pm\left(\frac{a}{p^m\Delta}\right) - p^w \rho^{-(m+1)} P^\pm\left(\frac{a}{p^{m-1}\Delta}\right),$$
$$\rho^2 - \lambda_p\rho + p^{w+1} = 0, \quad \Phi|T_p = \lambda_p\Phi. \tag{39}$$

The measure (39) *is clearly of moderate growth if* ord $\rho < 1$.

It would be interesting to widen the type of conditions under which (39) turns out to be of moderate growth. Here we only use the integrality of the $P^{\pm}(x)$, but it could be that the $P^{\pm}(a/p^m\Delta)$ are divisible by a power of p that grows with m, and that would compensate for the ρ^{-m}.

§10. p-adic Hecke functions

10.1. In this section we prove a stronger version of Theorem 1.5.

We keep all the notation of §§8 and 9, partially summarized in §1.4. In particular, χ is a primitive Dirichlet character modulo Δp^m, with $m \geq 0$; K is the p-adic field obtained by completing the field of values of $P^{\pm}(x)$ and of χ. The $K(\rho)$-measures $\mu_\Phi^\pm$ are constructed for a form $\Phi \in S_{w+2}^0$ which is an eigenfunction for all the T_n, $\Phi|T_p = \lambda_p\Phi$, and ρ is a root of $\rho^2 - \lambda_p\rho + p^{w+1} = 0$.

Under these conditions the following holds:

10.2. Theorem. *If the measure* $\mu_\Phi^\pm$ *is of moderate growth, then*

$$\frac{\Delta p^m}{\omega^\pm G(\chi)}\int_0^{i\infty}\Phi_\chi(z)\,dz=\rho^m\int_{Z_\Delta^*}\mu_\Phi^\pm\chi^*, \tag{40}$$

where the left-hand side of (40) *is an algebraic integer, considered as belonging to* K; *on the right of* (40) $\chi^*(x)=\chi^{-1}(-x)$; *the sign* $\pm$ *(on both sides) is to be taken according as* $\chi(-1)=\pm 1$.

To deduce Theorem 1.5 from (40), we must note that

$$\int_0^{i\infty}\Phi_\chi(z)\,dz=-\frac{1}{2\pi i}L_\Phi(1,\chi),$$

put χ in the form $\chi_0\chi_1$, where χ_1 is the wild component of χ, write the p-adic character χ_1 in the form $\chi_{(t)}$, where $t=\chi_1(1+q)-1$ according to §8.3, and finally apply Theorem 8.7 to the right-hand side.

Proof of Theorem 10.2. To compute the left-hand side of (40), we make use of the standard formula

$$\Phi_\chi(z)=\frac{G(\chi)}{\Delta p^m}\sum_{b \bmod \Delta p^m}\chi^*(b)\Phi\left(z+\frac{b}{\Delta p^m}\right)$$

(see for example [3], Lemma 9.4). This gives

$$\frac{\Delta p^m}{G(\chi)}\int_0^{i\infty}\Phi_\chi(z)\,dz=\sum_{b \bmod \Delta p^m}\chi^*(b)\int_{b/\Delta p^m}^{i\infty}\Phi(z)\,dz$$

$$=\frac{1}{2}\sum_{b \bmod \Delta p^m}\left(\chi^*(b)\int_{b/\Delta p^m}^{i\infty}+\chi^*(-b)\int_{-b/\Delta p^m}^{i\infty}\right)\Phi(z)\,dz.$$

The formulas (34) and (35) show at once that

$$\frac{\Delta p^m}{G(\chi)\,\omega^\pm}\int_0^{i\infty}\Phi_\chi(z)\,dz=\sum_{b \bmod \Delta p^m}\chi^*(b)P^\pm\left(\frac{b}{\Delta p^m}\right). \tag{41}$$

To compute the right-hand side of (40) we note that χ^* is the constant function on the "intervals" $b+(\Delta p^m)$, so that $\int_{Z_\Delta^*}$ is just the same as the corresponding Riemann sum:

$$\rho^m\int_{Z_\Delta^*}\mu_\Phi^\pm\chi^*=\rho^m\sum_{b \bmod \Delta p^m}\chi^*(b)\,\mu_\Phi^\pm(b+(\Delta p^m)) \tag{42}$$

$$=\sum_{b \bmod \Delta p^m}\chi^*(b)P^\pm\left(\frac{b}{\Delta p^m}\right)-p^w\rho^{-1}\sum_{b \bmod \Delta p^m}\chi^*(b)P^\pm\left(\frac{b}{\Delta p^{m-1}}\right).$$

Thus to show that (41) and (42) coincide we must check that

$$\forall\, m \geqslant 0 \qquad \sum_{b \bmod \Delta p^m} \chi^*(b)\, P^{\pm}\left(\frac{b}{\Delta p^{m-1}}\right) = 0.$$

For this we break the sum up into partial summations, indexed by the classes $b \bmod p^{m-1}$ (this is even possible when $m = 0$!), and note that $P^{\pm}(b/\Delta p^{m-1})$ depends only on such a class, and that

$$\sum_{b' \equiv b \bmod \Delta p^{m-1}} \chi^*(b') = 0,$$

since χ^* is a primitive character. (This is where the primitivity of χ is important.)

Received 23/JUNE/73

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Translated by M. REID