Problem of Coleman-Mazur on p-adic families of L-functions

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Abstract

For a prime number $p \geq 5$, consider a primitive cusp eigenform $f = f_k$ of weight $k \geq 2$, $f = \sum_{n=1}^{\infty} a_n q^n$, and consider a family of cusp eigenforms $f_{k'}$ of weight $k' \geq 2$,

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$k' \mapsto f_{k'} = \sum_{n=1}^{\infty} a_n(k')q^n$, containing f for $k' = k$, such that the Fourier coefficients $a_n(k')$ are given by certain p-adic analytic functions $k' \mapsto a_n(k')$ for $(n,p) = 1$, and let $\alpha_p(k')$ be a Satake p-parameter of $f_{k'}$.

In "The Eigencurve" (1998), R.Coleman and B.Mazur stated the following problem:

Given a prime p and Coleman's family $\{f_{k'}\}$ of cusp eigenforms of a fixed positive slope $\sigma = \mathrm{ord}_p(\alpha_p(k')) > 0$, to construct a two variable p-adic L-function interpolating on k' the Amice-Vélu p-adic L-functions $L_p(f_{k'})$.

A solution (2003) is described using the Rankin-Selberg method and the theory of p-adic integration with values in a p-adic algebra $\mathcal{A}$.

Our p-adic L-functions are p-adic Mellin transforms of certain $\mathcal{A}$-valued measures. Such measures come from Eisenstein distributions with values in certain Banach $\mathcal{A}$-modules $\mathcal{M}^\dagger = \mathcal{M}^\dagger(N;\mathcal{A})$ of families of overconvergent forms over $\mathcal{A}$.

Another approach, based on overconvergent families of modular symbols, was developed by Glenn Stevens. Applications of these results to the p-adic Birch and Swinnerton-Dyer conjecture were discussed by P.Colmez (Bourbaki talk, June 2003, [Colm03]).

Contents

0 Statement of the problem of Coleman-Mazur

This report is on [PaTV] by A.P., *Two variable p-adic L functions attached to eigenfamilies of positive slope*, Invent. Math. v. 154, N3 (2003), pp. 551 - 615.

The Tate field $\mathbb{C}_p$

Fix a prime p, and let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ be the Tate field (the completion of the field of p-adic numbers)

We fix an embedding $i_p : \overline{\mathbb{Q}} \to \mathbb{C}_p$, and view algebraic numbers as p-adic numbers via i_p.

A primitive cusp eigenform f

$$f = f_k = \sum_{n \geq 1} a_n q^n \in \mathcal{S}_k(\Gamma_0(N), \psi),$$

(where $q = e(z) = \exp(2\pi i z), \mathrm{Im}(z) > 0$)

A primitive cusp eigenform $f = f_k$ of weight $k \geq 2$ for $\Gamma_0(N)$ with a Dirichlet character $\psi \pmod N$.

The special values of the L-function attached to f at $s = 1, \cdots, k-1$:

$L_f(s,\chi) = \sum_{n\geq 1} \chi(n) a_n n^{-s}$,	where $1 - a_p X + \psi(p) p^{k-1} X^2 = (1-\alpha X)(1-\alpha' X)$ is the Hecke polynomial
(χ are Dirichlet characters)	α and α' are called the Satake parameters of f

Periods of f

Following a known theorem of Manin [Ma73], there exist two non-zero complex constants $c^+(f), c^-(f) \in \mathbb{C}^\times$ (the *periods* of f) such that for all $s = 1, \cdots, k-1$ and for all Dirichlet characters χ of fixed parity, $(-1)^{k-s}\chi(-1) = \pm 1$, the normalized special values are *algebraic numbers*:

$$L_f^*(s,\chi) = \frac{(2i\pi)^{-s}\Gamma(s) L_f(s,\chi)}{c^\pm(f)} \in \overline{\mathbb{Q}}. \tag{0.1}$$

A family of slope $\sigma > 0$ of cusp eigenforms $f_{k'}$ of weight $k' \geq 2$ containing f

$$k' \mapsto f_{k'} = \sum_{n=1}^{\infty} a_n(k')q^n \in \overline{\mathbb{Q}}[\![q]\!]$$

1) the Fourier coefficients $a_n(k')$ of $f_{k'}$
and the Satake p-parameter $\alpha_p(k')$ are given by certain p-adic analytic functions $k' \mapsto a_n(k')$ for $(n, p) = 1$
2) the slope is *constant and positive*:
$\mathrm{ord}(\alpha_p(k')) = \sigma > 0$

A model example of a p-adic family (not cusp and $\sigma = 0$): Eisenstein series

$$a_n = \sum_{d|n} d^{k'-1}, f_{k'} = E_{k'}$$

the $f_{k'}$ the Fourier coefficients $a_n(k')$ of $E_{k'}$
and one of the Satake p-parameters $\alpha_p(k') = 1$
$\mathrm{ord}_p(\alpha_p(k')) = \mathrm{ord}_p(1) = 0$

The existence of families of slope $\sigma > 0$: R.Coleman, [CoPB]

He gave an example with $p = 7$, $f = \Delta$, $k = 12$

$a_7 = \tau(7) = -7 \cdot 2392, \sigma = 1$, and

a program in PARI for computing such families is contained in [CST98] (see also the Web-page of W.Stein, `http://modular.fas.harvard.edu/`)

The Problem, ***see [Co-Ma] R. Coleman, B. Mazur, The eigencurve. Galois representations in arithmetic algebraic geometry, (Durham, 1996), London Math. Soc. Lecture Note Ser., 254, at p.6***

Given a p-adic analytic family $k' \mapsto f_{k'} = \sum_{n=1}^{\infty} a_n(k')q^n \in \overline{\mathbb{Q}}[\![q]\!]$ of positive slope $\sigma > 0$, to construct a two-variable p-adic L-function interpolating $L^*_{f_{k'}}(s, \chi)$ on (s, k').

Known cases:

- One-variable case ($k = k'$ is fixed, $\sigma > 0$), treated in [Am-Ve] by Y. Amice, J. Vélu, in [Vi76] by M.M. Višik, and in [MTT] by B. Mazur; J. Tate; J. Teitelbaum
- $\sigma = 0$ (H.Hida) ("ordinary families") (see in [Hi93] H. Hida, Elementary theory of L-functions and Eisenstein series, London Mathematical Society Student Texts. **26**, Cambridge University Press, 1993

- Special values of L-functions attached to families f_k of Yu.I. Manin and M. M.Vishik, [Ma-Vi]
  $$f_k = \sum_{\mathfrak{a}\subset \mathcal{O}_K} \lambda^{k-1}(\mathfrak{a})q^{N\mathfrak{a}}$$
  and of N.M.Katz, [Kat]), which are are certain *ordinary families*

  they correspond to powers of a grössen-character λ of an imaginary quadratic field K at a *splitting prime* p, (resp. to grössencharacters of type A_0 of the idèle class group $\mathbb{A}_K^*/K^*$ (in the sense of Weil [We56],) of a CM-field K.

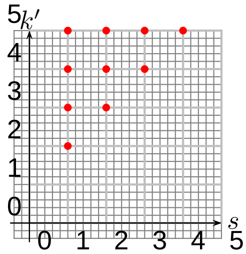
Motivation:

comes from the conjecture of Birch and Swinnerton-Dyer, see in [Colm03] , Colmez, P.: La conjecture de Birch et Swinnerton-Dyer p-adique. Séminaire Bourbaki. [Exposé No.919] (Juin 2003). For a cusp eigenform $f = f_2$, corresponding to an elliptic curve E by Wiles [Wi], we consider a family containing f.

One can try to approach $k = 2, s = 1$ from the other direction, taking $k' \to 2$, instead of $s \to 1$, this leads to a formula linking the derivative over s at $s = 1$ of the p-adic L-function with the derivative over k' at $k' = 2$ of the p-adic analytic function $\alpha_p(k')$, see in [CST98]:

$$\boxed{L'_{p,f}(1) = \mathcal{L}_p(f) L_{p,f}(1)}$$

with $\mathcal{L}_p(f) = -2\dfrac{d\alpha_p(k')}{dk'}\Big|_{k'=2}$



The validity of this formula needs the existence of our two variable L-function!

Our method

is a combination of the Rankin-Selberg method with the theory of p-adic integration with values in p-adic Banach algebras $\mathcal{A}$ and the spectral theory of Atkin's U-operator: $U = U_p : \mathcal{A}[\![q]\!] \to \mathcal{A}[\![q]\!]$ defined by:

$$U\left(\sum_{n\geq 1} a_n q^n\right) = \sum_{n\geq 1} a_{pn} q^n \in \mathcal{A}[\![q]\!].$$

Here $\mathcal{A} = \mathcal{A}(\mathcal{B})$ is a certain p-adic Banach algebra of functions on an open analytic subspace $\mathcal{B}$ on the weight space $X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$. This is an *analytic space over* $\mathbb{C}_p$, which consists of all continuous characters of a certain profinite group Y over $\mathbb{Z}_p^*$.

The classical analogue of the weight space is the complex plane

$$\mathbb{C} = \mathrm{Hom}_{cont}(\mathbb{R}_+^*, \mathbb{C}^*), s \mapsto (y \mapsto y^s).$$

The weights k' correspond to certain points in the neighborhood $\mathcal{B}$ of the given weight k. Any series $f = \sum_{n\geq 1} a_n q^n \in \mathcal{A}[\![q]\!]$ produces a family of q-expansions $\left\{f_{k'} = ev_{k'}(f) = \sum_{n\geq 1} ev_{k'}(a_n) q^n \in \mathbb{C}_p[\![q]\!]\right\}$, which can be classical modular forms in $\overline{\mathbb{Q}}[\![q]\!]$.

- We construct first an analytic function $\mathcal{L}_\mu : X \to \mathcal{A} = \mathcal{A}(\mathcal{B})$ as the Mellin transform

$$\mathcal{L}_\mu(x) = \int_Y x d\mu$$

of a certain measure μ on our profinite group Y with values in $\mathcal{A}$.

- For each $s \in \mathcal{B}$, there is the evaluation homomorphism $ev_s : \mathcal{A}(\mathcal{B}) \to \mathbb{C}_p$; we obtain $\mathcal{L}_\mu(x, s)$ by evaluation of an $\mathcal{A}$-valued integral:

$$\mathcal{L}_\mu(x, s) = \mathcal{L}_\mu(x)(s) = ev_s \left(\int_Y x d\mu \right) \quad (x \in X, \quad \mathcal{L}_\mu(x) \in \mathcal{A}).$$

This gives a p-adic analytic L-function in two variables $(x, s) \in X \times \mathcal{B} \subset X \times X$.

- We check an equality relating the algebraic numbers $L^*_{f_{k'}}(s, \chi)$ $(s = 1, \cdots, k' - 1)$ with the values $\mathcal{L}_\mu(x, k')$ at certain arithmetic characters $x \in X$.

Another approach (Glenn Stevens, unpublished)

uses overconvergent families of modular symbols, see [Ste]. As noted by Stevens, it yields a formula for the derivative at $s = k - 1$ of the p-adic L-function of $f_{k'}$.

1 p-adic integration and the p-adic weight space

Consider $Y = \varprojlim_v Y_v, \quad Y_v = (\mathbb{Z}/Np^v\mathbb{Z})^\times$	(a profinite group, endowed with a projection $y_p : Y \to \mathbb{Z}_p^\times$)
$X = X_N = \mathrm{Hom}_{\mathrm{cont}}(Y, \mathbb{C}_p^\times) \ni \chi, y_p^n$, where	(the p-adic weight space, which is a $\mathbb{C}_p$-analytic group)
$\chi \bmod Np^v\mathbb{Z} : (\mathbb{Z}/Np^v\mathbb{Z})^\times \to \mathbb{C}_p^\times$	(a Dirichlet character)
$y_p : Y \to \mathbb{Z}_p^\times$	(the canonical projection, a p-adic character of Y)

The p-adic weight space $X = X_N = \mathrm{Hom}_{\mathrm{cont}}(Y, \mathbb{C}_p^\times)$ has the following analytic structure over $\mathbb{C}_p$:

$$X \xrightarrow{\sim} \mathrm{Hom}((\mathbb{Z}/Np\mathbb{Z})^\times, \mathbb{C}_p^\times) \times \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{C}_p^\times)$$

where $Y \cong (\mathbb{Z}/Np\mathbb{Z})^\times \times \Gamma$, $\Gamma = (1 + p\mathbb{Z}_p)^\times$, is a procyclic group of generator $\gamma = 1 + p$, and there is a unique decomposition $y_p(y) = \epsilon(y) \cdot \langle y \rangle$ with the Teichmüller character $\epsilon : Y \to \mu_{p-1} \cong (\mathbb{Z}/p\mathbb{Z})^\times \subset \mathbb{Z}_p^\times$ and $\langle y \rangle \in \Gamma$, and we see that X is a finite cover of the p-adic unit disc:

$$X \twoheadrightarrow \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{C}_p^\times) \xrightarrow{\sim} \mathfrak{U} = \{t \in \mathbb{C}_p \mid |t-1|_p < 1\} \cong \{\chi_t : \gamma \mapsto t \mid t \in \mathfrak{U}\}.$$

Notation

$(k,\psi) = y_p^k \psi \in X$	is a point on the weight space X,
	were we view $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}_p^\times$ as a locally constant on Y
$\mathcal{A}$	(a p-adic Banach algebra)
V	(a Banach $\mathcal{A}$-module)
$\mathcal{C}(Y,\mathcal{A})$	(the $\mathcal{A}$-Banach algebra of *continuous functions* on Y)
$\cup$	
$\mathcal{C}^{loc-const}(Y,\mathcal{A})$	(the $\mathcal{A}$-algebra of *locally constant functions* on Y)

DEFINITION 1.0.1 *a) A distribution $\mathcal{D}$ on Y with values in V is an $\mathcal{A}$-linear form*

$$\mathcal{D} : \mathcal{C}^{loc-const}(Y,\mathcal{A}) \to V, \quad \varphi \mapsto \int_Y \varphi \mathcal{D}.$$

b) A measure μ on Y with values in V is a continuous $\mathcal{A}$-linear form

$$\mu : \mathcal{C}(Y,\mathcal{A}) \to V, \quad \varphi \mapsto \int_Y \varphi \mu.$$

Admissible measures of Amice-Vélu

A more delicate notion of an h-admissible measure was introduced in [Am-Ve] by Y. Amice, J. Vélu (see also [MTT], [Vi76]):

Definition 1.0.2

a) For $h \in \mathbb{N}, h \geq 1$ let $\mathcal{P}^h(Y, \mathcal{A})$ denote the $\mathcal{A}$-module of locally polynomial functions of degree $< h$ of the variable $y_p : Y \to \mathbb{Z}_p^\times \hookrightarrow \mathcal{A}^\times$; in particular,

$$\mathcal{P}^1(Y, \mathcal{A}) = \mathcal{C}^{loc-const}(Y, \mathcal{A})$$

(the $\mathcal{A}$-submodule of locally constant functions). Let also denote $\mathcal{C}^{loc-an}(Y, \mathcal{A})$ the $\mathcal{A}$-module of locally analytic functions, so that

$$\mathcal{P}^1(Y, \mathcal{A}) \subset \mathcal{P}^h(Y, \mathcal{A}) \subset \mathcal{C}^{loc-an}(Y, \mathcal{A}) \subset \mathcal{C}(Y, \mathcal{A}).$$

b) Let V be a normed $\mathcal{A}$-module with the norm $|\cdot|_{p,V}$. For a given positive integer h an h-admissible measure on Y with values in V is an $\mathcal{A}$-module homomorphism

$$\tilde{\Phi} : \mathcal{P}^h(Y, \mathcal{A}) \to V$$

such that for fixed $a \in Y$ and for $v \to \infty$ the following growth condition *is satisfied:*

$$\left| \int_{a+(Np^v)} (y_p - a_p)^{h'} d\tilde{\Phi} \right|_{p,V} = o(p^{-v(h'-h)}) \tag{1.1}$$

$$\text{for all} \quad h' = 0, 1, \ldots, h-1, a_p := y_p(a)$$

The condition (1.1) allows one to integrate all locally-analytic functions: there exists a unique extension of $\tilde{\Phi}$ to $\mathcal{C}^{loc-an}(Y, \mathcal{A}) \to V$ (via the embedding $\mathcal{P}^h(Y, \mathcal{A}) \subset \mathcal{C}^{loc-an}(Y, \mathcal{A})$). The integral is defined using generalized Riemann sums : take the beginning of the Taylor expansion of a locally-analytic function $\phi \in \mathcal{C}^{loc-an}(Y, \mathcal{A})$ (of order $h-1$) instead of just values of a function ϕ.

The p-adic Mellin transform and two variable p-adic analytic functions

Any h-admissible measure $\tilde{\mu}$ on Y with values in a p-adic Banach algebra $\mathcal{A}$ can be caracterized by the logarithmic growth $o(\log^h(\cdot))$ of its *Mellin transform* $\mathcal{L}_{\tilde{\mu}}(x)$ (see [Am-Ve], [Vi76], [HaH]):

$$\mathcal{L}_{\tilde{\mu}} : X \to \mathcal{A}, \text{ defined by } \mathcal{L}_{\tilde{\mu}}(x) = \int_Y x(y) d\tilde{\mu}(y),$$

where $x \in X,\ \mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A},\ X \subset \mathcal{C}^{loc-an}(Y, \mathcal{A})^\times$

Key property of h-admissible measures $\tilde{\mu}$: its Mellin transform $\mathcal{L}_{\tilde{\mu}}$ is analytic with values in $\mathcal{A}$.

Moreover, if $\mathcal{A}$ is a p-adic Banach algebra of functions on the weight space X, then for each s with (s, ψ) from a neigbourhood $\mathcal{B}$ of $(k, \psi) \in X$, we obtain by evaluation at (s, ψ) a $\mathbb{C}_p$-linear form

$$ev_s(\mu) = \mu(s) : \mathcal{C}^{loc-an}(Y, \mathbb{C}_p) \to \mathbb{C}_p.$$

This function produces a p-adic analytic function in two variables $(x, s) \in X \times \mathcal{B} \subset X \times X$:

$$\mathcal{L}_\mu(x, s) = \mathcal{L}_\mu(x)(s) = \int_Y x d\mu(s) \quad (x \in X, \;\; \mathcal{L}_\mu(x) \in \mathcal{A}).$$

EXAMPLE 1.0.3 ([AM-VE], [MTT], [VI76]) *For a primitive cusp eigenform* $f = f_k = \sum_{n\geq 1} a_n q^n \in \mathcal{S}_k(\Gamma_0(N), \psi)$, *of weight* $k \geq 2$ *for* $\Gamma_0(N)$ *with a Dirichlet character* ψ *and positive slope* $\sigma = \mathrm{ord}_p(\alpha)$ *put* $h = [\sigma] + 1$ *(where* $\sigma < k-1$, *and* $1 - a_p X + \psi(p)p^{k-1}X^2 = (1-\alpha X)(1-\alpha' X)$ *as above).*

Then there exists an h*-admissible* $\mathbb{C}_p$*-valued measure* $\tilde\mu = \tilde\mu_{\alpha,f}(y)$ *on* Y *such that for all couples* (j, χ) *with* $0 \leq j \leq k-2$, *and for any nontrivial primitive Dirichlet character* $\chi \bmod p^v$ *satisfying* $\chi\xi(-1) = (-1)^{k-1-j}$, *there is the following equality (in* $\mathbb{C}_p$*):*

$$\int_Y \chi(y)\, y_p^j \, d\tilde\mu = i_p \left(\frac{p^{vj} G(\chi)}{\alpha^v} L_f^*(1+j, \bar\chi) \right) \quad \big(= \mathcal{L}_{\tilde\mu}(\chi\, y_p^j)\big), \tag{1.2}$$

where $G(\chi)$ *is the Gauss sum of the character* $\chi \bmod p^v$, *and* $L_f^*(1+j, \bar\chi)$ *is given by a choice of periods (0.1).*

In other words, the complex L*-values (1.2) attached to* f *coincide with the values* $\mathcal{L}_{\tilde\mu}(\chi\, y_p^j)$ *of the* p*-adic Mellin transform of* $\tilde\mu$*.*

2 Coleman's families

The proof of the existence of families of slope $\sigma > 0$ by R.Coleman, [CoPB], uses the following ideas:

Notation

$[K : \mathbb{Q}_p] < \infty$	– a finite extension of $\mathbb{Q}_p$ contaning all the Fourier coefficients $i_p(a_n)$ of f
$\mathcal{A} = \mathcal{A}_K(\mathcal{B})$	– *the K-Banach algebra of rigid-analytic functions in a neighbourhood $\mathcal{B}$ of $(k, \psi) \in X$*
$ev_{k'} : \mathcal{A} \to K$	– the evaluation map defined for all $(k', \psi) \in \mathcal{B}$ (a neigbourhood around $(k, \psi) \in X$).
$\mathcal{M}^\dagger(N; \mathcal{A}) = \bigcup_{v \geq 1} \mathcal{M}^\dagger(Np^v, \psi; \mathcal{A}) \subset \mathcal{A}[\![q]\!]$	– a Banach $\mathcal{A}$-module of overconvergent families of modular forms: it can be generated by some $g = \sum_{n=1}^{\infty} b_n q^n \in \mathcal{A}[\![q]\!]$ such that $ev_{k'}(g) \in K[\![q]\!]$ are classical cusp eigenforms for all k' with (k', ψ) in a neigbourhood $\mathcal{B}$ around $(k, \psi) \in X$.

Coleman proved:

- The operator U acts as a completely continuous operator on each $\mathcal{A}$-submodule $\mathcal{M}^\dagger(Np^v;\mathcal{A}) \subset \mathcal{A}[\![q]\!]$ (i.e. it is a limit of finite-dimensional operators) $\Longrightarrow$ there exists the Fredholm determinant $P_U(T) = \det(Id - T \cdot U) \in \mathcal{A}[\![T]\!]$

- there is a version of the Riesz theory: for any inverse root $\alpha \in \mathcal{A}^*$ of $P_U(T)$ there exists an eigenfunction g, $Ug = \alpha g$. such that $ev_{k'}(g) \in K[\![q]\!]$ are classical cusp eigenforms for all k' such that (k', ψ) is in a neigbourhood $\mathcal{B}$ around $(k, \psi) \in X$ (see in [CoPB])

Definition 2.0.1

a) *A function $g \in \mathcal{M}^\dagger(Np^v;\mathcal{A}) \subset \mathcal{A}[\![q]\!]$ is called Coleman's family if $Ug = \alpha g$, and the functions $ev_{k'}(g) \in K[\![q]\!]$ are cusp eigenforms for all k' such that (k', ψ) is in a neigbourhood $\mathcal{B}$ around (k, ψ) in the p-adic weight space X, and $\mathrm{ord}_p(\alpha(k')) = \sigma > 0$ is constant and positive, where $\alpha(k') = ev_{k'}(\alpha) \in K \cap i_p(\overline{\mathbb{Q}})$*

b) *Let $f_{k'} \in \overline{\mathbb{Q}}[\![q]\!]$ denote the primitive cusp eigenform attached to $ev_{k'}(g) \in K[\![q]\!]$. Then the family $\{f_{k'}\}$ of classical primitive cusp forms is also called Coleman's family.*

REMARK 2.0.2 *Hida's families correspond to $\sigma = 0$, they were constructed in [Hi86] (see also [Hi93]).*

There exist analogues of Hida's families in the Siegel modular case (see [Bue], [Hi04]).

In the ordinary case such p-adic families of Siegel modular forms were studied by K.Buecker (Dissertation of Cambridge University, UK, 1994, under the direction of Prof. R. Taylor, see in [Bue]), and by J.Tilouine and E.Urban [Ti-U]. A more general approach is developed in new Hida's book [Hi04].

3 Main results

MAIN THEOREM 3.0.1 *Consider a nonzero rigid-analytic function* $\alpha = \alpha(s) \in \mathcal{A}^\times$ *defined in an affinoid neigbourhood* $\mathcal{B}$ *around* $(k, \psi) \in X$*, and Coleman's family*

$$f = \left\{ f_{k'} = \sum_{n=1}^{\infty} a_n(k') q^n \right\} \in \mathcal{A}[\![q]\!]$$

(with coefficients in the affinoid algebra $\mathcal{A} = \mathcal{A}(\mathcal{B})$ *of* $\mathcal{B}$*) attached to the family of the eigenvalues* $\alpha(k')$*. Suppose that the slope* $\mathrm{ord}_p(\alpha) = \sigma > 0$ *is fixed for all* $\alpha = \alpha(k')$ *with* (k', ψ) *in* $\mathcal{B}$*, and put* $h = [\sigma] + 1$*.*

Then there exists an h*-admissible* $\mathcal{A}$*-valued measure* $\tilde{\mu} = \mu_{\alpha,f}$ *on* Y *such that for all couples* (j, χ) *with* $0 \le j \le k' - 2$, $k' > 2\sigma + 2$*, any primitive Dirichlet character* $\chi \bmod p^v$ *satisfying* $\chi\xi(-1) = (-1)^{k'-1-j}$*, there is the following equality for the* $\mathcal{A}$*-valued integral at* $s = k'$

$$ev_{k'} \left(\int_Y \chi(y)\, y_p^j \, d\tilde{\mu} \right) = i_p \left(R_{k'} \cdot \frac{p^{vj} G(\chi)}{\alpha_p(k')^v} L^*_{f_{k'}}(1 + j, \bar{\chi}), \right) \tag{3.1}$$

where $G(\chi)$ *is the Gauss sums of* $\chi \bmod p^v$*, and* $R_{k'} \in \mathbb{Q}^\times$ *is an elementary factor coming from an explicit choice of periods* $c^{\pm}(f_{k'})$*.*

Let us fix an auxiliary non-trivial Dirichlet character ξ mod p such that $i_p(\mathbb{Q}(\xi)) \subset K$ and assume the following *non-vanishing condition*:

$$L_{f_{k'}}(k'-1,\xi) \neq 0 \qquad (nv_\xi)$$

for all $k' > 2\sigma + 2$ with (k',ψ) in an affinoid neighbourhood $\mathcal{B}$ around $(k,\psi) \in \mathcal{W}$. Note that for all $k' > 3$ we have $L_{f_{k'}}(k'-1,\xi) \neq 0$ in view of the absolute convergence of the Euler product $L_{f_{k'}}(s,\xi)$ for $\mathrm{Re}(s) > \frac{k'+1}{2}$.

Let us use the following choice of periods:

$$c^{\pm}(f_{k'}) = \frac{(-2i\pi)^{k'-1}\langle f_{k'}, f_{k'}\rangle_{Np}}{\Gamma(k'-1)L_{f_{k'}}(k'-1,\bar{\xi})}, \text{ where } \xi(-1) = \pm(-1)^j. \tag{3.2}$$

It is known from [Ra52] and [Sh77] that the numbers

$$\frac{L_f(1+j,\bar{\chi})L_f(k'-1,\bar{\xi})}{\pi^{k'+r}\langle f_{k'}, f_{k'}\rangle_{Np}} \text{ are algebraic for all } j \in \mathbb{Z} \text{ with } 0 \leq j \leq k'-2,$$

$\chi\xi(-1) = (-1)^{k'-1-j}$ (here $\langle f_{k'}, f_{k'}\rangle_{Np}$ denotes the Petersson scalar product given by

$$\int_{\Gamma_0(Np)\backslash H} \mid f_{k'} \mid^2 y^{k'-2} dx\, dy, \quad H = \{z = x+iy \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}, z = x+iy).$$

A key ingredient in our construction is the use of a linear form

$$\ell_\alpha : \mathcal{M}(Np, \psi, \overline{\mathbb{Q}}) \to \overline{\mathbb{Q}}^\times,$$

such that $\alpha \in \mathbb{Q}^\times$, $\ell_\alpha(U_p h) = \alpha \ell_\alpha(h)$ for all $h \in \mathcal{M}(Np, \psi, \overline{\mathbb{Q}})$, and $1 - a_p X + \psi(p) p^{k-1} X^2 = (1-\alpha X)(1-\alpha' X)$ for a primitive cusp eigenform $f = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_k(\Gamma_0(N), \psi, \overline{\mathbb{Q}})$ of weight $k \geq 2$ for $\Gamma_0(N)$ with a Dirichlet character $\psi \pmod N$. One can define such linear form by $\ell_\alpha : h \longmapsto \dfrac{\langle f^0, h\rangle}{\langle f^0, f_0\rangle}$, where f_0 is an eigenfunction of U_p: $f_0 | U_p = \alpha f_0$,

$$f_0 = \sum_{n\geq 1} a_n q^n - \alpha \sum_{n \geq 1} a_n q^{pn} = \sum_{n\geq 1} a(f_0, n) q^n \in \mathcal{S}_k(\Gamma_0(Np), \psi, \overline{\mathbb{Q}}), \text{ and}$$

$$f^0 = f_0^\rho\Big|_k \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}, \; f_0^\rho = \sum_{n \geq 1} \overline{a}(f_0, n) q^n \in \mathcal{S}_k(\Gamma_0(Np), \overline{\psi}, \overline{\mathbb{Q}})$$

is an eigenfunction of the adjoint operator U_p^*, and $\langle f^0, f_0\rangle / \langle f, f\rangle \in \overline{\mathbb{Q}}^\times$ (see [Go-Ro]).

THEOREM 3.0.2 *Under the assumptions and notations of Theorem 3.0.2 there exists a unique p-adic analytic function on $X \times \mathcal{B}$ (of two variables x, s),*

$$\mathcal{L}_{\alpha,f}(\cdot_1, \cdot_2, \xi, f) : X \times \mathcal{B} \to \mathbb{C}_p \tag{3.3}$$

such that
i) for any fixed $(s, \psi) \in \mathcal{B}$, the function $\mathcal{L}_{\alpha,f}(x, s; \xi, f)$ of the variable x is $\mathbb{C}_p$-analytic and has the logarithmic growth $o(\log^h(x))$,
ii) for each couple (χ, j) with $0 \le j \le k' - 2$, $k' > 2\sigma + 2$ and any primitive Dirichlet character $\chi \bmod p^v \in X^{\mathrm{tors}}$ with values in $K^\times$ satisfying $v \ge 2$, $\chi\xi(-1) = (-1)^{k'-1-j}$, the special value $\mathcal{L}(\chi y_p^j, k'; \xi, f_{k'})$ is given by the image under i_p of the algebraic number

$$R_{k'} \cdot \frac{p^{vj} G(\chi)}{\alpha_p(k')^v} L^*_{f_{k'}}(1 + j, \bar{\chi}),$$

where $G(\chi)$ is the Gauss sums of $\chi \bmod p^v$, and $R_{k'} \in \mathbb{Q}^\times$ is an elementary factor coming from the explicite choice of periods $c^\pm(f_{k'})$ given by (3.2).

The function (3.3) answers the question of Coleman–Mazur. The proof uses the *Mellin transform* $\mathcal{L}_{\tilde{\mu}}(x) = \int_Y x(y) d\tilde{\mu}(y)$, which is an $\mathcal{A}$-valued analytic function on X.

4 Construction of the admissible measure $\tilde{\mu} : \mathcal{P}^h(Y, \mathcal{A}) \to \mathcal{A}$

Recall that by Definition 1.0.2, an *h-admissible measure* on a profinite group Y with values in an $\mathcal{A}$-module V is given as an $\mathcal{A}$-module homomorphism

$$\tilde{\mu} : \mathcal{P}^h(Y, \mathcal{A}) \to V,$$

satisfying a certain growth condition (1.1), where V be a normed $\mathcal{A}$-module with the norm $|\cdot|_{p,V}$, h a given positive integer.

This means that $\tilde{\mu}$ is given by sequence $\{\mu_j\}$ of certain distributions on Y, in such a way that for $j = 0, 1, \cdots, h-1$ and for all compact open subsets $U \subset Y$ one has

$$\int_U y_p^j d\tilde{\mu} = \mu_j(U). \tag{4.1}$$

In terms of $\{\mu_j\}$, the growth condition (1.1) takes the form: for $t = 0, 1, \cdots, h-1$

$$\left|\int_{a+(Np^v)} (y_p - a_p)^t d\tilde{\mu}\right|_p \tag{4.2}$$
$$= \left|\sum_{j=0}^{t} \binom{t}{j} (-a_p)^{t-j} \mu_j(a + (Np^v))\right|_p = o(p^{v(h-t)}) \text{ for } v \to \infty.$$

We construct $\{\mu_j\}$ out of the algebraic special values $L^*_{f_{k'}}(1+j,\chi)$ in such a way that the equality (3.1) of the Main Theorem 3.0.1 is satisfied:

$$ev_{k'}\left(\int_Y \chi(y)\, y_p^j\, d\mu_\alpha(y; f)\right) = i_p\left(R_{k'} \cdot \frac{p^{vj} G(\chi)}{\alpha_p(k')^v} L^*_{f_{k'}}(1+j,\bar{\chi}),\right)$$

We construct the distributions $\mu_j = \mu_{f,\alpha,j}$ out of three more simple objects:

$$\mu_{f,\alpha,j} = \ell_\alpha(\pi_\alpha(\Phi_j)), \quad (j = 0, 1, \cdots, k-2)$$

- Φ_j is a sequence of modular distributions on Y with values in an $\mathcal{A}$-module $\mathcal{M} = \mathcal{M}(N, \psi; \mathcal{A})$ of overconvergent families of modular forms (it has infinite rank):

  $$\mathcal{M}(N, \psi; \mathcal{A}) := \bigcup_{v \geq 0} \mathcal{M}^\dagger(Np^v, \psi; \mathcal{A}), \text{ where } \mathcal{M}^\dagger(Np^v, \psi; \mathcal{A}) = \mathcal{M}_k^\dagger(\Gamma_0(Np^v), \psi; \mathcal{A})$$

  (*the modular forms $\Phi_j(\chi)$ are products of certain classical Eisenstein series in $\mathcal{A}[\![q]\!]$*)

- π_α is the canonical projector over the characteristic $\mathcal{A}$-submodule $\mathcal{M}^\alpha = \mathcal{M}^\alpha(\mathcal{A})$ of Atkin's operator $U\left(\sum_{n\geq 0} b_n q^n\right) = \sum_{n \geq 0} b_{pn} q^n$

  (KEY POINT: THE $\mathcal{A}$-MODULE $\mathcal{M}^\alpha(\mathcal{A})$ IS LOCALLY FREE OF FINITE RANK)

- $\ell_\alpha \in \mathrm{Hom}_\mathcal{A}(\mathcal{M}^\alpha, \mathcal{A})$ is a $\mathcal{A}$-linear form (coming from the *method of Rankin-Selberg*, and interpolating over $\mathcal{M}^\alpha$ the Petersson scalar product with $h \in \mathcal{M}^\alpha$, as in Section 3: $h \longmapsto \frac{\langle f^0, h\rangle}{\langle f^0, f_0\rangle}$, normalized by the equality $\ell_\alpha(g) = 1$ for Coleman's eigenfunction $g = f_0$; $\langle f^0, f_0\rangle$ is studied in [Go-Ro]).

5 Criterion of admissibility

THEOREM 5.0.1 *Let* $0 < |\alpha|_p < 1$ *and* $h = [\mathrm{ord}_p \alpha] + 1$. *Suppose that there exists a positive integer* $\varkappa$ *such that the following conditions are satisfied: for all* $j = 0, 1, \cdots, \varkappa h - 1$ *and* $v \geq 1$,

$$\Phi_j(a + (Np^v)) \in \mathcal{M}(Np^{\varkappa v}) \text{ (the level condition)} \tag{5.1}$$

and the following estimate holds: for all $w \geq \max(\varkappa v, 1)$ *and for all* $t = 0, 1, \cdots, \varkappa h - 1$

$$U^w \sum_{j=0}^{t} \binom{t}{j} (-a_p)^{t-j} \Phi_j(a + (Np^v)) \equiv 0 \bmod p^{-vt} \text{(the divisibility condition)} \tag{5.2}$$

Then the linear form

$$\tilde{\Phi}^\alpha : \mathcal{P}^{h\varkappa}(Y, \overline{\mathbb{Q}}) \to \mathcal{M}^\alpha \subset \mathcal{M} \tag{5.3}$$

given by $\tilde{\Phi}^\alpha(\delta_{a+(Np^v)} y_p^j) := \pi_\alpha(\Phi_j(a + (Np^v)) \left(= \int_{a+(Np^v)} y_p^j d\tilde{\Phi}^\alpha \right)$
(for all $j = 0, 1, \cdots, h\varkappa - 1$*), is an* $h\varkappa$*-admissible measure.*

Proof uses the commutative diagramm:

$$\begin{array}{ccccc} \mathcal{M}^{\dagger}(Np^{v+1},\psi;\mathcal{A}) & \xrightarrow{\pi_{\alpha,v}} & \mathcal{M}^{\dagger\alpha}(Np^{v+1},\psi;\mathcal{A}) & & \\ U^v\big\downarrow & & \big\downarrow\wr\, U^v & & \\ \mathcal{M}^{\dagger}(Np,\psi;\mathcal{A}) & \xrightarrow[\pi_{\alpha,0}]{} & \mathcal{M}^{\dagger\alpha}(Np,\psi;\mathcal{A}) & = \mathcal{M}^{\dagger\alpha}(Np^{v+1},\psi;\mathcal{A}) \end{array} \tag{5.4}$$

The existence of the projectors $\pi_{\alpha,v}$ comes from Coleman's Theorem A.4.3 [CoPB].

On the right: U acts on the locally free $\mathcal{A}$-module $\mathcal{M}^{\alpha}(Np^{v+1},\mathcal{A})$ via the matrix:

$$\begin{pmatrix} \alpha & \cdots & \cdots & * \\ 0 & \alpha & \cdots & * \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & \cdots & \alpha \end{pmatrix} \text{ and } \alpha \in \mathcal{A}^{\times} \implies \text{ this is an isomorphism over } \mathrm{Frac}(\mathcal{A}),$$

and one controls the denominators of the modular forms of all levels v by the relation:

$$\pi_{\alpha,v}(h) = U^{-v}\pi_{\alpha,0}(U^v h) =: \pi_{\alpha}(h) \tag{5.5}$$

The equality (5.5) can be used as the definition of π_{α}. The growth condition (1.1) for $\pi_{\alpha}(\Phi_j)$ is deduced from the congruences (5.2) using the relation (5.5) between modular forms.

6 Modular Eisenstein distributions $\Phi_j : \mathcal{C}^{loc-const}(Y) \to \mathcal{M}(N, \psi; \mathcal{A})$

Let us fix an auxiliary Dirichlet character $\xi \mod p$, $\xi(-1) = \pm 1$, and use the method of Rankin-Selberg for the convolution

$$\begin{aligned} D(s,f,g) =& L_N(2s+2-k-l, \psi\overline{\xi\chi}) \sum_{n=1}^{\infty} a_n b_n n^{-s}, \text{ where} \\ b_n =& \sigma_{l-1,\bar\chi,\bar\xi}(n) = \sum_{d|n, d>0} \bar\chi(d)\bar\xi(n/d) d^{l-1}, \end{aligned} \tag{6.1}$$

are the Fourier coefficients of an Eisenstein series $g = \sum_{n=0}^{\infty} b_n q^n$ of weight l (and of Dirichlet character $\bar\chi\bar\xi$) if $\chi\xi(-1) = (-1)^l$, so that

$$L_g(s) = \sum_{n=1}^{\infty} b_n n^{-s} = L(s-l+1, \bar\chi) L(s, \bar\xi).$$

The Rankin lemma (cf. [Ra52]) expresses $D(s, f, g)$ through the function

$$L_f(s-l+1, \bar\chi) L_f(s, \bar\xi). \tag{6.2}$$

Let us define the modular distributions Φ_j on a profinite group $Y = \varprojlim_v (\mathbb{Z}/Mp^v\mathbb{Z})^\times$ (for some suitable M, divisible by N) in such a way that the modular form $\Phi_j(\chi) \in \mathcal{A}[\![q]\!]$ is a product of two Eisenstein series:
$ev_{k'}(\Phi_j(\chi)) = (-1)^j E_{k'-1-j}(\xi, \chi) E_{1+j}(\psi\overline{\xi\chi}) =: \Phi_{j,k'}(\chi)$.
Explicitely, the Fourier coefficients of Φ_j (for $j = 0, \ldots, k'-2$) are given by

$$\Phi_j(a + Mp^v) = \sum_{b \in Y_{Mp^v}} \psi\bar{\xi}(b) \sum_{n \geq 0}^{\infty} \sum_{n_1+n_2=n} A_j(n_1, ab)_v B_j(n_2, b)_v q^n \in \mathcal{A}[\![q]\!], \text{ where} \quad (6.3)$$

$$A_j(n_1, ab)_v(k') = \sum_{\substack{d_1 | n_1 \\ (n_1/d_1) \equiv ab \bmod Mp^v}} \xi(d_1) \operatorname{sgn}(d_1) d_1^{k'-2-j} \quad (6.4)$$

$$B_j(n_2, b)_v(k') = \sum_{\substack{d_2 | n_2 \\ d_2 \equiv b \bmod Mp^v}} \operatorname{sgn}(d_2)(n_2/d_2)^j \quad \text{for} \quad n_2 > 0.$$

(Note that the last series has constant coefficients). One verifies coefficient-by-coefficient that the distributions Φ_j satisfy the level condition with $\varkappa = 1$, and the divisibility condition (5.1), (5.2):

Main congruence: For all $w \geq w_0(v)$ one has

$$U^w \sum_{j=0}^{t} \binom{t}{j} (-a_p)^{t-j} \Phi_j(a + (Mp^v)) \tag{6.5}$$

$$= \sum_{j=0}^{t} \binom{t}{j} (-a_p)^{t-j} \sum_{n\geq 0}^{\infty} \sum_{n_1+n_2=p^w n} (-1)^j A_j(n_1, ab)_v B_j(n_2, b)_v q^n \stackrel{?}{\equiv} 0 (\mathrm{mod} p^{tv}).$$

Let us fix n_1 et n_2 with $n_1 + n_2 = p^w n$, $d_1|n_1$ and $d_2|n_2$ with $(n_1/d_1) \equiv ab \bmod Mp^v$ et $d_2 \equiv b \bmod Mp^v$, and write only the terms which depend on j:

$$\sum_{j=0}^{t} \binom{t}{j} (-a)^{t-j} (-1)^j d_1^{k'-2-j} \left(\frac{n_2}{d_2}\right)^j = d_1^{k'-2} \left(-a - \left(\frac{n_2}{d_1 d_2}\right)\right)^t \tag{6.6}$$

$$\equiv d_1^{k-2} d_2^{-t} \left(-ad_2 + \left(\frac{n_1}{d_1}\right)\right)^t \equiv 0 \bmod p^{vt}$$

The congruence (6.6) is then satisfied for all $w \geq v(k'-1) > tv$ because $p \nmid d_2$ and

$$d_1^{k'-2-j} d_2^j \left(-\frac{n_2}{d_1 d_2}\right)^j \equiv d_1^{k'-2} \left(\frac{n_1}{d_1}\right)^j \bmod p^{tv}.$$

7 Algebraic $\mathcal{A}$-linear form $\ell_\alpha : \mathcal{M}(N,\psi;\mathcal{A})^\alpha \to \mathcal{A}$

Let us describe a linear form ℓ_α on the locally free module $\mathcal{M}(\mathcal{A};N,\psi)^\alpha = \pi_\alpha(\mathcal{M}(\mathcal{A};N,\psi))$ of finite rank.

Let us use a basis $\{g^i\}$ of $\mathcal{M}(\mathcal{A};N,\psi)^\alpha$ over the field of fractions $\mathrm{Frac}(\mathcal{A})$, such that $g^1 = g$ is fixed Coleman's eigenvector as above, and g^i are eigenfunctions of all Hecke operators T_l, $(l \nmid Np)$.

Define

$$\ell_\alpha(h) = x_1, \text{ where } h = \sum_i x_i g^i, x \in \mathcal{A} \text{ (the first coordinate of } h \in \mathcal{M}(\mathcal{A};N,\psi)^\alpha)$$

An explicit evaluation shows:

$$ev_{k'}(\ell_\alpha(h)) = \ell_{\alpha(k')}(h_{k'}), \text{ where } h_{k'} = ev_{k'}(h) \in \mathcal{M}_{k'}(N,\psi).$$

The R.H.S can be computed for classical modular forms $h_{k'}$ through the (normalized) Petersson scalar product, moreover, $\ell_\alpha(g) = 1$.

8 Proof of Main Theorem 3.0.1

Take the admissible measure $\tilde{\mu}_\alpha := \ell_{\alpha,f}(\tilde{\Phi}^\alpha)$, with $\tilde{\Phi}^\alpha$ constructed by the admissibility criterium of Theorem 5.0.1 out of products of Eisenstein series Φ_j and the linear form $\ell_{\alpha,f}$ (the Petersson product over $\mathcal{A}$). Let us compute the integrals

$$ev_{k'}\left(\int_Y \chi y_p^j \, d\tilde{\mu}_{\alpha,f}\right) = ev_{k'}(\ell_\alpha(\pi_\alpha(\Phi_j(\chi))) = ev_{k'}(\ell_\alpha(U^{-v}\pi_{\alpha,0}U^v\Phi_j(\chi))) \quad (8.1)$$
$$= \ell_{\alpha(k')}(\pi_{\alpha(k')}\Phi_{j,k'}(\chi)) = \alpha(k')^{-v}\frac{\langle f_{k'}, U^v\Phi_{j,k'}(\chi)\rangle}{\langle f_{k'}, f_{k'}\rangle}$$

for primitive Dirichlet characters χ mod p^v, using the relation (5.5): $\pi_\alpha(h) = U^{-v}\pi_{\alpha,0}(U^v h)$, where $\Phi_{j,k'} = ev_{k'}(\Phi_j) = (-1)^j E_{k'-1-j}(\xi,\chi)E_{1+j}(\psi\overline{\xi\chi})$. The value (8.1) can be computed using the Rankin–Selberg convolution:

$$L_{f_{k'}}(s-l+1,\bar{\chi})L_{f_{k'}}(s,\bar{\xi}) = L_N(2s+2-k'-l,\psi\overline{\xi\chi})\sum_{n=1}^{\infty} a_n(k')b_n n^{-s}, \quad (8.2)$$

where $b_n = \sigma_{l-1,\bar{\chi},\bar{\xi}}(n) = \sum_{d|n,d>0}\bar{\chi}(d)\bar{\xi}(n/d)d^{l-1}$, are the Fourier coefficients of an Eisenstein series $g = \sum_{n=0}^{\infty} b_n q^n$ of weight l with character $\bar{\chi}\bar{\xi}$) (if $\chi\xi(-1) = (-1)^l$).

Put $s = k' - 1$, $l = k' - 1 - j$, $j = 0, \cdots k' - 2$ with $k' > 2 + j$, into (8.2):

$$L_{f_{k'}}(1+j,\bar{\chi})L_{f_{k'}}(k'-1,\bar{\xi}) = L_N(1+j,\psi\overline{\xi\chi})\sum_{n=1}^{\infty} a_n(k')b_n n^{-k'+1}.$$

Using this equality, the R.H.S. of (8.1): can be computed using the Rankin–Selberg integral in the form:

$$ev_{k'}(\ell_\alpha(\pi_\alpha(\Phi_j(\chi))) = t_{k'} \cdot \frac{p^{\nu j}G(\chi)}{\alpha(k')^\nu} L^*_{f_{k'}}(1+j,\bar{\chi}),\ c^\pm(f_{k'}) = \frac{(-2i\pi)^{k'-1}\langle f_{k'}, f_{k'}\rangle}{\Gamma(k-1)L_{f_{k'}}(k-1,\overline{\xi})},$$

where $G(\chi)$ is the Gauss sum of the character $\chi \bmod p^\nu$ and $t_{k'} \in \mathbb{Q}^\times$ is an explicit elementary constant. Then one applies directly theorem 5.0.1 (the admissibility criterion) with $\varkappa = 1$, and the congruences (6.5) in order to obtain the required h-admissibles measures $\tilde{\mu} = \mu_{f,\alpha}$ in the form $\mu_{f,\alpha} = \ell_{f,\alpha}(\tilde{\Phi}^\alpha)$ (given by the sequence of the distributions $\Phi_j^\alpha = \pi_\alpha(\Phi_j)$).

Note that this method gives also an alternative proof of the result of Yu.I.Manin on the algebraicity (0.1).

After having $\tilde{\Phi}^\alpha$, we construct the required ($\mathcal{A}$-valued) h-admissibles measures $\tilde{\mu} = \tilde{\mu}_{f,\alpha}$ in the form $\tilde{\mu}_{f,\alpha} = \ell_\alpha(\tilde{\Phi}^\alpha)$, as explained above.

9 Open questions and remarks

9.1 Families of symmetric squares

Take Coleman's family $k' \mapsto f_{k'} = \sum_{n=1}^{\infty} a_n(k') q^n \in \overline{\mathbb{Q}}[\![q]\!]$ of slope $\sigma > 0$ of cusp eigenforms $f_{k'}$ of weight $k' \geq 2$ containing f, and consider the symmetric square L-function:

$$D(s, f_{k'}, \chi) = L(2s - 2k' + 2, \psi^2\chi^2) \sum_{n=1}^{\infty} \chi(n) a_{n^2}(k') n^{-s} = \tag{9.1}$$

$$\prod_{l \ prime} \left\{(1 - \chi(l)\alpha_l^2(k')l^{-s})(1 - \chi(l)\alpha_l(k')\beta_l(k')l^{-s})(1 - \chi(l)\beta_l^2(k')l^{-s})\right\}^{-1}$$

- *Holomorphy* of the function: (9.1) G.Shimura, [Sh75]
- *Algebraicity for critical values* of the function (9.1) Don Zagier, [Za77] J.Sturm, [St80]
- *Admissible p-adic L-functions* attached to (9.1) A.Dąbrowski, D.Delbourgo, [Da-De]

Question:

To construct two variable p-adic symmetric squares attached to Coleman's families.

For ordinary families this was done by Hida, and for Coleman's families this is the topic of the PhD Thesis of B.Gorsse, (Institut Fourier, Grenoble). He uses Cohen-Zagier Eisenstein series of half integral weight, and the admissibility criterion of Theorem 5.0.1 with $\varkappa = 2$. See also [Go-Ro] for a related algebraic computation of a certain Petersson product.

Related techniques were used by W.Kim (Berkeley) in [Kim], who developed the method of Hida [Hi81], and suggested a conjectural description of the zeroes of such L-function in terms of the ramification points of the eigencurve.

An altenative approach was suggested by Don Zagier, using the construction in [Za77] *Modular Forms whose Fourier Coefficients involve Zeta-Functions of Quadratic Fields*, In: Modular Functions. V, Springer-Verlag, Lect. Notes in Math. N° 627 (1977), p. 106-168 (for the ordinary case, see [Gue]: P.Guerzhoy, *Jacobi-Eisenstein series and p-adic interpolation of symmetric squares of cusp forms*, Annales de l'Institut Fourier (1995)).

9.2 Families of triple products

Consider the $\mathcal{A}$-module

$$\mathcal{M} := \bigcup_{m \geq 0} \mathcal{M}_k(\Gamma_1(\mathcal{A}; Np^v))^{\otimes 3}$$

Take three Coleman's families

$$k' \mapsto f_{k'} = \sum_{n=1}^{\infty} a_n(k')q^n, \; k' \mapsto g_{k'} = \sum_{n=1}^{\infty} b_n(k')q^n, \; k' \mapsto h_{k'} = \sum_{n=1}^{\infty} c_n(k')q^n \in \overline{\mathbb{Q}}[\![q]\!]$$

of cusp eigenforms $f_{k'}, g_{k'}, h_{k'}$ of weight $k' \geq 2$ containing f, g, h.

Let $L(f \otimes g \otimes h, s)$ be the triple L attached to $f \otimes g \otimes h \in \mathcal{S}_k(\Gamma_1(N))^{\otimes 3}$ with a nonzero eigenvalue $\alpha\beta\gamma$. Let us use the restriction on the diagonal $\Phi = E^3_{k'}(z_1, z_2, z_3) \in \mathcal{M}$ of the Siegel-Eisenstein distribution (see [PaSE]) viewed as a formal Fourier series. One obtains distributions Φ_j on Y^3 with values $\mathcal{M}$ using the action of certain differential operators on the modular form Φ (see [PTr]).

Put $l_{f_{k'} \otimes g_{k'} \otimes h_{k'}, \alpha(k')\beta(k')\gamma(k')}(\Phi_j) := i_p \left(\dfrac{\langle f_{k'} \otimes g_{k'} \otimes h_{k'}, \Phi_j^{\alpha\beta\gamma} \rangle}{\langle f_{k'}, f_{k'} \rangle \langle g_{k'}, g_{k'} \rangle \langle h_{k'}, h_{k'} \rangle} \right)$

THEOREM 9.2.1 *Put* $H = [2\mathrm{ord}_p(\alpha\beta\gamma)] + 1$. *There exist a sequence of distributions* $l_{f\otimes g\otimes h,\alpha\beta\gamma}(\Phi_j)$ *on* Y^3 *with values in* $\mathcal{M}^\alpha \subset \mathcal{M}$ *giving (via the admissibility criterion of Theorem 5.0.1 as above) an* H*-admissible measure,*

$$l_{f_{k'}\otimes g_{k'}\otimes h_{k'},\alpha(k')\beta(k')\gamma(k')}(\tilde{\Phi}^{\alpha(k')\beta(k')\gamma(k')})$$

such that the integrals

$$l_{f_{k'}\otimes g_{k'}\otimes h_{k'},\alpha(k')\beta(k')\gamma(k')}(\Phi)(\chi_1 \otimes \chi_2 \otimes \chi_3)$$

on the products of Dirichlet characters $\chi_1 \otimes \chi_2 \otimes \chi_3$ *coincide with the special values*

$$L^*(f_{k'}(\chi_1) \otimes g_{k'}(\chi_2) \otimes h_{k'}(\chi_3), k' + j), \quad (j = 0, \ldots, k' - 2),$$

where the normalisation of L^* *includs at the same time some Gauss sums, Petersson scalar products, powers of* π *and of* $\alpha(k')\beta(k')\gamma(k')$*, and a certain finite Euler product.*

(A JOINT WORK IN PROGRESS WITH S.BOECHRER: we use the Siegel-Eisenstein measure, constructed in [PaSE], and the admissibility criterion of Theorem 5.0.1 with $\varkappa = 2$).

9.3 Other special values in families

In [CourPa] we gave a conceptual explanation of the p–adic properties satisfyied by the special values of the standard L-function $\mathcal{D}(s, f, \chi)$, where f is a Siegel cusp form of an even degree m and of weight $k > 2m + 2$, χ is a varying Dirichlet character. We have shown that these admissible measures can be lifted to arithmetical nearly holomorphic Siegel modular forms studied by G.Shimura [Sh2000]. This lifting is given by a universal sequence $\Phi_s^{\pm}(\chi)$ of distributions with values in arithmetical nearly holomorphic Siegel modular forms (for critical pairs (s, χ), see Proposition 5.4). It would be interesting to extend these lifting results to Siegel cusp eigenforms of odd degree, using the method of Böcherer-Schmidt [BöSch].

9.4 Some advantages of the new p-adic method

The construction can be splitted in several independent steps:

1) Construction of distributions Φ_j (on a profinite or adelic space Y like $Y = \mathbb{A}_K^*/K^*$ for a number field K) with values in an infinite dimensional modular tower $\mathcal{M}(\psi)$ over complex numbers (or in an $\mathcal{A}$-module of infinite rank over some algebra $\mathcal{A}$).

2) Application of a canonical projector of type π_α onto a finite dimensional subspace $\mathcal{M}^\alpha(\psi)$ of $\mathcal{M}^\alpha(\psi)$ (or over locally free $\mathcal{A}$-module of finite rank over some algebra $\mathcal{A}$):
$\boxed{\pi_\alpha(g) = (U^\alpha)^{-v}\pi_{\alpha,0}(U^v(g)) \in \mathcal{M}^\alpha(\Gamma_0(Np), \psi, \mathbb{C})}$ (this works only for nonzero α!)

   (this is the α-characteristic projector of $g \in \mathcal{M}(\Gamma_0(Np^{v+1}), \psi, \mathbb{C})$ (independant of v)).

3) One proves the admissibility criterium 5.1 saying that the sequence $\pi_\alpha(\Phi_j)$ of distributions with values in $\mathcal{M}^\alpha(\psi)$ determines an h-admissible measure $\tilde{\Phi}$ with values in this finite dimensional space for a suitable h (determined by the slope $\mathrm{ord}_p(\alpha)$).

4) Application of a linear form ℓ of type $g \mapsto \langle f^0, \pi_\alpha(g)\rangle / \langle f, f\rangle$ produces distributions $\mu_j = \ell(\pi_\alpha(\Phi_j))$, and (automatically) an admissible measure: the growth condition is automatically satisfied starting from congruences between modular forms $\pi_\alpha(\Phi_j)$

5) One shows that certain integrals $\mu_j(\chi)$ of the distributions μ_j coincide with certain L-values; however, these integrals are not necessary for the construction of measures (already done at stage 4).

6) One shows a resultat on uniqueness for the constructed h-admissibles measures: they are determined by many of their integrals over Dirichlet characters (not all), for example, only over Dirichlet characters with sufficiently large conductor (this stage is not necessary, but it is nice to have uniqueness of in the construction), see [Hu].

7) If we are lucky, we can prove a functional equation for the constructed measure μ (using the uniqueness in 6), and using a functional equation for the L-values (over complex numbers, comuted at stage 5), for example, for Dirichlet characters with sufficiently large conductor (again, this stage is not necessary, but it is nice to have a functional equation)

   This strategy is applicable in various cases (described above), cf. [PaJTNB], [Puy], [Go02].

9.5 Remarks on modular forms of positive slope

According to R.Coleman, F. Gouvêa and B. Mazur, the structure of modular forms of a given positive slope is more complicated than in the ordinary case, even for elliptic modular forms (see the theory of "ferns" in [Gou-Ma], and [Co-Ma]). Our key point is that in order to describe a good p-adic behaviour of p-adic L-functions one needs to fix not only the slope but also the eigenvalue itself, see also [PaTV].

This would be extremely important for constructions of rigid–analytic families of Coleman type [CoPB] in the Siegel modular case, and the corresponding families of p-adic L-functions. Notice that the structure of weights is more complicated in the Siegel modular case due to vector-valued modular forms. The results of [CourPa] show that a description of p-adic families depends not only on a positive slope, but one needs visibly some additional and more subtle parameters, given probably by an analogue of "ferns" in [Gou-Ma], which could provide a good understanding of overconvergency in the Siegel modular case. In the future, it would be interesting to combine our method in [CourPa] with geometric methods of Faltings–Chai [Fa-Ch90] and of Coleman–Mazur [Co-Ma].

Relation to the context of Wiles' proof

Note that the problem of construction of families of modular forms is closely related to the context of Wiles' proof [Wi] which is based on a Galois cohomological construction of p-adic families of classical elliptic modular forms. It seems that a natural thing would be to try to extend constructions of such families to other classes of modular forms; the paper [CourPa] gives an example: a canonical lift of previously known p–adic distributions to distributions with values in an appropriate subspace of arithmetical Siegel modular forms. This lift depends on a choice of a non-zero Satake parameter, and it produces families by integration of arithmetical characters.

Applications of this theory to construction of certain new p-adic families of modular forms (families of Siegel-Eisenstein series, families of theta-series with spherical polynomials...) is based on the following main sources:

- *Serre's theory* of p-adic modular forms as certain formal q-expansions, [Se2].
- *Shimura's theory of arithmeticity for nearly holomorphic forms*, [Sh2000].
- *Hida's theory* of p-adic modular forms and p-adic Hecke algebras, [Hi93].
- *Constructions of p-adic Siegel-Eisenstein series, and of p-adic Klingen–Eisenstein series* by the author, [PaSE].

Note that the eigenspaces $\mathcal{M}^{(\alpha)}$ of U are contained in the primary subspaces $\mathcal{M}^{\alpha}$, and they where used by D. Kazhdan, B. Mazur, C.-G. Schmidt, see [KMS2000], in the p-ordinary case via a p–adic limit procedure. Notice that we do not need a p–adic limit procedure, and we treat the general case of any positive slope.

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