Analytic constructions of p-adic L-functions and Eisenstein series

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New ways of constructing complex and *p*-adic *L*-functions

In March 2012, Marc Hindry, Philippe Lebacque, Michael Tsfasman, and Alexey Zykin invited me to this fourth conference "Zeta Functions".

Many thanks and gratitude to the organizers for this occasion both to contribute again to the continuity of the interest to the theory of zeta functions, and to indicate productive directions to take research from here.
I would be interested in new developpments in the theory of complex and *p*-adic zeta functions, especially related to the areas of autmorphic forms and geometry.

At present stage of science and education it is important to find possibilities to support our exchange in the study and in further developpment of zeta functions.

It would be very useful to continue our efforts in this direction. On this occasion I wish to point out that in July 1-5, 2013 in Grenoble, a Conference "Journées Arithmétiques" will be held, in which the theme of zeta functions will be widely presented.

Welcome to Grenoble in July 2013!

p-adic Siegel-Eisenstein series and related geometry

Let us consider the symplectic group $\Gamma = \mathrm{Sp}_m(\mathbb{Z})$ (of $(2m \times 2m)$-matrices), and prove that the Fourier coefficients $a_h(k)$ of the original Siegel-Eisenstein series E_k^m admit an explicit p-adic meromorphic interpolation on k where h runs through all positive definite half integral matricies for $\det(2h)$ not divisible by p, where

$$E_k^m(z) = \sum_{(c,d)/\sim} \det(cz+d)^{-k} = \sum_{\gamma \in P \backslash \Gamma} \det(cz+d)^{-k}$$

$$= \sum_{h \in B_m} a_h \exp(\mathrm{tr}(hz))$$

on the Siegel upper half plane $\mathfrak{H}_m = \{z = {}^t z \in M_m(\mathbb{C}) | \mathrm{Im}(z) > 0\}$ of degree m, (c,d) runs over equivalence classes of all coprime symmetric couples, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs over equivalence classes of Γ modulo the Siegel parabolic $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

p-adic Siegel-Eisenstein series and related geometry

The homogeneuos space $X = \{(c, d)/ \sim\} = P\backslash \mathrm{Sp}_m$ and its *p*-adic points admit Siegel's coordinates

$$\nu = \det(c) \text{ and } \mathfrak{R} = c^{-1}d$$

defined on the main affine subset given by $\det(c) \in \mathrm{GL}_1$, which is used in the construction.

I try also to present various applications: to *p*-adic *L*-functions, to Siegel's Mass Formula, to *p*-adic analytic families of automorphic representations.

Eisenstein series are basic automorphic forms, and there exist several ways to construct them via group theory, lattice theory, Galois representations, ...

For me, the Eisenstein series is the main tool of analytic constructions of complex and *p*-adic *L*-functions, in particular via the doubling method, see [PSR], [GRPS], [Boe85], [Shi95], [Boe-Schm],..., greatly thanks to Ilya Piatetski-Shapiro and his collaborators.

4

General strategy

For any Dirichlet character χ mod p^v consider Shimura's "involuted" Siegel-Eisenstein series assuming their absolute convergence (i.e. $k > m + 1$):

$$E_k^*(\chi, z) = \sum_{(c,d)/\sim} \chi(\det(c)) \det(cz+d)^{-k} = \sum_{0 < h \in B_m} a_h(k, \chi) q^h.$$

The two sides of the equality produce dual approaches: geometric and algebraic. The Fourier coefficients can be computed by Siegel's method (see [St81] [Shi95], . . .) via the singular series

$$a_h(E_k^*(\chi, z)) \tag{1}$$

$$= \frac{(-2\pi i)^{mk}}{2^{\frac{m(m-1)}{2}} \Gamma_m(k)} \sum_{\mathfrak{R} \bmod 1} \chi(\nu(\mathfrak{R})) \nu(\mathfrak{R})^{-k} \det h^{k - \frac{m+1}{2}} e_m(h\mathfrak{R})$$

The orthogonality relations mod p^v produce two families of distributions (notice that terms in the RHS are invariant under sign changes, and (3) is algebraic after multiplying by the factor in (1)):

$$\frac{1}{\varphi(p^v)} \sum_{\chi \bmod p^v} \bar{\chi}(b) \sum_{(c,d)/\sim} \frac{\chi(\det(c))}{\det(cz+d)^k} = \sum_{\substack{(c,d)/\sim \\ \det(c) \equiv b \bmod p^v}} \frac{\operatorname{sgn}(\det(c))^k}{\det(cz+d)^k}$$

$$\tag{2}$$

$$\frac{1}{\varphi(p^v)} \sum_{\chi \bmod p^v} \bar{\chi}(b) \sum_{\mathfrak{R} \bmod 1} \frac{\chi(\nu(\mathfrak{R})) e_m(h\mathfrak{R})}{\nu(\mathfrak{R})^k} = \sum_{\substack{\mathfrak{R} \bmod 1 \\ \nu(\mathfrak{R}) \equiv b \bmod p^v}} \frac{e_m(h\mathfrak{R}) \operatorname{sgn} \nu(\mathfrak{R})^k}{\nu(\mathfrak{R})^k}$$

$$\tag{3}$$

The use of Iwasawa theory and pseudomeasures

We express the integrals of Dirichlet characters θ mod p along the distributions (3) through the reciprocal of a product of L-functions, and elementary integral factors. The result turns out to be an Iwasawa function of the variable $t = (1 + p)^k - 1$ divided by a distinguished polynomial provided that $\det h$ is not divisible by p. Thus the second family (3) comes from a unique pseudomeasure μ_h^* which becomes a measure after multiplication by an explicit polynomial factor (in the sense of the convolution product).

Then we deduce that (2) determines a unique pseudomeasure with coefficients in $\mathbb{Q}[\![q^{B_m}]\!]$ whose moments are given by those of the coefficients (3) (after removing from the Fourier expansion $f(z) = \sum_{h \geq 0} a_h e_m(hz)$ all h with $\det h$ divisible by p):

$$\sum_{h>0,\, p \nmid \det h} a_h q^h = p^{-m(m+1)/2} \sum_{\substack{h_0 \bmod p \\ p \nmid \det h_0}} \sum_{x \in S \bmod p} e_m(-h_0 x/p) f(z + (x/p)).$$

In this way a p-adic family of Siegel-Eisenstein series is geometrically produced.

The text of this talk is closely related to that in
http://hal.archives-ouvertes.fr/hal-00688525 arXiv: 1204.3878

Contents:

Generalities about p-adic L-functions

There exist two kinds of L-functions
- Complex-analytic L-functions (Euler products)
- p-adic L-functions (Mellin transforms L_μ of p-adic measures)

Both are used in order to obtain a number (L-value) from an automorphic form. Usually such a number is algebraic (after normalization) via the embeddings

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \quad \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p.$$

How to define and to compute p-adic L-functions? We use Mellin transform of a $\mathbb{Z}_p$-valued distribution μ on a profinite group

$$Y = \varprojlim_i Y_i, \ \mu \in Distr(Y, \mathbb{Z}_p) = \mathbb{Z}_p[[Y]] = \varprojlim_i \mathbb{Z}_p[Y_i] =: \Lambda_Y$$

(the Iwasawa algebra of Y).

$$L_\mu(x) = \int_Y x(y) d\mu, \quad x \in X_Y = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$$

(the Mellin transform of μ on Y).

Examples of p-adic measures and L-functions

- $Y = \mathbb{Z}_p$, $X_Y = \{\chi_t : y \mapsto (1 + t)^y\}$. The Mellin transform $L_\mu(\chi_t) = \int_{\mathbb{Z}_p} (1 + t)^y d\mu(y)$ of any measure μ on $\mathbb{Z}_p$ is given by the Amice transform, which is the following power series

$$A_\mu(t) = \sum_{n \geq 0} t^n \int_{\mathbb{Z}_p} \binom{y}{n} d\mu(y) = \int_{\mathbb{Z}_p} (1 + t)^y d\mu(y),$$

  e.g. $A_{\delta_m} = (1 + t)^m$. Thus, $Distr(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p[\![T]\!]$.

- $Y = \mathbb{Z}_p^* = \Delta \times \Gamma = \{y = \delta(1 + p)^z, \delta^{p-1} = 1, z \in \mathbb{Z}_p\}$
  $X_Y = \{\theta\chi_{(t)} \mid \theta \bmod p, \chi_{(t)}\}$, where $\chi_{(t)}((1 + p)^z) = (1 + t)^z$,
  Δ is the subgroup of roots of unity, $\Gamma = 1 + p\mathbb{Z}_p$.
  The p-adic Mellin transform $L_\mu(\theta\chi_{(t)}) = \int_{\mathbb{Z}_p^*} \theta(\delta)(1 + t)^z \mu(y)$
  of a measure μ on $\mathbb{Z}_p^*$ is given by the collection of Iwasawa
  series $G_{\theta,\mu}(t) = \sum_{n \geq 0} a_{n,\theta} t^n$, where $(1 + t)^z = \sum_{n \geq 0} \binom{z}{n} t^n$,
  $a_{n,\theta} = \sum_{\delta \bmod p, \ n \geq 0} \theta(\delta) t^n \cdot \int_{\mathbb{Z}_p} \binom{z}{n} \mu(\delta(1 + p)^z).$

- A general idea is to construct p-adic L-functions directly from Fourier coefficients of modular forms (or from the Whittaker functions of automorphic forms).

Mazur's p-adic integral

For any choice of a natural number $c \geq 1$ not divisible by p, there exists a p-adic measure μ_c on $\mathbb{Z}_p^*$, such that the special values

$$\zeta(1-k)(1-p^{k-1}) = \frac{\int_{\mathbb{Z}_p^*} y^{k-1} d\mu_c}{1-c^k} \in \mathbb{Q}, (k \geq 2 \text{ even })$$

produce the Kubota-Leopoldt p-adic zeta-function $\zeta_p : X_p \to \mathbb{C}_p$ (where $X_p = X_{\mathbb{Z}_p^*} = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$) as the p-adic Mellin transform

$$\zeta_p(x) = \frac{\int_{\mathbb{Z}_p^*} x(y) d\mu_c(y)}{1-cx(c)} = \frac{L_{\mu_c}(x)}{1-cx(c)},$$

with a single simple pole at $x = x_p^{-1} \in X_p$, where $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ the Tate field, the completion of an algebraic closure of the p-adic field $\mathbb{Q}_p$, $x \in X_p$ (a $\mathbb{C}_p$-analytic Lie group), $x_p(y) = y \in X_p$, and $x(y) = \chi(y)y^{k-1}$ as above.
Explicitly: Mazur's measure is given by $\mu_c(a + p^v \mathbb{Z}_p)$
$= \frac{1}{c}\left[\frac{ca}{p^v}\right] + \frac{1-c}{2c} = \frac{1}{c}B_1(\{\frac{ca}{p^v}\}) - B_1(\frac{a}{p^v})$, $B_1(x) = x - \frac{1}{2}$,
see [LangMF], Ch.XIII.

Meromorphic p-adic continuation of $\frac{1}{\zeta(1-k)(1-p^{k-1})}$

For any odd prime p take the Iwasawa series $G_{\theta,c}(t)$ of Mazur's measure μ_c where θ is a character $\mathrm{mod}\, p$,

$$G_{\theta,c}(t) := \int_{\mathbb{Z}_p^*} \theta(y)\chi_{(t)}(\langle y \rangle)\mu_c = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Z}_p[\![t]\!], \text{ and}$$

$\chi_{(t)} : (1+p)^z \mapsto (1+t)^z$, $\langle y \rangle = \dfrac{y}{\omega(y)}, \omega$ the Teichmüller character. Mazur's integral of the character $y^{k-1} = \omega^{k-1} \cdot \chi_{(t)}$ shows that $\theta = \omega^{k-1}, (1+t) = (1+p)^{k-1}$

$$\zeta(1-k)(1-p^{k-1}) = \frac{G_{\theta,c}((1+p)^{k-1}-1)}{1-c^k}. \qquad (4)$$

By the Weierstrass preparation theorem we have a decomposition

$$G_{\theta,c}(t) = U_{\theta,c}(t)P_{\theta,c}(t)$$

with a distinguished polynomial $P_{\theta,c}(t)$ and invertible power series $U_{\theta,c}(t)$. The inversion of (4) for any even $k \geq 2$ gives :

$$\frac{1}{\zeta(1-k)(1-p^{k-1})} = G_{\theta,c}((1+p)^{k-1}-1)^{-1}(1-c^k).$$

The answer: for any prime $p > 2$ and even $k \geq 2$

is the following Iwasawa function on $t = t_k = (1+p)^k - 1$ divided by a distingushed polynomial:

$$\frac{1}{\zeta(1-k)(1-p^{k-1})} = \frac{U^*_{\theta,c}((1+p)^{k-1}-1)(1-c^k)}{P_{\theta,c}((1+p)^{k-1}-1)} \qquad (5)$$
$$= \frac{U^*_{\theta,c}((1+t_k)(1+p)^{-1}-1)(1-c^k)}{P_{\theta,c}((1+t_k)(1+p)^{-1}-1)}$$

which is meromorphic in the unit disc of the variable $t = (1+p)^k - 1$ with a finite number of poles (expressed via roots of $P_{\theta,c}$) for $\theta = \omega^{k-1}$, and

$$U^*_{\theta,c}((1+p)^{k-1}-1) := 1/U_{\theta,c}((1+p)^{k-1}-1).$$

The above formula immediately extends to all Dirichlet L-functions of characters χ mod p^v as the following Iwasawa function divided by a polynomial:

$$\frac{1}{L(1-k,\chi)(1-\chi(p)p^{k-1})} = \frac{U^*_{\theta,c}(\chi(1+p)(1+p)^{k-1}-1)(1-\chi(c)c^k)}{P_{\theta,c}(\chi(1+p)(1+p)^{k-1}-1)}$$

where
$$U^*_{\theta,c}(\chi(1+p)(1+p)^{k-1}-1) := \frac{1}{U_{\theta,c}(\chi(1+p)(1+p)^{k-1}-1)}$$

Illustration: numerical values of $\zeta(1-2k)^{-1}(1-p^{2k-1})^{-1}$ for $p = 37$

```
gp >zetap1(p,n)= -2*n/(bernfrac(2*n)*(1-p^(2*n-1)+O(p^5)));
gp > p=37;
gp > for(k=1,(p-1)/2, print(2*k, zetap1(p,k)))
```

$2k$	$\zeta(1-2k)^{-1}(1-p^{2k-1})^{-1}$
2	$25 + 24*37 + 24*37^2 + 24*37^3 + 24*37^4 + O(37^5)$
4	$9 + 3*37 + 9*37^3 + 3*37^4 + O(37^5)$
6	$7 + 30*37 + 36*37^2 + 36*37^3 + 36*37^4 + O(37^5)$
8	$18 + 6*37 + O(37^5)$
10	$16 + 33*37 + 36*37^2 + 36*37^3 + 36*37^4 + O(37^5)$
12	$8 + 25*37 + 28*37^2 + 23*37^3 + O(37^5)$
14	$25 + 36*37 + 36*37^2 + 36*37^3 + 36*37^4 + O(37^5)$
16	$6 + 16*37 + 31*37^2 + 29*37^3 + 20*37^4 + O(37^5)$
18	$3 + 4*37 + 10*37^2 + 32*37^3 + 25*37^4 + O(37^5)$
20	$11 + 13*37 + 19*37^2 + 36*37^3 + 12*37^4 + O(37^5)$
22	$1 + 26*37 + 15*37^2 + 35*37^3 + 9*37^4 + O(37^5)$
24	$16 + 28*37 + 24*37^2 + 27*37^3 + 31*37^4 + O(37^5)$
26	$4 + 17*37 + 25*37^2 + 25*37^3 + 19*37^4 + O(37^5)$
28	$22 + 36*37 + 8*37^2 + 4*37^3 + 33*37^4 + O(37^5)$
30	$22 + 5*37 + 35*37^2 + 9*37^3 + 5*37^4 + O(37^5)$
32	$36*37^{-1} + 28 + 3*37 + 19*37^2 + 18*37^3 + O(37^4)$
34	$20 + 37 + 30*37^2 + 15*37^3 + 22*37^4 + O(37^5)$
36	$36*37 + 29*37^2 + 35*37^3 + 5*37^4 + 37^5 + O(37^6)$

Fourier expansion of the Siegel-Eisenstein series

has the form

$$E_k^m(z) = \sum_{\gamma \in P \backslash \Gamma} \det(cz + d)^{-k} = \sum_{h \in B_m} a_h q^h,$$

where $a_h = a_h(k) = a_h(E_k^m)$, $q^h = e^{2\pi i \operatorname{tr}(hz)}$, h runs over semi-definite half integral $m \times m$ matrices.

The rationality of the coefficients a_h was established in Siegel's pioneer work [Si35] in connection with a study of local densities for quadratic forms. Siegel expressed $a_h(k)$ as a product of local factors over all primes and ∞.
In a difficult later work [Si64b] Siegel proved the boundedness of their denominators, and S.Boecherer [Boe84] gave a simplified proof of a more precise result in 1984. M.Harris extended the rationality to wide classes of Eisenstein series on Shimura varieties [Ha81], [Ha84]. Their relation to the Iwasawa Main Conjecture and p-adic L-functions on the unitary groups was established in [HLiSk].

Explicit p-adic continuation of $a_h(k)$

as Iwasawa functions on $t = (1+p)^k - 1$ divided by distinguished polynomials. Let $a_h^{(p)}(k)$ denote the p-regular part of the coefficient $a_h(k)$ (i.e. with the Euler p-factor removed from the product). Namely, for any even k, $a_h^{(p)}(k) = a_h(E_k^m)$ times

$$\begin{cases} 1/((1-p^{k-1})(1+\psi_h(p)p^{k-\frac{m}{2}-1})\prod_{i=1}^{(m/2)-1}(1-p^{2k-2i-1})) \\ = (1-\psi_h(p)p^{k-\frac{m}{2}-1})/((1-p^{k-1})\prod_{i=1}^{m/2}(1-p^{2k-2i-1})), & m \text{ even} \\ 1/((1-p^{k-1})\prod_{i=1}^{(m-1)/2}(1-p^{2k-2i-1})), & m \text{ odd}, \end{cases}$$

where the p-correcting factor is a p-adic unit, and

$$\psi_h(n) := \left(\frac{\det(2h)(-1)^{m/2}}{n} \right).$$

Theorem (1) (A.P., 2012)

Let h be any positive definite half integral matrix with $\det(2h)$ not divisible by p. Then there exist explicitly given distinguished polynomials $P_{\theta,h}^E(T) \in \mathbb{Z}_p[T]$ and Iwasawa series $S_{\theta,h}^E(T) \in \mathbb{Z}_p[\![T]\!]$ such that the p-regular part $a_h^{(p)}(k)$ of the Fourier coefficient $a_h(k)$ admit the following p-adic meromorphic interpolation on all even k with $\theta = \omega^k$ fixed

$$a_h^{(p)}(k) = \frac{S_{\theta,h}^E((1+p)^{k-1}-1)}{P_{\theta,h}^E((1+p)^{k-1}-1)}$$

with a finite number of poles expressed via the roots of $P_{\theta,h}^E(T)$ where the denominator depends only on $\det(2h) \bmod 4p$ and $k \bmod p-1$.

Computation of the Fourier coefficients

Recall that Siegel's computation of the coefficients $a_h = a_h(E_k^m)$:

$$E_k^m(z) = \sum_{\gamma \in P \backslash \Gamma} \det(cz + d)^{-k} = \sum_{h \in B_m} a_h q^h$$

is based on the Poisson summation formula giving the equality (see [Maa71], p.304):

$$\sum_{a \in S_m} \det(z + a)^{-k} = \frac{(-2\pi i)^{mk}}{2^{\frac{m(m-1)}{2}} \Gamma_m(k)} \sum_{h \in C_m} \det(h)^{k - \frac{m+1}{2}} e^{2\pi i \mathrm{tr}(hz)},$$

where $\Gamma_m(k) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(s - \frac{j}{2})$, $q^h = e^{2\pi i \mathrm{tr}(hz)}$, h runs over the set C_m of positive definite half integral $m \times m$ symmetric matrices, and a runs over the set S_m of integral $m \times m$ symmetric matrices, see [Si39], p.652, [St81], p.338.

Formulas for the Fourier coefficients for $\det(2h) \neq 0$

$$a_h(E_k^m) = \frac{(-2\pi i)^{mk} \Gamma_m^{-1}(k)}{\zeta(k) \prod_{i=1}^{[m/2]} \zeta(2k - 2i)}$$

$$\times \det(2h)^{k - \frac{m+1}{2}} M_h(k) \begin{cases} L(k - \frac{m}{2}, \psi_h), & m \text{ even}, \\ 1, & m \text{ odd}. \end{cases}$$

The integral factor $M_h(k) = \prod_{\ell \in P(h)} M_\ell(h, \ell^{-k})$ is a finite Euler product, extended over primes ℓ in the set $P(h)$ of prime divisors of all elementary divisors of the matrix h. The important property of the product is that for each ℓ we have that $M_\ell(h, t) \in \mathbb{Z}[t]$ is a polynomial with integral coefficients.

Notice the L-factor $L(k - \frac{m}{2}, \psi_h)$ depends on the index h of the Fourier coefficient; this makes a difference to the case of odd m; the case of $GL(2)$ corresponds to $m = 1$.

Proof: the use of the normalized Siegel-Eisenstein series

defined as in [Ike01], [PaSE] and [Pa91] by

$$\mathcal{E}_k^m = E_k^m(z) 2^{m/2} \zeta(1-k) \prod_{i=1}^{[m/2]} \zeta(1-2k+2i),$$

I show that it produces a nice p-adic family, namely:

Proposition (2)

(1) For any non-degenerate matrice $h \in C_m$ the following equality holds

$$a_h(\mathcal{E}_k^m) = 2^{-\frac{m}{2}} \det h^{k-\frac{m+1}{2}} M_h(k) \qquad (6)$$

$$\times \begin{cases} L(1-k+\frac{m}{2}, \psi_h) C_h^{\frac{m}{2}-k+(1/2)}, & m \text{ even}, \\ 1, & m \text{ odd}, \end{cases}$$

where C_h is the conductor of ψ_h.

(2) for any prime $p > 2$, and $\det(2h)$ not divisible by p, define the p-regular part $a_h(\mathcal{E}_k^m)^{(p)}$ of the coefficient $a_h(\mathcal{E}_k^m)$ of $\mathcal{E}_k^m$ by introducing the factor $\begin{cases} (1 - \psi_h(p)p^{k-\frac{m}{2}-1}), & m \text{ even}, \\ 1, & m \text{ odd}. \end{cases}$

Then $a_h(\mathcal{E}_k^m)^{(p)}$ is a p-adic analytic Iwasawa function of $t = (1+p)^k - 1$ for all k with ω^k fixed, and divided by the elementary factor $1 - \psi_h(c_h)c_h^{k-\frac{m}{2}}$.

17

Proof of (1) of Proposition 2

Proof of (1) is deduced like at p.653 of [Ike01] from the Gauss duplication formula

$$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = 2^{1-s}\sqrt{\pi}\Gamma(s),$$

the definition

$$\Gamma_m(k) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(s - \frac{j}{2})$$

and the functional equations

$$\zeta(1-k) = \frac{2(k-1)!}{(-2\pi i)^k}\zeta(k),$$

$$\zeta(1-2k+2i) = \frac{2(2k-2i-1)!}{(-2\pi i)^{2k-2i}}\zeta(2k-2i),$$

$$L(1-k+\frac{m}{2}, \psi_h) = \frac{2(k-\frac{m}{2}-1)!}{(-2\pi i)^{k-\frac{m}{2}}}L(k-\frac{m}{2}, \psi_h)C_h^{k-\frac{m}{2}-\frac{1}{2}}$$

Proof of (2) of Proposition 2

is then deduced easily :

Notice that for any $a \in \mathbb{Z}_p^*$, the function of $t = (1+p)^k - 1$

$$k \mapsto a^k = \omega(a)^k \langle a \rangle^k = \omega(a)^k (1+p)^{k \frac{\log \langle a \rangle}{\log (1+p)}} \tag{7}$$

$$= \omega(a)^k (((1+p)^k - 1) + 1)^{\frac{\log \langle a \rangle}{\log (1+p)}}$$

$$= \omega(a)^k \sum_{n=0}^{\infty} \binom{\frac{\log \langle a \rangle}{\log (1+p)}}{n} t^n$$

is a p-adic analytic Iwasawa function denoted by $\tilde{a}(t) \in \mathbb{Z}_p[\![t]\!]$, of $t = (1+p)^k - 1$ with ω^k fixed, where $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$.

Then Mazur's formula applied to $L(1 - k + \frac{m}{2}, \psi_h)(1 - \psi_h(p)p^{k - \frac{m}{2} - 1})$ shows that this function is a p-adic analytic Iwasawa function of $t = (1+p)^k - 1$ with ω^k fixed (a single simple pole may occur at $k = \frac{m}{2}$ only if $\omega^{k - \frac{m}{2}}$ is trivial).

Proof of Main Theorem

Let us use the equality

$$E_k^m = \mathcal{E}_k^m(z) \cdot \frac{2^{-m/2}}{\zeta(1-k)\prod_{i=1}^{[m/2]}\zeta(1-2k+2i)}$$

and the properties of the normalized series $\mathcal{E}_k^n(z)$ in Proposition 2.
First let us compute the reciprocal of the product of L-functions

$$\zeta(1-k)\prod_{i=1}^{[m/2]}\zeta(1-2k+2i)$$

using the above: for even $k \geq 2$,

$$\zeta(1-k)^{-1}(1-p^{k-1})^{-1} = \frac{U_{\theta_k,c}^*((1+p)^{k-1}-1)(1-c^k)}{P_{\theta_k,c}((1+p)^{k-1}-1)} \quad (8)$$

$$\zeta(1-2k+2i)^{-1}(1-p^{2k-2i-1})^{-1}$$
$$= \frac{U_{\theta_{2k-2i},c}^*((1+p)^{2k-2i-1}-1)(1-c^{2k-2i})}{P_{\theta_{2k-2i},c}((1+p)^{2k-2i-1}-1)} \quad (9)$$

which is meromorphic in the unit disc with a finite number of poles
(expressed via roots of P_θ) for $\theta_k = \omega^{k-1}$.

Proof of Main Theorem (continued)

Let us use again the notation $1 + t = (1 + p)^k$ with $k \in \mathbb{Z}_p$.

$$\frac{2^{-m/2}}{\zeta(1 - k)(1 - p^{k-1}) \prod_{i=1}^{[m/2]} \zeta(1 - 2k + 2i)(1 - p^{2k-2i-1})}$$
$$= \frac{U_{\omega^k}^E(t)}{P_{\omega^k}^E(t)} \tag{10}$$

where the **numerator** is (an Iwasawa function) $U_{\omega^k}^E(t) =$

$$U_{\theta_k,c}^* \left(\frac{1+t}{1+p} - 1 \right)(1 - c^k) \prod_{i=1}^{[m/2]} U_{\theta_{2k-2i},c}^* \left(\frac{(1+t)^2}{(1+p)^{2i+1}} - 1 \right)(1 - c^{2k-2i}),$$

and

$$P_{\omega^k}^E(t) = P_{\theta,c} \left(\frac{1+t}{1+p} - 1 \right) \prod_{i=1}^{[m/2]} P_{\theta_{2k-2i},c} \left(\frac{(1+t)^2}{(1+p)^{2i+1}} - 1 \right)$$

is the **polynomial denominator** which depends only on k mod $p - 1$.

Proof of Main Theorem: control over the conductor of ψ_h

Mazur's formula applied to $L(1 - k + \frac{m}{2}, \psi_h)(1 - \psi_h(p)p^{k-\frac{m}{2}-1})$ (in the numerator) shows that for all h with $\det(2h)$ not divisible by p,

$$L(1 - k + \frac{m}{2}, \psi_h)(1 - \psi_h(p)p^{k-\frac{m}{2}-1}) \qquad (11)$$

$$= \frac{G_{\theta,h}((1+p)^{k-\frac{m}{2}-1} - 1)}{1 - \psi_h(c_h)c_h^{k-\frac{m}{2}}}$$

which is meromorphic in the unit disc with a possible single simple pole at $k = \frac{m}{2}$ for all k with $\theta = \omega^{k-1}$. It comes from Mazur's measure on the finite product $\prod_{\ell \in P_h} \mathbb{Z}_\ell^*$ extended over primes ℓ in the set $P_h = P(h) \cup \{p\}$; recall that $P(h)$ is the set of prime divisors of all elementary divisors of the matrix h as above.

Indeed, for any choice of a natural number $c_h > 1$ coprime to $\prod_{\ell \in P_h} \ell$, there exists a p-adic measure $\mu_{c_h, h}$ on $\mathbb{Z}_p^*$, such that the special values

$$L(1 - k + \frac{m}{2}, \psi_h)(1 - \psi_h(p)p^{k-1-\frac{m}{2}}) = \frac{\int_{\mathbb{Z}_p^*} y^{k-\frac{m}{2}-1} d\mu_{c_h,h}}{1 - \psi_h(c_h)c_h^{k-\frac{m}{2}}}$$

$$:= (1 - \psi_h(c_h)c_h^{k-\frac{m}{2}})^{-1} \int_{\prod_{\ell \in P_h} \mathbb{Z}_\ell^*} \psi_h(y)y_p^{k-\frac{m}{2}-1} d\mu_{c_h},$$

Proof of Main Theorem (continued)

where Mazur's measure μ_{c_h} extends on the product $\prod_{\ell \in P_h} \mathbb{Z}_\ell^* \xrightarrow{y_p} \mathbb{Z}_p^*$

(see §3,Ch.XIII of [LangMF]):

$$\mu_{c_h}(a + (N)) = \frac{1}{c_h}\left[\frac{c_h a}{N}\right] + \frac{1 - c_h}{2c_h} = \frac{1}{c_h}B_1(\{\frac{c_h a}{N}\}) - B_1(\frac{a}{N})$$

for any natural number N with all prime divisors in P_h.
The regularizing factor is the following Iwasawa function which
depends on c_h mod 4 and k mod $p - 1$:

$$1 - \psi_h(c_h)c_h^{k-\frac{m}{2}} = 1 - (\psi_h\omega^{k-\frac{m}{2}})((\frac{(1+p)^k}{(1+p)^{\frac{m}{2}}} - 1) + 1)^{\frac{\log\langle c_h\rangle}{\log(1+p)}}$$

$$\tag{12}$$

$$= 1 - (\psi_h\omega^{k-\frac{m}{2}})(c_h)\sum_{n=0}^{\infty}\binom{\frac{\log\langle c_h\rangle}{\log(1+p)}}{n}((\frac{(1+p)^k}{(1+p)^{\frac{m}{2}}} - 1)^n$$

$$= 1 - (\psi_h\omega^{k-\frac{m}{2}})(c_h)\sum_{n=0}^{\infty}\binom{\frac{\log\langle c_h\rangle}{\log(1+p)}}{n}(\frac{(1+t)}{(1+p)^{\frac{m}{2}}} - 1)^n \in \mathbb{Z}_p[\![t]\!],$$

where we write c_h in place of $i_p(c_h)$ and use the notation
$1 + t = (1+p)^k$. The function (12) is divisible by t or invertible in
$\mathbb{Z}_p[\![t]\!]$ according as $\omega^{k-\frac{m}{2}}\psi_{c_h}$ is trivial or not because
$t = 0 \iff k = 0$ and $1 + t = (1+p)^k$.

Elementary factors

Notation:

$$u_{c_h}(t) = \begin{cases} (1 - \psi_h(c_h)\tilde{c}_h(t))/t, & \text{if } \omega^{k - \frac{m}{2}}\psi_h \text{ is trivial,} \\ 1 - \psi_h(c_h)\tilde{c}_h(t), & \text{otherwise.} \end{cases}$$

By (12) we have that $u_{c_h}(t) \in \mathbb{Z}_p[\![t]\!]^*$, and we denote by $u_{c_h}^*(t)$ its inverse. Moreover, (6) gives the elementary factor

$$\mathcal{M}_h((1+p)^k - 1) = 2^{-\frac{m}{2}} \det h^{k - \frac{m+1}{2}} \prod_{\ell | P(h)} M(h, \ell^{-k}) C_h^{k - \frac{m+1}{2}}$$

which is also an Iwasawa function as above:

$$\mathcal{M}_h((1+p)^k - 1) = \mathcal{M}_h(t) \in \mathbb{Z}_p[\![t]\!].$$

Proof of Main Theorem: the numerator

It follows that

$$a_h^{(p)}(k) = \frac{S_{\theta,h}^E((1+p)^k - 1)}{P_{\theta,h}^E((1+p)^k - 1)} = \frac{S_{\theta,h}^E(t)}{P_{\theta,h}^E(t)},$$

where

$$S_{\theta,h}^E = u_{c_h}^*((1+p)^k - 1)\mathcal{M}((1+p)^k - 1)$$
$$\times U_{\theta,h}((1+p)^{k-\frac{m}{2}-1} - 1)(1 - c_h^k)U_{\theta_k,c}^*((1+p)^{k-1} - 1)$$
$$\times \prod_{i=1}^{[m/2]} U_{\theta_{2k-2i},c}^*((1+p)^{2k-2i-1} - 1)(1 - c_h^{2k-2i})$$
$$= u_{c_h}^*(t)\mathcal{M}(t)U_{\theta,h}((1+t)(1+p)^{-\frac{m}{2}-1} - 1)$$
$$\times (1 - \tilde{c}_h(t))U_{\theta_k,c_h}^*((1+t)(1+p)^{-1} - 1)$$
$$\times \prod_{i=1}^{[m/2]} U_{\theta_{2k-2i},c_h}^*((1+t)^2(1+p)^{-2i-1} - 1)(1 - \tilde{c}_h^2(t)c_h^{-2i}),$$

Proof of Main Theorem (end)

The denominator is the following distinguished polynomial

$$P_{\theta,h}^{E}((1+p)^{k-1}-1) = (1+((1+p)^{k-1}-2)\delta(\omega^{k-\frac{m}{2}}\psi_{c_h}))$$

$$\times P_{\theta_k,c_h}((1+p)^{k-1}-1) \prod_{i=1}^{[m/2]} P_{\theta_{2k-2i},c_h}((1+p)^{2k-2i-1}-1)$$

$$= (1+(t-1)\delta(\omega^{k-\frac{m}{2}}\psi_{c_h}))P_{\theta_k,c_h}((1+t)(1+p)^{-1}-1)$$

$$\times \prod_{i=1}^{[m/2]} P_{\theta_{2k-2i},c}((1+t)^2(1+p)^{-2i-1}-1), \text{ where}$$

$$\delta(\omega^{k-\frac{m}{2}}\psi_{c_h}) = \begin{cases} 1, & \text{if } \omega^{k-\frac{m}{2}}\psi_{c_h} \text{ is trivial,} \\ 0, & \text{otherwise,} \end{cases} \text{ so that}$$

$$1+(t-1)\delta(\omega^{k-\frac{m}{2}}\psi_{c_h}) = \begin{cases} t, & \text{if } \omega^{k-\frac{m}{2}}\psi_{c_h} \text{ is trivial,} \\ 1, & \text{otherwise.} \end{cases} .$$

It remains to notice that different choices of c_h coprime to $p\det(2h)$ give the same polynomial factors $P_{\theta,h}^{E}$ (up to invertible Iwasawa function). Indeed they all give the same single simple zero. ∎

Interpretation: Mellin transform of a pseudomeasure

Pseudomeasures were introduced by J.Coates [Co] as elements of the fraction field $\mathcal{L}$ of the Iwasawa algebra. Such a pseudomeasure is defined by its Mellin transform which is a ring homomorphism and we can extend it by universality (the extension of the integral along measures in $\Lambda = \mathbb{Z}_p[[T]]$ to the whole fraction field $\mathcal{L}$). The p-adic meromorphic function

$$a_h^{(p)}(k) = \frac{S_{\theta,h}^E((1+p)^k - 1)}{P_{\theta,h}^E((1+p)^k - 1)} = \frac{S_{\theta,h}^E(t)}{P_{\theta,h}^E(t)},$$

is attached to an explicit pseudomeasure:

$$\rho_h^E = \frac{\mu_h^E}{\nu_h^E}, \quad \frac{S_{\theta,h}^E(t)}{P_{\theta,h}^E(t)} = \frac{\int_{\mathbb{Z}_p^*} \theta\chi_{(t)}\mu_h}{\int_{\mathbb{Z}_p^*} \theta\chi_{(t)}\nu_h}$$

- $\mathcal{S}(x) = \int_{\mathbb{Z}_p^*} x\mu_h^E$ is given by the collection of Iwasawa functions $S_{\theta,h}(t) = \int_{\mathbb{Z}_p^*} \theta\chi_{(t)}\mu_h^E$ (the numerator),
- $\mathcal{P}(x) = \int_{\mathbb{Z}_p^*} x\nu_h^E$ is given by the collection of polynomials $P_{\theta,h}(t) = \int_{\mathbb{Z}_p^*} \theta\chi_{(t)}\nu_h^E$ (the denominator).

Pseudomeasure ρ as a family of distributions

A pseudomeasure ρ can be described as a certain family of distributions, parametrized by the set X_p of p-adic characters. For any $x \in X_p$ we have a distribution given by the formula

$$\rho_{h,x}^E(a + (p^v)) = \frac{1}{\varphi(p^v)} \sum_{\chi \bmod p^v}' \chi(a)^{-1} \frac{\mathcal{S}^E(\chi x)}{\mathcal{P}^E(\chi x)}$$

where $'$ means that the terms with $\mathcal{P}(\chi x) = 0$ are omited. It follows that

$$\int_{\mathbb{Z}_p^*} \chi \rho_{h,x}^E = \begin{cases} \dfrac{\mathcal{S}^E(\chi x)}{\mathcal{P}^E(\chi x)}, & \text{if } \mathcal{P}^E(\chi x) \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\mathcal{S}_h^E(\chi x) = S_{(\chi x)_\Delta, h}^E(\chi x(1 + p) - 1) = S_{\theta, h}^E((1 + p)^t - 1),$$
$$\theta = (\chi x)_\Delta, (\chi x)(1 + p) = 1 + t,$$
$$\mathcal{P}_h^E(\chi x) = P_{(\chi x)_\Delta, h}^E(\chi x(1 + p) - 1) = P_{\theta, h}^E((1 + p)^t - 1).$$

A geometric construction: Siegel's method and duality

For any Dirichlet character χ mod p^ν consider Shimura's "involuted" Siegel-Eisenstein series assuming their absolute convergence (i.e. $k > m + 1$):

$$E_k^*(\chi, z) = \sum_{(c,d)/\sim} \chi(\det(c)) \det(cz+d)^{-k} = \sum_{h \in B_m} a_h(E_k^*(\chi,z)) q^h$$

The series on the left is geometrically defined, and the Fourier coefficients on the right can be computed by Siegel's method (see [St81] [Shi95], ...) via the singular series

$$a_h(E_k^*(\chi, z)) \tag{13}$$
$$= \frac{(-2\pi i)^{mk}}{2^{\frac{m(m-1)}{2}} \Gamma_m(k)} \sum_{\mathfrak{R} \bmod 1} \chi(\nu(\mathfrak{R})) \nu(\mathfrak{R})^{-k} \det h^{k - \frac{m+1}{2}} e_m(h\mathfrak{R})$$

If $\chi = \chi_0$ mod p is trivial and $p \nmid \det h$ then

$$a_h(E_k^*(\chi_0, z)) \tag{14}$$
$$= \frac{(-2\pi i)^{mk}}{2^{\frac{m(m-1)}{2}} \Gamma_m(k)} \sum_{\mathfrak{R} \bmod 1} \chi_0(\nu(\mathfrak{R})) \nu(\mathfrak{R})^{-k} \det h^{k - \frac{m+1}{2}} e_m(h\mathfrak{R}) =$$

$$a_h(E_k^m) \times \begin{cases} (1 - p^{-k})(1 + \psi_h(p) p^{-k + \frac{m}{2}}) \prod_{i=1}^{(m/2)-1}(1 - p^{-2k+2i}), & m \text{ even} \\ \\ (1 - p^{-k}) \prod_{i=1}^{(m-1)/2}(1 - p^{-2k+2i}), & m \text{ odd} \end{cases}$$

A geometric construction (continued)

The formula (14) means that the series $E_k^*(\chi_0, z)$ coincides with E_k^m after removing h with det h divisible by p and normalizing by the factor in (14). Moreover, the Gauss reciprocity law shows that the normalizing factor depends only on det h mod $4p =$ det h_0 mod $4p$, where $h_0 \equiv h$ mod $4p$ runs through a representative system. Let us denote this factor by $C^+(h_0, k, 4p)$: for the trivial character $\chi = \chi_0$ mod p and det h not divisible by p

$$a_h(E_k^*(\chi_0, z)) = a_h(E_k^m)C^+(h_0, k, 4p), \text{ where} \qquad (15)$$

$$C^+(h_0, k, 4p) = \begin{cases} (1 - p^{-k})(1 + \psi_h(p)p^{-k+\frac{m}{2}}) \prod_{i=1}^{(m/2)-1}(1 - p^{-2k+2i}), & m \text{ even} \\ \\ (1 - p^{-k}) \prod_{i=1}^{(m-1)/2}(1 - p^{-2k+2i}), & m \text{ odd.} \end{cases}$$

From the Fourier coefficients to modular forms:
If we remove in the Fourier expansion $E_k^m(z) = \sum_{h>0} a_h e_m(hz)$ all terms with det h divisible by p, the equality of Fourier coefficients (15) transforms to the equality of the series

$$E_k^*(\chi_0, z) = (4p)^{-m(m+1)/2} \sum_{\substack{h_0 \bmod 4p \\ p \nmid \det h_0}} C^+(h_0, k, 4p) \times \qquad (16)$$

$$\sum_{x \in S \bmod 4p} e_m(-h_0 x/4p) E_k^m(z + (x/4p)).$$

A geometric construction

Let us apply the interpolation theorem (Theorem 1) to all the coefficients

$$a_h^{(p)}(k) = a_h(E_k^m)C^-(h_0, k, 4p), \text{ where} \tag{17}$$

$$C^-(h_0, k, 4p) = \begin{cases} \dfrac{1-\psi_h(p)p^{k-\frac{m}{2}-1}}{(1-p^{k-1})\prod_{i=1}^{m/2}(1-p^{2k-2i-1})}, & m \text{ even} \\[2em] \dfrac{1}{(1-p^{k-1})\prod_{i=1}^{(m-1)/2}(1-p^{2k-2i-1})}, & m \text{ odd,} \end{cases}$$

and (16) becomes a "geometric-algebraic equality" of two families of modular forms

$$E_k^*(\chi_0, z) = (4p)^{-m(m+1)/2} \sum_{\substack{h_0 \bmod 4p \\ p \nmid \det h_0}} C^+(h_0, k, 4p) \times \tag{18}$$

$$C^-(h_0, k, 4p) \sum_{x \in S \bmod 4p} e_m(-h_0 x/4p)E_k^m(z + (x/4p)).$$

A geometric construction (end)

We deduce by the orthogonality that

$$\sum_{x' \in S \text{ mod } 4p} e_m(-h_0 x'/4p) E_k^*(\chi_0, z + (x'/4p)) = \tag{19}$$

$$C^+(h_0, k, 4p) \sum_{x \in S \text{ mod } 4p} e_m(-h_0 x/4p) E_k^m(z + (x/4p)).$$

Each series $C^-(h_0, k, 4p) \sum_{x \in S \text{ mod } 4p} e_m(-h_0 x/4p) E_k^m(z + (x/4p))$ in (18) determines a unique pseudomeasure with coefficients in $\bar{\mathbb{Q}}[\![q^{B_m}]\!]$ whose moments are given by those of the coefficients (17) (the unicity means that a pseudomeasure is determined by its Mellin transform). It is also a family of distributions geometrically defined by the series

$$\frac{C^-(h_0, k, 4p)}{C^+(h_0, k, 4p)} \sum_{x \in S \text{ mod } 4p} e_m(-h_0 x/4p) E_k^*(\chi_0, z + (x/4p)).$$

p-adic version of Minkowski-Siegel Mass constants.

An application of the construction is the p-adic version of Siegel's Mass formula. It expresses the Mass constant through the above product of L-values. This product can be viewed as the proportionality coefficient between two kinds of Eisenstein series in the symplectic case extending Hecke's result (1927) of the two kinds of Eisenstein series and the relation between them. However, there is no direct analogue of Hecke's computation in the symplectic case. Thus this mass constant admits an explicit product expression through the values of the functions (5) at $t_j = (1 + p)^j - 1$, for $j = k$, and $j = 2, 4, \ldots, 2k - 2$.

Recall that ([ConSl98], p.409)

> unimodular latticies have the property that there are explicit formulae, the mass formulae, which give appropriately weighted sums of the theta-series of all the inequivalent latticies of a given dimension. In particular, the numbers of inequivalent latticies is given by Minkowski-Siegel Mass constants for unimodular latticies.

Application to Minkowski-Siegel Mass constant (continued)

In the particular case of even unimodular quadratic forms of rank $m = 2k \equiv 0 \pmod 8$, this formula means that there are only finitely many such forms up to equivalence for each k and that, if we number them $Q_1, \ldots, Q_{h_k}$, then we have the relation

$$\sum_{i=1}^{h_k} \frac{1}{w_i} \Theta_{Q_i}(z) = m_k E_k$$

where w_i is the number of automorphisms of the form Θ_{Q_i} is the theta series of Q_i, E_k the normalized Eisenstein series of weight $k = m/2$ (with the constant term equal to 1),
The dimension of lattices is $2k$ and the Mass formula express an identity of a sum of weighted theta functions and a Siegel-Eisenstein series of weight k, multiplied by the Mass constant

$$m_k = 2^{-k} \zeta(1-k) \prod_{i=1}^{k-1} \zeta(1-2k+2i) = (-1)^k \frac{B_k}{2k} \times \prod_{j=1}^{k-1} \frac{B_{2j}}{4j}$$

which is related the above normalising coefficient.
gp > mass(4)
% = 1/696729600
gp > mass(8)
% = 691/277667181515243520000

Application to Minkowski-Siegel Mass constant (continued)

The present result says that the p-regular part of $1/m_k$ is a product of values of the p-adic meromorphic functions (5) at $t_j = (1+p)^j - 1$, $j = k$ and $j = 2, 4, \ldots, 2k-2$.

It is known that the rational number m_k becomes very large rapidly, when k grows (using the functional equation). It means that the denominator of $1/m_k$ becomes enormous. The explicit formula (10) applied to the reciprocal of the product of L-functions as above shows that these are only irregular primes which contribute to the denominator, and this contribution can be evaluated for all primes knowing the Newton polygons of the polynomial part P_θ, which can be found directly from the Eisenstein measure. Precisely, for the distingushed polynomial $P(t) = P_\theta(t) = a_d t^d + \cdots + a_0$, $\mathrm{ord}_p a_d = 0$, and $\mathrm{ord}_p a_i > 0$ for $0 \leq i \leq d-1$, and $\mathrm{ord}_p(t_j) = \mathrm{ord}_p j + 1$, where $t_j = (1+p)^j - 1$ for $j = k$ and $j = 2, 4, \ldots, 2k-2$. Then

$$\mathrm{ord}_p P(t_j) = \min_{i=0,\ldots,d} \left(\mathrm{ord}_p a_{i,k} + i(\mathrm{ord}_p j + 1) \right).$$

the values $\mathrm{ord}_p a_{i,k}$ for $0 \leq i \leq d$ come from the Iwasawa series in the denominator in the left hand side of (10). Also, it gives an important information about the location of zeroes of the polynomial part as in (10)). However $P(t_j) \neq 0$ in our case because all the L-values in question do not vanish.

Application to Minkowski-Siegel Mass constant (numerical illustration)

```
for(k=1,10,print(2*k, factor(denominator(1/mass(2*k)))))
2 1
4 1
6 1
8 [691, 1]
10 [691, 1; 3617, 1; 43867, 1]
12 [131, 1; 283, 1; 593, 1; 617, 1; 691, 2; 3617, 1;
43867, 1]
14 [103, 1; 131, 1; 283, 1; 593, 1; 617, 1; 691, 1;
3617, 1; 43867, 1; 6579 31, 1; 2294797, 1]
16 [103, 1; 131, 1; 283, 1; 593, 1; 617, 1; 691, 1;
1721, 1; 3617, 2; 9349, 1; 43867, 1; 362903, 1;
657931, 1; 2294797, 1; 1001259881, 1]
18 [37, 1; 103, 1; 131, 1; 283, 1; 593, 1; 617, 1;
683, 1; 691, 1; 1721, 1; 3617, 1; 9349, 1; 43867, 2;
362903, 1; 657931, 1; 2294797, 1; 305065927, 1;
1001259881, 1; 151628697551, 1]
20 [103, 1; 131, 1; 283, 2; 593, 1; 617, 2; 683, 1;
691, 1; 1721, 1; 3617, 1; 9349, 1; 43867, 1; 362903,
1; 657931, 1; 2294797, 1; 305065927, 1; 1001259881, 1;
151628697551, 1; 154210205991661, 1;
26315271553053477373, 1]
```

Methods of constructing p-adic L-functions

Our long term purposes are to define and to use the p-adic L-functions in a way similar to complex L-functions via the following methods:

(1) Tate, Godement-Jacquet;

(2) the method of Rankin-Selberg;

(3) the method of Euler subgroups of Piatetski-Shapiro and the doubling method of Rallis-Böcherer (integral representations on a subgroup of $G \times G$);

(4) Shimura's method (the convolution integral with theta series), and

(5) Shahidi's method.

There exist already advances for (1) to (4), and we are also trying to develop (5).

We use the Eisenstein series on classical groups and p-adic integral of Shahidi's type for the reciprocal of a product of certain L-functions.

Link to Shahidi's method in the case of SL(2) and regular prime p

The starting point here is the Eisenstein series

$$E(s, P, f, g) = \sum_{\gamma \in P \backslash G} f_s(\gamma g),$$

on a reductive group G and a maximal parabolic subgroup $P = MU^P$ (decomposition of Levi).
This series generalizes

$$E(z, s) = \frac{1}{2} \sum \frac{y^s}{|cz + d|^{2s}}, \quad (c, d) = 1.$$

Here f_s is an appropriate function in the induced representation space $I(s, \pi) = Ind_{P_\mathbb{A}}^{G_\mathbb{A}}(\pi \otimes |det_M(\cdot)|_\mathbb{A}^s))$, see (I.2.5.1) at p. 34 of [GeSha].

Computing a non-constant term (a Fourier coefficient)

of this Eisenstein series provides an analytic continuation and the functional equation for many Langlands L functions $L(s, \pi, r_j)$.

In this way the ψ-th Fourier coefficient (with ψ of type $\psi(x) = \exp(2\pi inx), n \in \mathbb{N}, n \neq 0$) of the series $E(s, P, f, e)$ is determined by the Whittaker functions W_v in the form (see [GeSha], (II.2.3.1), p.78):

$$E^{\psi}(e, f, s) = \prod_{v \in S} W_v(e_v) \prod_{j=1}^{m} \frac{1}{L^S(1 + js, \pi, r_j)},$$

where r_j are certains fundamental representations of the dual group $^L M$.

Theorem (a complex version)

With the data $G = SL(2)$, $M = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\} \cong \mathrm{GL}_1$, $\pi = I$, and ψ a non-trivial character of the group $U(\mathbb{A})/U(\mathbb{Q})$, $U = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\} \cong \mathbb{G}_a$, let $E^\psi(s, f, e) = \int E(s, f, n)\psi(n)dn$, the integration on the quotient space of $U(\mathbb{A})$ by $U(\mathbb{Q})$. Then the first Fourier coefficient has the form

$$E^\psi(s, f, e) = W_\infty(s)\frac{1}{\zeta(1+s)},$$

for a certain Whittaker function $W_\infty(s)$ (see [Kub], p.46).

Theorem (a p-adic version, work in progress with S.Gelbart, S.Miller, F.Shahidi)

Let p be a regular prime. Then there exists an explicitly given distribution μ^* on $\mathbb{Z}_p^*$ such that for all $k \geq 3$ and for all primitive Dirichlet characters χ mod p^v with $\chi(-1) = (-1)^k$ one has

$$\int_{\mathbb{Z}_p^*} \chi y_p^k \mu^* = \frac{1}{(1 - \chi(p)p^{k-1})L(1-k, \chi)},$$

where $L(s, \chi)$ is the Dirichlet L-function. More precisely, the distribution μ^* can be expressed through the non-constant Fourier coefficients of a certain Eisenstein series Φ^*.

Remark. Using Siegel's method for the symplectic groups GSp_m, and for all primes p, this result also follows from Main Theorem (1) by specializing to the case of regular p and $m = 1$.

Further applications: we only mention the proof of the p-adic Miyawaki Modularity Lifting Conjecture

by pullback of families Siegel modular forms (jointly with Hisa-Aki Kawamura), see [Kawa], [Palsr11].

Ikeda's constructions ([Ike01], [Ike06]) extend the doubling method to pullbacks of cusp forms instead of pullbacks of Eisenstein series.

The use of the The Eisenstein family $\mathcal{E}_k^{(n)}$ as above plays a crucial role in Ikeda's work: the idea was to substitute the Satake parameter $\alpha_p(k)$ of a cusp form in place of the parameter k in the Siegel-Eisenstein family.
Both p-adic and complex analytic L-functions are produced in this way.
Thus obtained cuspidal p-adic measures generalize the Eisenstein measure, and produce families of cusp forms.
A version of this construction produces Klingen-Eisenstein series and Langlands Eisenstein series, see [PaSE] (p-adic Peterson product of a cusp form with a pullback of the constructed family), more recently used by Skinner-Urban [MC].

For genus two, my student P.Guerzhoy found a p-adic version of the holomorphic Maass-Saito-Kurokawa lifting [Gue], answering a question of E.Freitag.

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