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MODULAR FORMS

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In this survey there are included results of recent years, concerning the theory of modular forms and representations connected with them of adele groups and Galois groups. There is discussed the hypothetical principle of functoriality of automorphic forms and other conjectures of Langlands concerning automorphic forms and the L-functions connected with them.

The choice of title for this survey may not seem entirely successful: is it really possible, within the limits of a small paper, to elucidate all aspects of the theory of modular forms, recently enduring a period of heavy development (cf. the foreword to Lang's book [33] and the survey of Fomenko [41]). Hence we restrict ourselves only to those aspects of it which are directly connected with the theory of representations and L-functions. This approach allows us to explain the connection between one-dimensional and multidimensional modular forms from the point of view of the general principle of functoriality of automorphic forms, and also the connection of modular forms with representations of Galois groups of extensions of global and local fields. In our view, precisely these connections motivate the fundamental interest in modular forms. We have touched on here only papers of the last 3-4 years, turning to older papers only when necessary; one can become acquainted with earlier results in this domain through the survey [41], which, together with Lang's book [33], contains a detailed account of the latest achievements in the theory of one-dimensional (classical) modular forms. Our account is in some measure superficial: the reason for this is the technicality and complexity of the basic methods of the contemporary theory of automorphic forms, a complete picture of which is given by the materials of the summer schools taking place in Antwerp (1972) [174] and Bonn (1976) [175], the symposium on L-functions, automorphic forms and representations in Corvallis, (1977) [58] and the conference on automorphic forms in number theory in Oberwolfach (1979) [48].

For the convenience of the reader we recall the connection of the classical theory of modular forms with representation theory, and also the more general concept of automorphic form on a reductive group. We note that a better account of the foundations of the classical theory can be found in Rankin's book [191] (cf. also the references in [41]), and the recent book of Weil [240] recalls the enduring value of the classical traditions in the theory of elliptic and modular functions.

In the last part of the survey there are noted the most interesting, from our point of view, achievements of recent years relating to other areas of the theory of modular forms.

1. Modular Forms and L-Functions. Connection with the Theory of Group Representations

Classical modular forms are introduced as functions on the upper complex half plane $H = \{z \in C \mid \mathrm{Im}(z) > 0\}$. Let Γ be a congruence-subgroup of the modular group $SL_2(Z)$, i.e., $\Gamma \supset \Gamma_N$, for some integer $N \geq 0$, where

$$\Gamma_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\, N) \right\}$$

is the principal congruence-subgroup of level N. The group $G_R^+ = GL_2^+(R)$ of matrices with positive determinant acts on H by linear-fractional transformations $z \to (az+b)/(cz+d) = \sigma(z)$, $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_R^+$.

A holomorphic function $f : H \to C$ is called a modular form of weight k with respect to the group Γ, if

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1) the condition of automorphicity

$$f((az+b)/(cz+d))(cz+d)^{-k}=f(z) \tag{1.1}$$

holds for elements $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$,

2) f is regular at parabolic vertices $P \in Q \cup i\infty$ (fixed points of parabolic elements of the group Γ); this means that for any element $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbf{Z})$ the function $f((az+b)/(cz+d))(cz+d)^{-k}$ admits an expansion in a Fourier series in nonnegative powers $q^{1/N}+e(z/N)$ (by tradition $q=e(z)=\exp(2\pi iz)$). In particular,

$$f(z)=\sum_{n=0}^{\infty}a_n e(nz/N), \tag{1.2}$$

f is called a parabolic form, if f vanishes at parabolic points (i.e., in the Fourier series from 2) only positive powers $q^{1/N}$) [33, 41, 191].

The C-linear space of modular (parabolic) forms of weight k with respect to Γ is denoted by $M_k(\Gamma)$ (respectively, $S_k(\Gamma)$).

Fundamental attention in our survey is given to the investigation of Dirichlet series of the form

$$L_f(s)=\sum_{n=1}^{\infty}a_n n^{-s}=\frac{(2\pi/N)^s}{\Gamma(s)}\int_0^{\infty}f(iy)\,y^{s-1}dy \tag{1.3}$$

the Mellin transforms of f, and also their generalizations, connected with the consideration of multidimensional automorphic forms.

Interest in the study of series of the form (1.3) is connected with the following properties of $L_f(s)$:

1) In the space of modular forms $M_k(\Gamma)$ there exists a basis, consisting of forms f such that the arithmetic functions of the form $n \to a_n$ are multiplicative $a_{nm}=a_n a_m$ (for $(n, m)=1$), here the Dirichlet series $L_f(s)$ admits an expansion as an Euler product of p-factors, corresponding to prime numbers p, and the coefficients a_n are algebraic integers.

2) If f is a modular form, then the Dirichlet series $L_f(s)$, convergent in some right half plane, admits a meromorphic continuation to all $s\in\mathbf{C}$ and satisfies a certain functional equation, connecting $L_f(s)$ and $L_f(k-s)$. Here $L_f(s)$ is an entire function, if f is a parabolic form [114, 33].

Properties 1) and 2) were established by Hecke. As an illustration we consider the example of the Ramanujan parabolic form:

$$\Delta(z)=q\prod_{n=1}^{\infty}(1-q^n)^{24}=\sum_{m=1}^{\infty}\tau(m)\,q^m,$$

$\Delta(z)\in S_{12}(SL_2(\mathbf{Z}))$, the Dirichlet series $L_\Delta(s)=\sum_{n=1}^{\infty}\tau(n)\,n^{-s}$ converges absolutely if $\mathrm{Re}(s)>13/2$, decomposes in the Euler product

$$L_\Delta(s)=\prod_p[1-\tau(p)\,p^{-s}+p^{11-2s}]^{-1} \quad (p-\text{prime numbers}),$$

extends to an entire function of order one on **C**, which satisfies the functional equation [33, 114]:

$$(2\pi)^{-s}\Gamma(s)\,L_\Delta(s)=(2\pi)^{s-12}\Gamma(12-s)\,L_\Delta(12-s). \tag{1.4}$$

3) The property of series (1.2) being a modular form is characterized by the analytic properties of the series $L_f(s,\chi)=\sum_{n=1}^{\infty}\chi(n)\,a_n n^{-s}$ (χ is the Dirichlet character) [243] (in particular, their functional equations).

Property 3) is called, in the theory of Dirichlet series connected with modular forms, the inverse theorem of Hecke–Weil, and property 2) is the direct theorem. Properties 1)-3) for the series $L_f(s)$ and their generalizations are discussed in Section 2.

4) The Euler products $L_f(s)$ are connected with zeta-functions, having algebrogeometric and arithmetic origins. Thus, if $f \in S_2(\Gamma)$, then $f(z)dz$ defines a differential on the Riemann surface H/Γ, which corresponds to a complete algebraic curve X_Γ, defined over the field of algebraic numbers. Here the Dirichlet series $L_f(s)$ is a factor of the Hasse–Weil zeta-function of the curve X_Γ [90, 169]. The properties of divisibility of the coefficients a_n are connected with the structure of the set of rational points of the curve X_Γ in finite extensions of the field Q [137, 203]. The values $L_f(s)$ for integral s (for example, for s = 1) are also connected with rational points (conjecture of Birch–Swinnerton–Dyer [67, 171]). The value of this connection for Diophantine geometry is illustrated by two recent achievements in the arithmetic of elliptic curves. Mazur [171] proved the conjecture on the uniform boundedness of the torsion of elliptic curves over $\mathbf{Q}$. The torsion group $E(\mathbf{Q})^{tors}$ of the elliptic curve E, defined over $\mathbf{Q}$, can be isomorphic only to one of the fifteen groups: Z/mZ ($m \le 10$, $m = 12$), $Z/2Z \times Z/2\nu Z$ ($\nu \le 4$); here all these possibilities are realized. Coates and Wiles [67] proved part of the Birch–Swinnerton–Dyer conjecture for elliptic curves E with complex multiplication by elements of the one class imaginary quadratic field K: If the group E(K) is infinite, then the Hasse–Weil zeta-function L(E/K, s) vanishes for s = 1. Another example is connected with representations of the Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. Serre and Deligne [76] made correspond to modular forms f of weight 1, two-dimensional complex representations $\rho_f:\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathbf{C})$, under which the Euler product $L_f(s)$ is identified with the Artin L-series of the representation ρ_f. Generalizations of this example are discussed in Sec. 3.

5) All generalizations of the series $L_f(s)$, which were mentioned above, are connected with the passage from modular forms of one variable to modular forms of several variables, more generally to automorphic forms. In addition there emerge close connections between modular forms of one and several variables, which are often described in the forms of identities connecting the corresponding L-functions.

These connections can be combined in the framework of a general principle of funtoriality of automorphic forms, to the discussion of which Sec. 4 is devoted.

Properties 1)-5) are more naturally reformulated in the language of representation theory [19, 61]. The first general connection between representation theory and automorphic forms was noted by Gel'fand and Fomin [20], although examples of the use of group representations in the theory of modular forms occur already in the works of Hecke [114]. We recall briefly how one can formulate the classical theory of Hecke with the help of representation theory. First we note that the automorphicity condition (1.1) is equivalent with the invariance of the function f with respect to the subgroup $\Gamma \subset G_{\mathbf{R}}^+$, if the action $f \to f|k[\sigma]$ is defined by the formula:

$$(f|_k[\sigma])(z) = j(\sigma, z)^{-k} f(\sigma(z)), \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbf{R}}^+,$$

where $j(\sigma, z) = |\det \sigma|^{-1/2}(cz + d)$ is the factor of automorphicity.

It is easy to verify that if $\sigma \in GL_2^+(\mathbf{Q})$, then $f|_k[\sigma]$ is a modular form with respect to the congruence-subgroup $\sigma^{-1}\Gamma\sigma \cap SL_2(\mathbf{Z})$ (possibly of another level). For $\sigma \in G_{\mathbf{Q}}^+ = GL_2^+(\mathbf{Q})$ we consider the double coset $\Gamma\sigma\Gamma = \bigcup_{i=1}^{\mu} \alpha_i \Gamma$ (here the left cosets $\alpha_i \Gamma$ are disjoint and their number is finite).

Then if $f \in M_k(\Gamma)$, then the linear combination $\sum_{i=1}^{\mu} f|_k[\alpha_i]$ now belongs to $M_k(\Gamma)$, which allows one to define the Hecke operators on $M_k(\Gamma)$ with the help of double cosets. In the case $\Gamma = SL_2(\mathbf{Z})$ for $\sigma = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ we set

$$T_k(p) f = p^{k/2-1} \sum_{i=1}^{\mu} f|_k[\alpha_i].$$

According to Hecke, the existence of an Euler expansion for $L_f(s)$ is equivalent to the fact that f is an eigenfunction of all Hecke operators. The eigenfunctions are constructed with the help of the Petersson scalar product on $S_k(\Gamma)$:

$$(f, g)_\Gamma = \int_{H/\Gamma} f(z) \overline{g(z)} y^{k-2} dx dy$$

(here $z = x + iy$, $y > 0$, H/Γ is the fundamental domain of Γ). The operators $T_k(n)$ commute with one another and are normal operators with respect to the scalar product introduced, which allows one to find an orthogonal basis of the space $S_k(\Gamma)$, consisting of common eigenfunctions of the operators $T_k(n)$ [33, 114].

Now we consider the C-linear space $\Omega(f)$, spanned by the set $\{f|_k[\sigma], \sigma \in G_Q^+\}$. This gives a representation of the group G_Q^+. One proves that $\Omega(f)$ is (algebraically) irreducible if and only if the Dirichlet series $L_f(s)$ has an Euler expansion (this follows from Hecke's theory, since the algebraic irreducibility of $\Omega(f)$ is equivalent with the fact that f is an eigenfunction of the Hecke operators) (cf. [19, 185]).

We consider the completion $\overline{G_Q^+}$ in the topology whose basis is the set of congruence-subgroups. Then $\overline{G_Q^+} = \{g \in \prod_p GL_2(\mathbf{Q}_p) \mid \det g_p = r > 0, r \in \mathbf{Q}\}$; where $g = \prod_p g_p$, $\mathbf{Q}_p$ is the p-adic numbers (the p are prime numbers). $\overline{G_Q^+}$ acts on $\Omega(f)$, since any element of $\Omega(f)$ is invariant with respect to some congruence-subgroup. One can show [19, 86], that the representation π of the group $\overline{G_Q^+}$ on $\Omega(f)$ admits an expansion as a tensor product $\pi_f = \otimes_p \pi_{p,f}$, where $\pi_{p,f}$ is a representation of the group $GL_2(\mathbf{Q}_p)$, while almost all $\pi_{p,f}$ are irreducible.

Instead of the group $\overline{G_Q^+}$ it is more convenient to consider the adele group

$$GL_2(\mathbf{A}) = \{g = g_\infty \prod_p g_p \mid g_\infty \in GL_2(\mathbf{R}),\quad g_p \in GL_2(\mathbf{Q}_p),$$
$$\text{here } g_p \in GL_2(Z_p) \text{ for almost all } p\},$$

and instead of the functions f on H, the functions $\tilde{f}$ on the group $GL_2(R)$:

$$\tilde{f}(g) = \begin{cases} f(g(i))\, j(g, i)^{-k}, & \text{if } \det g > 0, \\ f(g(-i))\, j(g, -i)^{-k}, & \text{if } \det g < 0, \end{cases}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Here if $f \in M_k(\Gamma)$, then

$$\tilde{f}(\gamma g) = \tilde{f}(g), \quad \text{if} \quad \gamma \in \Gamma,$$
$$\tilde{f}(xg) = e^{-ik\theta(x)} \tilde{f}(g), \text{ if x is rotation by an angle } \theta(x).$$

Whence it follows that the function $\tilde{f}$ can be considered in the same way as a function on the homogeneous space

$$\Gamma(N) \setminus GL_2(\mathbf{R}) = GL_2(\mathbf{Q}) \setminus GL_2(\mathbf{A}) / U^{(N)},$$

(where $U^{(N)} = \{g = 1 \cdot \prod_p g_p \in GL_2(\mathbf{A}),\ g_p \in GL_2(\mathbf{Z}_p);\ g_p \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\bmod N\mathbf{Z}_p)$ for $p \mid N\}$), or as a function on the adele group $GL_2(A)$; here the action of elements of G_Q^+ on f goes into the action on $\tilde{f}$ by left translations:

$$(\tilde{f}|_k[\sigma])(g) = \tilde{f}(\sigma g).$$

Modular forms f of weight k with respect to the group

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid c \equiv 0 \,(\bmod N)\right\}$$

with Dirichlet character $\psi(\bmod N)$, i.e., forms f of level N, satisfying the condition

$$f\left(\frac{az+b}{cz+d}\right) = \psi(d)(cz+d)^k f(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

go into functions $\tilde{f}$ such that

$$\tilde{f}(zg) = \tilde{\psi}(z) \tilde{f}(g),$$

where $z \in Z_A \cong A^*$ (the center of $GL_2(A)$, $\tilde{\psi}$ is a character of the group of adele classes $\tilde{\psi}: A^*/Q^* \to C^*$, extending ψ. Here we use the notation $f \in M_k(N, \psi)$ (or f is of type (N, k, ψ)).

For $f \in M_k(N, \psi)$ we consider the C-linear space $\Omega(f)$ of functions on $GL_2(Q) \setminus GL_2(A)$, generated by left translations of $\tilde{f}$ by means of $GL_2(A)$. The representation $\pi_{\tilde{f}}$ of the group $GL_2(A)$ on $\Omega(\tilde{f})$ is irreducible, if π_f is irreducible and $\pi_f \cong \pi_\infty \otimes (\otimes \pi_{p,\tilde{f}})$, where π_∞ is a representation of $GL_2(R)$, and the representation $\pi_{p,\tilde{f}}$, is equivalent to the representation $\pi_{p,f}$. $\Omega(\tilde{f})$ can be considered as a subrepresentation of the regular representation of $GL_2(A)$ in continuous functions on $GL_2(A)$ (such representations are called automorphic). Here, if f is parabolic, then $\Omega(\tilde{f}) \subset L_0^2(\tilde{\psi})$, where $L_0^2(\psi)$ is the space of measurable functions h on $GL_2(A)$, square-integrable on $Z_A GL_2(Q) \setminus GL_2(A)$ with respect to the Haar measure on $GL_2(A)$, satisfying the parabolicity condition:

$$\int_{U_Q \setminus U_A} h(ug)\, du = 0$$

for any subgroup U_A, conjugate to

$$N_A = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in A \right\} \text{ over Q (for almost all g).}$$

The Hecke operators act on elements of $\Omega(\tilde{f})$ as operators of integral convolution with functions from the Hecke algebra $\mathcal{H}_A$ ($\mathcal{H}_A$ is the algebra with respect to convolution of continuous complex functions on $GL_2(A)$ with compact support, bilaterally invariant with respect to a maximal compact subgroup $K \subset GL_2(A)$. Instead of a representation of the group $GL_2(A)$ one can hence consider the corresponding representation of the Hecke algebra $\mathcal{H}_A$ [60-62].

Jacquet and Langlands [123] as the starting point for the construction of L-functions took irreducible admissible representations of the groups $GL_2(Q_p)$. Admissibility means that the vectors of the representation space K_p are finite (where $K_p = GL_2(Z_p)$), i.e., all elements of the representation space, obtained from a fixed vector by application of elements of a maximal compact subgroup K_p, lie in a finite-dimensional vector space [61]. To each such representation π_p corresponds some diagonal element $h_p = \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}$ in the group $GL_2(C)$ [61]. In the special case when $\pi_p \cong \mathrm{Ind}(\mu_1 \otimes \mu_2)$ is the representation induced from a one-dimensional representation of the group of diagonal matrices: $\mu_1 \otimes \mu_2 \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \mu_1(x)\mu_2(y)$, where $\mu_1, \mu_2 : Q_p^* \to C^*$ are unramified characters of Q_p^*, the element h_p is equal to $\begin{pmatrix} \mu_1(p) & 0 \\ 0 & \mu_2(p) \end{pmatrix}$. If $\pi = \pi_\infty \otimes \pi_p$ is an irreducible admissible representation of $GL_2(A)$, then the L-function $L(s, \pi)$ is introduced as the Euler product $L(s, \pi) = \prod_p L_p(s, \pi_p)$, where $L_p(s, \pi_p) = \det(I - h_p p^{-s})^{-1} = [(1 - \alpha_p p^{-s})(1 - \beta_p)p^{-s}]^{-1}$ (in this definition, for simplicity we have omitted Γ, the factor corresponding to $L_\infty(s, \pi_\infty)$).

If $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(N, \psi)$ is an eigenfunction of the Hecke operators, $a_1 = 1$, then the representation $\pi_{p,f}$ corresponds to the p-factor

$$L_p(s, \pi_{p,f}) = [1 - a_p p^{-s-(k-1)/2} + \psi(p)\, p^{-2s}]^{-1}$$

and $L_f(s) = L(s + (k-1)/2, \pi_f)$. Here $h_p \in SL_2(C)$, if the character ψ is trivial. In general, if the elements of the center Z_A act trivially in some representation π, then $h_p \in SL_2(C)$; in this case π can be considered as a representation of the group $PGL_2(A)$. For irreducible admissible automorphic representations π, Jacquet and Langlands constructed an analytic continuation of the functions $L(s, \pi)$ and we get for them a functional equation of the form:

$$L(s, \pi) = \varepsilon(s) L(1-s, \tilde{\pi}), \tag{1.5}$$

where $\tilde{\pi}$ is the representation, contragradient to π, and ε is the factor $\varepsilon(s) = \prod_p \varepsilon_p(s)$ which plays the role of the constant of the functional equation [238]. For the functions $L(s, \pi_f)$ this functional equation goes into the Hecke functional equation (of type (1.4)).

Interesting classes of Euler products are connected with finite-dimensional representations r of the group $GL_2(C)$:

$$L(s, \pi, r) = \prod_p L_p(\pi_p, s, r), \tag{1.6}$$

where $L_p(s, \pi_p, r) = \det(I - r(h_p)p^{-s})^{-1}$. Products of the form (1.6) are absolutely convergent if Re(s) is sufficiently large.

Hypothetically [150] such L-functions admit analytic continuation and satisfy some functional equation. This conjecture is proved only in a few special cases:

1) $r = \mathrm{Sym}^i(\mathrm{St})$ (i = 2, 3, 4) (symmetric powers of the standard representation $\mathrm{St}: GL_2(C) \to GL_2(C)$). The representation $r = \mathrm{Sym}^2(\mathrm{St})$ is isomorphic with the adjoint representation; this case is analyzed by Gelbart and Jacquet [93]. If $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(N, \psi)$ is an eigenfunction of the Hecke operators, $a_1 = 1$, then $L(s, \pi_f, r)$ coincides with the symmetric square of the Hecke series $L(s, \pi_f) = L_f(s + (k-1)/2)$, while

$$L(s, \psi) L(s, \pi_f, r) = L(2s, \psi^2) \sum_{n=1}^{\infty} a_n^2 n^{-s(k-1)/2}, \qquad (1.4)$$

where $L(s, \psi) = \sum_{n=1}^{\infty} \psi(n) n^{-s}$ is the Dirichlet L-series. The cases $r = \mathrm{Sym}^3\mathrm{St}$, $\mathrm{Sym}^4\mathrm{St}$ are analyzed in [89].

In this example the L-function $L(s, \pi_f, \mathrm{Sym}^2 r)$ gets an interesting interpretation as the Mellin transformation of some automorphic form on the group $GL_3(A)$ (cf. Sections 2, 4).

For a discussion of the analytic properties of $L(s, \pi_f, \mathrm{Sym}^n r)$ and their connection with the Sato–Tate conjecture on the uniformity of the distribution of arguments of the eigenfunctions h_p, cf. the survey of Fomenko [41]. We give only a result of Kurokawa: if $f \in S_1(N, \varepsilon)$ and $m \geq 3$, then the Dirichlet series $\sum_{n=1}^{\infty} a_n^m n^{-s}$ and $\sum_{n=1}^{\infty} a_{n^m} n^{-s}$ can be meromorphically continued to $\{s \in C \mid \mathrm{Re}(s) > 0\}$, while the line Re(s) = 0 is the natural boundary of meromorphicity [147] (cf. with formula (1.7)!).

2) Shahidi [205] investigated the analytic properties of $L(s, \pi, r)$ in the case when π is a parabolic representation of $PGL_2(A)$, r is an irreducible four-dimensional representation of the group $SL_2(C)$.

We note that analogous constructions can be made in more generality, replacing the field **Q** by an arbitrary global field F. Here if F is completely real, then automorphic representations of $GL_2(A_F)$ (where A_F is the ring of adeles of the field F) correspond to Hilbert modular forms [35, 194].

Hecke's theory also admits a generalization to this more general case.

1) The theorem on the existence of a basis of eigenfunctions of the Hecke operators in the space of parabolic forms is now reformulated as the theorem of Gel'fand–Harder [19, 107] that the space $L_0^2(\omega)$ (for the precise definition, cf. Sec. 2), where ω is a Hecke character (a homomorphism $\omega: A_F^*/F^* \to C^*$), is a countable sum of irreducible admissible representations of $GL_2(A_F)$, while each of them occurs with finite multiplicity. This theorem is valid in considerably more generality (for connected reductive algebraic groups over a global field F).

2) Direct Theorem [123]: The L-function $L(s, \pi)$ of an automorphic representation admits meromorphic continuation to $s \in C$ and satisfies a functional equation (of the type (6)).

3) Inverse Theorem [123, 238]: the representation π is automorphic, if all functions of the form $L(s, \pi \otimes \chi)$ admit meromorphic continuation to $s \in C$, satisfy a functional equation, and have analytic properties of the given type; here χ is the Hecke character.

Another formulation of the inverse theorem is given by Li [158]. In the case F = **Q**, an interesting refinement of the inverse theorem was obtained by Razar [195].

The theory of Atkin–Lehner [33, 41] can be interpreted as multiplicity one theorem [186]: the multiplicity of irreducible admissible representations of the Hecke algebra in $L_0^2(\psi)$ does not exceed one. Moreover, a strong multiplicity one theorem holds: An automorphic representation π, occurring in $L_0^2(\psi)$ (parabolic representation), is uniquely determined by giving almost all local factors π_v [186].

The classical conjecture of Ramanujan–Petersson that for an eigenfunction of the Hecke operators $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(N, \psi)$, $a_1 = 1$ one has the estimate $|a_p| < 2p^{(k-1)/2}$, can be reformulated in the general case as the assertion that the eigenvalues of elements $h_v \in GL_2(C)$, corresponding to components π_v of the parabolic representation $\pi = \otimes_v \pi_v$ (v runs through the valuations of the field F), are in absolute value equal to 1. V. G. Drinfel'd [24] proved the generalized conjecture for parabolic forms on $GL_2(A_F)$ over global fields of positive characteristic. Earlier, Deligne proved this conjecture for F = Q [71].

2. Automorphic Forms and L-Functions. Connection with the Theory of Group Representations

We consider the more general case of a linear reductive group G over a global field E. G_A will denote the locally compact group of points of G in the ring $A = A_F$ of adeles of the field F, G_F is the discrete subgroup of F-rational points of G, K is a maximal compact subgroup of G_A, Z_A is the adele points of the center of G (cf. [19, 60, 62]).

An automorphic form f on G is introduced as a continuous function $f: C_A \to C$, satisfying the conditions:

a) f is invariant with respect to right translations by elements of G_F;

b) f is K-finite;

c) there exists a Hecke character $\psi: A_F^*/F^* \to C^*$ such that

$$f(zg) = \psi(z) f(g)$$

($z \in Z_A$, $g \in G_A$) in the case when $Z_A \cong A^*$;

d) the function $x \to f(x \cdot y)$ on the group $G_\infty = G(F \otimes R)$ of Archimedean points of G (i.e., points in $F \otimes R$) is annihilated by some ideal of finite codimension in the algebra of biinvariant differential operators on G_∞ (such a function is automatically real-analytic, since it is annihilated by an elliptic differential operator);

e) f satisfies some diminution condition;

*) the form f is called parabolic if

$$\int_{U_A/U_E} f(ux)\,dx = 0,$$

where U denotes the unipotent radical of any parabolic F-subgroup in G (cf. [60, 62]).

Automorphic representations of G_A (or of the Hecke algebra $\mathcal{H}_A$) are defined as representations, lying in the regular representation of G_A. Here the parabolic forms f lie in the space $L_0^2(\psi)$, consisting of measurable functions h on G_A/G_F, satisfying condition c), such that $x \to h(x)|\psi(\det x^{-1})|^{1/2}$ is square integrable on $G_A/Z_A G_F$, and satisfying the parabolicity condition (for almost all x).

Parabolic representations of G_A are defined as subrepresentations of $L_0^2(\psi)$ (cf. [60]).

By the symbol $\mathfrak{A}(G/F)$ we denote the set of equivalence classes of irreducible admissible representations of G_A. Langlands proved [153] that each $\pi \in \mathfrak{A}(G/F)$ is a component of a representation, induced from some parabolic $\sigma \in \mathfrak{A}(M/F)$, where M is a Levi F-subgroup of a parabolic F-subgroup of bG.

L-functions of irreducible admissible representations of the group G_A are introduced with the help of expansions $\pi = \otimes_v \pi_v$, where v runs through the set of valuations of the field F, π_v is a representation of the group $G(F_v)$ (of points in the v-completion) (or a representation of the local Hecke algebra $\mathcal{H}_v$) (cf. [19, 86]). Here with almost all π_v (except for a finite set of valuations $v \in S$) one can associate the conjugacy class of a semisimple element h_v in the Langlands group ${}^LG$ (${}^LG$ is a certain reductive linear group over C; for $G = GL_n$ the element $h_v \in GL_n(C)$; here the group $GL_n(C)$ coincides with a connected component of the Langlands group ${}^LG$). L-functions of representations π are introduced as Euler products of the form:

$$L(s, \pi) = \prod_{v \notin S} L(s, \chi_v), \quad L(s, \chi_v) = \det(I - Nv^{-s} h_v)^{-1}$$

(here N_v is the number of elements of the residue field for the valuation v). More generally, for finite-dimensional representations r of the group ${}^LG$ one can define

$$L(s, \pi, r) = \prod_{v \notin S} L(s, \chi_v, r), \quad L(s, \chi_v, r) = \det(I - Nv^{-s} r(h_v))^{-1}.$$

Both products converge absolutely for sufficiently large Re(s), if π is an automorphic representation (cf. [150] for parabolic π, [60] in the general case).

In the general case the dual of the Langlands group ${}^LG$ of a reductive algebraic group G is constructed with the help of the root data [221, 60]:

$$\psi_0(G)=(X^*(T), \Delta, X_*(T), \Delta^\vee)$$

of the group G; here T is a maximal torus of G (over the separable closure F^s of the ground field F), $X^*(T)$ ($X_*(T)$) is the group of characters (respectively, of one-parameter subgroups) of the torus T, $\Delta(\Delta^\vee)$ is a basis of the root system $\Phi(G, T)$ with respect to the torus T (respectively, a dual basis of coroots). The connected component ${}^LG^0$ of the Langlands group is defined as the reductive group over C, whose root data are obtained by inversion $\psi_0 \to \psi_0^\vee$, i.e., are isomorphic to the collection of data of the form:

$$\psi_0(G)^\vee=(X_*(T), \Delta^\vee, X^*(T), \Delta).$$

If G is simple, then up to a central isogeny G is characterized by one of the types A_n, B_n, ..., G_2 of the Killing–Cartan classification. It is known that the map $\psi_0(G)$ $\psi_0(G)\vee$ permutes the types B_n and C_n, and the remaining types remain fixed. Thus, if $G = Sp_{2n}$ (respectively, GSp_{2n}), then ${}^LG^0 = SO_{2n+1}(C)$ (respectively, ${}^LG^0 = Spin_{2n+1}(C)$). The group ${}^LG$ is defined as the semidirect product of ${}^LG^0$ by the Galois group of a certain extension of the field F, over which G splits, i.e., the torus T becomes isomorphic to $(GL_1)^r$. Such a semidirect product is introduced with the help of the action of the Galois group $\Gamma_F = Gal(F^s/F)$ on the group ${}^LG^0$, which is defined with the help of the action of Γ_F on the set of maximal tori, defined over F^s [60].

The group ${}^0G^L$ can be considered also over other fields, among them over global and local ones.

The construction of the classes h_v for many reductive groups is contained in [60] and is based on the detailed study of representations of reductive groups over local fields. We shall not dwell on this, referring the reader to the survey of Cartier [64].

There are several conjectures about analytic properties of L(s, π, r), verified in certain special cases.

(A) If $\pi\in\mathfrak{A}(G/F)$, then L(s, π, r) admits a meromorphic continuation to $s\in C$.

(B) One can define local L- and ε-factors at all points such that one has the functional equation:

$$L(s, \pi, r)=\varepsilon(s, \pi, r)L(1-s, \tilde{\pi}, r),$$

where $\tilde{\pi}$ is the representation contragradient to π [60, 228].

(C) In a certain number of cases it is proved that:

(*) if π is parabolic, r is irreducible and nontrivial, then L(s, π, r) is an entire function.

Property (*) does not always hold. Hypothetically, (*) is violated only when π is "lifted" from a parabolic representation of a reductive group H (the lift of automorphic forms will be discussed in Sec. 4) and the restriction of r to the image of ${}^LH$ in ${}^LG$ contains the trivial representation.

In the case $G = GL_n$, $r = r_n$ is the standard representation of $GL_n(C)$, properties B) and C) are established in [123] for n = 2 and in [101] for n > 2. Recently it was shown [223], that analogous results are valid for L-functions of automorphic representations, not necessarily parabolic.

Recently the theorem on multiplicity one was carried over to the case of $G = GL_n$ over a global field [186], which is closely connected with the theory of nonvanishing: L(s, π, r_n) $\neq 0$, if Re(s) = 1 and π is a parabolic representation of $GL_n(A_F)$ (Jacquet and Shalika [126]). We note that for n = 1, F = Q this is a classical theorem of Dirichlet. For n = 2 the theorem was established by Rankin in the case when F = Q and π corresponds to the parabolic form of Ramanajan (cf. [41]).

For n = 3 the inverse theorem is proved [124]: if all L-functions of the form L(s, $\pi\otimes\chi$, r_n) (where χ is a Hecke character, π is an irreducible admissible representation) extend holomorphically to $s\in C$, then the representation π can be realized in parabolic forms. It is noted that for $n \geq 4$ the analogous result no longer holds: hypothetically for the realizability of π in parabolic forms it is necessary to require the holomorphicity of all L-functions of the form L(s, $\pi\otimes\sigma$), where σ is any parabolic representation of $GL_j(A)$, $1 \geq j - r + 2$.

As also in the case n = 2, the proof depends essentially on the fact that the function L(s, π) admits an integral representation with the help of a certain function on $GL_3(A)$, which turns out to be a parabolic form (analog of the Mellin transform). The construction is based on the theory of models of Whittaker for local and global representations.

Jacquet [121] established properties analogous to A)-C) for automorphic representations of the group $G = GL_2 \times GL_2$ and $r = r_2 \otimes r_2$. In particular, there are defined local factors $L(s, \pi_1 \otimes \pi_2, r)$ and $\varepsilon(s, \pi_1 \otimes \pi_2, r)$ where π_i (i = 1, 2) are two irreducible admissible infinite-dimensional representations of $GL_2(F)$, F is a non-archimedean local field. However, such factors were found explicitly only for certain pairs of representations. The results of Jacquet were refined by Li [159, 160], who explicitly calculated the factors in terms of a sum of the form:

$$\sum_{f(\chi)=m} \varepsilon\left(\frac{s+1}{2}, \pi_1\otimes\chi, r_2\right) \varepsilon\left(\frac{s+1}{2}, \pi_2\otimes\chi^{-1}, r_2\right), \tag{2.1}$$

where $f(\chi)$ denotes the exponential conductor of the Dirichlet character $\chi: F^* \to C^*$ (an integer). Such sums turned out to be closely connected with the interesting concept of n-closeness of irreducible supercuspidal representations π_i (i = 1, 2). Let ω_i be central quasicharacters of the representations π_i, obtained from the restriction of π_i to the center of $GL_2(F)$. It is known that π_1 is isomorphic to π_2 [123] if and only if: 1) $\omega_1 = \omega_2$ and 2) $\varepsilon(s, \pi_1 \otimes \chi) = \varepsilon(s, \pi_2 \otimes \chi)$ for all quasicharacters χ; here if π_1 and π_2 are isomorphic, then they have identical conductors $f(\pi_1) = f(\pi_2)$. In the case when $f(\pi_1) = f(\pi_2)$, one can introduce a finer concept than n-closeness, where π_1 and π_2 are ∞-close if and only if π_1 is isomorphic with π_2 [160].

By definition, π_1 is n-close to π_2, if $f(\pi_1) = f(\pi_2)$ and n is the largest integer m such that

$$\varepsilon(s, \pi_1)\varepsilon(s, \pi_2\otimes\chi) = \varepsilon(s, \pi_2)\varepsilon(s, \pi_1\otimes\chi)$$

for all characters $\chi: F^* \to C^*$ with conductor $f(\chi) \leq m$. Tests are given for the n-closeness of two representations π_1 and π_2 in terms of sums of the form (2.1); in interesting cases such sums are explicitly calculated. The results are applied [159] for the explicit calculation of constants and local factors of the functional equation of the convolution of two L-series, connected with primitive parabolic forms (parabolic representations of GL_2 over Q). Let $f_1(z) = \sum_{n=1}^{\infty} a_1(n) q^n$, $f_2(z) = \sum_{n=1}^{\infty} a_2(n) q^n$ be the two parabolic forms of weights k_1, k_2 with respect to the groups $\Gamma_0(N_1)$, $\Gamma_0(N_2)$ with Dirichlet characters $\nu_1 \bmod N_1$, $\nu_2 \bmod N_2$, respectively. The Dirichlet series

$$L_{f_1,f_2}(s) = L(2s, \varepsilon) \sum_{n=1}^{\infty} a_1(n) \overline{a_2(n)} n^{-\left(s+\frac{k_1+k_2}{2}-1\right)} \tag{2.2}$$

(where $L(s, \varepsilon) = \sum_{n=1}^{\infty} \varepsilon(n) n^{-s}$, $\varepsilon = \nu_1 \bar{\nu}_2 \bmod N$, $N =$ l.c.m. (N_1, N_2)) is called the convolution of the L-series connected with f_1 and f_2. The series $L_{f_1, f_2}(s)$ were first considered by Rankin, who constructed the analytic extension of such series and got for them a functional equation in the special case N = 1. In the general case Li [159] established a functional equation, connecting $L_{f_1, f_2}(s)$ and $L_{\overline{f_1}, \overline{f_2}}(1-s)$, where

$$\bar{f}_i(z) = \sum_{n=1}^{\infty} \overline{a_i(n)}\, e(nz).$$

The constants and local factors of this functional equation are calculated completely explicitly and are connected with the action of the Atkin–Lehner involutions [57] (W-operators) on the modular forms with respect to the group $\Gamma_0(N)$. W-operators correspond to each prime divisor $q \mid N$ and carry a primitive normalized form f of level N into another normalized form of the same type, multiplied by a certain number $\lambda_q(f)$ ("pseudo-eigenvalue" of the W-operator on f). The constants of the functional equation of the series (2.2) are calculated [159] in terms of products of numbers of the form $\lambda_q(f_i)$; in its own right, $\lambda_q(f_i)$ is described in [57] in terms of a Gaussian sum. In special cases this functional equation was obtained earlier [36, 41].

In the considerably more general case $G = GL_m \times GL_n$, $r = r_m \otimes r_n$ over a functionally global field F one can define [122] local L- and ε-factors. Hypothetically (Jacquet dixit, [60]) here properties A) and B) hold and also holomorphicity (excluding the case when m = n, $\pi = \pi_1 \otimes \pi_2$ and π_1 is contragradient to π_2; here π_i are automorphic representations of GL_m).

Interesting classes of L-functions, connected with symplectic groups, were introduced and studied by Andrianov [1-6, 47]. In this case $G = GSp_{2n} = \{g \in GL_{2n} \mid g^t J_n g = r(g) J_n\}$, where $r(g) \in GL_1$,

$$J_n = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

E_n is the identity matrix of size $n \times n$. The dual Langlands group ${}^LG^0$ in this case coincides with the universal covering $Spin_{2n+1}(C)$ of the orthogonal group $SO_{2n+1}(C)$. We shall use the standard representation of the orthogonal group

$$St_{2n+1}: SO_{2n+1}(C) \hookrightarrow GL_{2n+1}(C),$$

where $SO_{2n+1}(C) = \{g \in L_{2n+1}(C) \mid g^t G_n g = G_n\}$

$$G_n = \begin{pmatrix} 0 & E_n & 0 \\ & & \vdots \\ E_n & 0 & 0 \\ 0\ldots & 0 & 1 \end{pmatrix}.$$

If π is an irreducible admissible representation of the group G over the global field F, $\pi = \otimes_v \pi_v$, then for almost all v the representation π_v corresponds to the conjugacy class of a semisimple element h_v in $Spin_{2n+1}(C)$, whose image in $SO_{2n+1}(C)$ is the diagonal matrix

$$\operatorname{diag}\{\alpha_{1,v}, \ldots, \alpha_{n,v}, \alpha_{1,v}^{-1}, \ldots, \alpha_{n,v}^{-1}, 1\} \in SO_{2n+1}(C).$$

The element h_v itself in the spinor representation ρ_n of dimension 2^n of the group $Spin_{2n+1}(C)$ (or of the group $SO_{2n+1}(C)$) can be described by a diagonal matrix of the form:

$$\operatorname{diag}\{\beta_{0,v}, \beta_{0,v}\alpha_{1,v}, \ldots, \beta_{0,v}\alpha_{i_1,v}\alpha_{i_2,v}\ldots\alpha_{i_r,v}, \ldots\},$$

here for each $r \leq n$ one considers all products of the form $\beta_{0,v}\alpha_{i_1,v}\alpha_{i_2,v}\ldots\alpha_{i_r,v}$, $1 \leq i_1 < i_2 < \ldots < i_r \leq n$, and the number $\beta_{0,v}$ up to sign by the normalizing condition:

$$\beta_{0,v}^2\alpha_{1,v}\ldots\alpha_{n,v} = 1.$$

Here $|\det h_v| = 1$.

The element h_v is defined uniquely up to the action of the Weyl group Γ generated by permutations of the coordinates $\alpha_{1,v}, \ldots, \alpha_{n,v}$ and by maps

$$\beta_{0,v} \mapsto \beta_{0,v}\alpha_{i,v}, \quad \alpha_{i,v} \mapsto \alpha_{i,v}^{-1}, \quad \alpha_{j,v} \mapsto \alpha_{j,v} \quad (j \neq 0, i;\ i = 1, \ldots, n).$$

For automorphic L-functions of the form $L(s, \pi_f, r)$, where π_f is an automorphic representation of the group G over the field **Q**, connected with a parabolic Siegel form with respect to $\Gamma_n = Sp_{2n}(Z)$, $r = \rho_n$, $n = 2$, Andrianov [4] established properties A)-B) and investigated for which parabolic forms f one has the holomorphicity property C). Evdokimov [26] and Matsuda [170] extended these results to the case of congruence-subgroups of Γ_n. We note that for $n = 1$, $GSp_2 = GL_2$, $Spin_3(C) = GL_2(C)$, $\rho_1 = r_2$ (standard representation), and the L-functions $L(s, \pi, r_2)$ are studied by Jacquet and Langlands [123].

The general case (for $n = 2$) is analyzed in [180], where there are studied the L-functions corresponding to irreducible admissible representations of the groups $G = GSp_4$ and $G = GSp_4 \times GL_2$ over a global field F, and the representations of L-groups ${}^LG$ of the form: $r = \rho_2$, $r = \rho_2 \times \rho_2$ (we recall that r_2 is the standard representation of GL_2). For the case of a functional field F in [180] properties A) and B) are established, and for number fields F only the part relating to analytic continuation is proved. We note that for $n \geq 3$ the analytic properties of the functions $L(s, \pi, \rho_2)$ have not yet been investigated.

Andrianov and Kalinin [5, 47] studied analytic properties of standard zeta-functions of Siegel modular forms, which have the form $L(s, \pi_f, St_{2n+1})$, where π_f is an automorphic representation of G over **Q**, connected with the Siegel modular form f of genus n with respect to the congruence-subgroup

$$\Gamma_0^n(q) = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbf{Z}) \,\middle|\, C \equiv 0 \pmod{q}\right\}.$$

We note that in the case $n = 1$, L-functions of the form $L(s, \pi_f, St_3)$ coincide with symmetric squares of Hecke series; the holomorphicity of such L-functions is investigated by Shimura [41] (cf. also [38]). In [5] there is constructed the meromorphic continuation of the function $L(s, \pi_f, St_{2n+1})$ for any even n; in the special case $q = 1$ under certain additional restrictions it is proved that these L-functions are holomorphic, if one excludes a finite number of poles, and satisfy a functional equation of the type B). The case $n = 2$ was analyzed earlier by Andrianov [47] and Gritsenko [23].

Orthogonal groups of quadratic forms of an odd number of variables are dual according to Langlands [60, 150] to symplectic ones: If $G = SO_{2n+1}$, then the dual group ${}^LG^0$ is $Sp_{2n+1}(C)$ (for $n = 2$, however, the groups

$Sp_4(C)$ and $SO_5(C)$ are locally isomorphic: $Sp_4(C) \cong Spin_5(C)$).

L-functions connected with irreducible admissible representations of orthogonal groups of the form $G = SO_{2n+1}$ are studied in the case when the ground field F is a (global) functional field (cf. [181]), and r is the standard representation of the symplectic group ${}^LG = Sp_{2n}(C)$. Here properties A)–C) are established.

We dwell in more detail on the case of L-functions connected with Siegel modular forms. We recall that the Siegel modular functions of genus n are introduced as holomorphic functions $f: H_n \to C$ on the Siegel upper half plane of genus $n \geq 1$ [4]:

$$H_n = \{Z = X + iY \in M_n(\mathbf{C}),\ {}^tZ = Z, \text{ positive definite}\}$$

so that:

1) for each element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(Z)$

$$\det(CZ+D)^{-k} f(M\langle Z\rangle) = f(Z), \quad Z \in H_n,$$

where $M\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$ is an analytic automorphism of H_n; k is called the weight of f;

2) f(Z) is bounded in each domain of the form

$$\{Z = X + iY \in H_n,\ Y \geqslant cE_n,\ c > 0\}.$$

The symbol M_k^n will denote the complex space of Siegel modular forms of genus n of weight k with respect to $\Gamma_n = Sp_{2n}(Z)$. Analogous definitions are introduced for congruence-subgroups $\Gamma \subset \Gamma_n$.

Each modular form $f \in M_k^n$ has a Fourier expansion of the form:

$$f(Z) = \sum_{N \in \mathfrak{N}_n,\ N \geqslant 0} a(N)\, e(\mathrm{Tr}(NZ)),$$

where Tr(NZ) is the trace of the matrix NZ,

$$\mathfrak{N}_n = \{N = (n_{ij}) \in M_n(\mathbf{Q}),\ {}^tN = N,\ n_{ii},\ 2n_{ij} \in \mathbf{Z}\}$$

is the set of symmetric half-integral matrices of order n.

Parabolic forms are introduced as $f \in M_k^n$ such that $a(N) = 0$, if $\det N = 0$, and form a linear space S_k^n.

Since H_n is a homogeneous space of the group $Sp_n(R) \to H_n$, the image of the Haar measure on $Sp_n(R)$ under the map $Sp_n(R) \to H_n$ defines uniquely up to a constant factor a volume element, invariant with respect to $Sp_n(R)$:

$$\tilde{d}Z = (\det Y)^{-(n+1)} \prod_{\alpha < \beta} dx_{\alpha\beta} \prod_{\alpha < \beta} dy_{\alpha\beta} \quad (Z = X + iY).$$

For each pair of modular forms f, $f_1 \in M_k^n$ the measure on H_n

$$f(Z) f_1(Z) (\det Y)^k \tilde{d}Z$$

is invariant; here the integral

$$(f, f_1) = \int_{D_n} f(Z) \overline{f_1(Z)} (\det Y)^k \tilde{d}Z \tag{2.3}$$

(where D_n is some fundamental domain of the group Γ_n) converges absolutely if at least one of the forms f, f_1 is parabolic, independent of the choice of fundamental domain and defines a nondegenerate Hermitian pairing. The orthogonal complement E_k^n to S_k^n in M_k^n is called the space of Eisenstein series of genus n of weight k (cf. [4, 29]): $M_k^n = S_k^n \oplus E_k^n$.

L-functions are introduced with the help of Hecke operators; these operators correspond to double cosets of the form $\Gamma_n g \Gamma_n$, where

$$g \in S^{(n)} = \{g \in M_{2n}(\mathbf{Z}) \mid {}^tgJ_ng = r(g)J_n,\ r(g) = 1, 2, \ldots\}.$$

By definition

$$T_k(\Gamma_n g \Gamma_n) f = r(g)^{nk - n(n+1)/2} \sum_{i=1}^{\mu} f|_k[\sigma_i]$$

(where $\Gamma_n g \Gamma_n = \bigcup_{i=1}^{\mu} \Gamma_n \sigma_i$, and for $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S^{(n)}$

$$(f|_k[\sigma])(Z) = \det(CZ+D)^{-k} f((AZ+B)(CZ+D)^{-1}).$$

The Hecke algebra $L^{(n)}$ is defined as the free C-module, generated by double cosets $\Gamma_n g \Gamma_n$; multiplication in $L^{(n)}$ is introduced so that the product of double cosets corresponds to the product of operators $T_k(\Gamma_n g \Gamma_n)$: the map

$$(\Gamma_n g \Gamma_n) \mapsto T_k(\Gamma_n g \Gamma_n)$$

defines a representation of $L^{(n)}$ on the space M_k^n, where S_k^n is an invariant subspace.

We set

$$T(m) = \sum_{r(g)=m} (\Gamma_n g \Gamma_n) \in L^{(n)}.$$

The local Hecke algebra $L_p^{(n)}$ (p is a prime) is introduced as the subalgebra of $L^{(n)}$, generated by elements of the form: $(\Gamma_n g \Gamma_n)$ with $r(g) = p^{\delta}$.

The representation $L_p^{(n)}$ on $S_k^{(n)}$ decomposes with respect to the characters of the algebra $L_p^{(n)}$ with the help of the scalar product (2.3). All characters $\lambda: L_p^{(n)} \to C$ can be written thus: we choose in the double coset $\Gamma_n g \Gamma_n$ triangular representatives g_i of the left cosets $\Gamma_n g \Gamma_n = \bigcup_{i=1}^{\mu} \Gamma_n g_i$

$$g_i = \begin{pmatrix} p^{\delta_i} {}^t D_i^{-1} & B_i \\ 0 & D_i \end{pmatrix}, \quad D_i = \mathrm{diag}(p^{d_{i1}}, \ldots, p^{d_{in}}).$$

Then (cf. [1, 4]) there exist complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that

$$\lambda(\Gamma_n g \Gamma_n) = \sum_{i=1}^{u} \alpha_0^{\delta_i} \prod_{j=1}^{n} (\alpha_j p^{-j})^{d_{ij}}.$$

For characters λ, obtained from the representation $L_p^{(n)}$ on M_k^n one has the relation:

$$\alpha_0^2 \alpha_1 \ldots \alpha_n = p^{kn-n(n+1)/2}.$$

We set $\beta_0 = \alpha_0 p^{-(2kn-n(n+1))/4}$, so the collection of p-parameters $\beta_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ defines a semisimple element h_p in $\mathrm{Spin}_{2n+1}(C)$ (and in $SO_{2n+1}(C)$) (cf. above).

To each eigenfunction $f \in M_k^n$ of the Hecke algebra $L^{(n)}$, thus corresponds a collection of p-parameters $\alpha_{0,p}, \alpha_{1,p}, \ldots, \alpha_{n,p}$ and eigenvalues $\lambda_f(\Gamma_n g \Gamma_n)$:

$$T_k((\Gamma_n g \Gamma_n)) f = \lambda_f(\Gamma_n g \Gamma_n) f,$$

in particular:

$$T_k(m) f = \lambda_f(m) f.$$

Andrianov proved [4] that the Dirichlet series $D_f(s) = \sum_{m=1}^{\infty} \lambda_f(m) m^{-s}$, which is absolutely convergent on some right half plane, admits an expansion as an Euler product

$$D_f(s) = \prod_p \left(\sum_{\delta=0}^{\infty} \lambda_f(p^{\delta}) p^{-\delta s} \right) = \prod_p D_{p,f}(s),$$

where each p-factor $D_{p,q}(s)$ is a rational-fraction of p^{-s}:

$$D_{p,f}(s) = P_{p,f}(p^{-s}) Q_{p,f}(p^{-s})^{-1},$$

where $P_{p,f}(t)$ and $Q_{p,f}(t)$ are polynomials with real coefficients of degree $2^n - 2$ and 2^n, respectively; the polynomial $Q_{p,f}(t)$ has the form:

$$Q_{p,f}(t) = 1 - \lambda_f(p) t + \ldots + p^{2^{n-1}(nk-n(n+1)/2)} t^{2^n} = (1-\alpha_{0p} t) \prod_{r=1}^{n} \prod_{1 \leq i_1 < i_2, \ldots < i_r \leq n} (1 - \alpha_{0,p} \alpha_{i_1,p} \ldots \alpha_{i_r,p} t).$$

The zeta-funtion of the modular form f is the Euler product:

$$Z_f(s) = \prod_p Q_{p,f}(p^{-s})^{-1}.$$

In our earlier notation

$$Z_f(s)=L(s+(2kn-n(n+1))/4, \pi_f, \rho_n).$$

The proof of the theorem on analytic continuation and the functional equation of $Z_f(s)$ in the case n = 2 is based on the connection between the Fourier coefficients $a(N)$ of the form f and the eigenvalues $\lambda_f(m)$ of the Hecke operators, discovered by Andrianov [4]. This connection is that the function $Z_f(s)$ can be represented in the form of a linear combination of series of the form:

$$R_N(s)=R_{N,f}(s)=\sum_{m=1}^{\infty} a(mN)\,m^{-s},$$

where $N=\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ is an arbitrary positive definite half-integral matrix. Here the Dirichlet series $R_N(s)$ can be obtained by integrating the restriction of the form f to the three-dimensional real domain $H_N \subset H_2$. On H_N there acts a certain discrete arithmetic subgroup Γ_N of the group $SL_2(C)$. The proof of the theorem on the analytic properties of $Z_f(s)$ follows from the properties of Eisenstein series which are automorphic forms with respect to Γ_n on H_N.

In the case of arbitrary n the connection between eigenvalues of Hecke operators and Fourier coefficients is studied in [1], where there is also investigated the interesting question about the action of the Hecke operators on theta-series of genus n, i.e., functions of the form:

$$\Theta_A^{(n)}(Z)=\sum_{X\in M_{m,n}(Z)} e(\mathrm{Tr}\,({}^tXAXZ)/2)=\sum_B r_A(B)\,e(\mathrm{Tr}\,(BZ)/2).$$

Here $Z\in H_n$, B runs through all integral matrices of order n with even diagonal, satisfying the conditions: ${}^tB = B$, $B \geq 0$ and $r_A(B)$ denotes the number of integral representations of the quadratic form with matrix B/2 by a quadratic form with matrix A/2.

Freitag [88] solved the question about the invariance of the spaces of theta series of genus n of quadratic forms A of given type with respect to Hecke operators, based on the theory of singular modular forms [188]. Andrianov gave an effective expression for the image of any theta-series of a quadratic form with an even number of variables under the action of the Hecke operators in the form of a linear combination of theta-series of the same type. The coefficients of such a linear combination are expressed explicitly in terms of trigonometric sums; from these formulas it follows in particular, that all eigenvalues of suitable systems of generators of the Hecke algebras $L^{(n)}$ are algebraic integers.

There are explicitly calculated the expansions into factors of Hecke polynomials for genus n (cf. also [2, 27]). It is proved, in particular, that the standard zeta-functions

$$L(s, \pi_f, \mathrm{St}_{2n+1})=L(f, s)=\prod_p L_p(f, s),$$

where $L_p(f, s)=\left[(1-p^{-s})\prod_{i=1}^{n}(1-\alpha_{i,p}p^{-s})(1-\alpha_{i,p}^{-1}p^{-s})\right]^{-1}$ can also be expressed in terms of Fourier coefficients, more precisely, in terms of series of the form

$$\sum_{M\in SL_n(Z)\setminus M_n^+(Z)} a(MN\,{}^tM)(\det M)^{-(s+k-1)}$$

(cf. [1, 5]). This allows us to write the integral representation for L(f, s) in the form of the integral convolution of f with a certain theta-series of genus n. The proof of the theorem on the analytic properties of L(f, s) is obtained with the help of a modification of Rankin's method [41, 47].

Another class of zeta-functions, connected with Siegel modular forms, was investigated by Arakawa [49, 50]. If

$$f(Z)=\sum_{N\in\mathfrak{n}_n} a(N)\,e(\mathrm{Tr}\,(NZ))\in M_k^n$$

is a Siegel modular form of genus n, then one can define a Dirichlet series, using only the coefficients $a(N)$.

For a positive-definite symmetric matrix N we denote by the symbol $\varepsilon(N)$ the order of the finite group $\{U\in SL_n(Z) \mid {}^tUNU = N\}$. It is proved that the Dirichlet series

$$D_n(f, s)=\sum_{N>0} a(N)\,\varepsilon(N)^{-1}(\det N)^{-s}$$

(where the summation is taken over classes of $SL_n(Z)$-equivalent positive definite half-integral symmetric matrices) admits a meromorphic continuation to $s \in C$ and satisfies a certain functional equation; there are also calculated the residues of the function $D_n(f, s)$ at its finite number of poles.

3. Automorphic Forms and Artin's Conjecture

An important class of arithmetic L-functions is made up of the Artin L-series, connected with complex representations of Galois groups of extensions of global fields.

If K is a Galois extension of the global field F with Galois group Gal(K/F) and

$$\sigma: \mathrm{Gal}(K/F) \to GL_n(C)$$

is an n-dimensional complex representation of Gal(K/F), then the Artin L-function is introduced as the Euler product:

$$L(s, \sigma) = \prod_v L(s, \sigma_v)$$

over all valuations v of the field F. Here σ_v denotes the restriction of σ to the decomposition group of Gal(K/F) for the valuation v, while for valuations v, unramified in K,

$$L(s, \sigma_v) = \det(I - \sigma(\mathrm{Fr}_v)\, Nv^{-s})^{-1},$$

where Fr_v is the Frobenius element over v (cf. [89]).

Artin Conjecture. If σ is irreducible and nontrivial, then $L(s, \sigma)$ extends to an entire function on $s \in C$.

For representations, induced from a one-dimensional representation of a subgroup (monomial representations) $L(s, \sigma)$ reduces to the function $L(s, \chi)$, where χ is a Hecke character of finite order (E. Artin); in the general case Brauer proved the meromorphic extandability of $L(s, \sigma)$ to $s \in C$, using integral virtual expansions of characters of representations σ with respect to characters of monomial representations.

A new approach to the proof of Artin's conjecture was suggested by Langlands, who assumed that $L(s, \sigma)$ coincide with L-functions of irreducible parabolic representations of GL_n over F. The validity of Artin's conjecture would then follow from the direct theorem of Jacquet (cf. Sec. 2 and [101]).

In the case n = 2, the image of σ(Gal (K/F)) in $PGL_2(C)$ can only be isomorphic (cf. [33]) to the following groups:

1) the dihedral group; in this case σ is monomial,

2) A_4 (tetrahedral case),

3) S_4 (octahedral case),

4) A_5 (icosahedral case).

Langlands proved the conjecture on the coincidence of L-functions in case 2), 3) (cf. the account in [89]).

The validity of Artin's conjecture in case 4) remains an open question, although Buhler (cf. [63]) gave an example of a representation of type 4), for which Artin's conjecture is valid. Here F = **Q**, K is the splitting field of the polynomial

$$x^5+10x^3-10x^2+35x-18.$$

Buhler's result is based on a construction of Serre and Deligne (cf. [76]), which associated with each primitive modular form f of type (N, 1, ψ) with odd Dirichlet character ψ mod N some two-dimensional complex representation ρ_f:Gal (**Q**/D) $\to GL_2(C)$. In Buhler's example N = 800.

Langlands' conjecture also relates to a wider class of L-functions, connected with representations of Weil groups. For a finite extension K/F of a local or global field, the Weil group $W_{K/F}$ is defined as a certain extension of the group Gal(K/F) by C_K, where $C_K = K^*$, if K is a local field, $C_K = A_{K^*}/K^*$ (the group of adele classes of K), if K is global; if the group G(K/F) is Abelian, then cohomologically this extension is given by the fundamental class $\alpha(K/F) \in H^2(G(K/F), C_K)$ from class field theory (cf. [228, 60]). The Weil group W_F is defined as the projective limit of the groups $W_{K/F}$.

In the case when F is a local nonarchimedean field or global functional field, W_F admits the following description. We consider the homomorphism $v: \mathrm{Gal}(\bar{F}/F) \to \tilde{Z}$, where $\tilde{Z}$ is identified with the help of the Frobenius element Φ with the Galois group of the algebraic closure of the residue field (respectively, the field of constants). Then W_F is identified with the subgroup of elements of the group $\mathrm{Gal}(\bar{F}/F)$, whose image in $\hat{Z}$ lies in Z, and the topology on W_F is induced by the inclusion of W_F in $\mathrm{Gal}(\bar{F}/F) \times Z$.

The L-functions of representations of Weil groups are introduced by analogy with Artin L-series as products of certain local factors. Such L-functions admit meromorphic continuation to the entire complex plane and satisfy a functional equation with suitably defined ε-factors (cf. [228, 73]). Unfortunately, the ε-factors are explicitly defined only for one-dimensional representations. In the general case one has an existence and uniqueness theorem for ε-factors; there are also individual descriptions of ε-factors for certain classes of representations (cf. [72, 73]).

The L-functions so defined include as special cases: a) the Abelian L-series of Hecke with grössencharaktere, b) the nonabelian L-functions of Artin.

Local Langlands Conjecture. Let F be a local field, $n \geq 1$. There exists a one-to-one correspondence between classes of isomorphic irreducible n-dimensional complex representations σ of the Weil group W_F and irreducible parabolic (supercuspidal) representations π of the group $GL_n(E)$, under which the L- and $\tilde{\varepsilon}$-factors of corresponding representations $\pi \otimes \chi$ and $\sigma \otimes \chi$ coincide for any Hecke character χ (cf. [70, 148, 245, 231, 232]).

For n = 1 Langlands' conjecture is equivalent with local class field theory [70].

For n = 2 this conjecture is proved (cf. [231, 232]).

An interesting generalization of the local Langlands conjecture was made by Tate [228]. Instead of the group W_F he proposed to consider the Weil–Deligne group scheme W_F (cf. [72]), which is the semidirect product of W_F on G_a, where G_a is an additive group scheme on which W_F acts according to the rule: $wxw^{-1} = \|w\| x$, and $\|w\| = q^{-v(w)}$, q is the order of the residue field. A representation of W_F', on which the geometric Frobenius element Φ acts semisimply, we call Φ-semisimple. For such representations one can also define L- and ε-factors.

Conjecture. There exists a natural bijection between the classes of isomorphic Φ-semisimple representations of W_F' of degree n and the irreducible admissible representations of $GL_n(F)$ (not only parabolic ones!).

Such a generalization of the local Langlands conjecture is motivated by the recent results of Bernshtein and Zelevinskii [9] on representations of GL_n over a local nonarchimedean field. This conjecture is also proved in [231] for n = 2.

A new approach to the proof of Langlands' conjecture for n > 2 was proposed by Koch [134].

The local Langlands conjecture can be generalized to the case of arbitrary reductive groups [60, 162]. Here instead of complex representations of the Weil group one considers classes of homomorphisms $\alpha: W_F' \to {}^LG$ over the Galois group, satisfying certain additional conditions. If $\Phi(G)$ is the set of such classes, $\Pi(G(F))$ is the set of irreducible admissible representations of G(F), then hypothetically there exists a partition of $\Pi(G(F))$ into nonempty disjoint sets Π_φ, parametrized by elements $\varphi \in \Phi(G)$. Here the elements of a fixed set Π_φ are called L-nondistinct: to all of them correspond identical L-functions (cf. [60, 156, 206]).

The Global Langlands Conjecture can be formulated in terms of λ-adic representations. Let F be a global field with Weil group W_F. If E is a number field, $[E:Q] < \infty$ with nonarchimedean valuation λ, then a λ-adic representation is a homomorphism of topological groups $\rho_\lambda: W_F \to GL_{E_\lambda}(V_\lambda)$, where E_λ is the completion of E, V_λ is a finite-dimensional vector space over E_λ, and the group $GL_{E_\lambda}(V_\lambda)$ is considered in the λ-adic topology.

A system $\{\rho_\lambda\}$ (λ are points of the field E) is called compatible, if for almost all valuations v of the field F and almost all λ the characteristic polynomials of elements $\rho_\lambda(\Phi_v)$ do not depend on λ and have coefficients in E; here Φ_v denotes the Frobenius element $\Phi_v \in W_F$, corresponding to the nonarchimedean valuation v (cf. [228]).

The compatibility condition of representations ρ_λ allows one to define for them L-functions analogous to the Artin L-functions (cf. [228]).

A wide class of examples of λ-adic representations is connected with the action of the Galois group on points of finite order of elliptic curves and Abelian manifolds defined on F. If E is an elliptic curve, defined over Q, E_{l^n} is the kernel of the multiplication by l^n in the group of points $E(\bar{Q})$ over the algebraic closure $\bar{Q}$ of the field Q, then $E_{l^h} \cong (Z/l^h Z)^2$ as an Abelian group. The action of the Galois group $Gal(\bar{Q}/Q)$ on the Tate module $T_l(E) = \varprojlim E_{l^n} \cong Z_l^2$ gives an l-adic representation

$$\rho_l : Gal(\bar{Q}/Q) \to GL_2(Z_l),$$

which is unramified for almost all p. Here if Φ_p is the Frobenius p-element, then the trace of the matrix $\rho_l(\Phi_p)$ is equal to a_p, where $N_p = 1 - a_p + p$ is the number of points on the reduction of the curve E modulo p. The theory of l-adic cohomology makes it possible to interpret $T_l(E)$ as the one-dimensional l-adic cohomology of the curve E; a more systematic method of constructing λ-adic representations consists of considering the action of Galois groups on the λ-adic cohomology of algebraic manifolds X over F. The L-functions of such λ-adic representations are factors of the Hasse–Weil zeta function of the manifold X [71, 65, 149].

A larger class of λ-adic representations is given by the theory of motifs (cf. [72, 228]). Local nonarchimedean factors of L-functions, connected with motifs, are constructed according to Artin, at Archimedean points, using Hodge structures (cf. 72, 228]).

The global Langlands conjecture for GL_n can now be formulated as the assertion of the existence of a correspondence (bijection) between equivalence classes of systems of compatible λ-adic n-dimensional representations of the Weil group W_F, of irreducible admissible representations of $GL_n(A_F)$ and motifs of rank n over F. [228]; such a correspondence must preserve L-functions.

As in the local case too, this conjecture can be generalized to arbitrary reductive groups; here instead of n-dimensional λ-adic representations one considers continuous homomorphisms $\rho_\lambda : W_F \to {}^LG(E_\lambda)$ of the Weil group W_F into the λ-adic points of the Langlands group (as already remarked, the Langlands group ${}^LG$ can be defined over a number field).

For the group $G = GL_1$ the Langlands conjecture includes global class field theory, the generalization of which to the noncommutative case corresponds to the passage from GL_1 to GL_n, $n \geq 2$.

Drinfel'd (cf. [25, 78]) proved the Langlands conjecture for GL_2 over functional fields. We formulate his result precisely. Let F be a global field of characteristic $p > 0$. For each number field E we denote by the symbol $\Sigma_1(E)$ the set of classes of isomorphic systems of compatible absolutely irreducible two-dimensional λ-adic representations of the Weil group W_F; we set

$$\Sigma_1 = \varinjlim_{E} \Sigma_1(E).$$

By the symbol Σ_2 we denote the set of classes of isomorphic irreducible parabolic representations of GL_2, (A_F), defined over $\bar{Q}$. Let $E \subset \bar{Q}$, $[E:Q] < \infty$, $\rho = \{\rho_\lambda\} \in \Sigma_1$, $\pi \in \Sigma_2$. We call π compatible with ρ, if for some λ and almost all points v of the field F

$$L(s - 1/2, \pi_v) = \det(I - \rho_\lambda(\Phi_v) N_v^{-s})^{-1},$$

where Φ_v is the geometric Frobenius element at v, $\Phi_v \in W_F$, $L(s, \pi_v)$ is the Jacquet–Langlands L-function. Now $\rho \in \Sigma_1$ and $\pi \in \Sigma_2$ we call compatible, if for some E and $\rho_E \in \Sigma_1(E)$ ρ is the image of ρ_E and ρ_E is compatible with π. We denote by Γ the set of pairs $(\rho, \pi) \in \Sigma_1 \times \Sigma_2$ such that π is compatible with ρ.

<u>THEOREM A.</u> Γ is the graph of a bijection $\Sigma_1 \xrightarrow{\sim} \Sigma_2$ for irreducible admissible representations Σ_2.

The surjectivity of the projection $\Gamma \to \Sigma_1$ was proved earlier by Deligne [72].

The injectivity of the projection $\Gamma \to \Sigma_1$ follows from the multiplicity one theorem (cf. Sec. 1), and the injectivity of the projection $\Gamma \to \Sigma_2$ is a consequence of the theorem of Chebotarev on density of prime ideals (cf. [203]).

The proof of the surjectivity of the projection $\Gamma \to \Sigma_2$ is based on the study of interesting algebrogeometric objects, EH-bundles and their spaces of modules. Let X be a smooth projective model of the function field F considered, $\bar{X} = X_{F_q} \otimes \bar{F}_q$, FH-bundles are certain vector bundles over $\bar{X}$, provided with additional structures, connected with the action of the geometric Frobenius element. With the help of the Selberg trace formula one studies the trace of the Frobenius morphism, acting on the cohomology of schemes of modules of FH-bundles; this leads to an explicit construction of λ-adic representations, connected with π, and proves Theorem A. Such λ-adic representations are connected with motifs, arising from the one-dimensional cohomology of spaces of modules of FH-bundles [141].

THEOREM B. Let π be an irreducible unitary representation of $GL_2(A_F)$, lying in the space of parabolic forms. Then for each point v of the field F, π_v does not belong to the complementary series. (The complementary series consists of induced representations of the form $\mathrm{Ind}(\mu\otimes\nu)$, where μ and ν are quasi-characters, but not characters.)

This theorem proves the Ramanujan conjecture over F (cf. Sec. 1).

In the case when F is a number field, the general Langlands conjecture is still very far from proof even in the case n = 2. The first advance in this domain is connected with the theory of Deligne and Serre: a construction is given of a system of compatible λ-adic representations $\overset{\sim}{\to}\pi$, lying in the space of parabolic forms (in the case $F = \mathbf{Q}$). First Deligne (cf. [41]) constructed compatible systems of l-adic representations, connected with parabolic forms with respect to $SL_2(Z)$. Let $f(z)=\sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma)$, $a_1=1$, where $\Gamma = SL_2(Z)$. The space $S_k(\Gamma)$ is one-dimensional for k = 12, 16, 18, 20, 22, 26. Deligne proved that for any prime l there exists an l-adic representation

$$\rho_l: \mathrm{Gal}(K_l/\mathbf{Q}) \to GL_2(\mathbf{Z}_l)$$

(K_l is the maximal algebraic extension of Q, ramified only at l) such that for prime $p \neq l$ the image of the Frobenius p-element Fr_p has characteristic polynomial $X^2 - a_p X + p^{k-1}$. Then Ribet generalized Deligne's construction to the case of congruence-subgroups (cf. [197, 137, 225]). In the language of the theory of representations of $GL_2(A)$ the construction of λ-adic representations was suggested by Langlands [151].

The construction of parabolic forms corresponding to systems of l-adic representations on Tate modules of elliptic curves, constitutes the contents of the famous conjecture of A. Weil on the uniformization of elliptic curves E over Q by modular forms. This conjecture is proved only in certain special cases (for example, for curves with complex multiplication, cf. Gelbart's survey [90] on other results in this direction). In connection with this, we note the result of Belyi [7]: any algebraic curve, defined over an algebraic number field, admits a covering by a projective line, ramified only over the three points 0, 1, ∞, and defined over a number field; from this result it follows, in particular, that elliptic curves over Q admit uniformization by modular forms with respect to subgroups of finite index in $SL_2(Z)$ (not necessarily congruence-subgroups).

Langlands [149] suggested an approach to the explicit construction of a correspondence between automorphic representations and representations of the Weil groups, preserving L-functions. This approach is based on one category-theoretic result [199]: any Abelian category with tensor products and direct sums, provided with a "fiber functor" with values in the category of finite-dimensional vector spaces over the field F, is equivalent with the category of finite-dimensional representations of a certain reductive algebraic group over F. In particular, the existence of such a group is assumed for the category of automorphic representations of $G(A_F)$; here the above-indicated correspondence is given by a homomorphism of this group into the "algebraic hull" of the Weil group [228]. Interesting applications of this construction are connected with the study of Shimura manifolds [65, 139, 155, 149], whose zeta-functions can hypothetically be expressed in terms of L-functions of automorphic representations.

4. Lift of Automorphic Forms

In the last 10 years (starting with the paper of Doi and Naganuma [79]) there have appeared many examples of connections between different classes of automorphic forms, among them, between modular forms of one and of several variables [46, 55, 68, 84, 96, 125, 165, 183, 189, 246]. As a rule, such connections are naturally described in the language of L-functions, connected with automorphic forms. All these examples can be combined in the domain of a general principle of functoriality of automorphic forms, introduced by Langlands.

To formulate this principle, we consider a connected reductive group G over a global (or local) field F, and let ${}^LG$ be the dual Langlands group (cf. Sec. 2). As remarked, ${}^LG$ is the semidirect product of a connected reductive group ${}^LG^0$ over C (or over a number field E) by the Galois group $\mathrm{Gal}(F_1/F)$ of a certain extension F_1 of the field F:

$$1 \to {}^LG^0 \to {}^LG \overset{p_1}{\to} \mathrm{Gal}(F_1/F) \to 1.$$

If H is another connected reductive group over F, ${}^LH$ is the Langlands group,

$$1 \to {}^LH^0 \to {}^LH \overset{p_2}{\to} \mathrm{Gal}(F_2/F) \to 1,$$

then we call the homomorphism $u: {}^LH \to {}^LG$ an L-homomorphism, if the restriction of u to ${}^LH^0(C)$ is a complex-analytic homomorphism into ${}^LG^0(C)$, where $F_1 \subset F_2$ and the diagram

$$\begin{array}{ccc} {}^LH & \xrightarrow{u} & {}^LG \\ & \searrow^{\varkappa} \quad \swarrow_{p_1} & \\ & Gal(F_1/F) & \end{array}$$

is commutative (here $\varkappa$ is the composition of the projection p_2 and the natural map $Gal(F_2/F) \to Gal(F_1/F)$).

Functoriality Principle. Let F be a global field, $u: {}^LH \to {}^LG$ be an L-homomorphism, $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of the group $H(A_F)$, where for almost all v, $h_v \in {}^LH$ corresponds to the representation π_v of the group $H(F_v)$. Then there exists an irreducible admissible representation $u_*(\pi) = \pi' = \otimes_v \pi_v'$ of the group $G(A_F)$ such that for almost all v the class $h_v' = u(h_v)$ corresponds to the irreducible admissible representation $\pi_v' = u_*(\pi_v)$ of the group $G(F_v)$ (for a more precise formulation, cf. [60]).

Applying the global Langlands conjecture (cf. Sec. 3, [60]), one can also formulate the functoriality principle in the language of λ-adic representations [228].

We consider some examples.

The first two examples are connected with extension of the ground field F. Let F'/F be a finite extension, G' be a connected reductive group over F, $G = R_{F'/F}G'$ be the connected reductive group over F, obtained from G' by restriction of scalars a la Weil (here for any commutative F-algebra B

$$G(B) = G'(B'), \text{ где } B' = B \otimes_F F').$$

Then the L-group ${}^LG$ can be obtained with the help of induction from the group ${}^LG'$ [60]. More precisely, we consider in the group $\Gamma = Gal(F^s/F)$ the subgroup of finite index $\Gamma_1 = Gal(F^s/F')$, corresponding to the extension F'. The group Γ acts on ${}^LG^0$, and the group Γ_1 on ${}^LG'^0$ (cf. Sec. 2). We consider the induced group

$$\mathrm{Ind}_{\Gamma_1}^{\Gamma}({}^LG'^0) = \prod_{\sigma \in \Gamma_1 \setminus \Gamma} {}^LG'^0,$$

which is the product of $|\Gamma_1 \setminus \Gamma|$ copies of the group ${}^LG'^0$. Here on $\mathrm{Ind}_{\Gamma_1}^{\Gamma}({}^LG'^0)$ the group Γ acts with the help of the action of Γ on the set of right cosets (permutation of factors) and the action of Γ_1 on each factor. Then there exists a natural isomorphism (preserving the action of Γ):

$${}^LG^0 \xrightarrow{\sim} \mathrm{Ind}_{\Gamma_1}^{\Gamma}({}^LG'^0).$$

1) Change of Basis. Let F'/F be a finite Galois extension, H be an F-splitting reductive group over F, $G = R_{F'/F}H$. We consider the natural L-homomorphism $u: {}^LH \to {}^LG$, whose restriction to ${}^LH^0$ is the diagonal inclusion. In this case $G(A_F)$ and $G(F)$ are canonically isomorphic with $H(A_{F'})$ and $H(F')$, so the functoriality principle goes into the following problem: to connect with an automorphic representation of $H(A_F)$ an automorphic representation of $H(A_{F'})$.

This problem is solved by Langlands for $H=GL_2$ and a cyclic extension F'/F of prime degree [97, 140]; a description is given of the image and fibers of the map u_*. This result generalizes the preceding investigations of Doi and Naganuma [79, 41], Jacquet [121] (in the case of quadratic extensions F'/F), Saito [200], Shintani [220, 44] (cf. also other descriptions of u_* in [54, 55, 68, 84]).

Here the connection between π and $\pi' = u_*\pi$ is intuitively interpreted with the help of λ-adic representations $\rho_\lambda': W_{F'} \to GL_2(F_\lambda)$, are obtained with the help of the restriction

$$(Gal(F^s/F') \hookrightarrow Gal(F^s/F))$$

of the representations $\rho_\lambda: W_F \to GL_2(E_\lambda)$, corresponding to π, to the subgroup.

2) L-series with grössencharaktere. Let F'/F be a Galois extension of degree n, $H = R_{F'/F}GL_1$, $G = GL_n$. Here ${}^LH = {}^LH^0 \times Gal(F'/F)$, ${}^LG_0 = GL_n(C)$, and there exists a natural inclusion $u: {}^LH \hookrightarrow {}^LG$, under which ${}^LH^0 \cong GL_1 \times \ldots \times GL_1$ goes into the subgroup of diagonal matrices (maximal torus T in $GL_n(C)$, and $Gal(F'/F)$ into the subgroup of permutation matrices (more precisely, into the normalizer of the maximal torus). Automorphic representations of H are identified with Hecke characters $A^*_{F'}$: $H(A_F) = GL_1(A_{F'}) = A^*_{F'}$.

The functoriality principle reduces in this case to the question of whether L-series $L(s, \chi)$ with grössencharaktere χ are L-functions of automorphic representations π of the group G.

If $n = 2$, $F = \mathbf{Q}$, E' is imaginary quadratic, then this was proved by Hecke [114], here π is connected with a parabolic automorphic form. If $n = 2$, $F = \mathbf{Q}$, F' is real quadratic, then Maass proved that π is connected with a nonholomorphic automorphic form [163, 166]. For $n = 3$ this is proved in [125], cf. also [60].

3) $H = GL_2$, $G = GL_3$, ${}^LH = GL_2(C)$, ${}^LG = GL_3(C)$ $u: GL_2(C) \to GL_3(C)$ is the adjoint representation (or symmetric square of the standard one). In this case the problem of lifting is solved by Gelbart and Jacquet [93].

In the language of λ-adic representations this result means that λ-adic representations corresponding to $u_*\pi$ are symmetric squares of λ-adic representations corresponding to π.

4) Let M be a Levi F-subgroup of a parabolic F-subgroup $P \subset G$. Then ${}^LM$ is naturally imbedded in ${}^LG$, and the imbedding $u: {}^LM \to {}^LG$ is an L-homomorphism [60].

Langlands [152] constructed in a large number of cases $u_*(\pi)$ for parabolic representations π of the group $M(A_F)$ with the help of analytic continuation and residues of Eisenstein series.

5) Let $H = PGL_2 \times PGL_2$, $G = Sp_4$. In this case

$$^{L}H = SL_2(C) \times SL_2(\mathbf{C}),\ ^{L}G = Sp_4(\mathbf{C}) \cong Spin_5(\mathbf{C}) .$$

We consider the homomorphism $u: SL_2(C) \times SL_2(C) \to Sp_4(C)$ defined by the formula:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_2 & 0 & c_2 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}.$$

If $\pi_1 = \otimes_v \pi_{1,v}$, $\pi_2 = \oplus_v \pi_{2,v}$ are two irreducible admissible automorphic representations of the group $PGL_2(A_F)$, then they determine a representation $\pi_1 \otimes \pi_2$ of the group $H(A_F)$, and according to the functoriality principle there must exist an irreducible automorphic representation $u_*(\pi_1 \otimes \pi_2)$ of the group $Sp_4(A_F)$.

Andrianov and Maass [46, 167] considered the case when $F = \mathbf{Q}$, π_1 is connected with some parabolic form $f(z) = \sum_{n=2}^{\infty} \omega(n) q^n$ of weight $2k - 2$ with respect to $SL_2(Z)$, characteristic with respect to the Hecke operators:

$$\sum_{n=1}^{\infty} \omega(n) n^{-s} = \prod_p [1 - \omega(p) p^{-s} + p^{2k-3-2s}]^{-1},$$

and π_2 is connected with the (nonholomorphic) Eisenstein series of weight 2:

$$E_2(z) = \frac{1}{8\pi y} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

where $\sigma_1(n) = \sum_{d/n} d.$

To the representations $\pi_{1,p}$ correspond classes $h_{1,p} = \begin{pmatrix} \alpha_p & 0 \\ 0 & \alpha_p^{-1} \end{pmatrix} \in SL_2(C)$, where $(1 - \alpha_p p^{-s})(1 - \alpha_p{}^{-1} p^{-s}) = 1 - \omega(p) p^{-s-(k-1)/2} + p^{-2s}$, and to the representations $\pi_{2,p} = \begin{pmatrix} p^{1/2} & 0 \\ 0 & p^{-1/2} \end{pmatrix}$, classes (cf. Sec. 1). It is proved that the representation $\pi' = u_*(\pi_1 \otimes \pi_2)$ exists, is given by the Siegel modular parabolic form Φ of genus 2 of weight k with respect to $Sp_4(Z)$, where the L-function of the representation π' coincides with the zeta-function of the form Φ: $Z_\Phi(s) = L(s - k + 3/2, \pi', \rho_2)$ (cf. Sec. 2). Here the image of u_* is completely described for $f \in S_{2k-2}(SL_2(Z))$, which allows one to formulate the result in the form of a certain theorem about modular descent.

More precisely, in the space M_k^2 of Siegel modular forms of weight $k > 0$ with respect to $Sp_4(Z)$ one considers the subspace D_k, consisting of parabolic forms

$$\Phi(Z) = \sum_{N \in \mathfrak{N}_2} A(N)\, e(\mathrm{Tr}(NZ)),$$

satisfying Maass' condition [167]:

$$A\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = \sum_{g/a,\, b,\, c;\, g>0} g^{k-1} A\begin{pmatrix} 1 & b/2g \\ b/2g & ac/g^2 \end{pmatrix}.$$

It is proved that D_k is invariant with respect to all Hecke operators, acting in M_k^2. Whence it follows that there exists a basis $\Phi_1, \Phi_2, \ldots, \Phi_r$ in D_k, consisting of eigenfunctions of the Hecke operators. Let Φ be one of these forms $Z_\Phi = \prod_p Q_{p,\Phi}(p^{-s})^{-1}$.

THEOREM. For any prime number p,

$$Q_{p,\Phi}(t) = (1 - p^{k-2}t)(1 - p^{k-1}t)(1 - \omega(p)t + p^{2k-3}t^2).$$

If one defines the numbers $\omega(n)$ ($n = 1, 2, \ldots$) by

$$\sum_{n=1}^{\infty} \omega(n) n^{-s} = \prod_p [1 - \omega(p)p^{-s} + p^{2k-3-2s}]^{-1},$$

then the series

$$f(z) = \sum_{n=1}^{\infty} \omega(n)\, e(nz)$$

defines a parabolic form of weight $2k - 2$ with respect to $SL_2(Z)$.

This theorem proves the Kurokawa conjecture [146], introduced in connection with the study of the eigenvalues of the Hecke operators, acting on the Siegel parabolic forms of genus 2, and the generalized Ramanujan–Petersson conjecture on the estimation of such eigenvalues. For parabolic forms Φ of arbitrary genus n with respect to Γ_n of weight k, characteristic with respect to the Hecke operators: $T_k(m)\Phi = \lambda_\Phi(m)\Phi$, this generalization consists of the following. Let $Q_{p,\Phi}(t)$ be a polynomial of degree 2^n giving a p-factor of the zeta-function $Z_\Phi(s)$ of the form Φ: $Z_\Phi(s) = \prod_p Q_{p,\Phi}(p^{-s})^{-1}$. Then Φ satisfies the generalized Ramanujan conjecture if the absolute values of the zeros of the polynomial $Q_{p,\Phi}(t)$ are equal to $p^{-n(2k-n-1)/4}$ for all p. Kurokawa gave examples of parabolic forms of genus 2, which do not satisfy the generalized Ramanujan conjecture and assumed that such forms are obtained with the help of lifting parabolic forms of genus 1, described above.

In connection with example 5) we note the papers of Yoshida [244, 246], proposing an explicit construction, putting in correspondence a pair of certain modular forms of genus 1 with a Siegel modular form of genus 2. This construction is based on the use of theta-series, connected with a definite quaternion algebra over Q.

Maas [165] proved an analog of Kurokawa's conjecture for Siegel modular forms with respect to $Sp_4(Z)$ with systems of multiplicators. In this case the corresponding one-dimensional parabolic forms are automorphic forms with respect to the Hecke group $G(\sqrt{2})$, generated by the elements

$$\begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

As already noted (cf. Sec. 1), the generalized Ramajujan conjecture can be reformulated in the language of representation theory. According to this interpretation [119], the local constituents π_v of parabolic representations $\pi = \otimes_v \pi_v$ of the reductive group G must be moderately increasing (this is a condition on the growth of the matrix coefficients of the representation). In the case of anisotropic groups, however, this conjecture is not corroborated [119]. In the case of splitting groups also there are counterexamples: in [119] such examples are given for $G = Sp_4$; the construction is based on the theory of dual reductive pairs and the Weil representations [91, 118, 141, 190, 224]: the consideration of the pair (Sp_4, O_2) gives an imbedding of automorphic forms on O_2 in automorphic forms on Sp_4.

The construction of [119] is parallel to the construction of automorphic forms, corresponding to Hecke characters of quadratic fields, which corresponds to the consideration of the dual pair (SL_2, O_2) (cf. Example 2).

An interesting method of lifting automorphic forms with the help of theta-functions was proposed also by Oda [183] and Kudla [143]. We consider an indefinite quadratic form A (over the field Q of rational numbers) of signature (p, q): $A: Y \to A[Y] = {}^tYAY$, $Y \in Q^n$. Let $L \subset Q^n$ be a Z-lattice, $A(L) \subseteq 2Z$ and let X be the space of all majorants of the quadratic form A, i.e., those matrices $R \in M_n(R)$, $n = p + q$, such that ${}^tR = R$, R is positive definite and $RA^{-1}R = A$. We set $SO(A) = \{g \in SL_n(R) \mid {}^tgAg = A\}$, $\Gamma_L = \{U \in SO(A) \mid UL = L\}$, so X is a symmetric space of the orthogonal group SO(A), and $SO(A) \cap SO_n(R)$ is a maximal compact subgroup in SO(A). We consider the theta-function

$$\theta(z, R) = v^{q/2} \sum_{l \in L} e^{i\pi(uA + ivR)[l]}$$

of two variables $(z, R) \in H \times X$, where $z = u + iv \in H$ $v > 0$. Then

$$\theta(z, R[U]) = \theta(z, R), \quad \text{if} \quad U \in \Gamma_L;$$

$$\theta(\gamma z, R) = (cz + d)^{\frac{p-q}{2}} \theta(z, R),$$

if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \,\middle|\, a \equiv d \equiv 1 \bmod N,\ c \equiv 0 \bmod N \right\},$$

$$N = 2 \det A.$$

Now if $\varphi(z)$ is a function on H and $\psi(R)$ is a function on X, which transform with respect to z and R just like $\theta(z, R)$, then the integrals

$$\theta_\varphi(R) = \int\limits_{H/\Gamma_1(N)} \varphi(z) \overline{\theta(z, R)} v^{(p-q)/2} \frac{du dv}{v^2},$$

$$\theta^\psi(z) = \int\limits_{X/\Gamma_L} \psi(R) \theta(z, R)\, dR$$

define maps of the spaces of modular forms on H (with respect to $\Gamma_1(N) \subset SL_2(Z)$ into the space of automorphic forms on X (with respect to Γ_L) and conversely [143]. The case $(p, q) = (2, n-2)$ is analyzed in more detail in [183]; for $n = 4$ as a consequence one gets the results of Zagier [248] on change of basis for GL_R over a real quadratic extension of Q.

A powerful instrument in the study of the lift of automorphic forms is the Selberg trace formula [149, 155]; this formula is the strongest generalization of the connection between characters of irreducible representations and conjugacy classes of elements, well-known for finite groups. For GL_2 over a global field the trace formula is in the book [123], Sec. 6; a better account (in a more general situation) is given by Gelbart and Jacquet [94].

We note the papers [11, 141, 227], where the case of GL_2 over an imaginary quadratic field is analyzed in more detail, and also [110, 120, 140, 97, 201], where GL_2 over a completely real field is considered. For parabolic forms of type (N, k, χ) (cf. Sec. 1) an elegant account of the trace formula for Hecke operators is given by Zagier (in Lang's book [33], N = 1) and Oesterle [182]. Arthur [52, 53] studied the generalization of the trace formula to the case of arbitrary reductive groups; for $G = GL_3$ interesting results in this direction were obtained earlier by Venkov [10]. Trace formulas are closely connected with analytic continuation of Eisenstein series [29, 15, 152]. One of the most important applications of the trace formula is the calculation of the dimensions of spaces of automorphic forms [69, 117, 179, 202]; an interesting application is connected with divisibility properties of coefficients of modular forms [136, 137, 138]. For other applications, connected, in particular, with analytic properties of the Selberg zeta-function, cf. [11-13, 22, 31, 115, 234].

In conclusion we note the most interesting results, in our view, relating to other aspects of the theory of modular forms, not entering into this survey.

1) Modular forms of half-integral weight [41, 204, 135, 233] and their interpretation as automorphic forms on the metaplectic group (two-sheeted covering of SL_2, [92, 95, 96]. Generalization to the case of n-sheeted coverings of SL_2 of the results of Shimura [41]: [51, 87].

Solution of Kummer's problem on the distribution of the sign of cubical Gaussian sums [113] with the help of the cubical analog of theta-series, automorphic forms on the three-sheeted covering of SL_2 [74, 184].

The connection of automorphic forms on the metaplectic group and on orthogonal groups [189].

2) The generation of spaces of modular forms by theta-series [81, 103, 230, 236]. The solution of the basis problem in spaces of modular forms of type (N, k, χ) [83, 116], allowing one to give an algorithm for finding a basis in such spaces (in the form of an algorithm for ECM) [187].

The theory of theta-series, connected with indefinite quadratic forms [163, 235], theta-series of genus n [40, 132, 164], and also theta-series of completely real fields [82, 84].

Connection with quadratic forms; new formulas for the number of representations of integers by quadratic forms [8, 14, 21, 34, 106, 208, 226].

3) The values of L-series at integral points. The construction of p-adic L-functions of Jacquet–Langlands for GL_2 over a completely real field: cf. the surveys of Yu. I. Manin [35, 169]. The case of GL_2 over an imaginary quadratic field is analyzed by Kurchanov [32].

The development of the theory of nonarchimedean integration [16, 18, 35, 129, 130, 161] and the theory of modular symbols [42, 56, 112, 241], applications to the arithmetic of modular curves [171, 172]; the generalization to the case $G = GL_n$ [56].

The transcendence of periods of parabolic forms [59]. Integrals of Eisenstein series and values of the Riemann zeta-function $\zeta(s)$ at odd positive points [102].

The method of Zagier [247]. The values at integral points of L-series with Hecke characters [218, 219] and their generalizations; the connection with the arithmetic of modular forms of Hilbert and automorphic forms on unitary and orthogonal groups [207, 209, 212, 214-217]. Applications of Rankin's method of convolution to the calculation of values of L-functions at integral points [36-39, 100, 109, 213, 216, 217, 222].

Generalization of the Chowla-Selberg formula [66] for calculating periods of integrals of Abelian manifolds with complex multiplications [105, 242].

General conjecture on the values of L-functions made by Deligne [75].

4) Congruence and divisibility of coefficients of modular forms and modular functions. Connection with l-adic representations [44, 77, 80, 111, 133, 136-138, 192, 197, 198, 222, 225]. Generalization to the case of Siegel modular forms [145, 178].

5) Modular forms and analytic number theory: cf. the survey of Moreno [176], and also [12, 15, 22, 28, 31, 45, 98-100, 177].

6) Modular forms in positive characteristic and nonarchimedean modular forms [24, 25, 104, 108, 131, 136, 157].

7) Connection with the theory of simple finite groups (in particular, with the "Fisher–Griss monster") and the description of their characters: [127, 128, 229]. The connection of modular forms with coding theory, lattices, and packings of spheres, cf. [14], the survey of Mahler [168], and the literature cited in it.

8) Differentiation of modular forms. Constructions of nonlinear differential operators, acting in spaces of modular forms (in the spirit of the papers of Kuznetsov [30] and Rankin [193]): [43, 85].

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