Computing L-values and Petersson products via algebraic and p-adic modular forms

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Explicit Theory of Automorphic Forms
Department of Mathematics Tongji University

March 24-28, 2014, Shanghai China

Summary of recent results, trends and possible future activities

In November 2013 Haigang Zhou invited me to this conference.
I would like to report on some recent results (2011-13) and have speculations or ideas of future directions of research, trends and possible future activities in this area.
Many thanks to the organizers for this occasion both to discuss some explicit methods in the theory of Siegel modular forms, and to point to some directions to take research from here.

The use of the Maass-Shimura differential operator and the Gauss-Manin connection for computing with nearly-holomorphic Siegel modular forms and proving congruences between them will be discussed.
Constructing p-adic measures via congruences between algebraic modular forms using the canonical projection method is indicated.
Doubling method and Rankin-Selberg method ans their algebraic and p-adic versions are discussed.
New examples of complex and p-adic families of Siegel modular forms will be mentioned.

Using algebraic and p-adic modular forms in computations

There are several methods to compute various L-values attachted to Siegel modular forms using Petersson products of nearly-holomorphic vector valued Siegel modular forms :
the Rankin-Selberg method,
the doubling method (pull-back method).

Let $D(s, f, \chi)$ be the standard zeta function of a Siegel cusp eigenform $f \in \mathcal{S}_k^n(\Gamma)$ of genus n (with local factors of degree $2n+1$) and χ be a Dirichlet character.

Theorem (the case of even genus n (Courtieu-Panchishkin), via the Rankin-Selberg method) gives a p-adic interpolation of the normailzed critical values $D^*(s, f, \chi)$ using Andrianov-Kalinin integral representation of these values $1+n-k \le s \le k-n$ through the Petersson product $\langle f, \theta\delta^r E\rangle$ where δ^r is a certain composition of Maass-Shimura differential operators.

Theorem (the general case (by Boecherer-Schmidt), via the doubling method) uses Boecherer–Garrett–Shimura identity (a pull-back formula)

A pull-back formula

allows to compute the critical values through certain double Petersson product by integrating over $z \in \mathbb{H}_n$ the identity:

$$\Lambda(l+2s,\chi)D(l+2s-n,f,\chi)f = \langle f(w), E^{2n}_{l,\nu,\chi,s}(\mathrm{diag}[z,w])\rangle_w.$$

Here $k = l+\nu$, $\nu \geq 0$, $\Lambda(l+2s,\chi)$ is a product of special values of Dirichlet L-functions and Γ-functions, $E^{2n}_{l,\nu,\chi,s}$ a higher twist of a Siegel-Eisenstein series on $(z,w) \in \mathbb{H}_n \times \mathbb{H}_n$ (see [Boe85], [Shi94]). A p-adic construction uses congruences for the L-values, expressed through the Fourier coefficients of the Siegel modular forms and nearly-modular forms.

We indicate a new approach of computing the Petersson products and L-values, using an embedding of algebraic nearly holomorphic modular forms into p-adic modular forms.

A recent discovery by Takashi Ichikawa (Saga University), [Ich12], [Ich13]

allows to embed nearly-holomorphic arithmetical vector valued Siegel modular forms into p-adic modular forms.
Via the Fourier expansions, the image of this embedding is often represented by certain quasimodular holomorphic forms like $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$. with algebraic Fourier expansions and with algebraic normalized CM-values.
This description provides many advantages, both computational and theoretical, in the study of algebraic parts of such Petersson products and L-values, which we would like to develop here.
This work is related to a recent preprint [BoeNa13] by S. Boecherer and Shoyu Nagaoka where it is shown that Siegel modular forms of level $\Gamma_0(p^m)$ are p-adic modular forms. Moreover they show that derivatives of such Siegel modular forms are p-adic. Parts of these results are also valid for vector-valued modular forms. They follow Serre [Se73], with some important modifications in the Siegel modular case.

Arithmetical nearly-holomorphic Siegel modular forms

Nearly-holomorphic Siegel modular forms over a subfield k of $\mathbb{C}$ are certain $\mathbb{C}^d$-valued smooth functions f of $Z = X + \sqrt{-1}Y \in \mathbb{H}_n$ given by the following expression $f(Z) = \sum_T P_T(S)q^T$, where T run through half-integral semi-positive matricies, $S = (4\pi Y)^{-1}$ a symmetric matrix, $q^T = \exp(2\pi\sqrt{-1}\mathrm{tr}(TZ))$, $P_T(S)$ are vectors of degree d whose entries are polynomials over k of the entries of S.
We use the notation

$$q^T = \exp(2\pi i \mathrm{tr}(TZ)) = \prod_{i=1}^{n} q_{ii}^{T_{ii}} \prod_{i<j} q_{ij}^{2T_{ij}}$$
$$\in \mathbb{C}[\![q_{11}, \ldots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$$

are used (with $q_{ij} = \exp(2\pi(\sqrt{-1}Z_{i,j})))$; they form a multiplicative semi-group so that $q^{T_1} \cdot q^{T_2} = q^{T_1+T_2}$, and one may consider f as a formal q-expansion. In this way one can introduce Siegel modular forms over an arbitrary ring A via elements of the semi-group algebra $A[\![q^{B_n}]\!]$.
Namely, $f \in S_e(Sym^2(A^n), A[\![q^{B_n}]\!]^d)$, where S_e denotes the A-polynomial mappings of degree e on symmetric matricies $S \in Sym^2(A^n)$ of order n with vector values in $A[\![q^{B_n}]\!]^d$.

Holomorphic projection of nearly-holomorphic Siegel modular forms

Recall a passage from nearly holomorphic to holomorphic Siegel modular forms preserving the Petersson product with a given $f \in \mathcal{S}^n_k$. For an algebra homomorphism $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$ over k, denote by $\mathcal{N}_\rho(k)$ the k-vector space of all $\mathbb{C}^d$-valued smooth functions which are nearly holomorphic over k with ρ-automorphic condition for $\Gamma(N)$. The elements of $\mathcal{N}_\rho(k)$ are nearly holomorphic Siegel modular forms over k of weight ρ, degree n, and level N.

Let $\rho = \det^{\otimes k} \otimes \rho_0$. By a structure theorem of Shimura (Prop. 14.2 at p.109 of [Sh00]), provided that k is large enough, for $h \in \mathcal{N}_\rho(k)$, $\boxed{h = \mathfrak{A}_{k,\rho_0}(h) + \Delta,}$ where $\mathfrak{A}_{k,\rho_0}(h) \in \mathcal{M}_\rho(k)$ is a holomorphic function and Δ is a finite sum of images of certain holomorphic functions under differential operators of Maass-Shimura type. Analytically $\mathfrak{A}_{k,\rho_0}(h)$ is the "holomorphic projection" of h.

Using Fourier expansions as p-adic modular forms

A method of computing with arithmetical nearly-holomorphic Siegel modular forms is based on the use of Ichikawa's mapping

$\iota_p : \mathcal{N}\rho \to \mathcal{M}p, \rho \overset{F_c}{\hookrightarrow} (\mathcal{R}_{g,p})^d$, where F_c is the Fourier expansion at a cusp c,

$$\mathcal{R}_{n,p} = \mathbb{C}_p[\![q_{11}, \dots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}.$$

Then the poynomial Fourier expansion of a nearly holomorphic form

$$f(Z) = \sum_T a_T(S) q^T \in \mathcal{N}\rho(\overline{\mathbb{Q}}),$$

over $\overline{\mathbb{Q}}$ becomes the Fourier expansion of an algebraic p-adic form over $i_p(\overline{\mathbb{Q}}) \subset \mathbb{C}_p$, whose Fourier coefficients can be obtained using Ichikawa's approach in [Ich13] by putting $S = 0$:

$$f \mapsto F_c(\iota_p(f)) = \sum_T a_T(0) q^T \in F_c(\mathcal{M}p, \rho).$$

Example. $f = \tilde{E}_2 = E_2 - \frac{3}{\pi y} = -12S + 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n$ gives the p-adic modular form $F_c(\iota_p(f)) = E_2 = \tilde{E}_2|_{S=0}$ over $\mathbb{Z}$, which is also a quasimodular form of weight 2.
There are nice relations like $D(E_2) = \frac{1}{12}(E_2^2 - E_4)$ [MaRo5]

Computing the Petersson products

The Petersson product $h(Z) = \sum_T b_T q^T \in \mathcal{M} \subset \mathcal{M}_\rho(\bar{\mathbb{Q}})$ by a given modular form $f(Z) = \sum_T a_T q^T \in \mathcal{M} \subset \mathcal{M}_\rho(\bar{\mathbb{Q}})$ gives a linear form

$$\ell_f : h \mapsto \frac{\langle f, h \rangle}{\langle f, f \rangle}$$

defined over a subring $R \subset \bar{\mathbb{Q}}$. Thus ℓ_f can be expressed through the Fourier coefficients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coefficients.

$$\ell_{T_i} : h \mapsto b_{T_i} \quad (i = 1, \ n).$$

It follows that $\ell_f(h) = \sum_i l_i b_{T_i}$.

Complex and p-adic L-functions

There exist two kinds of L-functions

- Complex-analytic L-functions (Euler products)
- p-adic L-functions (Mellin transforms L_μ of p-adic measures)

Both are used in order to obtain a number (L-value) from an automorphic form. Usually such a number is algebraic (after normalization) via the embeddings

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \quad \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p.$$

How to define and to compute p-adic L-functions?

Using the Mellin transform of a $\mathbb{Z}_p$-valued distribution μ on a p-adic group Y gives an analytic function on the group of p-adic characters

$$x \mapsto L_\mu(x) = \int_Y x(y) d\mu, \quad x \in X_Y = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*).$$

A general idea is to construct p-adic measures directly from Fourier coefficients of modular forms

Example: Mazur's p-adic integral

For any choice of a natural number $c \geq 1$ not divisible by p, there exists a p-adic measure μ_c on $\mathbb{Z}_p^*$, such that the special values

$$\zeta(1-k)(1-p^{k-1}) = \frac{\int_{\mathbb{Z}_p^*} y^{k-1} d\mu_c}{1-c^k} \in \mathbb{Q}, (k \geq 2 \text{ even })$$

produce the Kubota-Leopoldt p-adic zeta-function $\zeta_p : X_p \to \mathbb{C}_p$ (where $X_p = X_{\mathbb{Z}_p^*} = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$) as the p-adic Mellin transform

$$\zeta_p(x) = \frac{\int_{\mathbb{Z}_p^*} x(y) d\mu_c(y)}{1-cx(c)} = \frac{L_{\mu_c}(x)}{1-cx(c)},$$

with a single simple pole at $x = x_p^{-1} \in X_p$, where $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ the Tate field, the completion of an algebraic closure of the p-adic field $\mathbb{Q}_p$, $x \in X_p$ (a $\mathbb{C}_p$-analytic Lie group), $x_p(y) = y \in X_p$, and $x(y) = \chi(y)y^{k-1}$ as above.

Explicitly: Mazur's measure is given by $\mu_c(a + p^v\mathbb{Z}_p)$
$= \frac{1}{c}\left[\frac{ca}{p^v}\right] + \frac{1-c}{2c} = \frac{1}{c}B_1(\{\frac{ca}{p^v}\}) - B_1(\frac{a}{p^v}),\ B_1(x) = x - \frac{1}{2}$,
see [LangMF], Ch.XIII.

How to construct p-adic measures using the Fourier coefficients?

Suppose that we are given some L-function $L^*_f(s,\chi)$ attached to a Siegel modular form f and assume that for infinitely many "critical pairs" (s_j,χ_j) one has an integral representation $\boxed{L^*_f(s_j,\chi_j) = \langle f, h_j\rangle}$ with all $h_j = \sum_T b_{j,T} q^T \in \mathcal{M}$ in a certain finite-dimensional space $\mathcal{M}$ containing f and defined over $\bar{\mathbb{Q}}$.
We want to prove the following Kummer-type congruences:

$$\forall x \in \mathbb{Z}_p^* \sum_j \beta_j \chi_j x^{k_j} \equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j \frac{L^*_f(s_j,\chi_j)}{\langle f, f\rangle} \equiv 0 \mod p^N.$$

for any choice of $\beta_j \in \bar{\mathbb{Q}}$. Here $k_j = s_j - s_0$ or $k_j = -s_j + s_0$, according that there is $s_0 = \min_j s_j$ or $s_0 = \max_j s_j$.
According to N.Katz, then there exisits a bounded p-adic measure μ_f on $\mathbb{Z}_p^*$ such that

$$\int_{\mathbb{Z}_p^*} \chi_j x^{k_j}\, d\mu_f(x) = i_p\Big(\frac{L^*_f(s_j,\chi_j)}{\langle f, f\rangle}\Big).$$

Using the above expression for $\ell_f(h_j) = \sum_j l_{i,j} b_{j,T_i}$, the above congruences reduce to

$$\sum_{i,j} l_{i,j}\beta_j b_{j,T_i} \equiv 0 \mod p^N.$$

Reduction to a finite dimensional case

In order to prove the congruences

$$\sum_{i,j} l_{i,j}\beta_j b_{j,T_i} \equiv 0 \mod p^N.$$

in general we use the functions h_j which belong only to a certain infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{M} = \mathcal{M}(\overline{\mathbb{Q}})$

$$\mathcal{M}(\overline{\mathbb{Q}}) := \bigcup_{m \geq 0} \mathcal{M}_k(Np^m, \overline{\mathbb{Q}}).$$

Starting from the functions h_j, we use their caracteristic projection $\pi = \pi^\alpha$ on the characteristic subspace $\mathcal{M}^\alpha$ (of generalized eigenvectors) associated to a non-zero eigenvalue α Atkin's U-operator on f which turns out to be of fixed finite dimension so that for all j, $\pi^\alpha(h_j) \in \mathcal{M}^\alpha$.

From holomorphic to nearly holomorphic and p-adic modular forms

Next we explain, how to treat the functions h_j which belong to a certain infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{N} \subset \mathcal{N}_\rho(\overline{\mathbb{Q}})$ (of *nearly holomorphic modular forms*).

Usually, h_j can be expressed through the functions $\delta^{k_j}(\varphi_0(\chi_j))$ for a certain non-negative power k_j of the Maass-Shimura-type differential operator applied to a holomorphic form $\varphi_0(\chi_j)$.

Then the idea is to proceed in two steps:

1) to pass from the infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{N} = \mathcal{N}(\overline{\mathbb{Q}})$ of nearly holomorphic modular forms,

$$\mathcal{N}(\overline{\mathbb{Q}}) := \bigcup_{m \geq 0} \mathcal{N}_{k,r}(Np^m, \overline{\mathbb{Q}}) \text{ (of the depth } r).$$

to a fixed finite dimensional characteristic subspace $\mathcal{N}^\alpha \subset \mathcal{N}(Np)$ of U_p in the same way as for the holomorphic forms.

This step respects the Petersson products with a conjugate f^0 of an eigenfunction f_0 of $U(p)$:

$$\langle f^0, h \rangle = \alpha^{-m} \langle f^0, |U(p)^m h \rangle = \langle f^0, \pi^\alpha(h) \rangle.$$

From holomorphic to nearly holomorphic and p-adic modular forms (continued)

2) To apply Ichikawa's mapping $\iota_p : \mathcal{N}(Np) \to \mathcal{M}_p(Np)$ to a certain space $\mathcal{M}_p(Np)$ of p-adic Siegel modular forms. Assume algebraically,

$$h_j = \sum_T b_{j,T}(S) q^T \mapsto \kappa(h_j) = \sum_T b_{j,T}(0) q^T,$$

which is also a certain quasi-modular Siegel form. Under this mapping, computation become much easier, as the action of δ^{k_j} becomes simply a k_j-power of the Ramanujan Θ-operator

$$\Theta : \sum_T b_T q^T \mapsto \sum_T \det(T) b_T q^T,$$

in the scalar-valued case. In the vector-valued case such operators were studied in [BoeNa13].

After this step, proving the Kummer-type congruences reduces to those for the Fourier coefficients the quasimodular forms $\kappa(h_j(\chi_j))$ which can be explicitely evaluated using the Θ-operator.

How to compute with Siegel modular forms?

There are several types of Siegel modular forms (vector-valued, nearly-holomorphic, quasi-modular, p-adic). We consider modular forms defined over $\mathbb{Q}$, over a number field $\begin{matrix} k \subset \bar{\mathbb{Q}} \overset{i_\infty}{\hookrightarrow} \mathbb{C} \\ k \subset \bar{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \mathbb{C}_p \end{matrix}$ or over a ring $\mathcal{R}$, and attached to an algebraic representation $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$, for simplicity, attached to an algebraic representation $\rho_k = \rho_0 \otimes \det^{\otimes k}$ (like in [BoeNa13]).

We may take $\mathcal{R} = \mathbb{C}, \mathbb{C}_p, \Lambda = \mathbb{Z}_p[\![T]\!], \cdots$, and treat these modular forms as certain formal Fourier expansions over $\mathcal{R}$.

Let us fix the congruence subgroup Γ of a nearly holomorphic modular form $f \in \mathcal{N}_\rho$ and its depth r as the maximal S-degree of the poynomial Fourier Fourier coefficients $a_T(S)$ of a nearly holomorphic form

$$f = \sum_T a_T(S) q^T \in \mathcal{N}\rho(R),$$

over R, and denote by $\mathcal{N}_{\rho,r}(\Gamma, R)$ the R-module of all such forms. This module is locally-free of finite rank, that is, over the fraction field $F = Frac(R)$, it becomes a finite-dimensional F-vector space.

Types of modular forms

- $\mathcal{M}_\rho$ (holomorphic vector-valued Siegel modular forms attached to an algebraic representation $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$)
- $\mathcal{N}_\rho$ (nearly holomorphic vector-valued Siegel modular forms attached to ρ over a number field $k \subset \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$))
- $\mathcal{M}^\sharp_\rho$ (quasi-modular vector-valued forms attached to ρ)
- $\mathcal{M}^\flat_\rho$ (algebraic p-adic vector-valued forms attached to ρ over a number field $k \subset \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$)

Definitions and interrelations:

- $\mathcal{M}^\sharp_{\rho,r} = \kappa(\mathcal{N}_\rho) \subset \mathcal{R}^d_{n,\infty}$, where $\kappa : f \mapsto f|_{S=0} = \sum_T P(T,0)q^T$, where $\mathcal{R}_{n,\infty} = \mathbb{C}[\![q_{11}, \dots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$.
- $\mathcal{M}^\flat_{\rho,r}(R,\Gamma) = F_c(\iota_p(\mathcal{N}_{\rho,r}(R,\Gamma))) \subset \mathcal{R}^d_{n,p}$, where $\mathcal{R}_{n,p} = \mathbb{C}_p[\![q_{11}, \dots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$.

Let us fix the level Γ, the depth r, and a subring R of $\bar{\mathbb{Q}}$, then all the R-modules $\mathcal{M}_\rho(R,\Gamma)$, $\mathcal{N}_{\rho,r}(R,\Gamma)$, $\mathcal{M}^\sharp_{\rho,r}(R,\Gamma)$, $\mathcal{M}^\flat_{\rho,r}(R,\Gamma)$ are then locally free of finite rank.
In interesting cases, there is an inclusion $\mathcal{M}^\sharp_{\rho,r}(R,\Gamma) \hookrightarrow \mathcal{M}^\flat_{\rho,r}(R,\Gamma)$.
If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $k = 2$, $P = E_2$ is a p-adic modular form, see [Se73], p.211.

Review of the algebraic theory

Following [Ha81], consider the columns $Z_1, Z_2, \ldots, Z_n$ of Z and the $\mathbb{Z}$-lattice L_Z in $\mathbb{C}^n$ generated by $\{E_1, \ldots, E_n, Z_1, \ldots, Z_n\}$, where $E_1, \ldots, E_n$ are the columns of the identity matrix E. The torus $\mathcal{A}_Z = \mathbb{C}^n / L_Z$ is an abelian variety, and there is an analytic family $\mathcal{A} \longrightarrow \mathbb{H}_n$ whose fiber over the point Z is $\mathcal{A}_Z$.

Let us consider the quotient space $\mathbb{H}_n / \Gamma(N)$ of the Siegel upper half space $\mathbb{H}_n$ of degree n by the integral symplectic group

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \;\middle|\; \begin{matrix} A_\gamma \equiv D_\gamma \equiv 1_n \\ B_\gamma \equiv C_\gamma \equiv 0_n \end{matrix} \right\}$$

If $N > 3$, $\Gamma(N)$ acts without fixed points on $\mathcal{A} = \mathcal{A}_n$ and the quotient is a smooth algebraic family $\mathcal{A}_{n,N}$ of abelian varieties with level N structure over the quasi-projective variety $\mathcal{H}_{n,N}(\mathbb{C}) = \mathbb{H}_n / \Gamma(N)$ defined over $\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N-th root of 1.

For positive integers n and N, $\mathcal{H}_{n,N}$ is the moduli space classifying principally polarized abelian schemes of relative dimension n with symplectic level N structure.

De Rham and Hodge vector bundles

The fiber varieties $\mathcal{A}$ and $\mathcal{A}_{n,N}$ give rise to a series of vector bundles over $\mathbb{H}_n$ and $\mathcal{H}_{n,N}(\mathbb{C})$.

Notations

- $\mathcal{H}^1_{DR}(\mathcal{A}/\mathbb{H}_n)$ and $\mathcal{H}^1_{DR}(\mathcal{A}_{n,N}/\mathcal{H}_{n,N})$
  the relative algebraic De Rham cohomology bundles of dimension $2n$ over $\mathbb{H}_n$ and $\mathcal{H}_{n,N}$) respectively. Their fibers at $Z \in \mathbb{H}_n$ are $H^1 := \mathrm{Hom}_{\mathbb{C}}(L_Z \otimes \mathbb{C}, \mathbb{C})$ generated by α_j, β_j:

  $$\alpha_i(\sum_j a_j E_j + b_j Z_j) = a_i, \;\; \beta_i(\sum_j a_j E_j + b_j Z_j) = b_i \;\; (i = 1, \cdots, n).$$

- $\mathcal{H}^1_\infty$ the C^∞ vector bundle associated to $\mathcal{H}^1_{DR}$ (over $\mathbb{H}_n$ and $\mathcal{H}_{n,N}$). It splits as a direct sum $\mathcal{H}^1_\infty = \mathcal{H}^{1,0}_\infty \otimes \mathcal{H}^{0,1}_\infty$ and induces the Hodge decomposition on the De Rham cohomology of each fiber.
- The summand $\omega = \mathcal{H}^{1,0}_\infty$ is the bundle of relative 1-forms for either $\mathcal{A}/\mathbb{H}_n$ or $\mathcal{A}_{n,N}/\mathcal{H}_{n,N}$. Let us denote by $\pi : \mathcal{A}_{n,N} \to \mathcal{H}_{n,N}$ the universal abelian scheme with 0-section s, and by the Hodge bundle of rank n defined as

  $$\mathbb{E} = \pi_*(\Omega^1_{\mathcal{A}_{n,N}/\mathcal{H}_{n,N}}) = s^*(\Omega^1_{\mathcal{A}_{n,N}/\mathcal{H}_{n,N}})$$

- The bundle of holomorphic 1-forms on the base $\mathbb{H}_n$ or on $\mathcal{H}_{n,N}$, is denoted Ω.

Algebraic Siegel modular forms

are defined as global sections of $\mathbb{E}_\rho$, the locally free sheaf on $\mathcal{H}_{n,N} \otimes R$ obtained from twisting the Hodge bundle $\mathbb{E}$ by ρ.

Definition. Let R be a $\mathbb{Z}[1/N, \zeta_N]$-algebra. For an algebra homomorphism $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$ over R, define algebraic Siegel modular forms over R as elements of $\mathcal{M}_\rho(R) = H_0(\mathcal{H}_{n,N} \otimes R, \mathbb{E}_\rho)$, called of weight ρ, degree n, level N.

If $\rho = \det^{\otimes k} : \mathrm{GL}_n \to \mathbb{G}_m$, then elements of $\mathcal{M}_k(R) = \mathcal{M}_{\det^{\otimes k}}(R)$ are called of weight k. For $R = \mathbb{C}$, each $Z \in \mathbb{H}_n$, let $\mathcal{A}_Z = \mathbb{C}^n/(\mathbb{Z}^n + \mathbb{Z}^n \cdot Z)$ be the corresponding abelian variety over $\mathbb{C}$, and $(u_1, ..., u_n)$ be the natural coordinates on the universal cover $\mathbb{C}^n$ of $\mathcal{A}_Z$. Then $\mathbb{E}$ is trivialized over $\mathbb{H}_n$ by $du_1, ..., du_n$, and $f \in \mathcal{M}\rho(\mathbb{C})$ is a complex analytic section of $\mathbb{E}_\rho$ on $\mathcal{H}_{n,N}(\mathbb{C}) = \mathbb{H}_n/\Gamma(N)$. Hence f is a $\mathbb{C}^d$-valued holomorphic function on $\mathbb{H}_n$ satisfying the ρ-automorphic condition:

$$f(Z) = \rho(C_\gamma Z + D_\gamma)^{-1} \cdot f(\gamma(Z)) \left(Z \in \mathbb{H}_n, \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \right),$$

because $\mathcal{A}_Z \overset{\sim}{\to} \mathcal{A}_{\gamma(Z)}; {}^t(u_1, ..., u_n) \mapsto (CZ + D)^{-1} \cdot {}^t(u_1, ..., u_n)$, and γ acts equivariantly on the trivialization of $\mathbb{E}$ over $\mathbb{H}_n$ as the left multiplication by $(CZ + D)^{-1}$.

Algebraic Fourier expansion

can be defined algebraically using an algebraic test object over the ring $\mathcal{R}_n = \mathbb{Z}[\![q_{11}, \ldots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]]_{i,j=1,\cdots,n}$, where $q_{i,j}(1 \leq i, j \leq n)$ are variables with symetry $q_{i,j} = q_{j,i}$.
Mumford constructs in [Mu72] an object represented over $\mathcal{R}_n$ as

$$(\mathbb{G}_m)^n / \langle (q_{i,j})_{1,\cdots,n} \big| 1 \leq i \leq n \rangle, (\mathbb{G}_m)^n = \mathrm{Spec}(\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]).$$

For the level N, at each 0-dimensional cusp c on $\mathcal{H}^*_{n,N}$, this construction gives an abelian variety over

$$\mathcal{R}_{n,N} = \mathbb{Z}[1/N, \zeta_N][\![q_{11}^{1/N}, \ldots, q_{nn}^{1/N}]\!][q_{ij}^{\pm 1/N}]_{i,j=1,\cdots,n}$$

with a symplectic level N structure, and $\omega_i = dx_i/x_i$ $(1 \leq i \leq n)$ form a basis of regular 1-forms.
We may view algebraically Siegel modular forms as certain sections of vector bundles over $\mathcal{H}_{n,N}$. Using the morphism $\mathrm{Spec}(\mathcal{R}_{n,N}) \to \mathcal{H}_{n,N}$, $\mathbb{E}$ becomes $(\mathcal{R}_{n,N} \otimes R)^n$ in the basis $\omega_1, \ldots, \omega_n$.

Fourier expansion map and q-expansion principle

For an algebraic representation $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$, $\mathbb{E}_\rho$ becomes in the above basis ω_i

$$\mathbb{E}_\rho \times_{\mathcal{H}_{n,N}\otimes R} \mathrm{Spec}(\mathcal{R}_{n,N} \otimes R) = (\mathcal{R}_{n,N} \otimes R)^d.$$

For an R-module M, the space of Siegel modular forms with coefficients in M of weight ρ is defined as $\mathcal{M}_\rho(M) = H^0(\mathcal{H}_{n,N} \otimes R, \mathbb{E}_\rho \otimes_R M)$. Then the evaluation on Mumford's abelian scheme gives a homomorphism

$$F_c : \mathcal{M}_\rho(M) \to (\mathcal{R}_{n,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} M)^d$$

which is called the Fourier expansion map associated with c. According to [Ich13], Theorem 2, F_c satisfies the following

q-expansion principle:

If M' is a sub R-module of M and $f \in \mathcal{M}_\rho(M)$ satisfies that $F_c(f) \in (\mathcal{R}_{n,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} M')^d$, then $f \in \mathcal{M}_\rho(M')$.

Differential operators on modular forms, [Sh00],[Ich13],

Let $S_e(\mathrm{Sym}^2(R^n), R^d)$ be the R-module of all polynomial maps of $\mathrm{Sym}^2(R^n)$ into R^d homogeneous of degree e. For a $\mathbb{C}^d$-valued smooth function f of $Z = (z_{ij})_{i,j} = X + \sqrt{-1}Y \in \mathbb{H}_n$, consider $S_1(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}^d)$-valued smooth functions $(Df)(u)$, $(Cf)(u)$ $(u = (u_{ij})_{i,j} \in \mathrm{Sym}^2(\mathbb{C}^n))$ of $\mathbb{Z} \in \mathbb{H}_n$

$$(Df)(u) = \sum_{1 \le i \le j \le n} u_{ij} \frac{\partial f}{\partial(2\pi\sqrt{-1}z_{ij})}, \quad (Cf)(u) = (Df)((Z - \overline{Z})u(Z - \overline{Z})),$$

Let $\rho \otimes \tau^e : \mathrm{GL}_n(R) = \mathrm{GL}(R^n) \to \mathrm{GL}(S_e(\mathrm{Sym}^2(R^n), R^d))$ be the following R-homomorphism

$$[(\rho \otimes \tau^e)(\alpha)(h)](u) := \rho(\alpha)h({}^t\alpha \cdot u \cdot \alpha),$$

for $\alpha \in \mathrm{GL}_n(R), h \in S_e(\mathrm{Sym}^2(R^n), R^d)$, $u \in \mathrm{Sym}^2(R^n)$.
Then define $S_e(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}^n)$-valued analytic functions $C^e(f)$, D^e $C^e(f), D^e_\rho(f)$ of $Z \in \mathbb{H}_n$ inductively, so that

$$D^e_\rho(f) = (\rho \otimes \tau^e)(Z - \overline{Z})^{-1} C^e(\rho(Z - \overline{Z})f).$$

D^e_ρ coincides with $(2\pi\sqrt{-1})^{-e}$ times Shimura's differential operator; it acts on arithmetical nearly-holomorphic Siegel modular forms.

Arithmetical nearly-holomorphic Siegel modular forms

Let $f(Z) = \sum_T q(T, S) \cdot q^{T/N} \in \mathcal{N}^r_\rho(k)$ be a nearly holomorphic Siegel modular forms over k, of weight ρ, degree n, level N for is a subfield k of $\mathbb{C}$ containing ζ_N, $q^{T/N} = \exp(2\pi\sqrt{-1}\mathrm{tr}(TZ)/N)$, so that f is a $\mathbb{C}^d$-valued smooth function of $Z = X + \sqrt{-1}Y \in \mathbb{H}_n$, satisfying ρ-automorphic condition for $\Gamma(N)$ for an algebraic homomorphism $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$, namely
$f(\gamma(Z)) = \rho(C_\gamma Z + D_\gamma)f(Z) \left(Z \in \mathbb{H}_n, \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix}\right)$, where $q(T, S) \in \mathbb{C}^d$ are vectors whose entries are polynomials over k of degree r of the entries of the symmetric matrix $S = (4\pi Y)^{-1}$.
According to [Sh00], Chapter III, 12.10, if f satisfies the ρ-automorphic condition for $\Gamma(N)$, then $D^e_\rho(f)(u)$ satisfies the $\rho \otimes \tau^e$- automorphic condition: $D^e_\rho : \mathcal{N}_\rho \to \mathcal{N}_{\rho\otimes\tau^e}$ (defined over $\bar{\mathbb{Q}}$).
If f is arithmetical, $D^e_\rho(f)(u)$ is arithmetica and can be expressed through the Gauss-Manin connection ([Ha81], p.96) $\nabla = 1 \otimes d$, $\nabla(du_i) = \sum_j \beta_j dZ_{ij}$, $\nabla : H^1_{DR}(\mathcal{A}/\mathbb{H}_n) \to H^1_{DR}(\mathcal{A}/\mathbb{H}_n) \otimes \Omega^1(\mathbb{H}_n)$, using $H^1_{DR}(\mathcal{A}/\mathbb{H}_n) = \mathrm{Hom}_{\mathbb{C}}(L_Z \otimes \mathbb{C}, \mathbb{C}) \otimes \mathcal{O}_{\mathbb{H}_n}$. Recall that ∇ computes to which extent the sections du_i fail to have constant periods: $du_i = \alpha_i + \sum_j \beta_j Z_{ij}$. Also, ∇ can be algebraically defined.

Arithmeticity of Shimura's differential operator ([Ich12],[Ich13], [Ha81], §4, [Ka78])

Proposition (see 2.2 of [Ich13]). Let $\pi : \mathcal{A} \to \mathbb{H}_n$ be the analytic family of

$$\mathcal{A}_Z = \mathbb{C}^n/(\mathbb{Z}^n + \mathbb{Z}^n \cdot Z)(Z \in \mathbb{H}_n).$$

Then the normalized Shimura's differential operator D^e_ρ is obtained from the composition

$$\mathbb{E}_\rho \to \mathbb{E}_\rho \otimes (\Omega^1\mathbb{H}_n)^{\otimes e} \to \mathbb{E}_\rho \otimes (\mathrm{Sym}^2(\pi^*(\Omega^1_{\mathcal{A}/\mathbb{H}_n})))^{\otimes e},$$

the first map is given by the Gauss-Manin connection $\nabla : H^1_{DR}(\mathcal{A}/\mathbb{H}_n) \to H^1_{DR}(\mathcal{A}/\mathbb{H}_n) \otimes \Omega^1(\mathbb{H}_n)$ together with the projection onto $\mathbb{E} = H^{1,0}$ in the Hodge decomposition of $H^1_{DR}(\mathcal{A}/\mathbb{H}_n)$, $H^1_{DR}(\mathcal{A}/\mathbb{H}_n) \to \pi^*(\Omega^1_{\mathcal{A}/\mathbb{H}_n})$; the second map is given by the Kodaira-Spencer isomorphism

$$\Omega^1_{\mathbb{H}_n} \overset{\sim}{\to} \mathrm{Sym}^2(\pi^*(\Omega^1_{\mathcal{A}/\mathbb{H}_n})), \quad \frac{dq_{i,j}}{q_{i,j}} \leftrightarrow \omega_i\omega_j = du_i du_j (1 \le i, j \le n)$$

Computing with families of Siegel modular forms

Let $\Lambda = \mathbb{Z}_p[\![T]\!]$ be the Iwasawa algebra, and consider Serre's ring

$$\mathcal{R}_{n,\Lambda} = \Lambda[\![q_{11}, \ldots, q_{nn}]\!][q_{ij}^{\pm 1}]_{i,j=1,\cdots,n}.$$

For any pair (k, χ) as above consider the homomorphisms:

$$\kappa_{k,\chi} : \Lambda \to \mathbb{C}_p, \mathcal{R}^d_{n,\Lambda} \mapsto \mathcal{R}^d_{n,\mathbb{C}_p}, \text{ where } T \mapsto \chi(1+p)(1+p)^k - 1.$$

Definition (families of Siegel modular forms)

Let $\mathbf{f} \in \mathcal{R}^d_{n,\Lambda}$ such that for infinitely many pairs (k, χ) as above,

$$\kappa_{k,\chi}(\mathbf{f}) \in \mathcal{M}_{\rho_k}((i_p(\bar{\mathbb{Q}})) \overset{F_c}{\hookrightarrow} \mathcal{R}^d_{n,\mathbb{C}_p}$$

is the Fourier expansion at c of a Siegel modular form over $\bar{\mathbb{Q}}$.
All such $\mathbf{f}$ generate the Λ-submodule $\mathcal{M}_{\rho_k}(\Lambda) \subset \mathcal{R}^d_{n,\Lambda}$ of Λ-adic Siegel modular forms of weight ρ.
In the same way, the Λ-submodule $\mathcal{M}^\sharp_{\rho_k}(\Lambda) \subset \mathcal{R}_{n,\Lambda}$ of Λ-adic Siegel quasi-modular forms is defined.

Examples of families of Siegel modular forms

can be constructed via differential operators of Maass $\Delta = \det(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial z_{ij}})$, so that $\Delta q^T = \det(T)q^T$. Shimura's operator $\delta_k f(Z) = (-4\pi)^{-n}\det(Z-\bar{Z})^{\frac{1+n}{2}-k}\Delta(\det(Z-\bar{Z})^{k-\frac{1+n}{2}+1}f(Z)$ acts on q^T using $\rho_r : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(\wedge^r\mathbb{C}^n)$ and its adjoint ρ_r^*:

$$\delta_k(q^T) = \sum_{l=0}^{n}(-1)^{n-l}c_{n-l}(k+1-\frac{1+n}{2})\mathrm{tr}({}^t\rho_{n-l}(S)\rho_l^*(T))q^T,$$

where $c_{n-l}(s) = s(s-\frac{1}{2})\cdots(s-\frac{n-l-1}{2})$, $S = (2\pi i(\bar{z}-z))^{-1}$.

- Nearly holomorphic Λ-adic Siegel-Eisenstein series as in [PaSE] can be produced from the pairs $(-s,\chi)$: if s is a nonpositive integer such that $k+2s > n+1$,

$$E_k(Z,s,\chi) = \prod_{i=0}^{-s+1} c_n(k+2s+2i)^{-1}\delta_{k+2s}^{(-s)}(E_{k+2s}(Z,0,\chi)).$$

Examples of families of Siegel modular forms (continued)

- Ichikawa's construction: quasi-holomorphic (and p-adic) Siegel - Eisenstein series obtained in [Ich13] using the embedding ι_p

$$\iota_p(\pi^{ns}E_k(Z,s,\chi)) = \prod_{i=0}^{-s-1} c_n(k+2s+2i)^{-1} \sum_T \det(T)^{-s} b_{k+2s}(T) q^T,$$

  where
  $E_{k+2s}(Z,0,\chi) = \sum_T b_{k+2s}(T)q^T, k+2s > n+1, s \in \mathbb{Z}$.
- Normalized Siegel-Eisenstein series is an explicite family, (reported in Yale 2012, see [Yale], also [Ike01], [PaSE], [Pa91])

$$\mathcal{E}_k^n = E_k^n(z)2^{n/2}\zeta(1-k)\prod_{i=1}^{[n/2]}\zeta(1-2k+2i) = \sum_T a_T(\mathcal{E}_k^n)q^T,$$

  for any non-degenerate matrice T of quadratic character ψ_T:

$$a_T(\mathcal{E}_k^n) = 2^{-\frac{n}{2}} \det T^{k-\frac{n+1}{2}} M_T(k) \times \begin{cases} L(1-k+\frac{n}{2},\psi_T) C_T^{\frac{n}{2}-k+(1/2)}, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases}$$

  (C_T= cond(ψ_T), $M_T(k)$ a finite Euler product over $\ell | \det(2T)$).

Examples of families of Siegel modular forms (continued)

- Ikeda-type families of cusp forms of even genus [PaIsr11] (reported in Luminy, May 2011). Start from a p-adic family

$$\varphi = \{\varphi_{2k}\} : 2k \mapsto \varphi_{2k} = \sum_{n=1}^{\infty} a_n(2k) q^n \in \overline{\mathbb{Q}}[\![q]\!] \subset \mathbb{C}_p[\![q]\!],$$

  where the Fourier coefficients $a_n(2k)$ of the normalized cusp Hecke eigenform φ_{2k} and one of the Satake p-parameters $\alpha(2k) := \alpha_p(2k)$ are given by certain p-adic analytic functions $k \mapsto a_n(2k)$ for $(n, p) = 1$. The Fourier expansions of the modular forms $F = F_{2n}(\varphi_{2k})$ can be explicitly evaluated where $L(F_{2n}(\varphi), St, s) = \zeta(s) \prod_{i=1}^{2n} L(\varphi, s + k + n - i)$. This sequence provide an example of a p-adic family of Siegel modular forms.
- Ikeda-Myawaki-type families of cusp forms of $n = 3$, [PaIsr11] (reported in Luminy, May 2011).
- Families of Klingen-Eisenstein series extended in [JA13] from $n = 2$ to a general case
  (reported in Journées Arithmétiques, Grenoble, July 2013).

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