Representations of the orthogonal group O(n) vs representations of the special orthogonal group SO(n), over an arbitrary field

Let $O(n)$ and $SO(n)$ denote the split orthogonal linear algebraic group and its special subgroup, over some fixed field of characteristic not two. I am looking for a reference that explains how to describe the simple (finite-dimensional) representations of $O(n)$ in terms of the simple representations of $SO(n)$.

The relation between the complex representations of the corresponding compact Lie groups is explained in section VI.7 of Bröcker and tom Dieck's *Representations of Compact Lie Groups*. I believe the key statements made there also hold in the above algebraic situation, and I have worked this out in a fair amount of detail, but I am currently unwilling to believe that this has not already been done somewhere in the literature. Roughly, these statements are as follows:

- When n is odd, $O(n)$ is a direct product of $SO(n)$ with $\mathbb{Z}/2$. Thus, every simple $SO(n)$-representation can be lifted to two distinct $O(n)$-representations, and every $O(n)$-representation arises in this way. (See also this question regarding simple representations of products.)
- When n is even, $O(n)$ is only a semi-direct product of $SO(n)$ with $\mathbb{Z}/2$. In this case, only some simple $SO(n)$-representations can be lifted. Those that can be lifted can again be lifted to two distinct $O(n)$-representations. The remaining simple $SO(n)$-representations occur in pairs whose direct sum can be lifted to a unique simple $O(n)$-representation. All simple $O(n)$-representations arise in either of these ways.

Of course, in this question I am mainly interested in references concerning the case when n is even.

reference-request | rt.representation-theory | algebraic-groups

edited Apr 13 '17 at 12:58
Community ♦
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asked Jul 27 '13 at 10:50
Communicative Algebra
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For the time being, the "fair amount of detail" I am referring to may be found in Proposition 3.18 and in Section 4.2 of arXiv:1308.0796. – Communicative Algebra Aug 7 '13 at 7:49

3 For $k = \mathbb{C}$, it's explained in Section 5.5.5 of Goodman-Wallach's book "Symmetry, represenations, and invariants". – emiliocba Aug 13 '13 at 22:17

As long as the characteristic of the field of definition is *good* (not 2), as you assume, it doesn't seem to matter over which field the groups are defined and split. Representations will be studied over an algebraically closed field, where for n even the methods will rely on standard induction/restriction. Older group representation texts for physicists probably cover orthogonal groups, while Jantzen's book on algebraic groups may be overkill. – Jim Humphreys Aug 14 '13 at 17:34

P.S. As pointed out in a comment, the Springer GTM 255 by Goodman-Wallach is likely to be a reliable source for your purpose. I don't have that 2009 edition, but the first edition titled *Representations and Invariants of the Classical Groups* (Cambridge, 1998) has the relevant material in sections 5.2.2 and 10.2.5. – Jim Humphreys Aug 14 '13 at 19:00

@JimHumphreys: I'm not sure what you mean by "representations will be studied over an algebraically closed field"—I want the representations to be defined over the same field as the group itself. Do I need to make this more precise in the question? – Communicative Algebra Aug 15 '13 at 7:48